

Quantum quenches across continuous and first-order quantum transitions in one-dimensional quantum Ising models

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We investigate the quantum dynamics generated by quantum quenches (QQs) of the Hamiltonian parameters in many-body systems, focusing on protocols that cross first-order and continuous quantum transitions, both in finite-size systems and in the thermodynamic limit. As a paradigmatic example, we consider the quantum Ising chain in the presence of homogeneous transverse (g) and longitudinal (h) magnetic fields. This model exhibits a continuous quantum transition (CQT) at $g = g_c$ and $h = 0$, and first-order quantum transitions (FOQTs) driven by h along the line $h = 0$ ($g < g_c$). In the integrable limit $h = 0$, the system can be mapped onto a quadratic fermionic theory; however, any nonvanishing longitudinal field breaks integrability and the spectrum of the resulting Hamiltonian is generally expected to enter a chaotic regime. We analyze QQs in which the longitudinal field is suddenly changed from a negative value $h_i < 0$ to a positive value $h_f > 0$. We focus on values of h_f such that the spectrum of the post-QQ Hamiltonian $\hat{H}(g, h_f)$ lies in the chaotic regime, where thermalization may emerge at asymptotically long times. We study the out-of-equilibrium dynamics for different values of g , finding qualitatively distinct behaviors for $g > g_c$ (where the chain is in the disordered phase), for $g = g_c$ (QQ across the CQT), and for $g < g_c$ (QQ across the FOQT line).

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I. INTRODUCTION

The evolution of many-body systems in the presence of a time-dependent Hamiltonian is generally characterized by an out-of-equilibrium behavior, especially when the parameters cross values at which the system undergoes a classical (thermal) or quantum phase transition [1–10]. In such situations, large-scale modes are unable to equilibrate, even when the parameters are varied very slowly [9,11,12]. Hysteresis, spinodal-like and coarsening phenomena, critical aging, and Kibble-Zurek defect production are typical examples of nonequilibrium phenomena that are observed at first-order and continuous transitions, driven by thermal or quantum fluctuations (see, e.g., Refs. [3,9–17] and references therein).

In this paper, we investigate the quantum dynamics induced by instantaneous variations of the Hamiltonian parameters (quench dynamics) across first-order and continuous quantum phase transitions [6–10] in the thermodynamic limit. We focus on cases in which the postquench Hamiltonian has a chaotic spectrum [18–21], where the quantum system is expected to thermalize for typical initial states and for sufficiently long times (see, e.g., Refs. [21–39]). As a paradigmatic model, we consider the quantum nearest-neighbor Ising chain in the

presence of homogeneous transverse (g) and longitudinal (h) magnetic fields. The system exhibits a continuous quantum transition (CQT) at $g = g_c$ and $h = 0$ (with $g_c = 1$, in units of the ferromagnetic coupling J), and first-order quantum transitions (FOQTs) driven by the longitudinal field, along the line $h = 0$, for $g < g_c$. Any nonvanishing value of h breaks integrability of the corresponding Hamiltonian $\hat{H}(g, h)$, which can be mapped onto a quadratic fermionic system only for $h = 0$ [7,40]. Consequently, the spectrum of $\hat{H}(g, h)$ for $h \neq 0$ is generally expected to enter a chaotic regime, characterized by a Wigner-Dyson distribution of level spacings [21].

One of the simplest protocols to probe the out-of-equilibrium dynamics of many-body systems is the so-called quantum quench (QQ) (see, e.g., Refs. [41–48]). This is generally implemented by suddenly changing one of the Hamiltonian parameters, such as the transverse (g) or the longitudinal (h) field in quantum Ising chains. For example, when only h is varied while g is kept fixed, one starts from the ground state $|\Phi_0(g, h_i)\rangle$ of the Hamiltonian $\hat{H}(g, h_i)$ associated with an initial value of the parameter h_i , and then suddenly quenches it to a different value $h_f \neq h_i$, at a reference time $t = 0$. The subsequent quantum evolution is governed by the new Hamiltonian $\hat{H}(g, h_f)$, so that, in units of $\hbar = 1$,

$$|\Psi(t)\rangle = e^{-i\hat{H}(g, h_f)t} |\Phi_0(g, h_i)\rangle. \quad (1)$$

The post-QQ dynamics can reveal a wide range of interesting phenomena [see, e.g., Refs. [9,16,21,27,37,49–116] and references therein (this should not be regarded as an exhaustive

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list)). We mention: the post-QQ time dependence of quantum correlations and entanglement [49–64] and the eventual approach to stationary states that admit a microcanonical or canonical statistical characterization in the thermodynamic limit, when probed locally [21,27,65–68]; the emergence of preasymptotic regimes [69,70], especially when the QQ crosses a FOQT, potentially leading to (quasi)metastable states and spinodal-like (nucleation) phenomena [71–76]; the appearance of peculiar out-of-equilibrium scaling regimes at both CQTs and FOQTs in the limit of small variations across quantum transitions [77–89]; signatures of quantum transitions imprinted by the post-QQ dynamics [90–95]; the statistics of the work associated with the QQ [96–99]; prethermalization regimes preceding eventual thermalization [37,98,100,101]; dynamic singularities, often interpreted as dynamical phase transitions [102]; the role of dissipation arising from interactions with an external environment [103–105]; etc.

The post-QQ dynamics at CQTs develops an out-of-equilibrium finite-size scaling (OFSS) behavior for *soft* QQs across the transition [9,77], i.e., when the initial (h_i) and final (h_f) Hamiltonian parameters remain close to the critical point and are tuned to zero with the system size L as $h_i \sim h_f \sim L^{-y_h}$, where $y_h = \frac{15}{8}$ is the renormalization-group dimension of the longitudinal field at criticality. This condition ensures that the system remains in the critical region throughout the entire post-QQ evolution. Analogous considerations apply to FOQTs, where OFSS behaviors also emerge [9,10,77,79], the main difference residing in the fact that the key features of the OFSS are particularly sensitive to the choice of boundary conditions.

By contrast, for standard (hard) QQs, in which h is kept fixed as L increases, one does not expect to observe any OFSS controlled by the universality class of the equilibrium quantum transitions. This is because the energy injected into the system is extensive, i.e., $O(L)$, and therefore much larger than the energy scale $E_c \sim L^{-z}$ of the low-energy critical modes, where $z = 1$ is the dynamic exponent governing the finite-size gap $\Delta \sim L^{-z}$ at the critical point. Moreover, in nonintegrable systems which are expected to thermalize, local observables are generally smooth functions of the temperature, and thus are not expected to develop asymptotic long-time singularities after a QQ. This raises a key open question: Do any footprints of the universal low-energy features of quantum transitions persist in the post-QQ unitary dynamics at finite values of h , where the system should enter a chaotic regime and thermalize in the long-time limit?

In fact, as argued in Refs. [90–95], signatures of quantum transitions may appear in certain models under standard QQs. Standard QQs have been thoroughly investigated in quantum Ising chains in the absence of longitudinal magnetic field, therefore in the integrable regime, varying g across the CQT, that separates the quantum paramagnetic and ferromagnetic phases [49,50,56,57,61,90,94]. These QQs typically start from the ground state corresponding to an initial value g_i and then suddenly change it to $g \neq g_i$. In the thermodynamic limit, the time-dependent energy density, as well as the longitudinal and transverse magnetizations, develop a singular behavior in the long-time limit, when $g \rightarrow g_c$ [9,90,94]. Indeed, in a QQ from the ground state at $g_i > g_c$ to values of

$g < g_i$, the g derivative of the large-time transverse magnetization M_{tr} turns out to be discontinuous at $g = g_c$ [9,94]:

$$\lim_{g \rightarrow g_c^+} \frac{\partial M_{\text{tr}}(g_i, g)}{\partial g} - \lim_{g \rightarrow g_c^-} \frac{\partial M_{\text{tr}}(g_i, g)}{\partial g} = 1, \quad (2)$$

independently of g_i .¹ Nonanalytic behaviors around g_c have also been observed in the long-time behavior of the longitudinal magnetization following QQs from an initially ordered state, corresponding to the ground state at large h , when the post-QQ evolution is governed by the Ising Hamiltonian with $h = 0$. However, these nonanalyticities at g_c seem to be related to the specific spectral properties of the post-QQ Hamiltonian in the absence of a longitudinal field, which can be mapped onto a quadratic fermionic Hamiltonian [7,40].

In this paper we rather consider QQs of the longitudinal field h , focusing on regimes where the spectrum of the post-QQ Hamiltonian is chaotic. We investigate several features of the resulting out-of-equilibrium dynamics for various values of the transverse field g , observing different qualitative behaviors for $g > g_c$ (QQ in the disordered phase), for $g = g_c$ (QQ across the CQT), and for $g < g_c$ (QQ across the FOQT line). To this end, we present a detailed analysis of the spectrum and of several observables in the diagonal ensemble for lattice sizes up to $L = 20$: these calculations require exact diagonalization of the Hamiltonian. As for the post-QQ evolution of the system, we are able to obtain exact results for significantly longer chains, up to $L = 28$, and for quite long times, by using Lanczos methods followed by a fourth-order Runge-Kutta integration of the post-QQ unitary dynamics (the time step can be adjusted to achieve accuracies of the order of machine precision). Depending on the values of the parameters g and h , these times may reach several hundreds (in units of $\hbar/J$) and be sufficient to achieve the thermodynamic-limit behavior. To access such chain sizes and times, it is crucial to exploit the symmetries of the system, projecting onto the zero-momentum Hilbert subspace [75,117].²

The paper is organized as follows. In Sec. II we introduce the quantum Ising-chain Hamiltonian, define the QQ protocols, and the relevant observables. Section III presents an

¹For generic quantum XY chains, involving an additional anisotropy parameter $\gamma > 0$ [7,9], one finds

$$\lim_{g \rightarrow g_c^+} \frac{\partial M_{\text{tr}}(g_i, g)}{\partial g} - \lim_{g \rightarrow g_c^-} \frac{\partial M_{\text{tr}}(g_i, g)}{\partial g} = \gamma^{-1}. \quad (3)$$

This equation corrects a misprint appeared in Refs. [9,94].

²There are other numerical methods, such as tensor-network techniques (e.g., the iTEBD algorithm operating on an infinite matrix-product-state representation of the system wave function) that can be used to assess the post-QQ evolution directly in the thermodynamic limit. The caveat is that reachable times are limited by entanglement growth [particularly in highly entangling dynamics, where simulations can be typically considered reliable up to times $t = O(10^1)$ in units of the characteristic energy scale (see, e.g., Refs. [29,51,59,62,106,107,110,113]). One would thus need to carefully scrutinize the effects of a finite truncation of the bond links of the various tensors, rather than controlling finite-size effects. Our approach is complementary to these methods and turns out to be particularly useful for QQs crossing quantum transitions, such as those considered here.

analysis of the spectral properties of the system as a function of g and h , within the zero-momentum sector. The remaining Secs. IV, V, and VI contain a detailed numerical study of the dynamics of the system following QQs of the longitudinal field h , respectively in the disordered phase ($g > g_c$), across the CQT ($g = g_c$), and across the FOQT line ($g < g_c$). In Sec. VII we summarize the results and draw our conclusions. Finally, we include a few sections as Appendixes. In Appendix A, we discuss soft QQs across the CQT and FOQTs of the Ising chain, which give rise to peculiar OFSS behaviors controlled by the universal features of the crossed transition. In Appendix B, we present some results for a QQ in the presence of oppositely fixed boundary conditions in the small- g and small- h regimes, where a one-kink approximation holds, allowing us to perform computations for large systems.

II. MODEL AND QUANTUM QUENCH PROTOCOL

A. The quantum Ising chain

The nearest-neighbor quantum Ising chain in a transverse field g is a paradigmatic model showing a CQT and a *magnetic* FOQT line driven by the longitudinal field h . Its Hamiltonian reads as

$$\hat{H}(g, h) = - \sum_{x=1}^L [J \hat{\sigma}_x^{(1)} \hat{\sigma}_{x+1}^{(1)} + g \hat{\sigma}_x^{(3)} + h \hat{\sigma}_x^{(1)}], \quad (4)$$

where $\hat{\sigma}_x^{(k)}$ are Pauli matrices ($k = 1, 2, 3$) defined on the x th lattice site. We consider chains with L sites and periodic boundary conditions ($\hat{\sigma}_x^{(k)} = \hat{\sigma}_{L+x}^{(k)}$), which preserve translational invariance (no boundaries are present) and the $\mathbb{Z}_2$ symmetry when $h = 0$. Without loss of generality, we assume $J = 1$ and $g > 0$. We also set $\hbar = k_B = 1$.

At zero temperature and for $g = g_c = 1$, $h = 0$, the model (4) undergoes a CQT belonging to the two-dimensional Ising universality class (with critical exponents $\nu = 1$, $\eta = \frac{1}{4}$, and $z = 1$), separating a disordered phase ($g > g_c$) from an ordered one ($g < g_c$). For any $g < g_c$, the longitudinal field drives FOQTs along the $h = 0$ line, with the spontaneous breaking of the $\mathbb{Z}_2$ symmetry in the thermodynamic limit. For any $g < g_c$ and for $|h| \rightarrow 0$, the magnetized states $|+\rangle$ and $|-\rangle$ along the longitudinal direction are the ground states of the system for $h > 0$ and $h < 0$, respectively, giving rise to a discontinuous infinite-volume longitudinal magnetization m_0 [118]:

$$\lim_{h \rightarrow 0^\pm} \lim_{L \rightarrow \infty} M = \pm m_0, \quad m_0 = (1 - g^2)^{1/8}, \quad (5)$$

where

$$M = \langle \Phi_0(g, h) | \hat{M} | \Phi_0(g, h) \rangle, \quad \hat{M} \equiv \frac{1}{L} \sum_x \hat{\sigma}_x^{(1)}, \quad (6)$$

and $|\Phi_0(g, h)\rangle$ is the ground state.

B. Quantum quench protocols

We focus on the out-of-equilibrium quantum dynamics arising from QQs of the longitudinal-field parameter h , from $h_i \leq 0$ to $h_f > 0$, keeping the transverse-field parameter g fixed (for the sake of clarity, in the following we omit the g dependence in the formulas). We consider a standard QQ

protocol: the system is initially in the ground state $|\Phi_0(h_i)\rangle$. At $t = 0$, the longitudinal field is instantaneously changed to h_f and the quantum evolution is described by the density matrix

$$\hat{\rho}(t) = |\Psi(t)\rangle \langle \Psi(t)|, \quad (7)$$

where $|\Psi(t)\rangle$ is the solution of the Schrödinger equation

$$i\partial_t |\Psi(t)\rangle = \hat{H}(h_f) |\Psi(t)\rangle, \quad |\Psi(t=0)\rangle = |\Phi_0(h_i)\rangle. \quad (8)$$

The evolving state $|\Psi(t)\rangle$ can be written as a sum over the eigenstates $|\Phi_n(h_f)\rangle$ of the post-QQ Hamiltonian $\hat{H}(h_f)$,

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_n(h_f)t} |\Phi_n(h_f)\rangle, \quad (9)$$

where $c_n = \langle \Phi_n(h_f) | \Phi_0(h_i) \rangle$ and $E_n(h_f)$ are the eigenvalues of $\hat{H}(h_f)$.

To simplify the analysis, we consider a QQ with

$$h_i = -h < 0, \quad h_f = h > 0, \quad (10)$$

so that the QQ protocol has only one free parameter h . Along the FOQT line for $g < g_c$, we also consider QQ protocols starting from $h_i = 0^-$, where we specify that h_i approaches zero from below, because of the discontinuity (5) in the thermodynamic limit.

We mention that other types of QQs from magnetized states (corresponding to large h_i) to $h_f = 0$ have been already investigated [9,56,57,94], revealing the emergence of singular behaviors in the thermodynamic and long-time limits. In particular, the large-time asymptotic stationary value of the longitudinal magnetization is nonanalytic at $g = g_c$ [9,94]. However, as already noted in the Introduction, this phenomenon is expected to be related to the specific integrability properties of the post-QQ Hamiltonian when $h = 0$.

The system evolution for $t > 0$ can be monitored by evaluating the instantaneous longitudinal magnetization

$$M(t) = \text{Tr}[\hat{\rho}(t) \hat{M}] = \text{Tr}[\hat{\rho}(t) \hat{\sigma}_x^{(1)}], \quad (11)$$

where we used translational invariance. We also consider the excess bond energy,

$$K(t) = \text{Tr}[\hat{\rho}(t) \hat{K}] - \langle \Phi_0(h) | \hat{K} | \Phi_0(h) \rangle, \quad (12a)$$

$$\hat{K} = -\frac{1}{L} \sum_x \hat{\sigma}_x^{(1)} \hat{\sigma}_{x+1}^{(1)}. \quad (12b)$$

C. Remarks on the postquench evolution

The time dependence of a generic observable $\hat{O}$ can be written as

$$\begin{aligned} O(t) &= \text{Tr}[\hat{\rho}(t) \hat{O}] = \sum_{m,n} c_m^* c_n e^{i(E_m - E_n)t} O_{mn} \\ &= \sum_n w_n O_{nn} + \sum_{m \neq n} c_m^* c_n e^{i(E_m - E_n)t} O_{mn}, \end{aligned} \quad (13)$$

where

$$w_n = |c_n|^2 = |\langle \Phi_n(h) | \Phi_0(h_i) \rangle|^2, \quad \sum_n w_n = 1, \quad (14)$$

and $O_{mn} = \langle \Phi_m(h) | \hat{O} | \Phi_n(h) \rangle$. The coefficients w_n provide the overlaps of the initial ground state associated with h_i with the eigenstates of the Hamiltonian with the final value h .

Averaging the expectation value $O(t)$ over a sufficiently long time (assuming no degeneracies of the involved Hamiltonian eigenstates), the oscillating last term in the second line of Eq. (13) does not contribute. Therefore, one can define a time-independent *diagonal ensemble* density matrix [27]

$$\hat{\rho}_D = \sum_n w_n |\Phi_n(h)\rangle \langle \Phi_n(h)|, \quad (15)$$

which is expected to provide an effective approximation of the long-time average of few-body observables (see, e.g., Ref. [21]). Corresponding expectation values M_D and K_D can then be defined as

$$M_D = \text{Tr}[\hat{\rho}_D \hat{M}], \quad (16)$$

$$K_D = \text{Tr}[\hat{\rho}_D \hat{K}] - \langle \Phi_0(h) | \hat{K} | \Phi_0(h) \rangle, \quad (17)$$

with $\hat{M}$ and $\hat{K}$ defined as in Eqs. (6) and (12), respectively.

The post-QQ evolution of the state is characterized by the energy-density distribution

$$D(e) = \sum_n w_n \delta(e - e_n), \quad e_n = \frac{E_n(h) - E_0(h)}{L}, \quad (18)$$

where $E_n = \langle \Phi_n(h) | \hat{H}(h) | \Phi_n(h) \rangle$. The corresponding mean and variance are given by

$$\begin{aligned} \bar{e} &= \frac{\langle \Phi_0(h_i) | \hat{H}(h) | \Phi_0(h_i) \rangle - E_0(h)}{L}, \\ \sigma_e^2 &= \frac{\langle \Phi_0(h_i) | \hat{H}(h)^2 | \Phi_0(h_i) \rangle - \langle \Phi_0(h_i) | \hat{H}(h) | \Phi_0(h_i) \rangle^2}{L^2}. \end{aligned} \quad (19)$$

For a QQ, the normalized energy distribution $D(e)$ is generally expected to approach a Gaussian, independently of the specific nature of $\hat{H}(h_i)$, provided the system is sufficiently large (this is essentially due to the exponentially growing density of states with the system size) [21,32,33]. Moreover, the ratio $\sigma_e/\bar{e}$ is expected to decrease as $V^{-1/2}$ with increasing volume V [21], or equivalently as $L^{-1/2}$ for one-dimensional lattice systems, leading to an effective microcanonical energy window.

Since $\sigma_e/e \rightarrow 0$ in the large- L limit, only states with given energy density e are relevant. Assuming that all these states have the same probability, i.e., an effective asymptotic thermalization, the chain behaves as an effectively microcanonical statistical system in the large- L and large-time limits. If this is the case, one can define a corresponding effective inverse temperature β_{th} , which can be determined by considering a small subsystem, since its reduced density matrix should be proportional to the canonical-distribution density matrix $e^{-\beta_{\text{th}} \hat{H}_s}$, where $\hat{H}_s$ is the restriction of the Hamiltonian $\hat{H}$ to the subsystem. Explicitly, we consider a subsystem of size ℓ , with sites $x \in [1, \ell]$ with $1 \ll \ell \ll L$ (the choice of the first site is arbitrary, due to translation invariance) and with Hamiltonian

$$\hat{H}_s = - \sum_{x=1}^{\ell-1} \hat{\sigma}_x^{(1)} \hat{\sigma}_{x+1}^{(1)} - \sum_{x=1}^{\ell} [g \hat{\sigma}_x^{(3)} + h \hat{\sigma}_x^{(1)}], \quad (20)$$

with open boundary conditions. In practice, the effective temperature can be determined by imposing that

$$\frac{\langle \Phi_0(h_i) | \hat{H}(h) | \Phi_0(h_i) \rangle}{L} = \frac{1}{\ell} \frac{\text{Tr}[e^{-\beta_{\text{th}} \hat{H}_s(h)} \hat{H}_s(h)]}{\text{Tr}[e^{-\beta_{\text{th}} \hat{H}_s(h)}]}, \quad (21)$$

namely, that the canonical mean energy per site of the subsystem equals the energy per site of the full chain in the initial state.

The asymptotic thermalization requires that analogous relations should hold for the asymptotic large-time values of generic local observables, which can also be obtained as canonical averages at the effective inverse temperature β_{th} . For instance, the asymptotic long-time values of M should satisfy

$$\lim_{t \rightarrow \infty} M(t) \approx \frac{\text{Tr}[e^{-\beta_{\text{th}} \hat{H}_s(h)} \hat{M}_s]}{\text{Tr}[e^{-\beta_{\text{th}} \hat{H}_s(h)}]}, \quad (22)$$

where $\hat{M}_s = \ell^{-1} \sum_{x=1}^{\ell} \hat{\sigma}_x^{(1)}$, and analogously for the excess bond energy. If these equations yield significantly different values of β_{th} , thermalization does not hold. Note that, as we shall see, one may also obtain negative values of β_{th} , i.e., asymptotic stationary states locally described by Gibbs ensembles with negative temperature, giving rise to a peculiar type of equilibrium states (see Ref. [119] for a thorough discussion of the relevance and thermodynamic meaning of a negative β_{th} to describe statistical systems).

The degree of spectral delocalization of the post-QQ Hamiltonian $\hat{H}(h)$ can be quantified through the diagonal-ensemble entropy [22,23,28,30,31]

$$S_D = -\text{Tr}[\hat{\rho}_D \ln \hat{\rho}_D] = - \sum_n w_n \ln w_n, \quad (23)$$

with w_n defined in Eq. (14). This quantity vanishes if there exists a single element m for which $w_m = 1$, and $w_n = 0$ for $m \neq n$, while it reaches its maximal value $S_D = \ln D_{\mathcal{H}}$ when all w_n are equal ($D_{\mathcal{H}}$ is the dimension of the Hilbert subspace spanned by the initial state, taking into account the symmetries). Further information can be obtained from the large- L decay of the largest value of w_n ,

$$w_{\text{max}} = \text{Max}[w_n], \quad (24)$$

or from the average $w_{\text{max},k}$ of the largest k values of w_n . This tells us whether few eigenstates of the post-QQ Hamiltonian dominate the quantum evolution.

III. ANALYSIS OF THE SPECTRUM

Before studying the post-QQ out-of-equilibrium dynamics of the quantum Ising chain, it is instructive to first examine its spectrum for generic finite values of g and h , where the model is nonintegrable. We recall that integrability is recovered for $h = 0$ at any g , in which case the Hamiltonian can be exactly mapped onto a quadratic fermionic model via a Jordan-Wigner transformation [7,40], as well as in the trivial limits $g \rightarrow 0$, $g \rightarrow \infty$, and $h \rightarrow \infty$.

A. The zero-momentum sector

We consider chains with an even number L of sites and periodic boundary conditions. The dimension of the full Hilbert space is $\mathcal{D}(L) = 2^L$. Periodic boundary conditions preserve

translational invariance, allowing the spectrum to be decomposed into sectors with different momenta $p = 2\pi k/L$, where $k = 0, \dots, L-1$. Since the dynamics starts from a zero-momentum ground state, we can restrict our computations to the zero-momentum sector $\mathcal{H}_0$. The dimension $\mathcal{D}_0(L)$ of this sector exhibits a nontrivial dependence on L , which can be computed by counting the number of binary necklaces [120,121], i.e., of distinct cyclic arrangements of L colored beads, chosen from two available colors. One obtains

$$\mathcal{D}_0(L) = \frac{1}{L} \sum_{n \in \text{Div}(L)} 2^{L/n} \varphi_E(n), \quad (25)$$

where $\text{Div}(L)$ is the set of integer numbers that divide L , and $\varphi_E(n)$ is Euler totient function. For example, $\mathcal{D}_0 = 6, 14, 36, 108, 352, 1182, 4116$, for $L = 4, 6, 8, 10, 12, 14, 16$, respectively. Note that, in the large- L limit, $\mathcal{D}_0(L)$ approaches the asymptotic behavior

$$\lim_{L \rightarrow \infty} \frac{L \mathcal{D}_0(L)}{\mathcal{D}(L)} = 1, \quad \mathcal{D}(L) = 2^L. \quad (26)$$

The zero-momentum sector of the quantum Ising Hamiltonian (4) is invariant under the $\mathbb{Z}_2$ symmetry transformation $\hat{P}_r$ that corresponds to the coordinate reflection $x \rightarrow L+1-x$. As a consequence, the $p=0$ eigenstates of the Hamiltonian can be divided into two sets, belonging to the subsectors $\mathcal{H}_{0+}$ and $\mathcal{H}_{0-}$, with parity $P_r = \pm 1$ under reflection. Note that their dimensions $\mathcal{D}_{0\pm}(L)$ are not equal.³ The reflection symmetry of the system implies that, if the initial state is a $p=0$ state with parity $P_r = \pm 1$, the subsequent time evolution remains confined within the corresponding subspace $\mathcal{H}_{0\pm}$.

B. Chaotic quantum regime

The energy spectrum of a quantum many-body system can be characterized by the distribution of the differences between adjacent energy levels

$$s_n = E_{n+1} - E_n. \quad (29)$$

Their average value decreases exponentially with increasing L . Indeed, in the Hilbert subspace $\mathcal{H}_{0+}$,

$$\langle s \rangle = \frac{\sum_n s_n}{\mathcal{D}_{0+} - 1} = \frac{E_{\max} - E_{\min}}{\mathcal{D}_{0+} - 1} \sim L^2 2^{-(L-1)}, \quad (30)$$

where we used Eqs. (26) and (28), and the fact that $E_{\max} - E_{\min} \sim L$. For this reason, to obtain large- L predictions, one should consider the statistics $P(\{\tilde{s}_n\})$ for the rescaled quantities $\tilde{s}_n = s_n/\langle s_n \rangle$, so that spacings are normalized by their average value $\langle s_n \rangle$ in a local energy window, where the mean level density is set to unity, a procedure which is usually referred to as the unfolding of the spectrum [20]. In

³The dimensions $\mathcal{D}_{0\pm}(L)$ differ. For even L one finds

$$\mathcal{D}_{0+}(L) - \mathcal{D}_{0-}(L) = 3 \times 2^{L/2-1}. \quad (27)$$

This imbalance becomes asymptotically negligible in the large- L limit since

$$\lim_{L \rightarrow \infty} \frac{\mathcal{D}_{0+}(L) - \mathcal{D}_{0-}(L)}{\mathcal{D}_0} = 0, \quad (28)$$

where $\mathcal{D}_{0+} + \mathcal{D}_{0-} = \mathcal{D}_0$.

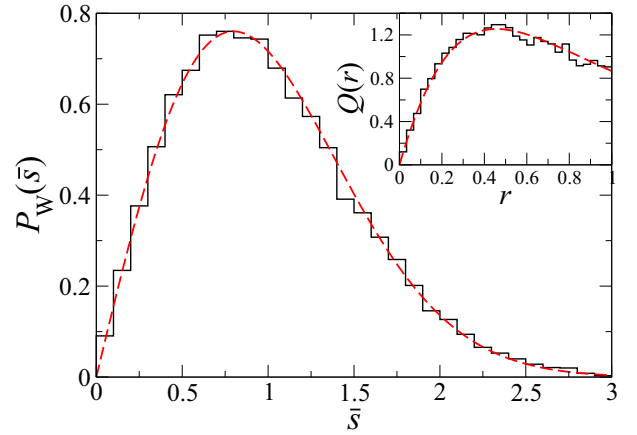


FIG. 1. The distribution of the level spacings s_n , normalized by the average value $\langle s \rangle$, for a quantum Ising chain with $L = 20$ spins and $g = h = 1$. We only consider states in the Hilbert subspace $\mathcal{H}_{0+}$, with dimension $\mathcal{D}_{0+} = 27012$. The dashed red curve shows the expected Wigner surmise for a GOE, given by Eq. (31). The inset displays the corresponding distribution of the ratios r_n for the same set of parameters, together with the analytic prediction (35) obtained from the Wigner surmise.

most situations, it is sufficient to normalize the spacings over the global energy average $\bar{s}_n = s_n/\langle s \rangle$.

The shape of the distribution $P(\{\tilde{s}_n\})$ allows one to distinguish between integrable and chaotic regimes (see, e.g., Ref. [21] for an extended discussion). Indeed, for models with time-reversal symmetry such as the quantum Ising chain (4), when the system is in a chaotic regime, $P(\{\tilde{s}_n\})$ is expected to follow the Wigner-Dyson (WD) level-spacing distribution associated with the Gaussian orthogonal ensemble (GOE). This distribution can be accurately approximated [25,34] by the Wigner surmise (exact for the random two-level problem):

$$P_W(s) = \frac{\pi s}{2} e^{-\pi s^2/4}, \quad (31)$$

with an average level spacing normalized to one. Another possibility (the Berry-Tabor conjecture [24]) is that the distribution of the energy-level separations follow a Poissonian distribution

$$P_P(s) = e^{-s}. \quad (32)$$

Note that a Poissonian distribution does not necessarily occur outside the chaotic regime for $h \neq 0$ (see Appendix B for an explicit example in the small- g and small- h regimes) as it only occurs when the Hamiltonian can be expressed as a sum of many independent contributions. In such case, the energy eigenvalues behave as sums of independent random variables, and consequently exhibit Poisson statistics [24].⁴

We emphasize that the WD distribution is expected to be observed in chaotic regimes only in the absence of symmetries, which may otherwise induce degeneracies in the

⁴For example, this occurs in the quantum Ising chain with Hamiltonian (4) and $h = 0$ since, after a Jordan-Wigner transformation, the Hamiltonian becomes a sum of independent quadratic fermionic terms [7,40].

spectrum [34,122,123]. Accordingly, for the Ising chain in Eq. (4), the analysis should be restricted to states within a single parity sector, $\mathcal{H}_{0+}$ or $\mathcal{H}_{0-}$. We performed such an analysis for generic values of g and h sufficiently far from the integrable limits of the model. For example, Fig. 1 shows that our numerical results obtained in the subspace $\mathcal{H}_{0+}$, for $g = h = 1$, nicely match the Wigner surmise (31). In contrast, computing the level-spacing distribution over the full $\mathcal{H}_0$ subspace would fail to reproduce the WD statistics, as it would reflect the underlying $\mathbb{Z}_2$ reflection symmetry which is present in $\mathcal{H}_0$. This would eventually lead to a variant distribution, appropriate for systems with this symmetry (see, e.g., Refs. [122,123]).

A useful parameter to quantify the closeness to the chaotic WD regime can be obtained from the average [26]

$$R = \langle r_n \rangle \quad (33)$$

of the ratios

$$r_n = \frac{\min[s_n, s_{n-1}]}{\max[s_n, s_{n-1}]} = \min\left[\frac{s_n}{s_{n-1}}, \frac{s_{n-1}}{s_n}\right]. \quad (34)$$

Contrary to the bare energy differences s_n , the ratios of consecutive level spacings are independent of the local density of states, thus unfolding is not necessary in this case. For that reason, the study of the statistics of the r_n has nowadays been established as the standard tool to ascertain the Hamiltonian spectral properties in quantum many-body systems. The corresponding distributions obtained from the Wigner surmise (31) and the Poissonian (32) statistics are respectively given by [34]

$$Q_W(r) = \frac{27}{4} \frac{r + r^2}{(1 + r + r^2)^{5/2}}, \quad Q_P(r) = \frac{2}{(1 + r)^2}. \quad (35)$$

The average quantity R takes the value [26,34] $R_W = 0.5307(1)$ for the level-spacing WD distribution of GOE (note that the approximate analytical Wigner surmise would give $R_W = 4 - 2\sqrt{3} \approx 0.5359$), and $R_P = 2 \ln 2 - 1 \approx 0.3863$ for the Poisson distribution.

In Fig. 2 we show some numerical results for the ratio R , as computed for the Ising chain with $g = 0.5$ (top), 1 (middle), and 1.5 (bottom), and several sizes, as a function of the longitudinal field h , for states in the sector $\mathcal{H}_{0+}$. The available data up to $L = 20$ provide a good evidence of convergence to the WD value R_W for large intervals of values of h , with oscillations that tend to attenuate with increasing L . In particular, we observe an apparent convergence for $0.3 \lesssim h \lesssim 1$ at $g = 0.5$, $0.1 \lesssim h \lesssim 1.5$ at $g = 1$, and $0.1 \lesssim h \lesssim 2$ at $g = 1.5$. As expected, deviations are found for small and large values of h , in view of the fact that the system approaches the integrable regime for $h = 0$ and $h \rightarrow \infty$. We also note that, with increasing L , the approach to the WD value R_W is faster for $g = 1$ and 1.5 than for $g = 0.5$, which may be explained by noting that the system tends to become integrable as soon as $g \rightarrow 0$. A reasonable hypothesis that may explain the behavior for small and large values of h is that the large- L convergence to the chaotic regime is nonuniform in h , so larger and larger sizes are necessary to observe convergence when approaching the integrability regimes occurring for $h \rightarrow 0$ and $h \rightarrow \infty$.

In the following Secs. IV, V, and VI discussing QQs, we will mostly concentrate on values of the field h for which

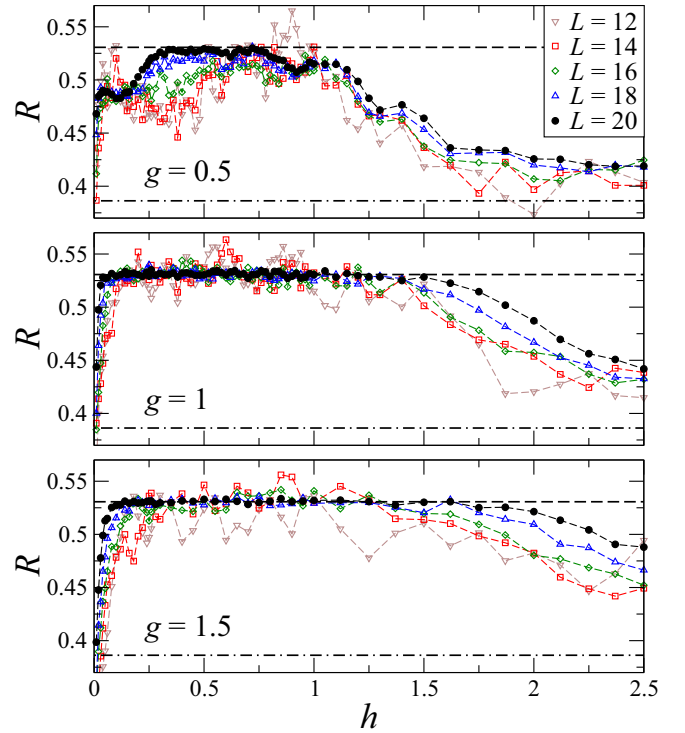


FIG. 2. The average value R of the ratios r_n [cf. Eqs. (33) and (34)] as a function of the longitudinal field h , for three fixed values of $g = 0.5$ (top), $g = 1$ (middle), and $g = 1.5$ (bottom), for the eigenstates in the sector $\mathcal{H}_{0+}$. Data are shown for different system sizes, as indicated in the legend. Horizontal lines correspond to the WD value $R_W = 0.5307$ (dashed line) and the Poisson value $R_P = 0.3863$ (dot-dashed line).

the system is chaotic (i.e., R is close to R_W). In these cases, asymptotic thermalization is expected to occur, provided that the initial ground state is somehow connected with the typical states of the post-QQ Hamiltonian. Otherwise, different dynamical regimes may emerge during the post-QQ evolution. When thermalization takes place, the predictions of the diagonal ensemble should agree with those of the microcanonical ensemble at the same energy density or, equivalently, with those of the canonical ensemble for a small subsystem at the corresponding temperature.

IV. QUANTUM QUENCHES IN THE DISORDERED PHASE

To avoid complications related to the presence of phase transitions, we first focus on the post-QQ dynamics in the disordered phase ($g > 1$). Specifically, we present results for $g = 1.5$, which is far from the CQT and the FOQT line. We consider a QQ protocol which starts from the ground state for $h_i = -h$, and then evolves with the Hamiltonian parameter $h_f = +h$ for $t > 0$ (the alternative protocol starting from $h_i = 0$ is not particularly interesting for $g > g_c$ and is therefore not considered). The time evolution of observables like the magnetization M [cf. Eq. (11)], and the excess bond energy K [cf. Eq. (12b)], is numerically computed up to $L = 28$, whereas the diagonal-ensemble observables M_D and K_D [cf. Eqs. (16) and (17)] are only obtained up to $L = 20$ since their determination requires the computation of the full spectrum of

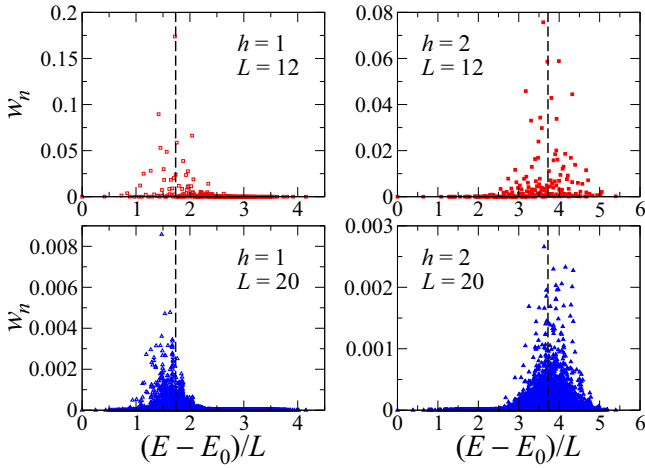


FIG. 3. The overlaps $w_n = |\langle \Phi_n(h) | \Phi_0(-h) \rangle|^2$ between the ground state $|\Phi_0(-h)\rangle$ of the Hamiltonian $\hat{H}(-h)$ and the eigenstates $|\Phi_n(h)\rangle$ with eigenvalues E_n of the post-QQ Hamiltonian $\hat{H}(h)$ vs the energy density $e_n = (E_n - E_0)/L$. Results for $h = 1$ (left panels) and $h = 2$ (right panels), and for system sizes $L = 12$ (upper panels) and $L = 20$ (lower panels). In all cases $g = 1.5$. The vertical lines indicate the average energy density $e = \sum_n w_n e_n \approx 1.735047$ (left) and $e \approx 3.72211379$ (right).

$\hat{H}(h)$. We primarily focus on chains with intermediate values of h (e.g., $h \approx 1$), since their spectrum is in the chaotic regime: for $g = 1.5$ and $L = 20$ a chaotic spectrum is observed for values of h in the interval $0.1 \lesssim h \lesssim 2$ (see the lower panel in Fig. 2).

A. The diagonal ensemble

To begin with, in Fig. 3 we plot the overlaps $w_n = |\langle \Phi_n(h) | \Phi_0(-h) \rangle|^2$ of the initial ground state $|\Phi_0(-h)\rangle$ with the eigenstates $|\Phi_n(h)\rangle$ in the relevant subsector $\mathcal{H}_{0+}$ of the Hilbert space of the post-QQ Hamiltonian $\hat{H}(h)$ for $h = 1$ (left) and $h = 2$ (right), and for two different sizes, $L = 12$ (top) and $L = 20$ (bottom). Our data show that the largest overlaps w_n correspond to larger energies as h increases (in Fig. 3, the corresponding mean energy density is indicated by a vertical dashed line), as expected. Moreover, as L increases, an increasing number of energy levels is close to the average energy and $\max_n w_n$ decreases.

To better quantify the delocalization of the values of w_n , we consider the diagonal-ensemble entropy defined in Eq. (23). In particular, we study the ratio

$$\bar{S}_D(L) \equiv \frac{S_D(L)}{\ln \mathcal{D}_{0+}(L)} \quad (0 \leq \bar{S}_D \leq 1), \quad (36)$$

where we recall that $\ln \mathcal{D}_{0+} \sim L$ in the large- L limit. Note that a value $\bar{S}_D \approx 1$ implies that the initial state is an approximately homogeneous superposition of the post-QQ Hamiltonian eigenstates in the Hilbert subspace $\mathcal{H}_{0+}$, while $\bar{S}_D \approx 0$ is obtained when only a finite number of states are relevant for large values of L . Figure 4 shows results for several values of h and chain sizes. The data up to $L = 20$ appear to approach the expected value $\bar{S}_D = 1$ as L increases, at least for $1 \lesssim h \lesssim 2$, although it is not straightforward to identify the

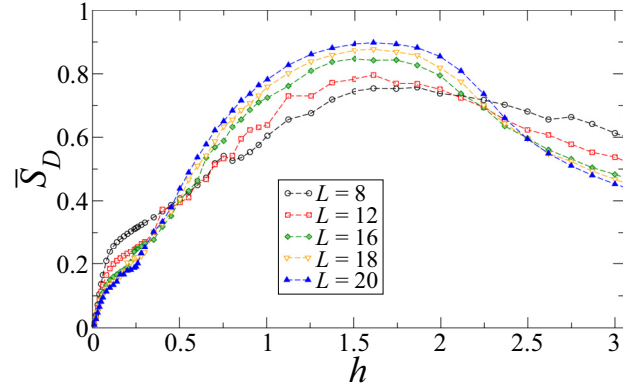


FIG. 4. The ratio $\bar{S}_D(L)$ between the diagonal-ensemble entropy $S_D(L)$ and its maximum value $\ln \mathcal{D}_{0+} \sim L$, as a function of the longitudinal field h , for different system sizes L . Here we set $g = 1.5$.

functional dependence with the size. In contrast, for $h \approx 0.5$, still belonging to the chaotic region, the values of $\bar{S}_D$ are significantly smaller. Nevertheless, convergence to $\bar{S}_D \rightarrow 1$ might still be possible for larger L . Indeed, the approach to the large- L limit should not be uniform as h decreases, potentially giving rise to nontrivial crossover effects for small h . Finally, note that the dependence of $\bar{S}_D$ on L is not monotonic with increasing h , likely because the system moves toward the integrability region.

As already discussed, the diagonal ensemble provides predictions for the asymptotic behavior of the magnetization and of the excess bond energy, which should converge to their diagonal-ensemble values M_D and K_D , defined in Eqs. (16) and (17). Data for M_D and K_D vs h are reported in Fig. 5 (top

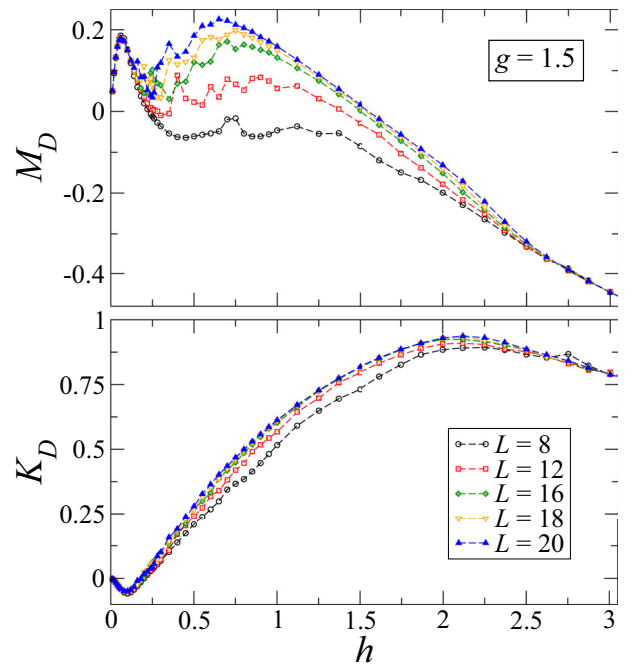


FIG. 5. The magnetization M_D (top) and excess-bond energy K_D (bottom) in the diagonal ensemble for $g = 1.5$ (disordered phase), for a QQ from $-h$ to h , and for different system sizes.

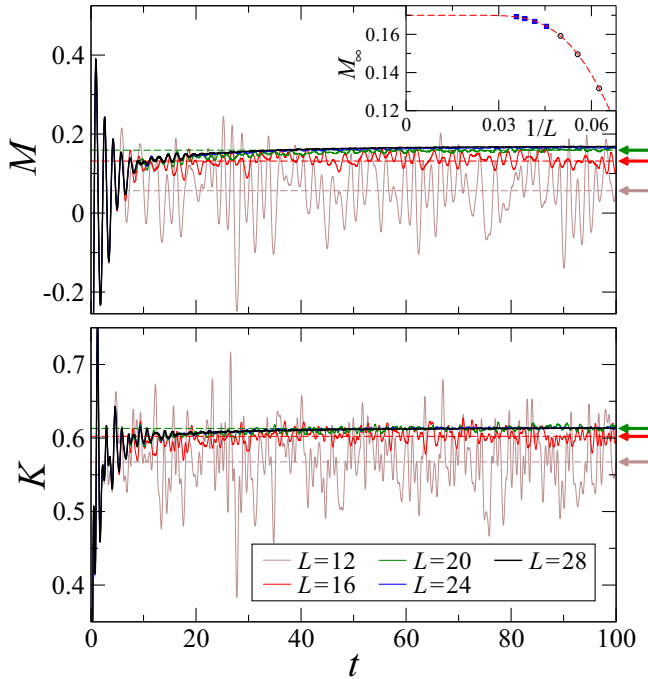


FIG. 6. Time behavior after a QQ from $h_i = -1$ to $h = 1$ for the magnetization (top) and the excess bond energy (bottom), at fixed $g = 1.5$, and for sizes up to $L = 28$ (see legend). Data up to $L = 20$ are compared against the asymptotic prediction of the diagonal ensemble (dashed horizontal lines and arrows with the corresponding color code). The inset shows the asymptotic long-time average $M_\infty(L)$ of the magnetization versus $1/L$, obtained either in the diagonal-ensemble (empty black circles) or by averaging $M(t)$ in the interval $t \in [150, 200]$ (filled blue squares); the dashed red line is a fit of the numerical data for $L \geq 16$, to $M_\infty(L) = M_\infty + ae^{-bL}$, which gives $M_\infty \approx 0.170$.

and bottom panels, respectively), for different system sizes. They appear to converge with increasing L , thus providing information on their thermodynamic limit. The convergence seems to be much faster for K_D , while size effects are larger for M_D , in particular around $h = 0.5$. It is, however, worth pointing out that the behavior of M_D and K_D is not monotonic with increasing h . In particular, the magnetization decreases for $h \gtrsim 1$ as L increases. This is related to the properties of the corresponding overlaps w_n between the initial state and the eigenstates of the post-QQ Hamiltonian, as their distributions move toward higher energies when increasing h , thus exhibiting an inversion of population (see Fig. 3).

B. The postquench evolution

We now discuss the post-QQ evolution of the magnetization $M(t)$ (top) and the excess bond energy $K(t)$ (bottom). Specifically, in Fig. 6 we show results for $h = 1$, and several sizes $L \leq 28$, up to relatively long times $t = O(10^2)$. Convergence has been ensured by setting a Runge-Kutta time step of $dt = 2.5 \times 10^{-3}$ and carefully verifying the stability of the results. With increasing L , the curves appear to converge to a large- L limit, which we interpret as the post-QQ evolution in the thermodynamic limit (see also footnote 2). Moreover, the approach to the long-time stationary values occurs with

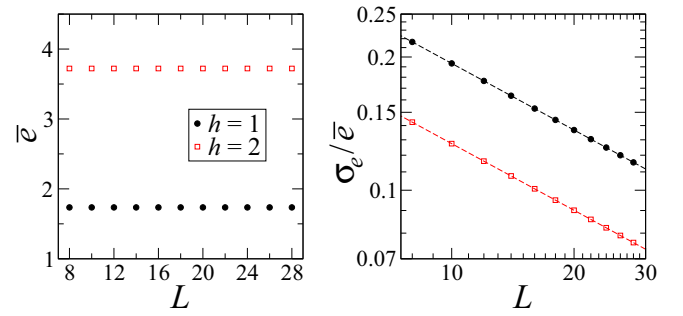


FIG. 7. The post-QQ mean energy per site $\bar{e}$ (left) and the normalized standard deviation $\sigma_e/\bar{e}$ (right) as a function of L , for two values of h and for fixed $g = 1.5$. In the left panel a linear scale is used, while in the right one data are plotted using a logarithmic scale on both axes.

fluctuations that decrease with increasing L . More quantitatively, the inset highlights the dependence on L of the asymptotic long-time average $M_\infty(L)$ of the magnetization, clearly converging to a finite value. To ensure reliable estimates of $M_\infty(L)$ up to $L = 28$, we performed time averages over the time interval $t \in [150, 200]$ (data not shown). The data shown in the inset of Fig. 6 suggest an exponential approach to the large- L limit, as $M_\infty(L) = M_\infty + ae^{-bL}$ (dashed red line is a fit to the numerical data with this formula).

We also compare the time dependence of $M(t)$ and $K(t)$ with the corresponding diagonal-ensemble estimates M_D and K_D [they are defined in Eqs. (16) and (17)], which are expected to provide the asymptotic large-time value (averaged over a sufficiently long-time interval). Our data are definitely compatible with these predictions (dashed lines in Fig. 6), up to lattice sizes $L = 20$: For sufficiently long times, M_D and K_D provide an excellent approximation of the time average of $M(t)$ and $K(t)$ for any L . Moreover, for the largest available values of L , the need of averaging over time in order to match the diagonal-ensemble predictions becomes less important, as fluctuations decrease.

C. Thermalization

The asymptotic stationary states, developed by the post-QQ unitary dynamics in the long-time and thermodynamic large- L limits exhibit several noteworthy properties. According to the thermalization scenario, the system should approach the statistical properties of a microcanonical ensemble at fixed energy density $e = \sum_n w_n e_n$. Equivalently, any sufficiently large subsystem should approach the canonical statistical behavior at a temperature corresponding to the given energy density, as discussed at the end of Sec. II C.

Here we present results of some of our simulations for fixed $g = 1.5$, considering longitudinal fields in the chaotic window $0.1 \lesssim h \lesssim 2$ identified for $L \approx 20$ (see the lower panel of Fig. 2). Figure 7 (left) displays the energy-density average $\bar{e}$ defined in Eq. (19) as a function of the chain length L , for two values of h . As is clearly visible, $\bar{e}$ varies very little with increasing L . We obtain $\bar{e} = 1.73504739$ for $h = 1$ and $L \geq 12$ and $\bar{e} = 3.72211379$ for $h = 2$ and $L \geq 8$. On the other hand, the ratio of the standard deviation σ_e defined in Eq. (19) and of

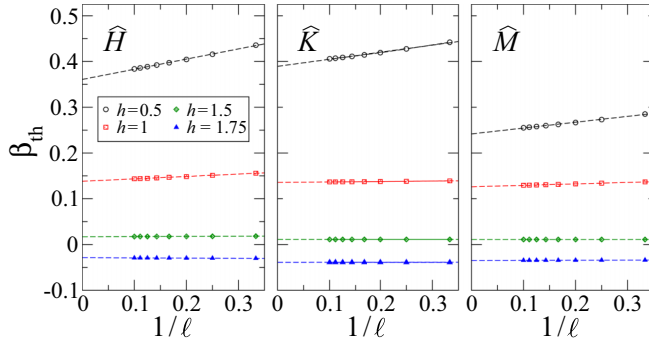


FIG. 8. The estimated effective inverse temperature β_{eff} assuming thermalization, using the energy (21) (left), K_D (middle), and M_D (right). Data are plotted against the inverse of the subsystem size ℓ , for $\ell \in [3, 10]$ and for a fixed system size $L = 20$. Here we fix $g = 1.5$, while the different sets are for various values of the longitudinal field h in the chaotic window (see legend). Straight dashed lines are linear fits in $1/\ell$ of the numerical data (symbols) to Eq. (37).

the average $\bar{e}$ decreases as $1/\sqrt{L}$ as expected [21] [see Fig. 7 (right)]. This confirms that, in the thermodynamic limit, the system approaches a microcanonical window.

We have also computed the effective inverse temperature β_{th} for subsystems of size $\ell \in [3, 10]$, using Eq. (21). Since the energy density is very stable, also β_{th} turns out to be essentially independent of L , with only a small residual dependence on ℓ , which can be parametrized as

$$\beta_{\text{th}}(\ell) = \beta_{\text{th}} + b/\ell, \quad (37)$$

as is visible from Fig. 8 (left). In fact, we recall that size corrections in systems with boundaries decrease as ℓ^{-1} (see, e.g., Ref. [124]). By fitting our numerical data for the largest available system size $L = 20$, we obtain the following estimates in the large- ℓ limit: $\beta_{\text{th}} \approx 0.364, 0.139, 0.017, -0.029$, respectively for $h = 0.5, 1, 1.5, 1.75$ (cf. dashed lines in left panel of Fig. 8).

Substantially consistent results are obtained by using the diagonal-ensemble values K_D and M_D for the excess bond energy and the magnetization (central and right panels of Fig. 8, respectively). However, these results appear to be less stable with increasing L , in particular at $h = 0.5$, presumably because the estimates K_D and M_D obtained for chains of size $L \leq 20$ do not show a complete convergence with respect to L (see Fig. 5). Namely, using the estimates of K_D for $L = 20$, we obtain $\beta_{\text{th}} \approx 0.394, 0.136, 0.011, -0.039$, for $h = 0.5, 1, 1.5, 1.75$ [we perform a linear extrapolation of the finite- ℓ data using Eq. (37), as in the energy-density analysis]. An analogous computation using M_D gives $\beta_{\text{th}} \approx 0.244, 0.127, 0.011, -0.035$, for the same values of h . An apparent larger discrepancy arises for M_D at $h = 0.5$, but this may be attributed to the pronounced finite-size corrections of M_D for this value of h (see Fig. 5, top panel). Nevertheless, we emphasize that the signs of β_{th} inferred from the various observables are consistent: for $h \lesssim 1.5$, we always find positive effective temperatures ($\beta_{\text{th}} > 0$), while the data for $h \gtrsim 1.75$ are compatible with an effective negative temperature ($\beta_{\text{th}} < 0$), indicating an inverted spectral population.

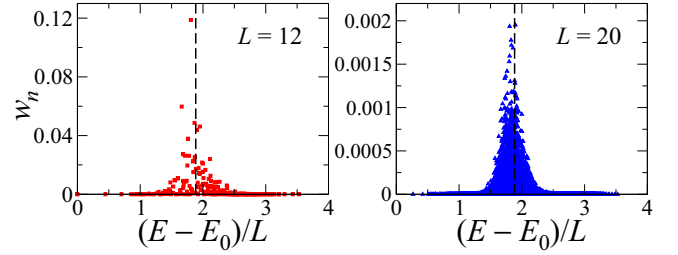


FIG. 9. The overlaps w_n among the initial ground state of the Hamiltonian $\hat{H}(-h)$ and the eigenstates of the post-QQ Hamiltonian $\hat{H}(h)$ vs the energy density $e_n = (E_n - E_0)/L$, for $g = 1$ and $h = 1$. The left and right panels report data for $L = 12$ and 20 , respectively. The vertical lines correspond to the energy-density average $\bar{e} \approx 1.8849419$.

The physical meaning of negative temperatures and the nature of the corresponding asymptotic stationary states have been discussed in Ref. [119].

The comparison of the effective inverse temperatures obtained from different local observables provides a robust evidence of thermalization, at least in the interval $0.5 \lesssim h \lesssim 2$, for system sizes $L \approx 20$. This interval is slightly smaller than the range $0.1 \lesssim h \lesssim 2$, in which the spectrum of $\hat{H}(g = 1.5, h)$ is clearly chaotic for $L \approx 20$, see the lower panel in Fig. 2. Indeed, for $h = 0.25$, fixing $L = 20$ and $\ell = 10$, the same analysis gives inconsistent results for the effective inverse temperature, namely, $\beta_{\text{th}} \approx 0.687, 1.773, 0.113$, determined from the energy density K_D , and M_D , respectively. We attribute this behavior to the limited system sizes accessible in our study.

V. QUANTUM QUENCHES ACROSS THE CQT

In this section, we study the QQ dynamics driven by a longitudinal field h at $g = 1$ (i.e., at the continuous Ising transition), focusing on QQs from $h_i = -h$ to $h_f = h$. We have not considered QQs starting from the ground state at $h_i = 0$, as this would introduce strong finite-size effects, due to the divergence of the correlation length at the critical point ($g = 1, h = 0$). As we shall see, the out-of-equilibrium post-QQ dynamical behavior of the chain is qualitatively similar to that observed in the disordered phase for $g > 1$, discussed in Sec. IV.

A. The diagonal ensemble

Figure 9 shows the overlaps $w_n = |\langle \Phi_n(h) | \Phi_0(h_i) \rangle|^2$ for $h = -h_i = 1$, in the $\mathcal{H}_{0+}$ subsector of the Hilbert space, for $L = 12$ (left) and $L = 20$ (right). As observed in Fig. 3 for $g = 1.5$, the data approach a Gaussian shape as L increases, centered around the energy-density average, which is essentially independent of L .

To quantify the delocalization of the w_n values, we again consider the diagonal-ensemble entropy defined in Eq. (23). Figure 10 reports the ratio $\bar{S}_D$, as defined in Eq. (36), as a function of h , for $g = 1$ and for several chain sizes up to $L = 20$. The behavior is qualitatively similar to the one observed for $g = 1.5$, with a nonmonotonic dependence on h . Also in this case, $\bar{S}_D$ is expected to converge to one in the

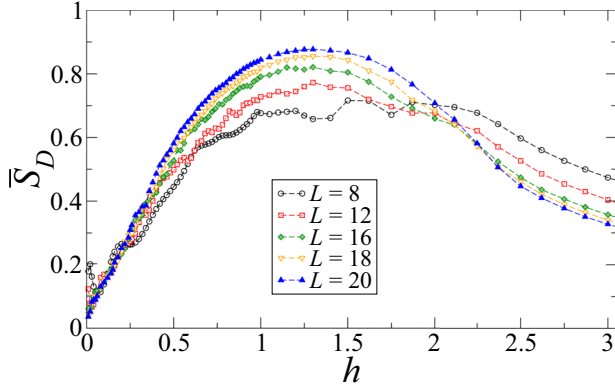


FIG. 10. The rescaled diagonal-ensemble entropy $\bar{S}_D(L)$, normalized by the logarithm of the dimension of the relevant Hilbert subspace $\mathcal{D}_{0+}$, as a function of h and for fixed $g = 1$ and different sizes (see legend).

large- L limit, provided that h lies in an extended intermediate region where integrability is clearly broken. Comparing Fig. 10 with Fig. 4, notice, however, a slight shift of this window toward smaller values of h (being roughly $0.5 \lesssim h < 2$, for $g = 1$). This likely indicates that, for $g = 1$, deviations from a fully chaotic behavior emerge earlier, as h increases, compared with the $g = 1.5$ case (see the central and lower panels of Fig. 2).

Results for the diagonal-ensemble magnetization M_D and excess bond energy K_D are shown in Fig. 11 for $L \leq 20$, as a function of the longitudinal field h . The behavior is qualitatively similar to that reported in Fig. 5 for $g = 1.5$, with a nonmonotonic dependence on h , which suggests an

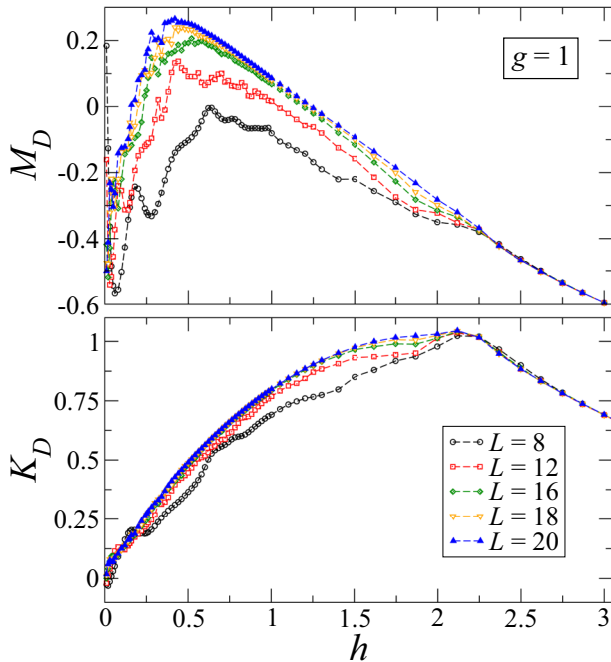


FIG. 11. The diagonal-ensemble magnetization M_D (top) and excess bond energy K_D (bottom) for $g = 1$, $h = -h_i = 1$, and sizes $L \leq 20$.

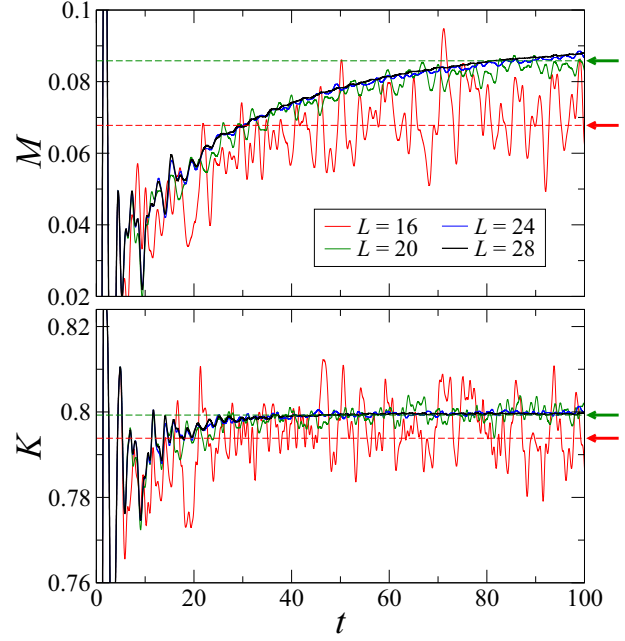


FIG. 12. Post-QQ time evolution of the magnetization (top) and the excess bond energy (bottom), for $g = 1$, $h = -h_i = 1$, different sizes (see legend). The dashed horizontal curves and arrows indicate the corresponding diagonal-ensemble predictions for $L \leq 20$.

inversion of the population across different energy levels (see also Fig. 9). A reasonable convergence with increasing L is visible for all values of h , although finite-size corrections of varying magnitude are present.

B. The postquench evolution

Turning to the analysis of the post-QQ evolution, Fig. 12 shows the time dependence of the magnetization (top) and of the excess bond energy (bottom) for $h = 1$. As L increases, the curves converge to what we consider the infinite-size limit, as already observed in the disordered phase (see Fig. 6). To better display this convergence, we use a small y-axis interval in both panels (thus, some large oscillations occurring for small values of t are not properly displayed) and report results only up to $t = 100$. It is, however, worth noting that it is not straightforward to predict the average long-time behavior. Although we expect it to be correctly predicted by the diagonal ensemble, our calculations in this ensemble are limited to $L \leq 20$. In fact, for the larger system sizes the dynamics is still dominated by a transient behavior (see, for instance, the blue and black curves for the magnetization for $L = 24$ and 28 , respectively). The duration of this transient depends both on the monitored observable and on the Hamiltonian parameters g and h . Moreover, the onset of the asymptotic long-time regime is pushed to progressively later times as L increases. Nonetheless, we verified that convergence with L is still clearly visible up to $t \approx 500$.

C. Thermalization

We have also repeated the analysis reported in Sec. IV C for QQs in the disordered phase, studying the asymptotic

thermalization for QQs at the critical point. Specifically, we have computed the effective inverse temperature β_{th} using energy, Eq. (21), and the diagonal-ensemble excess energy K_D and magnetization M_D . Across the entire chaotic window $0.1 \lesssim h \lesssim 1.5$ for systems of size $L \approx 20$ (cf. Fig. 2, middle), the results are qualitatively similar to those obtained for $g = 1.5$. In particular, once L is fixed, the residual dependence on the subsystem size ℓ is still accurately captured by Eq. (37).

As an illustration, for a system of size $L = 20$, we obtained the following large- ℓ estimates of the effective inverse temperature (data not shown): for $h = 0.5$, we found $\beta_{\text{th}} \approx 0.331, 0.334, 0.299$ using, respectively, the energy, the excess energy, and the magnetization; for $h = 1$, we found $\beta_{\text{th}} \approx 0.086, 0.085, 0.074$. They may be considered in substantial agreement. We also note the better agreement among the three estimates for $h = 0.5$, compared with the case at fixed $g = 1.5$. This is likely because for $g = 1$ the onset of the chaotic regime at relatively small values of h occurs more rapidly as L increases. Therefore, the analysis of the post-QQ dynamics across the CQT for systems up to $L = 20$ provides strong evidence of the eventual thermalization in the large-time limit, in the region $0.1 \lesssim h \lesssim 1.5$ where the spectrum of $\hat{H}(h)$ for $g = g_c$ develops a chaotic regime (see Fig. 2).

D. Comparison with QQs in the disordered phase

We conclude this section by comparing the $g = 1$ results with those obtained far from the critical point in the disordered phase (see Sec. IV, for the data at $g = 1.5$). The post-QQ evolutions of the magnetization and the excess bond energy, as well as their asymptotic values obtained from the diagonal ensemble, differ only quantitatively. We do not observe any qualitative difference in the post-QQ behavior, when keeping h fixed in the large- L limit, particularly when the spectrum of the post-QQ Hamiltonian lies in the chaotic regime. This confirms that these QQs, which involve an extended amount of energy exchange, do not effectively probe the low-energy large-scale modes responsible for the critical behavior unlike soft QQs, as thoroughly discussed in Appendix A 1.

VI. QUANTUM QUENCHES ACROSS FOQTS

The out-of-equilibrium quantum evolution induced by QQs across the FOQT line presents novel and unexpected features that warrant a detailed investigation. We primarily consider the paradigmatic case in which the transverse field is kept fixed at $g = 0.5$; no significant qualitative differences are expected as long as $g < 1$. Most of the results presented below refer to the quench protocol from $h_i = -h$ to $h_f = h$; we also examine an alternative protocol that starts from $h_i = 0^-$, finding no substantial differences. As discussed in Sec. III B, sizes up to $L = 20$ are sufficient to show that, for $g = 0.5$, the spectrum lies in the chaotic regime at least in the interval $0.3 \lesssim h \lesssim 1$ (see the upper panel of Fig. 2). As we shall show, the resulting dynamics is more complex than that observed for QQs in the disordered phase or across the CQT. Finally, in Sec. VID we discuss additional issues that arise in the diagonal-ensemble description following a QQ of the longitudinal field, when the transverse field is varied.

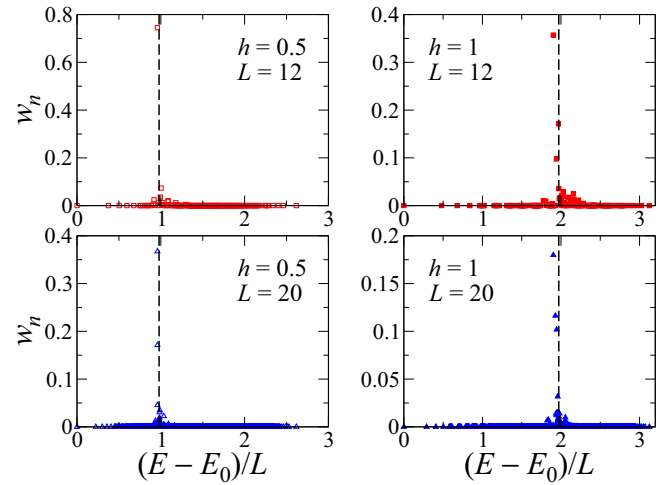


FIG. 13. The overlaps w_n between the ground state of the Hamiltonian $\hat{H}(-h)$ and the eigenstates of the post-QQ Hamiltonian $\hat{H}(h)$ vs the energy density $e_n = (E_n - E_0)/L$, at fixed $g = 0.5$ and for $h = 0.5$ (left panels) or $h = 1$ (right panels). In the upper panels we show results for $L = 12$, while in the lower panels we consider $L = 20$. Vertical lines correspond to the energy-density averages $e \approx 0.97935239$ (left) and 2.9693129 (right).

A. The diagonal ensemble

A first notable difference that emerges when crossing a FOQT, compared with the previously analyzed QQs, concerns the overlaps w_n between the initial ground state and the eigenstates of the postquench Hamiltonian $\hat{H}(h)$. Figure 13 displays the overlaps w_n , for $h = 0.5$ (left) and $h = 1$ (right), computed in the subsector $\mathcal{H}_{0+}$ of the Hilbert space, for $L = 12$ (top) and $L = 20$ (bottom). At first glance, it is evident that the overlaps have a sharp peak, with only few eigenvalues w_n being significantly different from zero (fewer than 10, approximately). This behavior stands in stark contrast with what observed for $g < 1$ and $g = 1$, reported in Figs. 3 and 9, respectively. Indeed, for $L = 20$ the y-axis values are here roughly two orders of magnitude larger than those in Figs. 3 and 9 (see the blue histograms). We also observe that the peak shifts toward higher energies as h increases, as observed in the previous cases. Notably, the presence of only a small number of relevant eigenstates of the post-QQ Hamiltonian, resulting in overlaps w_n with a pronounced maximum, is not a consequence of spectral chaos. Indeed, the same behavior is observed (see the Appendix) in the OFSS regime of a chain with oppositely fixed boundary conditions for small g and h , where the system is manifestly integrable. This feature is therefore only related with the presence of the FOQT for $h = 0$.

To quantify the delocalization of the w_n values, it is again useful to examine the diagonal-ensemble entropy defined in Eq. (23), and in particular the ratio $\tilde{S}_D$ in Eq. (36). Data for $g = 0.5$ are shown in Fig. 14, for various values of h and chain sizes up to $L = 20$. The resulting behavior is radically different from that observed for $g = 1.5$ (Fig. 4) and $g = 1$ (Fig. 10). In fact, after an initial region for $h \lesssim 0.2$, where $\tilde{S}_D$ appears to be nearly independent of L , wild oscillations as a function of h set in. These irregular features disappear

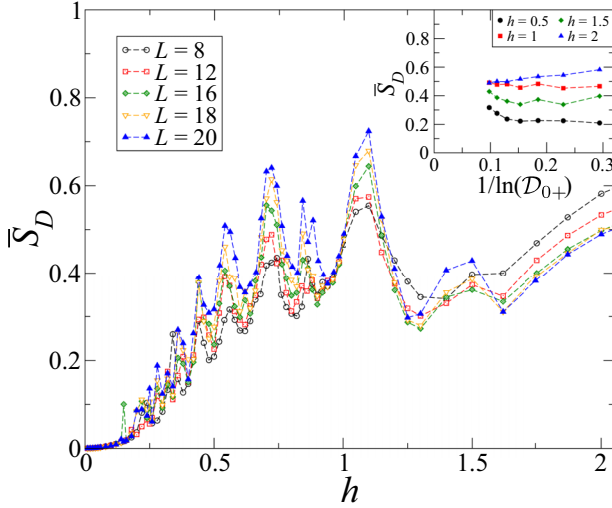


FIG. 14. The rescaled diagonal-ensemble entropy $\bar{S}_D(L)$, normalized by the logarithm of the dimension of the relevant Hilbert subspace $\mathcal{D}_{0+}$, as a function of h , for different sizes L . Here we fix $g = 0.5$. The inset presents the same data plotted against $1/\ln \mathcal{D}_{0+}$, for selected values of h , highlighting that numerical results for $L \leq 20$ remain far from the hypothetical thermodynamic-limit value of one, which would appear for $1/\ln \mathcal{D}_{0+} \rightarrow 0$.

only for large values of h ($h \gtrsim 2$, not shown). Moreover, the normalized diagonal-ensemble entropy typically remains far from one, even at $L = 20$ (see, in particular, the inset, showing results for four representative values of h), consistently with the very peaked overlaps w_n discussed above. Additional comments on the behavior of the overlaps w_n as a function of g are presented in Sec. VID.

A clear evidence of the presence of these singularities is visible in Fig. 15, where we report the magnetization (top) and the excess bond energy (bottom) as a function of h , obtained from the diagonal ensemble corresponding to the QQ from $-h$ to h . Unlike the cases for $g \geq 1$, we note that there is a large interval of values of h (i.e., $0.3 \lesssim h \lesssim 1.5$), where the diagonal-ensemble values M_D and K_D oscillate wildly as a function of h . This implies that small changes of h give rise to large changes of the asymptotic long-time behavior. A similar high sensitivity occurs also as a function of L , for some specific values of h in such interval (as is the case for $h = 0.5$ highlighted in Fig. 18 and discussed below). This behavior appears to be qualitatively different from that observed across the CQT and in the disordered phase.

It is likewise instructive to assess the robustness of the above features by considering an alternative QQ protocol, in which the system is initialized in the ground state of the quantum Ising Hamiltonian with an infinitesimal longitudinal field $h_i = 0^-$ (we typically take $h_i = -10^{-3}$, which is sufficient for $L \lesssim 20$). For $g < g_c$, this weak field explicitly breaks the $\mathbb{Z}_2$ symmetry of the model, thereby selecting a ground state with negative magnetization ($M < 0$). The system then evolves unitarily with a finite $h > 0$, while keeping g fixed. Some numerical results for the diagonal-ensemble magnetization and excess bond energy at $g = 0.5$ are shown in Fig. 16 as a function of h . Although we observe small quantitative

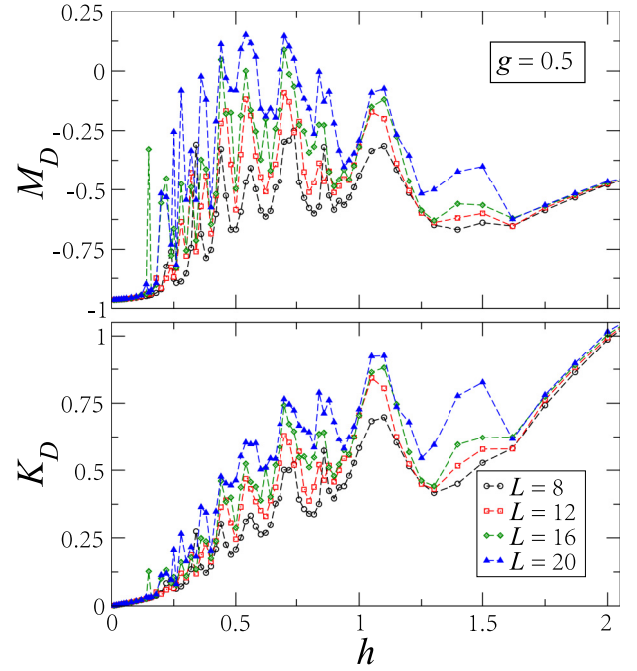


FIG. 15. The magnetization (top) and excess bond energy (bottom) across the FOQT for $g = 0.5$, obtained from the diagonal ensemble for QQs from $h_i = -h < 0$ to $h_f = h > 0$ and for different system sizes (see legend).

differences with the QQ protocol from $h_i = -h$ to h , especially for the largest displayed values of h , the behavior of M_D and K_D is qualitatively the same. In particular, both observables exhibit a pronounced sensitivity to h and L , which we identify as a hallmark of QQs crossing FOQTs. The independence of the results on the starting value of the longitudinal field, as long as it selects a negatively magnetized ground state, that is, for any $h_i < 0$, can be traced back to the fact that the corresponding ground states differ only over length scales of the order of the correlation length ξ , which is finite for $g < g_c$ (for $g = 0.5$ it is of order one in units of the lattice spacing).

B. The postquench evolution

The postquench evolution of the magnetization $M(t)$ for $g = 0.5$ is reported in Fig. 17, for system sizes up to $L = 28$ and for three distinct values of h (the QQ is from $h_i = -h$ to $h_f = +h$). For $h = 1.5$ (top), the dynamics is characterized by large oscillations around stable values, indicating that a few energy levels dominate the post-QQ evolution, even for the largest sizes considered (the data appear to converge in the large- L limit, at least up to $t \approx 100$). This can be easily explained by recalling that the value $h = 1.5$ is outside the chaotic region of the spectrum of the post-QQ Hamiltonian for $L = 20$ (see Fig. 2). For $h = 1$ (middle), the oscillations become less regular, although their amplitude still decreases as L increases. Finally, for $h = 0.5$ (bottom), the convergence to the asymptotic stationary state appears significantly delayed, compared to the other values of h . In fact, the curves for various system sizes at $t \gtrsim 50$ show no signs of convergence,

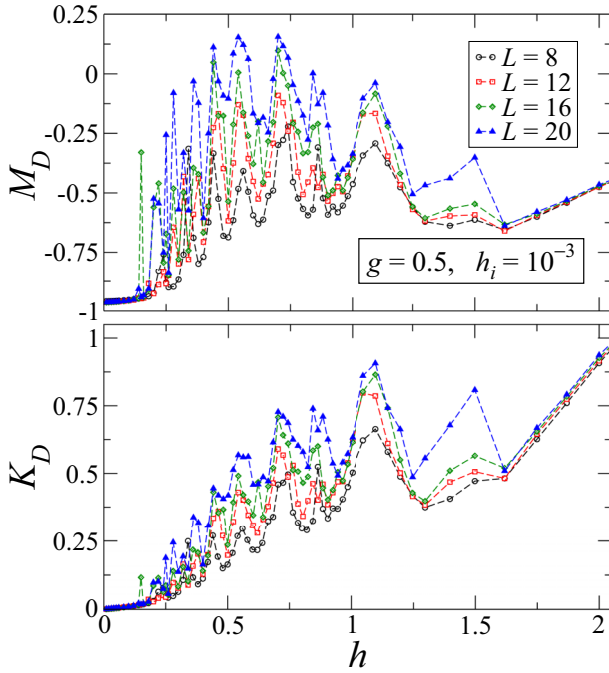


FIG. 16. The magnetization (top) and the excess bond energy (bottom) for $g = 0.5$, obtained from the diagonal ensemble for QQs from $h_i = -10^{-3}$ to $h_f > 0$ and for different system sizes (see legend).

indicating that sizes larger than $L = 28$ might be required to clarify the large- L and long- t behavior.

The above scenario is confirmed by comparing the post-QQ evolutions with the diagonal-ensemble predictions, shown in Fig. 18. In particular, for $h = 1$ (top), the curves on the scale of the figure are already close to convergence, both in time and with L , so that the resulting picture is analogous to those previously discussed for $g \geq 1$. In contrast, for $h = 0.5$ (bottom), while the curves for $L = 12$ and 16 appear to oscillate around the asymptotic long-time behavior predicted by the diagonal ensemble already for $L \approx 10$ (with fluctuations that decrease as L increases), this is not the case for $L = 20$. For that size, the transient time is significantly longer and appears to reach scales of order $t \sim 10^3$. Moreover, the asymptotic value exhibits a sudden jump as a function of L . These features are directly related to the spikes that may irregularly appear in Figs. 15 and 16, at seemingly random values of h in the interval $0.3 \lesssim h \lesssim 1.5$, and that also depend on the system size.

C. Thermalization issues

The apparent irregular behavior of M_D and K_D also reflects in the corresponding estimates of the effective inverse temperature, even in regimes where asymptotic thermalization would normally be expected. Recall that, for $L \approx 20$ and $g = 0.5$, a quantum chaotic dynamics is likely to occur for $0.3 \lesssim h \lesssim 1$ (see Fig. 2, top). Within this interval, we observe significant discrepancies among the estimated effective inverse temperatures β_{eff} obtained from different observables. As an illustrative example, for $g = h = 0.5$ and $L = 20$, we obtained the following large- ℓ estimates (data not shown):

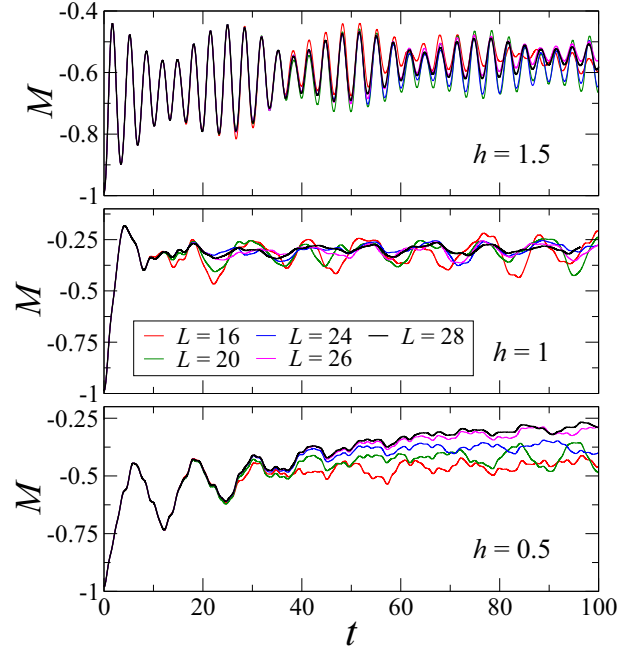


FIG. 17. Time dependence of the magnetization after a QQ from $h_i = -h$ to $h_f = h$, at fixed $g = 0.5$, and for $h = 1.5$ (top), $h = 1$ (middle), and $h = 0.5$ (bottom). The various curves are for different lattice sizes up to $L = 28$ (see legend).

$\beta_{\text{th}} \approx 0.318, 0.437, -0.277$ from the energy, the excess bond energy, and the magnetization, respectively. Similar inconsistencies occur at larger longitudinal fields; for instance, at $h = 0.8$, we found $\beta_{\text{th}} \approx 0.125, 0.265, 0.489$.

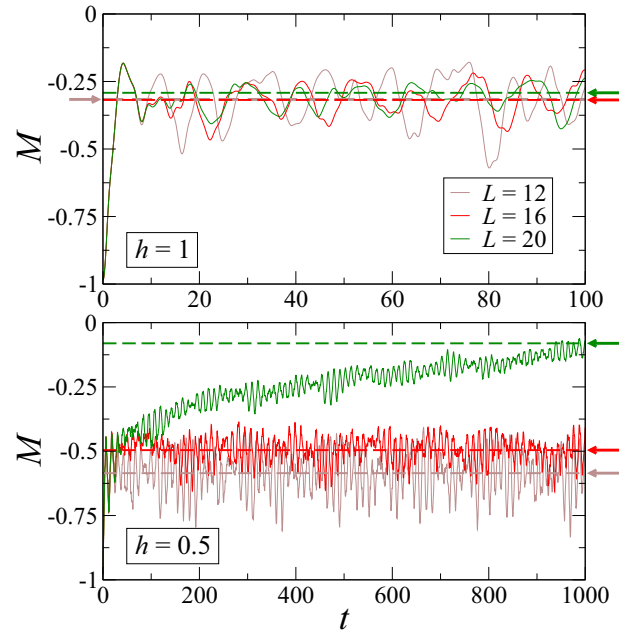


FIG. 18. Time dependence of the magnetization, compared with the diagonal-ensemble predictions (arrows and horizontal dashed lines with the corresponding color code), up to $L = 20$. We fix $g = 0.5$, while $h = 1$ (top) or $h = 0.5$ (bottom).

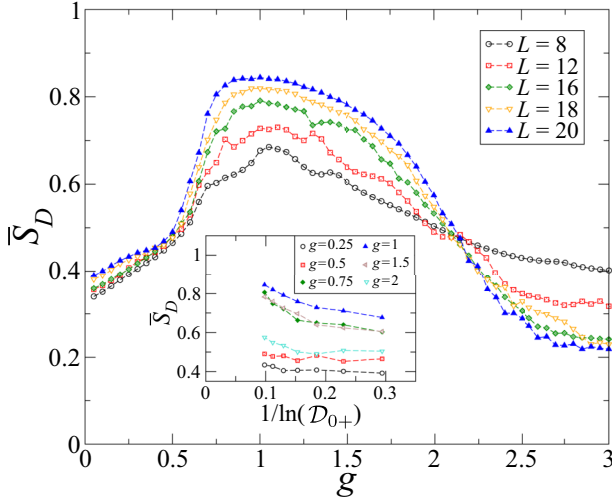


FIG. 19. The ratio $\bar{S}_D \equiv S_D(L)/\ln D_{0+}$ between the diagonal entropy and its maximum value $\ln D_{0+}$ as a function of g , after QQs from $-h$ to h with $h = 1$. The inset shows the same data for some fixed value of g (see legend) versus $1/\ln D_{0+} \sim 1/L$.

The discrepancies are striking, indeed, even the signs differ. These results indicate that, for system sizes up to $L \approx 20$, the post-QQ dynamics across FOQTs remains far from asymptotic thermalization, despite the postquench Hamiltonian already being in the chaotic regime. In this respect, the mere presence of spectral chaos in the post-QQ Hamiltonian should not be regarded as a sufficient condition for eventual thermalization. This suggests that QQ protocols across the FOQT line may not lead to thermalization even in the thermodynamic limit because the initial ground state corresponding to opposite longitudinal fields is not effectively connected with the typical states of the post-QQ Hamiltonian, when they are separated by a FOQT. Alternatively, much larger sizes might be required to observe thermal behavior, and the present results could simply reflect preasymptotic effects.

D. Comparison of results for various values of g

We conclude our numerical analysis by comparing results for different values of the transverse field g , when quenching the longitudinal field from $-h$ to $+h$, with $h = 1$ in the chaotic region. Figure 19 shows that the rescaled diagonal-ensemble entropy $\bar{S}_D$ exhibits a rather complex and nonmonotonic dependence on g at fixed L , reaching a maximum near the critical point ($g \approx 1$). Most importantly, $\bar{S}_D$ is significantly smaller for $g \lesssim 0.5$, i.e., deep in the FOQT region, at least for lattice sizes up to $L = 20$. In particular, the data around $g = 1$ (for $0.75 \lesssim g \lesssim 1.5$) appear to converge toward the maximal value $\bar{S}_D = 1$ as L increases, whereas for data deep in the FOQT region ($g \lesssim 0.5$) there is no indication of convergence to this value (see the inset, which displays $\bar{S}_D$ as a function of $1/\ln D_{0+}$, for several values of g). A similarly puzzling behavior emerges in the region $g \gtrsim 2$, where the apparent nonmonotonic trend likely signals a departure from the fully chaotic WD behavior at large transverse fields.

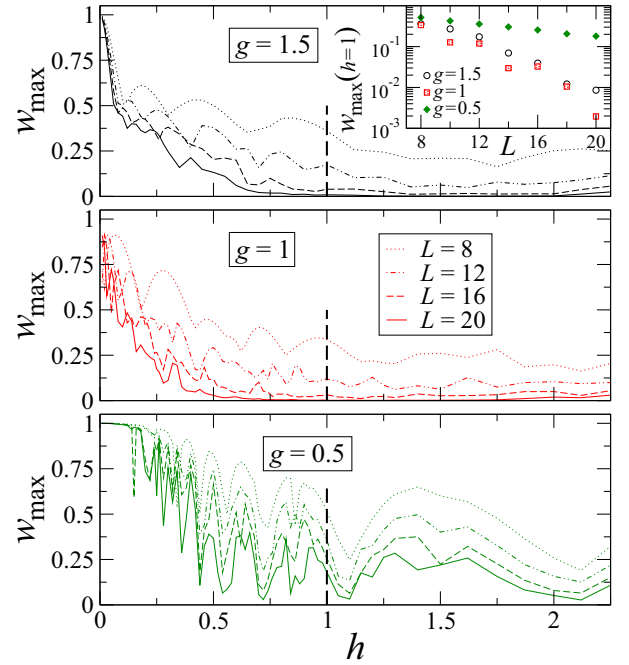


FIG. 20. The maximum overlap $w_{\max}$ as a function of h , for three values of g : 1.5 (top), 1 (middle), and 0.5 (bottom). Different curves are for various sizes (see legend). The inset in the top panel reports the dependence of $w_{\max}$ on L in a log-linear scale, for fixed $h = 1$ and the same values of g .

As already noted, the diagonal-ensemble entropy $\bar{S}_D$ also provides information about the overlaps w_n between the initial state and the eigenstates of the post-QQ Hamiltonian. Larger values of $\bar{S}_D$ indicate a more homogeneous distribution among the eigenstates (see, e.g., Fig. 3 for $g = 1.5$ and Fig. 9 for $g = 1$), whereas smaller values typically correspond to more sharply peaked distributions (see, e.g., Fig. 13 for $g = 0.5$). To quantify this different behavior, in Fig. 20 we report the maximum value $w_{\max}$ as a function of h , for $g = 1.5$ (top), $g = 1$ (middle), and $g = 0.5$ (bottom). On average, $w_{\max}$ decreases with increasing h : for instance, comparing the left and right panels in Fig. 3 (corresponding to $h = 1$ and 2), one sees that, at fixed L , the right-hand panels have smaller vertical scales and somewhat flatter distributions. Moreover, as expected, $w_{\max}$ decreases as L increases; however, the decay rate depends markedly on g , being significantly slower for $g = 0.5$ than for $g \geq 1$, although it remains at least exponential in L . This behavior is clearly illustrated in the inset of the top panel, which shows $w_{\max}$ versus L at fixed $h = 1$, for the three values of g . Similar conclusions are obtained when considering the average of the first few largest values of w_n .

VII. CONCLUSIONS

We have investigated the out-of-equilibrium quantum dynamics induced by QQs of the longitudinal field h in quantum Ising chains at fixed transverse field g , across the line $h = 0$. For this purpose, we focus on QQs across the CQT ($h = 0$ and $g = g_c = 1$) and the FOQT line ($h = 0$ and $g < g_c = 1$). The dynamics starts from the ground state of the Ising Hamiltonian $\hat{H}(g, h)$ with $h_i = -h < 0$; then, at time $t = 0$, the

longitudinal field is suddenly quenched to $h_f = h > 0$, while keeping g fixed. We analyze the subsequent unitary dynamics for $t > 0$, focusing on the thermodynamic limit, which is identified by examining the convergence of the post-QQ observables, such as the magnetization (11) and the excess bond energy (12), as the system size L increases. Our analysis primarily focuses on values of h for which the spectrum of the post-QQ Hamiltonian lies in the chaotic regime, as signaled by a level-spacing distribution following the WD statistics of the GOE. This occurs for sufficiently large (but not too large) values of h . Under these conditions, the system is generally expected to thermalize at long times, provided that the initial state, namely, the ground state at $h_i < 0$, is effectively connected with the typical eigenstates of $\hat{H}(g, h)$.

We mostly focus on QQs involving sudden changes of the longitudinal field h , such that the spectrum of the post-QQ Hamiltonian $\hat{H}(g, h > 0)$ lies in the chaotic regime. We investigate whether, under these more general conditions, qualitative signatures of the universal low-energy features of the transitions can still be observed in the post-QQ, energy-conserving, unitary dynamics. In particular, we analyze the system behavior under QQs across CQT ($g = 1$) and FOQTs ($g < 1$) for finite h in the large- L limit, comparing it with the behavior observed for QQs in the disordered phase ($g > 1$), in order to identify qualitatively distinct behaviors that may characterize QQs across CQTs and FOQTs.

We present a detailed analysis of the overlaps w_n defined in Eq. (14), comparing their behavior for QQs at fixed $g > g_c$, $g = g_c$, and $g < g_c$. A similar analysis has been performed for the corresponding diagonal-ensemble density matrix $\hat{\rho}_D$ defined in Eq. (15), which is expected to describe the long-time post-QQ evolution of local observables, such as the magnetization and the excess bond energy. Our data confirm that the diagonal-ensemble description becomes accurate in most cases after a relatively short time, i.e., without requiring exponentially large times, as L increases. When an oscillatory behavior is present, particularly for small chain sizes, the diagonal ensemble still provides an effective description of time averages taken over sufficiently large time windows. We also observe a general suppression of these oscillations as the chain size increases.

To compare the main features of the overlaps w_n in the disordered region, across the CQT, and across the FOQTs, we determine the diagonal-ensemble entropy S_D [cf. Eq. (23)], the largest values of w_n [cf. Eq. (24)], and the effective temperature, which can be defined assuming an eventual local thermalization [cf. Eq. (21)]. We do not observe significant qualitative differences between the overlaps w_n associated with QQs in the disordered region and across the CQT. This confirms that QQs, which involve an extended amount of energy exchange, do not effectively probe the low-energy large-scale modes responsible for the critical behavior unlike soft QQs.

On the other hand, the behavior at the FOQTs ($g < 1$) exhibits features that sharply distinguish it from the behavior for $g \gtrsim 1$. In particular, the diagonal-ensemble entropy S_D remains significantly smaller, at least for the sizes for which we could analyze the full Hamiltonian spectrum (see, for example, Fig. 19), while the decay of the largest values of w_n is considerably slower (cf. Fig. 20). Substantial differences

also emerge when comparing the observables obtained from the diagonal ensemble. The post-QQ long-time dynamics at FOQTs appears to be more intricate (at least in the chaotic regime, which is observed for $0.3 \lesssim h \lesssim 1$), showing a pronounced sensitivity to small changes of h . In addition, a stationary behavior is observed significantly later than in the QQ protocols at $g \geq 1$ (see Figs. 17 and 18). We also find significant discrepancies among the effective temperatures obtained from different local observables (such as the energy density, the excess bond energy, and the magnetization) using the diagonal ensemble up to $L = 20$, unlike for $g \geq 1$. This mismatch persists even in the region where the post-QQ Hamiltonian exhibits clear signatures of spectral chaos. These findings suggest that QQ protocols across the FOQT line may not lead to thermalization even in the thermodynamic limit, as the ground state on the opposite side of the FOQT may not be effectively connected with the typical eigenstates of the post-QQ Hamiltonian. This may be related to the effective separation of the relevant quantum states at first-order transitions in the thermodynamic limit. However, since the post-QQ dynamics is governed by a Hamiltonian with finite $h > 0$, we cannot exclude that much larger sizes are required to observe thermal behavior, or that long-lived prethermalization regimes exist, with genuine thermalization occurring only at sufficiently large sizes and at very long times.

Therefore, the behavior of QQs across the FOQTs of the quantum Ising chains cannot be considered as completely understood and we believe that it would deserve further investigation, to strengthen our understanding of the underlying dynamical mechanisms. In this respect, numerical studies of longer chains and time evolutions could be very useful. However, a significant improvement represents a very hard task, as the computations require either the calculation of the full spectrum to construct the diagonal-ensemble matrix density, or the time evolution over very long intervals.

We also mention that QQs across the FOQTs of quantum Ising models have been discussed in terms of spinodal-like quantum nucleation phenomena, assuming the emergence of some kind of metastabilities during the out-of-equilibrium evolution, which may be interpreted as the decay of a false vacuum (see, e.g., Refs. [72,73,113–116]). However, we believe that such interpretations may be appropriate only in some specific regimes, i.e., when the energy injected during the quench is small, or when the Hamiltonian parameters are varied quasiadiabatically, as in Kibble-Zurek protocols [9,11,12] (see, e.g., Refs. [75,76]).

FOQTs in quantum Ising models are characterized by a discontinuity in the magnetization [cf. Eq. (5)], when varying the longitudinal field. Different behaviors may arise in other systems featuring FOQTs with a discontinuity in the energy density, as in the quantum q -state Potts models for sufficiently large q [125–128] (in particular for $q > 4$ in one dimension), which are driven by a homogeneous transverse field. Therefore, it would be tempting to extend the study to QQs across FOQTs driven by external fields that do not break the global symmetry of the system.

We finally remark that all the numerical results presented here have been obtained for relatively small system sizes ($L \leq 28$). Using exact-diagonalization methods, we retained the full Hilbert-space subsector relevant to the quantum

dynamics, which prevented us from accessing larger sizes. Nonetheless, this allowed us to extract several quantitative results on the post-QQ behavior of the system in the thermodynamic limit, irrespective of the entanglement growth, particularly when crossing quantum transitions of different orders. The outcomes of this work suggest that, in some circumstances, achieving high accuracy may be more important than merely scaling to large sizes. Moreover, this observation may be useful in guiding experimental tests using, for instance, ultracold atoms in optical lattices [52], trapped ions [129], Rydberg atoms in arrays of optical microtraps [130], or even quantum-computing architectures based on superconducting qubits [131]. Recent experiments have begun to probe the dynamics, the excitation spectrum, as well as nonequilibrium scaling properties of quantum Ising-type chains [48,132–137], potentially opening new avenues to observe and investigate the post-QQ dynamical behavior anticipated here.

The authors of this paper were ordered alphabetically.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: SOFT QUENCHES ACROSS TRANSITIONS

In this Appendix we review the dynamic scaling behavior of soft QQs, in which the longitudinal magnetic field is small and decreases with increasing chain size [9,77]. We discuss the OFSS behavior, the underlying physical mechanism, and, most importantly, why these scaling behaviors are not expected to be recovered in the limit $h \rightarrow 0$ of the QQs considered in this work.

1. Across the CQT

At the CQT ($g = 1$) one observes a general scaling behavior if one considers the limits $L \rightarrow \infty$, $t \rightarrow \infty$, and $h \rightarrow 0$, keeping the scaling variables [9,138]

$$\Phi = hL^{y_h}, \quad \Theta = tL^{-z}, \quad \Psi = \Phi^{z/y_h} \Theta = h^{z/y_h} t \quad (\text{A1})$$

fixed (in the case $h_i \neq -h$, one should add a further dependence on the ratio $r_h = h_i/h$). Here $y_h = (d + z + 2 - \eta)/2$ is the renormalization-group dimension of the longitudinal field and z is the dynamic exponent that parametrizes the size dependence of the gap at the critical point. For the quantum Ising chain, $y_h = \frac{15}{8}$ and $z = 1$. A generic observable $A = \text{Tr}[\hat{\rho}(t)\hat{A}]$ is expected to behave as

$$A(t, h, L) \approx L^{-y_a} \tilde{A}(\Phi, \Theta) = h^{y_a/y_h} \mathcal{A}(\Phi, \Psi), \quad (\text{A2})$$

where y_a is the renormalization-group dimension of A . For instance, if $A = \hat{M}$ [cf. Eq. (6)] one has $y_m = d + z - y_h$ (for the quantum Ising chain, $y_m = \frac{1}{8}$ and $y_m/y_h = \frac{1}{15}$). The OFSS theory predictions have been confirmed by numerical results [77] and further extended to the Loschmidt echo, the bipartite entanglement entropy, and the work fluctuations [9,77,99].

From the above OFSS relations, one can also derive the scaling behavior in the thermodynamic limit [9], i.e., for $L \rightarrow \infty$ while keeping an appropriate combination of h and t fixed. This corresponds to taking $\Phi \rightarrow \infty$ in the scaling function $\mathcal{A}(\Phi, \Psi)$. Assuming the limit to be regular, one obtains

$$\lim_{L \rightarrow \infty} A(t, h, L) = A_\infty(t, h) \approx h^{y_a/y_h} \mathcal{A}_\infty(\Psi). \quad (\text{A3})$$

As anticipated above, this out-of-equilibrium scaling behavior requires tuning h to keep Ψ fixed in the long- t limit. Therefore, only the low-energy critical spectrum is substantially involved in the dynamics of such soft QQs. On the other hand, the infinite-volume dynamic behavior associated with standard QQs is not expected to reproduce the scaling behavior (A3) for any finite (even small) value of h . This is essentially due to the fact that in standard QQs an extensive amount of energy $\Delta E \sim L^d$ (where d is the spatial dimension of the system) is injected in the system, which is much larger than the energy scale $E_c \sim L^{-z}$ of the low-energy critical modes. Consequently, the post-QQ dynamics generally involves noncritical higher-energy levels, and the unitary energy-conserving evolution may not be significantly related to the quantum-critical features of the low-energy spectrum.

However, as discussed in Refs. [90–95], certain models can still exhibit signatures of the transition. Standard QQs have been extensively studied in the integrable quantum Ising and generalized XY models, under QQs of the transverse field g (keeping $h = 0$) across the transition [49,50,56,57,61,90,94]. In the thermodynamic limit, the asymptotic long-time limit of the energy density [90] and of the longitudinal and transverse magnetizations [9,94] still develop a singular dependence on the QQ parameter g around the critical value. However, these singularities are likely related with the specific spectral properties of the quantum Ising chain without a longitudinal field h , which is equivalent to a free-fermion system [7].

2. Across the FOQTs

Peculiar scaling behaviors are also observed at FOQTs (see, e.g., Refs. [8–10,124,139–166]). In particular, an OFSS arises when the system is driven across the first-order transition line at $h = 0$ and $g < g_c$ by varying the longitudinal field h [9,10,77]. The main difference, compared to CQTs, lies in the size dependence of the scaling variables, which also depends on the choice of boundary conditions. For the sake of clarity, here we focus on periodic boundary conditions, which do not favor any of the two magnetized phases. In this case the relevant scaling variables are

$$\Phi = \frac{2m_0 h L}{\Delta(L)}, \quad \Theta = \Delta(L) t, \quad (\text{A4})$$

where m_0 is the spontaneous magnetization defined in Eq. (5) and

$$\Delta(L) \approx 2\sqrt{\frac{1-g^2}{\pi L}} g^L \quad (\text{A5})$$

is the gap [118]. In the limit $L \rightarrow \infty$, $t \rightarrow \infty$, and $h \rightarrow 0$, keeping Φ and Θ fixed, an OFSS is observed. For instance, the magnetization behaves as

$$M(t, h, L) \approx m_0 \mathcal{M}(\Phi, \Theta), \quad (\text{A6})$$

$\mathcal{M}(\Phi, \Theta)$ being a scaling function. The OFSS equations hold both for $h_i = -h$ and for $h_i = 0$ (within OFSS one cannot distinguish between $h_i = 0$ and $h_i = 0^-$, as in the thermodynamic limit).

The OFSS behavior is observed in an exponentially small interval of h , i.e., for $|h| \sim L^{-1} \Delta(L)$, in which only the two quasidegenerate magnetized states are relevant for the dynamics. This allows one to compute [77,146] the function $\mathcal{M}(\Theta, \Phi)$. For QQs from $h_i = -h$ to h one has

$$\mathcal{M}(\Theta, \Phi) = -\frac{\Phi}{\sqrt{1+\Phi^2}} + \frac{2\Phi}{(1+\Phi^2)^{3/2}} \times [1 - \cos(\Theta\sqrt{1+\Phi^2})], \quad (\text{A7a})$$

while, for QQs from $h_i = 0$ to h , we obtain

$$\mathcal{M}(\Theta, \Phi) = \frac{\Phi}{(1+\Phi^2)} [1 - \cos(\Theta\sqrt{1+\Phi^2})]. \quad (\text{A7b})$$

Numerical results for the Ising chain with PBC [77] confirm the validity of the scaling ansatz (A6) and the above reported scaling functions obtained by the effective two-level model. The asymptotic OFSS behavior is approached quite rapidly, as size corrections turn out to decrease exponentially in L .

In the thermodynamic limit $L \rightarrow \infty$, while keeping an appropriate combination of h and t fixed [the analog of the variable Ψ defined in Eq. (A1)] the scaling functions become trivial, at variance with the CQT case, Eq. (A1). The same trivial limit is obtained when considering slow Kibble-Zurek protocols instead of a direct QQ [75,76]. As in the CQT case, this behavior arises because higher-energy states, which effectively decouple in the OFSS limit, play an important role in QQs.

For periodic boundary conditions the OFSS regime is rather trivial, as only two states are relevant. More complex OFSS behaviors arise with different boundary conditions. For instance, with oppositely fixed boundary conditions, kink states become the relevant low-energy excitations. In this case, OFSS behaviors still emerge [9,79], but now the variable $\Phi \propto hL^3$, reflecting the power-law behavior $\Delta \sim L^{-2}$ of the gap at the transition point.

APPENDIX B: QUANTUM QUENCHES ACROSS THE FOQT IN THE PERTURBATIVE SMALL- g REGIME

In this Appendix we analyze the quantum Ising system with a different set of boundary conditions, in the perturbative one-kink approximation. We consider an open chain of length L and fix the two boundary spins, restricting the Hilbert space to eigenstates of $\hat{\sigma}_1^{(1)}$ and $\hat{\sigma}_L^{(1)}$ with eigenvalues $+1$ and -1 , respectively. We consider small values of g and h , allowing us to study the dynamics within the one-kink subspace, as was already done for the Kibble-Zurek dynamics [75,76] (non-interacting ensembles of kink-antikink states have also been considered in Refs. [71,72,75,76,110,113–115,139,143,144]). More precisely, we focus on the infinite-volume limit keeping hL fixed, restricting our analysis to values satisfying $hL \lesssim 1$, and repeat the analyses reported in Sec. VI. We stress that the dynamics in this limit is not expected to belong to the chaotic regime. Therefore, the present results allow us to distinguish

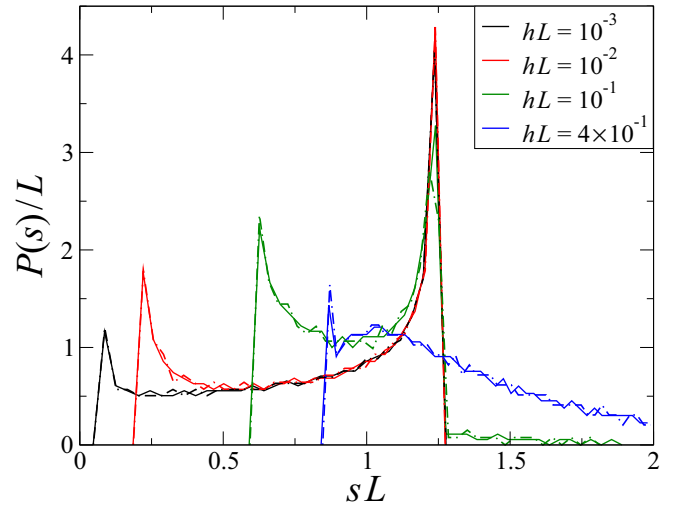


FIG. 21. The rescaled normalized distribution of the energy differences s_n for different values of hL and two system sizes, $L = 1000$ (solid lines) and $L = 800$ (dotted-dashed lines), for a chain with oppositely fixed boundary conditions in the one-kink perturbative limit. We set $g = \frac{1}{10}$.

which features are specific to the chaotic nature of the spectrum and which arise solely from the fact that the QQ crosses a FOQT.

In the limit considered here, the model is clearly integrable and thus the spacings between consecutive energy

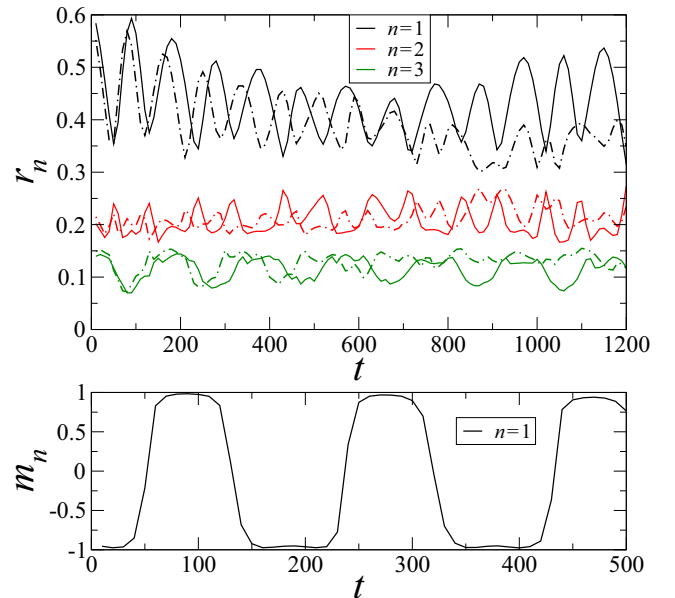


FIG. 22. (Top) The time evolution of the largest eigenvalues of the reduced density matrix for a subsystem of size $\ell = 6$, for two different values of L (continuous lines are for $L = 22$, dotted-dashed lines are for $L = 20$). (Bottom) The time evolution of the average magnetization $m_n = \langle r_n(t) | \hat{M} | r_n(t) \rangle$ for a subsystem of size $\ell = 6$; here $L = 22$. All calculations refer to a chain with oppositely fixed boundary conditions in the one-kink perturbative limit, with $g = \frac{1}{10}$.

levels should not follow the WD statistics. This is confirmed by our data. In Fig. 21 we show the distribution of the energy differences s_n , which exhibits the expected scaling behavior $P(s_n) = L^{-1}\pi(s_n L)$ as $L \rightarrow \infty$. Notably, the distribution displays two different shapes, depending on the value of the scaling variable hL . For $hL \lesssim 0.2$, it shows a two-peak shape, the higher one being insensitive of hL . In contrast, for $hL \gtrsim 0.2$, it develops a single peak at small values of s_n , followed by a long tail that decreases somewhat slowly. In both regimes, we find no evidence of a Poissonian behavior.

As already discussed in Sec. VI A, one of the main features of the postquench spectrum is a strongly peaked overlap w_n around a single energy value. This feature is also present in the one-kink limit. In this case as well, the normalized overlaps are sharply peaked, with a maximum that increases with L , indicating that the state of the system is a superposition of a few eigenstates of the postquench Hamiltonian. This conclusion is further supported by the behavior of the diagonal-ensemble entropy. We find that $S_D(L)$ is essentially independent of L , as L increases (at least within the considered range $50 \leq L \leq 1000$). For small values of hL , $S_D(L)$ is close to 1; it reaches a maximum of $S_D(L) \approx 1.7$ for $hL \approx 0.45$, and then slowly decreases as hL is further increased. Assuming that the relevant eigenstates are equally probable, this

corresponds to an effective participation of only about three to five states.

Finally, we analyze the main features of the dynamics of small subsystems. For this purpose, we consider a subsystem of size ℓ , as described in Sec. II C, and compute the corresponding reduced density matrix $\hat{\rho}(t)$. We then study its eigenvalues $r_1(t) > r_2(t) > r_3(t) > \dots$ and eigenvectors $|r_i(t)\rangle$, as a function of time. We find a remarkably simple structure, in which very few eigenvectors are relevant (see Fig. 22). Interestingly, the leading eigenvector behaves as if it were generated by a QQ in a single chain with open boundary conditions. Indeed, it oscillates between the two fully magnetized states, while its bond energy remains approximately time independent: $\langle r_1(t) | \hat{H}_s(0) | r_1(t) \rangle \approx -0.8$. The second-largest eigenvalue corresponds to an eigenvector that also has a large overlap with the fully magnetized state; accordingly, its magnetization is typically large and of opposite sign with respect to that of $|r_1(t)\rangle$. We have also checked whether the reduced density matrix can be rewritten as $e^{-\beta_{\text{th}} \hat{H}_s} / Z$, for some effective temperature β_{th} . Applying the procedure outlined in Sec. II C, we obtain a positive value for β_{th} . However, when this value is inserted back in the canonical-ensemble density matrix, the resulting eigenvalues $r_i(t)$ are quite different from the observed values. This indicated that, as expected, thermalization does not occur.

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