

Topological zero modes from the interplay of $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry

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Parity-time symmetry and anti-parity-time symmetry are complementary but generally incompatible. In this work, we investigate the coexistence of parity-time symmetry and anti-parity-time symmetry. These symmetries impose intriguing constraints on the spectrum, and the Bloch bands appear in opposite and conjugate tetrads. The bulk states, affected by the non-Hermiticity, are entirely complex; however, the topological edge states, protected by sublattice symmetry, are exactly zero. We demonstrate a two-leg ladder consisting of two Su-Schrieffer-Heeger chains connected by radiative and dissipative cross-couplings. The non-Hermiticity facilitates the formation of a $2\mathbb{Z}$ point gap topology, and the topological zero modes are robust against disorder.

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I. INTRODUCTION

Non-Hermiticity reshapes fundamental principles for topological phases of matter [1–6]. In non-Hermitian systems, the Bloch Hamiltonian may be unable to predict the edge states, and the bulk-boundary correspondence breaks down [7,8]. Under the open boundary condition (OBC), the imaginary gauge field causes the non-Hermitian skin effect [9], and all the eigenstates localize on the boundaries. Under the periodic boundary condition (PBC), the imaginary gauge field causes the non-Hermitian Aharonov-Bohm effect [10,11], and the spectrum becomes complex. Consequently, the spectrum under the PBC differs significantly from the spectrum under the OBC. To recover bulk-boundary correspondence, a non-Bloch topological invariant was developed [12–18]. On the complex energy plane, the braidings of complex bands reveal rich linked or knotted topology [19–23]. Additionally, non-Hermiticity increases the diversity of the band gap closing [24–34]. A topological phase transition may occur at the degenerate points (DPs) and also may occur at the exceptional points (EPs) [35–42]. The band gaps are classified as point gaps or line gaps. The tenfold Altland-Zirnbauer symmetry classification is generalized to the 38 symmetry classification [43,44].

Beyond the discrete symmetries for topological classification, parity-time ($\mathcal{PT}$) symmetry and anti- $\mathcal{PT}$ symmetry are pivotal in non-Hermitian physics [45–67]. In the $\mathcal{PT}$ -symmetric Su-Schrieffer-Heeger (SSH) model [68–71], the Berry phase associated with each band is quantized in an unbroken $\mathcal{PT}$ -symmetric phase and no longer quantized in

the broken $\mathcal{PT}$ -symmetric phase. Switching the gain and loss elements in every other unit cell of the $\mathcal{PT}$ -symmetric SSH model transforms the system into an anti- $\mathcal{PT}$ -symmetric SSH model, where the non-Hermiticity constructs and even induces nontrivial topology independently [72–75]. Notably, the $\mathcal{PT}$ -symmetric (anti- $\mathcal{PT}$ -symmetric) Hamiltonian commutes (anticommutes) with the combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) operator. In general, $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry cannot coexist, and they are studied separately [76–79]. However, the $\mathcal{P}$ operator is not uniquely defined, a subtlety that permits the simultaneous realization of both $\mathcal{PT}$ and anti- $\mathcal{PT}$ symmetries within a single non-Hermitian system.

In this work, we propose the coexistence of $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry, which imposes a unique spectral structure, $\{E, E^*, -E, -E^*\}$, in opposite and conjugate tetrads. The combination of $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry yields sublattice symmetry that guarantees quantized Zak phases associated with the Bloch bands. Remarkably, the degenerate edge modes are pinned to zero energy, although the bulk bands are always complex. A non-Hermitian two-leg ladder composed of two prototypical SSH chains coupled by radiative and dissipative cross-couplings is demonstrated. The non-Hermiticity helps with the formation of a $2\mathbb{Z}$ nontrivial topology, and the closing of a point band gap at zero energy signifies a topological phase transition. Moreover, $\mathcal{PT}$ symmetry, anti- $\mathcal{PT}$ symmetry, and sublattice symmetry exhibit robustness against coupling disorder. Intriguingly, disorder can independently induce nontrivial topology and zero-energy edge states.

II. MODEL

We consider a non-Hermitian two-leg ladder, as schematically shown in Fig. 1(a). The solid gray dots are the sites, and the lines and arrows are the couplings between the sites. The ladder legs are prototypical SSH chains with opposite couplings. The ladder rungs are a pair of non-Hermitian

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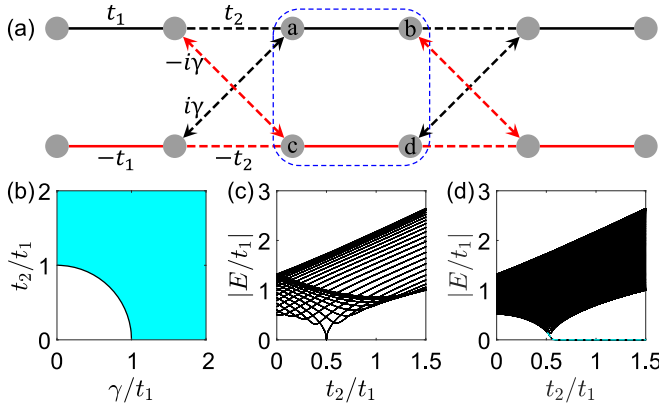


FIG. 1. (a) Schematic of a non-Hermitian two-leg ladder. The top leg and bottom leg are SSH chains connected by radiative and dissipative cross-couplings. (b) Phase diagram. The white (cyan) region denotes the topologically trivial (nontrivial) phase. (c) Spectrum under the PBC. (d) Spectrum under the OBC, where the dashed cyan curve highlights the fourfold degenerate topological zero modes. The parameters are $t_1 = 1$ and $\gamma = \sqrt{3}/2$.

cross-couplings, which present in a staggered manner. Cross-couplings with different orientations are conjugations.

The Hamiltonian of the two-leg ladder reads

$$H = H_t + H_c + H_b. \quad (1)$$

The ladder legs have the form

$$H_t = \sum_j (t_1 a_j^\dagger b_j + t_2 b_j^\dagger a_{j+1} + \text{H.c.}), \quad (2)$$

$$H_b = - \sum_j (t_1 c_j^\dagger d_j + t_2 d_j^\dagger c_{j+1} + \text{H.c.}), \quad (3)$$

and the ladder rung has the form

$$H_c = \sum_j i\gamma (a_{j+1}^\dagger d_j + d_j^\dagger a_{j+1}) - i\gamma (b_j^\dagger c_{j+1} + c_{j+1}^\dagger b_j), \quad (4)$$

where the creation and annihilation operators are $a_j^\dagger$ ($b_j^\dagger$) and a_j (b_j) for site j of sublattice A (B) on the top chain and the creation and annihilation operators are $c_j^\dagger$ ($d_j^\dagger$) and c_j (d_j) for site j of sublattice C (D) on the bottom chain. t_1 and t_2 are the nearest-neighbor couplings. $i\gamma$ and $-i\gamma$ are the radiative and dissipative next-nearest-neighbor cross-couplings. Notably, the couplings are all reciprocal; consequently, the Hamiltonian of the two-leg ladder is reciprocal, $H = H^T$.

The non-Hermitian two-leg ladder has a four-site unit cell, as indicated by the dashed blue square in the schematic. Applying the Fourier transformation $O_k = N^{-1/2} \sum_j e^{-ikj} O_j$ to sublattices $O = A, B, C, D$, the real-space Hamiltonian of the two-leg ladder [Eq. (1)] is rewritten into the Bloch Hamiltonian in momentum space $H = \sum_k H_k$, where the momentum is $k = 2\pi j/N$ and N is the total number of unit cells. In the basis $\{a_k^\dagger|\text{vac}\rangle, b_k^\dagger|\text{vac}\rangle, c_k^\dagger|\text{vac}\rangle, d_k^\dagger|\text{vac}\rangle\}$, the Bloch

Hamiltonian of the two-leg ladder has the form

$$H_k = \begin{pmatrix} 0 & t_1 + t_2 e^{-ik} & 0 & i\gamma e^{-ik} \\ t_1 + t_2 e^{ik} & 0 & -i\gamma e^{ik} & 0 \\ 0 & -i\gamma e^{-ik} & 0 & -t_1 - t_2 e^{-ik} \\ i\gamma e^{ik} & 0 & -t_1 - t_2 e^{ik} & 0 \end{pmatrix}. \quad (5)$$

Symmetries play important roles in topological physics. Notably, the Bloch Hamiltonian H_k has $\mathcal{PT}$ symmetry, defined by $(\mathcal{P}_S \mathcal{T}) H_k (\mathcal{P}_S \mathcal{T})^{-1} = H_k$, with the unitary operator $\mathcal{P}_S = \sigma_0 \otimes \sigma_x$, and the Bloch Hamiltonian H_k also has anti- $\mathcal{PT}$ -symmetry, defined by $(\mathcal{P}_A \mathcal{T}) H_k (\mathcal{P}_A \mathcal{T})^{-1} = -H_k$, with the unitary operator $\mathcal{P}_A = i\sigma_0 \otimes \sigma_y$, where $\mathcal{T} = \mathcal{K}$ is the complex conjugation operation, the Pauli matrices for spin-1/2 are $\sigma_{x,y,z}$, and the identity matrix is σ_0 . From the reciprocity, defined by $\mathcal{P}_R H_k^T \mathcal{P}_R^{-1} = H_k$, with the unitary operator $\mathcal{P}_R = \sigma_y \otimes \sigma_y$, $\mathcal{PT}$ symmetry equals the pseudo-Hermiticity $(\mathcal{P}_S \mathcal{P}_R) H_k^\dagger (\mathcal{P}_S \mathcal{P}_R)^{-1} = H_k$, and anti- $\mathcal{PT}$ symmetry equals the anti-pseudo-Hermiticity $(\mathcal{P}_A \mathcal{P}_R) H_k^\dagger (\mathcal{P}_A \mathcal{P}_R)^{-1} = -H_k$. Furthermore, the combination of $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry further gives rise to a sublattice symmetry defined by $\mathcal{S} H_k \mathcal{S}^{-1} = -H_k$, with the unitary operator $\mathcal{S} = \sigma_0 \otimes \sigma_z$.

These symmetries impose intriguing constraints on the energy spectrum of the Bloch Hamiltonian H_k . $\mathcal{PT}$ symmetry ensures E_k and E_k^* in pairs, anti- $\mathcal{PT}$ -symmetry ensures E_k and $-E_k^*$ in pairs, and sublattice symmetry ensures E_k and $-E_k$ in pairs. Consequently, the four Bloch bands appear in opposite and conjugate tetrads,

$$E_k, E_k^*, -E_k, -E_k^*. \quad (6)$$

The diagonalization of H_k yields the Bloch band energy, which has the explicit form $E_k = \alpha_k \alpha_{-k}$, with $\alpha_k = \sqrt{t_1 + t_2 e^{ik} + i\gamma e^{ik}}$. The band gap closing determines the topological phase transition, and the band gap closing is obtained from the relative position of the four bands in the complex plane. When $t_2 > t_1$, the four bands are separated. When $t_2 < t_1$, bands E_k and $-E_k$ touch bands E_k^* and $-E_k^*$ ($-E_k^*$ and E_k^*), respectively, on the real (imaginary) axis of the complex plane for $t_2^2 + \gamma^2 < t_1^2$ ($t_2^2 + \gamma^2 > t_1^2$) at momentum $\pm k_0$. Notably, $k_0 = \arccos(-t_2/t_1)$ is independent of non-Hermiticity γ . However, the momenta $\pm k_0$ are EPs when $t_2^2 + \gamma^2 = t_1^2$, where the four bands touch at zero energy; otherwise, $\pm k_0$ are DPs. The Bloch bands are fully complex except for the two DPs or EPs $\pm k_0$ when $t_2 \leq t_1$. At the DPs (EPs) $\pm k_0$, the two degenerate (coalesced) band energies are $\pm E_{k_0} = \pm \sqrt{t_1^2 - t_2^2 - \gamma^2}$, and they also become complex when $t_2 > t_1$.

The topological phase diagram in Fig. 1(b) shows the trivial and nontrivial regions, distinguished by the topological phase transition boundary at $t_2^2 + \gamma^2 = t_1^2$. Spectra under the PBC and OBC are depicted in Figs. 1(c) and 1(d), respectively. The absolute values of the energy highlight the band gap closing $E_{k_0} = 0$ for the topological phase transition. In the topologically nontrivial phase, zero-energy edge modes are indicated by the dashed cyan curve.

Figure 2 shows the spectra for fixed t_1 and γ for a series of distinct t_2 . The top panels show the four complex energy bands under the PBC as a function of the momentum k , and

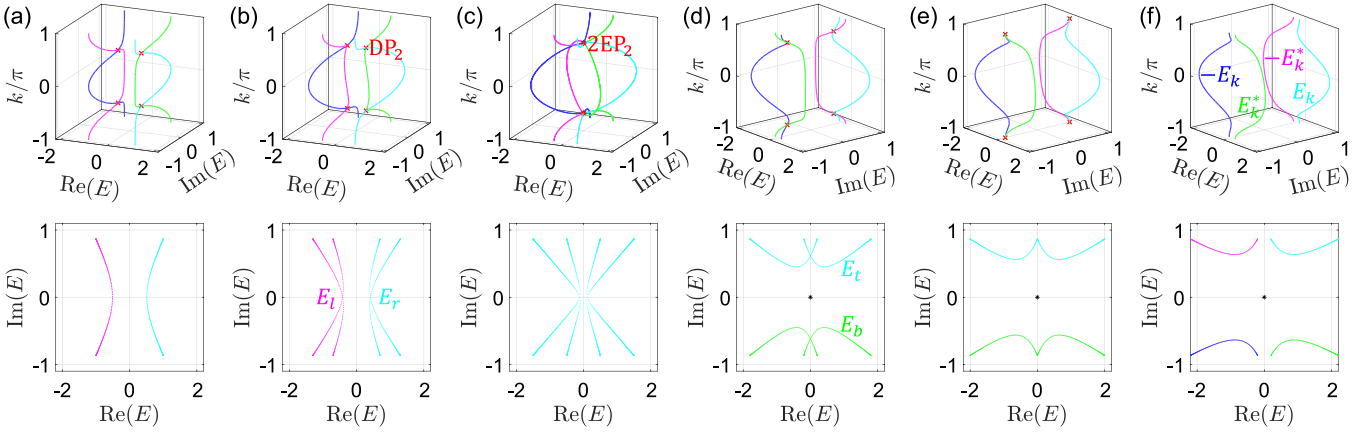


FIG. 2. Spectrum in the complex energy plane. The top row shows the spectrum under the PBC, where the red crosses and red asterisks denote the DP and EP, respectively. The bottom row shows the spectrum under the OBC, where the black asterisk marks the zero-energy edge modes with fourfold degeneracy. The parameters are $t_1 = 1$, $\gamma = \sqrt{3}/2$, and $t_2 = 0$ for (a), $t_2 = 0.3$ for (b), $t_2 = 0.5$ for (c), $t_2 = 0.8$ for (d), $t_2 = 1$ for (e), and $t_2 = 1.2$ for (f).

the bottom panels exhibit their corresponding OBC spectra projections on the complex plane $\text{Re}(E) - \text{Im}(E)$. In the complex plane, the spectra are mirror reflection symmetric about the line $\text{Re}(E) = 0$ and the line $\text{Im}(E) = 0$. The energy bands are separated into two parts (partly complex with DPs), become inseparable (the topological phase transition), and are separated into four parts (entirely complex). The topological phase transition is related to the band gap closing at zero energy. At the topological phase transition, the four bands of the two-leg ladder are connected at the EPs $\pm k_0$, and the band energies at the DPs change from being real on the line $\text{Im}(E) = 0$ to being imaginary on the line $\text{Re}(E) = 0$ and vice versa.

III. TOPOLOGICAL PHASES

Discrete symmetries for random matrices classify non-Hermitian topological phases and determine their types of topological invariants [43]. We discuss the discrete symmetries in the framework of non-Hermitian topological classification. Besides sublattice symmetry, the two-leg ladder also has time-reversal symmetry (TRS), particle-hole symmetry (PHS), $\text{TRS}^\dagger$, and $\text{PHS}^\dagger$. To be specific, the Bloch Hamiltonian H_k has TRS, defined by $T_+ H_k^* T_+^{-1} = H_{-k}$, with the unitary operator $T_+ = i\sigma_y \otimes \sigma_z$ and $T_+ T_+^* = -1$. The Bloch Hamiltonian H_k has PHS, defined by $C_- H_k^T C_-^{-1} = -H_{-k}$, with the unitary operator $C_- = \sigma_0 \otimes \sigma_z$ and $C_- C_-^* = 1$. The Bloch Hamiltonian H_k has $\text{TRS}^\dagger$, defined by $C_+ H_k^T C_+^{-1} = H_{-k}$, with the unitary operator $C_+ = \sigma_0 \otimes \sigma_0$ and $C_+ C_+^* = 1$. The Bloch Hamiltonian H_k has $\text{PHS}^\dagger$, defined by $T_- H_k^* T_-^{-1} = -H_{-k}$, with the unitary operator $T_- = i\sigma_y \otimes \sigma_0$ and $T_- T_-^* = -1$. Thus, the Bloch Hamiltonian H_k belongs to the DIII class with a point band gap in the non-Hermitian topological classification, and the topological phase is characterized by a $2\mathbb{Z}$ topological invariant.

To reveal the band topology in a geometric picture, we employ the analytic continuation method, in which the topology of the two-leg ladder is characterized by the geometry of its Bloch Hamiltonian winding around the degeneracy point

in a two-dimensional parameter space. Using this method, we demonstrate how a topologically trivial phase can evolve into a topologically nontrivial phase and vice versa. We replace the terms $e^{\pm ik}$ in the expression of the Bloch Hamiltonian H_k [Eq. (5)] by the terms $h_x \pm ih_y$, where independent parameters h_x and h_y are real numbers. The obtained non-Hermitian Hamiltonian $H(h_x, h_y)$ is the analytic continuation of the Bloch Hamiltonian H_k in the two-dimensional parameter space (h_x, h_y) . Notably, the four energy bands of the analytic continuation Hamiltonian $H(h_x, h_y)$ are

$$E(h_x, h_y), E^*(h_x, h_y), -E(h_x, h_y), -E^*(h_x, h_y). \quad (7)$$

The explicit form is $E(h_x, h_y) = \alpha(h_x, h_y)\alpha(h_x, -h_y)$, with $\alpha(h_x, h_y) = \sqrt{t_1 + t_2(h_x + ih_y)} + i\gamma(h_x + ih_y)$. The top row of Fig. 3 depicts the complex energy bands $E(h_x, h_y)$

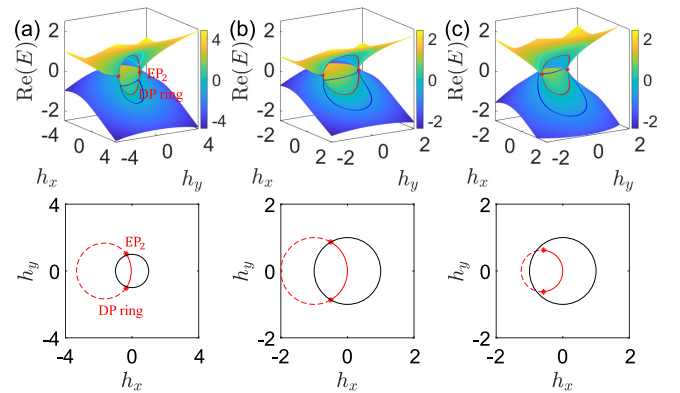


FIG. 3. The top row depicts the energy bands of $H(h_x, h_y)$. The value (color) represents the real (imaginary) part. The bottom row illustrates the topology from the geometric picture. The black unit circles indicate the Bloch Hamiltonian H_k , the red rings mark the DP ring, and the red asterisks denote the EP. The dashed part of the DP ring indicates that the band energy at the DP is imaginary, and the solid part of the DP ring indicates that the band energy at the DP is real. The parameters are $t_1 = 1$, $\gamma = \sqrt{3}/2$, and $t_2 = 0.3$ for (a), $t_2 = 0.5$ for (b), and $t_2 = 0.8$ for (c).

and $-E(h_x, h_y)$. The other two energy bands, $E^*(h_x, h_y)$ and $-E^*(h_x, h_y)$, being the complex conjugations of $E(h_x, h_y)$ and $-E(h_x, h_y)$, are not shown. The main features of $H(h_x, h_y)$ and H_k are encoded in the spectra.

The bottom row of Fig. 3 shows a geometric picture of the band topology; a topological phase transition process from trivial to nontrivial topology is displayed. In the parameter space (h_x, h_y) , the Bloch Hamiltonian H_k is a unit circle $h_x^2 + h_y^2 = 1$ centered at $(0,0)$. The unit circle winding around a pair of EPs corresponds to nontrivial topology. The pair of EPs is determined from the band gap closing $E(h_x, h_y) = 0$, given by $(h_x, h_y)_{\text{EP}} = [-t_1 t_2 / (t_2^2 + \gamma^2), \pm t_1 \gamma / (t_2^2 + \gamma^2)]$. The pair of EPs locates on a DP ring $(h_x, h_y)_{\text{DP}} = [(-t_1 \pm \sqrt{t_1^2 - 4t_2^2 h_y^2}) / 2t_2, h_y]$ under the condition $|h_y| \leq |t_1 / 2t_2|$. Separated by the pair of EPs, the degenerate levels on the left side of the DP ring are purely imaginary (marked with red dashed lines in Fig. 3), the degenerate levels on the right side of the DP ring are purely real (marked with red solid lines), and the degenerate levels at the pair of EPs are zero (marked with red asterisks).

The parameters in Fig. 3(a) are the same as those in Fig. 2(b). In this case, the unit circle intersects the solid part of the DP ring at two points. Consequently, the degenerate levels appear on the real axis of the complex plane. The pair of EPs is located outside the unit circle, yielding a trivial phase. The parameters in Fig. 3(b) are the same as in Fig. 2(c). In this case, the unit circle intersects the DP ring exactly at the pair of EPs. Consequently, the degenerate levels are zero. The pair of EPs lies on the unit circle, yielding a topological phase transition. The parameters in Fig. 3(c) are the same as in Fig. 2(d). In this case, the unit circle intersects the dashed part of the DP ring at two points. Consequently, the degenerate levels appear on the imaginary axis of the complex plane. The pair of EPs is located inside the unit circle, yielding a nontrivial phase. As the parameter t_2 gradually increases, the DP ring gradually moves inside the unit circle. The two EPs always simultaneously enter or exit the unit circle in the parameter space. The DP ring with a pair of EPs underlies the topological phase with a $2\mathbb{Z}$ topological invariant, distinguished from the topological phase with a $\mathbb{Z}$ topological invariant.

The two-leg ladder in the absence of the cross-couplings is two SSH chains ($\gamma = 0$). The topological phase transition occurs at $t_2 = t_1$, the region $t_2 > t_1$ has nontrivial topology, and the region $t_2 < t_1$ is a trivial phase. The crossing couplings alter the wave function distribution, affect the topological phase transition, and expand the nontrivial topology. Notably, the region $t_2^2 + \gamma^2 > t_1^2$ is topologically nontrivial, and the region $t_2^2 + \gamma^2 < t_1^2$ is topologically trivial. The non-Hermitian cross-couplings help the formation of nontrivial topology, and the non-Hermiticity can generate nontrivial topology even when the ladder legs of SSH chains reside in a trivial phase. Furthermore, robust edge states appear in the topologically nontrivial phase.

The geometric picture developed from the spectra of the analytic continuation of the Bloch Hamiltonian H_k well describes the topology of the Bloch bands. The topological phase transition and the appearance of edge states are correctly predicted. To further determine the number of

zero-mode edge states, a concrete topological invariant for the two-leg ladder is required.

IV. ZAK PHASE

We calculate the Zak phase to determine the topological invariant and verify the geometric picture in the topological characterization of the two-leg ladder. In non-Hermitian physics, the Berry connection associated with the Bloch band and the corresponding Zak phase are defined in the biorthogonal framework.

The Zak phase for the Bloch band n is obtained from the biorthogonal Berry connection, having the form

$$\Theta_n = -i \oint dk \langle \varphi_n | \partial_k | \psi_n \rangle, \quad (8)$$

where $|\psi_n\rangle$ and $|\varphi_n\rangle$ are the eigenstates of the Bloch Hamiltonian $H_k |\psi_n\rangle = E_n |\psi_n\rangle$ and the Hermitian conjugation of the Bloch Hamiltonian $H_k^\dagger |\varphi_n\rangle = E_n^* |\varphi_n\rangle$ for energies E_n and E_n^* , respectively, and the eigenstates satisfy the biorthogonal normalization $\langle \psi_m | \varphi_n \rangle = \delta_{mn}$. Notably, the four eigenstates associated with the four Bloch bands of the two-leg ladder H_k are

$$E_k \leftrightarrow |\psi_1\rangle = (1, \alpha_k / \alpha_{-k}, -1, \alpha_k / \alpha_{-k})^T / 2, \quad (9)$$

$$E_k^* \leftrightarrow |\psi_2\rangle = (1, \alpha_{-k}^* / \alpha_k^*, 1, -\alpha_{-k}^* / \alpha_k^*)^T / 2, \quad (10)$$

$$-E_k \leftrightarrow |\psi_3\rangle = (-1, \alpha_k / \alpha_{-k}, 1, \alpha_k / \alpha_{-k})^T / 2, \quad (11)$$

$$-E_k^* \leftrightarrow |\psi_4\rangle = (1, -\alpha_{-k}^* / \alpha_k^*, 1, \alpha_{-k}^* / \alpha_k^*)^T / 2. \quad (12)$$

The four eigenstates associated with the four Bloch bands of the Hermitian conjugation $H_k^\dagger$ are

$$E_k^* \leftrightarrow |\varphi_1\rangle = (1, \alpha_{-k}^* / \alpha_k^*, -1, \alpha_{-k}^* / \alpha_k^*)^T / 2, \quad (13)$$

$$E_k \leftrightarrow |\varphi_2\rangle = (1, \alpha_k / \alpha_{-k}, 1, -\alpha_k / \alpha_{-k})^T / 2, \quad (14)$$

$$-E_k^* \leftrightarrow |\varphi_3\rangle = (-1, \alpha_{-k}^* / \alpha_k^*, 1, \alpha_{-k}^* / \alpha_k^*)^T / 2, \quad (15)$$

$$-E_k \leftrightarrow |\varphi_4\rangle = (1, -\alpha_k / \alpha_{-k}, 1, \alpha_k / \alpha_{-k})^T / 2. \quad (16)$$

For the trivial phase $t_2^2 + \gamma^2 < t_1^2$, the spectrum includes two separated Bloch bands, E_r and E_l . Purely real degenerate levels exist between Bloch bands E_k and E_k^* ($-E_k$ and $-E_k^*$). In this sense, we can define two partial global Zak phases:

$$\Theta_r = \Theta_1 + \Theta_2, \quad \Theta_l = \Theta_3 + \Theta_4. \quad (17)$$

For nontrivial phases $t_2^2 + \gamma^2 > t_1^2$ and $t_2 \leq t_1$, the spectrum includes two separated Bloch bands, E_t and E_b . Purely imaginary degenerate levels exist between Bloch bands E_k and $-E_k^*$ ($-E_k$ and E_k^*). In this sense, we can define another two partial global Zak phases:

$$\Theta_t = \Theta_1 + \Theta_4, \quad \Theta_b = \Theta_2 + \Theta_3. \quad (18)$$

For nontrivial phases $t_2^2 + \gamma^2 > t_1^2$ and $t_2 > t_1$, the four Bloch bands are fully separated, and the Zak phase Θ_n for each band is well-defined. The numerical results of the Zak phases in the topologically trivial (nontrivial) phase are depicted in Fig. 4(a) [Figs. 4(b) and 4(c)]. In Fig. 4(a), the parameters satisfy

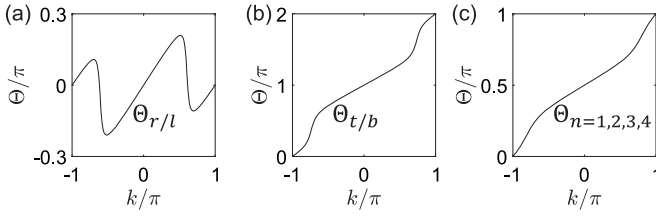


FIG. 4. Zak phase and partial global Zak phase. The parameters are $t_1 = 1$, $\gamma = \sqrt{3}/2$, and $t_2 = 0.3$ for (a), $t_2 = 0.8$ for (b), and $t_2 = 1.2$ for (c).

$t_2^2 + \gamma^2 < t_1^2$; the trivial phase has the partial global Zak phase $\Theta_{r/l} = 0$. In Fig. 4(b), the parameters satisfy $t_2^2 + \gamma^2 > t_1^2$ and $t_2 \leq t_1$; the nontrivial topology is characterized by the partial global Zak phase $\Theta_{t/b} = 2\pi$. The quantized change in the Zak phase reveals the $2\mathbb{Z}$ topological invariant. In Fig. 4(c), the parameters satisfy $t_2 > t_1$, all four Bloch bands become fully separated, and then $\Theta_{n=1,2,3,4} = \pi$. The band separation from Figs. 4(b) to 4(c) does not involve a topological phase transition. The nonzero global Zak phase $\sum_{n=1}^4 \Theta_n = 4\pi$ indicates that there are four edge states in the topologically nontrivial phase.

V. ZERO MODES

In topological lattices, non-Hermiticity enables various dissipative and radiative edge or interface states that are robust to disorder [80–94]. By contrast, the extremely long-lived exact zero-energy edge or interface states are relatively rare due to the presence of non-Hermiticity. The defect at the interface of two $\mathcal{PT}$ -symmetric SSH chains creates a zero mode [80,82]. The Hatano-Nelson model supports zero modes [9,14,24,94,95]. Furthermore, hidden zero modes may exist in the singular-value spectrum rather than the eigenvalue spectrum [96].

The sublattice symmetry of the two-leg ladder ensures that the topological edge modes have zero energy. In addition, the inversion symmetry ensures the validity of the conventional bulk-boundary correspondence [10]. In the topologically nontrivial phase, the two-leg ladder has fourfold degenerate topological zero modes under the OBC. Two zero modes localize at the left boundary, and the other two zero modes localize at the right boundary. The eigenstates of zero modes in the thermodynamic limit can be analytically obtained by solving the Hamiltonian of the two-leg ladder $H|\psi_L\rangle = 0$. The amplitudes of the left edge modes satisfy the following recurrence equations:

$$t_1 \psi_{L,j}^a + t_2 \psi_{L,j+1}^a - i\gamma \psi_{L,j+1}^c = 0, \quad (19)$$

$$-t_1 \psi_{L,j}^c - t_2 \psi_{L,j+1}^c + i\gamma \psi_{L,j+1}^a = 0. \quad (20)$$

Obviously, the amplitudes of the left edge modes satisfy $\psi_{L,j}^a = \pm \psi_{L,j}^c$. This relation substituted into the recurrence equations directly yields two analytical solutions, and the number of solutions is in accordance with the degeneracy of the left edge states. To be specific, the amplitudes of the n th unit cell of the left edge modes $|\psi_L\rangle$ are $\chi_{\pm}^n \{1, 0, \mp 1, 0\}$, and $\chi_{\pm} = -t_1/(t_2 \pm i\gamma)$ is the decay factor. Then, we notice that the decay rates of the two edge modes are exactly

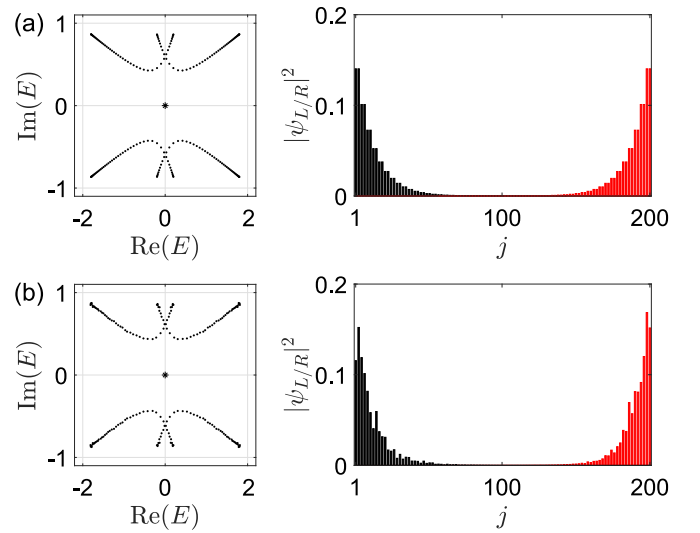


FIG. 5. Robust zero edge modes. (a) shows the spectrum under the OBC and zero edge modes in the absence of disorder. (b) shows the spectrum under the OBC and zero edge modes in the presence of disorder. All the couplings are randomly deviated by $[-10\%, 10\%]$. Left: Spectrum. Right: Distribution of zero edge modes. The parameters are $t_1 = 1$, $\gamma = \sqrt{3}/2$, and $t_2 = 0.8$.

identical, that is, $|\chi_+| = |\chi_-| = t_1/\sqrt{t_2^2 + \gamma^2}$. The existence of edge states requires the validity of $|\chi_{\pm}| < 1$, which indicates $t_2^2 + \gamma^2 > t_1^2$, which is the condition for the nontrivial topology. Notably, the two-leg ladder is inversion symmetric. Then, it is straightforward to obtain the right zero modes using the same approach, and the right zero modes $|\psi_R\rangle$ have similar expressions but decay from the right to the left. Moreover, the appearance of topological edge modes also aligns with the integer Zak phases associated with the Bloch bands. In summary, the Hermitian couplings support nontrivial topology when $t_2 > t_1$, and the non-Hermitian couplings support nontrivial topology when $\gamma > t_1$.

Figure 5(a) presents the complex energy spectrum and the spatial distributions of a pair of left and right edge states under the OBC in the absence of disorder. The spectrum exhibits mirror-reflection symmetry with respect to both the real and imaginary axes of the complex plane, and the edge states display mirror reflection about the center of the lattice. Figure 5(b) shows the complex energy spectrum and the spatial distributions of a pair of left and right edge states under the OBC in the presence of random coupling disorder, which follows a continuous uniform distribution within the range $[-10\%, 10\%]$. Remarkably, the symmetries are robust against random coupling disorder, and the spectrum retains its symmetry with respect to both the real and imaginary axes. The edge states have exactly zero energy, being immune to random coupling disorder, but the edge states do not display mirror reflection about the center of the lattice, and the decay rates in each unit cell are affected. We highlight that $\mathcal{PT}$ symmetry, anti- $\mathcal{PT}$ symmetry, and sublattice symmetry preserve and protect the topological zero modes in the presence of random coupling disorder. Furthermore, the topological zero modes emerge even when non-Hermitian cross-couplings are introduced as perturbations.

VI. EXPERIMENTAL REALIZATION

The non-Hermitian two-leg ladder involves four distinct types of couplings, originating from the intracell and intercell hopping processes along each leg $\pm t_{i=1,2}$, as well as the interleg cross-hopping processes between different sublattices $\pm i\gamma$. We name the configuration using the couplings $\{t_{i=1,2}, -t_{i=1,2}, i\gamma, -i\gamma\}$. To facilitate the realization in experiment, we further simplify the two-leg ladder to an alternative configuration. Notably, it is convenient to perform a local gauge transformation on the sites belonging to sublattice D , that is, $d_f^\dagger|\text{vac}\rangle \rightarrow -d_f^\dagger|\text{vac}\rangle$. The influence of the gauge transformation applied to the Bloch Hamiltonian is a unitary operation $(\sigma_0 \oplus \sigma_z)H_k(\sigma_0 \oplus \sigma_z)^{-1}$. Consequently, the unitary operation leaves the physical properties of the two-leg ladder completely unchanged; however, the spatial structure of the two-leg ladder changes into a more concise form. To be specific, the gauge transformation reshapes all hopping amplitudes relevant to sublattice D by an extra minus sign. Then, the intracell coupling $-t_1$ between sublattices C and D is transformed into t_1 , the intercell coupling $-t_2$ between sublattices C and D is transformed into t_2 , and the interleg cross-coupling $i\gamma$ between sublattices A and D is transformed into $-i\gamma$. To summarize, the four types of couplings, $\pm t_{i=1,2}$ and $\pm i\gamma$, change into only two types of couplings, $t_{i=1,2}$ and $-i\gamma$, after the gauge transformation is applied to the two-leg ladder. The original non-Hermitian two-leg ladder with the couplings $\{t_{i=1,2}, -t_{i=1,2}, i\gamma, -i\gamma\}$ is characterized by the equivalent non-Hermitian two-leg ladder with the couplings $\{t_{i=1,2}, t_{i=1,2}, -i\gamma, -i\gamma\}$. Therefore, the equivalent configuration greatly reduces the experimental implementation of the two-leg ladder.

We focus on the experimental realization of the equivalent configuration, which includes two identical prototypical SSH chains coupled through the dissipative coupling $-i\gamma$. We point out that the extended SSH chain [76,80,82,97,98] and the dissipative coupling $-i\gamma$ [59,61,63,76] have been extensively implemented and investigated across a wide range of physical platforms in experiments, such as coupled resonator optical waveguides, cold atoms in optical lattices, electronic circuits, and acoustic lattices. This progress indicates that the proposed non-Hermitian two-leg ladder can be well carried out on these physical platforms.

Without loss of generality, we discuss the experimental realization in representative coupled resonator optical waveguides, which is a promising recent physical platform for investigating non-Hermitian topology. In the coupled resonator optical waveguides, each ring resonator supports two degenerate modes—clockwise and counterclockwise—that are mutually decoupled. Therefore, either the clockwise or counterclockwise mode is described by the tight-binding model. The primary ring resonators play the role of the sublattices, and they are indirectly coupled via auxiliary ring resonators; the mode chiralities of the primary and auxiliary resonators are opposite. We note that topological edge and interface states in the non-Hermitian extension of the prototypical SSH chain were successfully demonstrated in coupled resonator optical waveguides [80,81,83,85].

Similarly, the prototypical SSH chain can be implemented in a simpler manner than the non-Hermitian SSH chain, and furthermore, causing dissipative coupling $-i\gamma$ between two prototypical SSH chains is also available with current experimental technology. The auxiliary resonators are designed to be resonant with the primary resonators and are evanescently coupled to them with the same coupling strength κ . When the auxiliary resonator is made lossy with an attenuation rate $\Gamma \gg \kappa$, the strong dissipation enables the adiabatic elimination of the light field of the auxiliary resonator, which effectively results in the dissipative coupling $-i\gamma$ between the two adjacent primary resonators at the coupling strength $\gamma = \kappa^2/\Gamma$ [99,100]. In addition, a topoelectrical circuit composed of RLC resonators is another ideal platform for the realization of a two-leg ladder due to its low cost and the well-developed experimental techniques [97].

VII. CONCLUSION

We proposed a non-Hermitian two-leg ladder with a point gap. The radiative and dissipative couplings connect the ladder legs. The coexistence of $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry causes the sublattice symmetry, which protects a quantized Zak phase and supports real zero-energy edge modes even though the bulk states are entirely complex. The inversion symmetry ensures the validity of bulk-boundary correspondence, and the topological phase is characterized by a $2\mathbb{Z}$ topological invariant. Furthermore, the non-Hermiticity plays a constructive role in the formation of nontrivial topology but does not alter the zero energy of the edge modes. Remarkably, the topological zero modes are robust against arbitrary coupling disorder and can be induced purely by disorder. Our findings highlight the interplay between $\mathcal{PT}$ symmetry and anti- $\mathcal{PT}$ symmetry in the creation of nontrivial topology and robust zero modes.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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