

Symmetry enhancement, symmetry-protected topological absorption, and duality in QED₃

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Quantum electrodynamics in 2 + 1 dimensions (QED₃) with two Dirac fermions displays time reversal symmetry, nontrivial SPT phases, and anomalies. The fate of this theory in its strongly coupled regime has been debated extensively. Surprisingly, we find that gluing together the phase diagrams of two standard Wilson-Fisher $O(4)$ theories suffices to reproduce all the SPT phases, anomalies, and semiclassical limits. A central mechanism behind it is “SPT absorption.” The patching of the $O(4)$ transitions makes very concrete predictions for the behavior of the theory in its strongly coupled limits; for instance, the $\theta = \pi$ sigma model with S^3 topology appears due to monopole condensation.

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Introduction. A $U(1)$ gauge field interacting with Dirac fermions in 2 + 1 space-time dimensions leads to a class of interesting, nontrivial, and experimentally relevant continuum quantum field theories (QFTs). Among them, there are the special theories that preserve time-reversal invariance for massless fermions. For brevity, we will refer to these as the QED₃ theories. The Lagrangian at the massless point is simply given by

$$L_{\text{kinetic}} = -\frac{1}{4e^2}F^2 + i\bar{\Psi}_i \not{D} \Psi^i, \quad (1)$$

where $i = 1, \dots, N_f$ is the number of Dirac fermions species. For this theory to exist, we must take N_f to be an even number (this is in order to avoid the breaking of time reversal symmetry by fermion loop diagrams [1][2]).

One basic question is to understand the low-energy physics and determine the phase diagram as a function of the fermion mass matrix $M_i^j \bar{\Psi}^i \Psi_j$. For sufficiently large N_f , calculations are possible in a $1/N_f$ expansion (as first demonstrated in Ref. [3]). There is a single second order transition at $M = 0$ and for nonzero M there are quantum Hall phases (as well as a phase with a $U(1)$ Nambu-Goldstone boson (NGB) due to monopole condensation [4] when the number of positive and negative eigenvalues of M_i^j coincide).

For small values of N_f , and in particular for the minimal nontrivial case $N_f = 2$, there are several speculations. One scenario is that the global $SU(2) \times U(1)$ symmetry is spontaneously broken [5]. A symmetry breaking pattern that was discussed extensively in the literature is $SU(2) \times U(1) \rightarrow U(1) \times U(1)$ which leads to 2 NGBs, and is due to the condensation of the mass operator $\bar{\Psi}^i \Psi_j$ [6]. Another more recent scenario postulates the existence of a 2nd order transition

with symmetry enhancements [7–10], see also the review [11]. These symmetry enhancement scenarios at a conformal fixed point are implausible in light of the bootstrap results [12] (and see references therein). Related work can be found in Refs. [13–16].

The impetus for this paper is a remarkable similarity between the scaling exponents of scalar operators in the rank q traceless symmetric irrep of $O(4)$ in the Wilson-Fisher (WF) fixed point, and charge q monopole operators in $N_f = 2$ QED₃ [17], which transform in the corresponding irrep upon the decomposition $SO(4) \rightarrow SU(2) \times U(1)$. The computation on the QED₃ side is via a $1/N_f$ expansion to subleading order [18–20], which was previously shown to be accurate for small values like $N_f = 4$ by comparison to conformal bootstrap results [21–23]. The $O(4)$ WF scaling dimensions were computed by a lattice simulation [24]. In Ref. [25], we see that the corresponding operators match for all values of q , and the match even gets better as q grows [26]. Furthermore, if one employs the web of dualities [8,27] away from its regime of validity then one finds a duality between $N_f = 2$ QED₃ and $O(4)$ WF [28]. Such a duality was actually suggested earlier in Ref. [29].

However, despite these tantalizing numerics, it cannot be that the massless QED₃ model flows to the $O(4)$ WF theory for several reasons, most importantly, because there is a ‘t Hooft anomaly involving time reversal symmetry in QED₃. Another reason is that the relevant singlet in QED₃ is $\bar{\Psi}_i \Psi^i$, which is time-reversal odd, while the relevant singlet in the WF theory is time-reversal even. Finally, the WF theory cannot match the SPT phases that we will explain later.

We will show that gluing two $O(4)$ WF theories can reproduce the SPT phases and the anomaly of QED₃! A crucial ingredient is that some SPT phases can be absorbed in the symmetry broken phase (SPT absorption). Another important ingredient is that a sigma model with S^3 target space can have a θ angle in 2 + 1 dimensions. Our picture clarifies in what sense QED₃ and $O(4)$ are dual and it suggests an explanation for why there is such a remarkable agreement between scaling

exponents. The large N_f expansion of the monopole scaling dimensions is performed in the regime where the massless theory flows to a CFT, while we are extrapolating to $N_f = 2$ where we claim the massless theory experiences spontaneous symmetry breaking, but the massive theory flows to the $O(4)$ CFT. It could be that this extrapolation happens to land on the $O(4)$ transition. Or it could be that the massless theory is very close in RG space to QED₃ with massive fermions, which is dual to $O(4)$ WF.

The scenario we describe can be viewed as an extension of the “quantum phase” paradigm in Refs. [30–34]. First of all, we propose that the massless theory breaks the symmetry as $SU(2) \times U(1) \rightarrow U(1)$ due to condensation of the $q = 1$ (elementary) monopole,

$$\langle \mathcal{M} \rangle \neq 0 \quad (2)$$

leading to 3 NGBs (unlike the previously discussed symmetry breaking pattern, due to the condensation of a fermion bilinear). These 3 NGBs live on a target space of S^3 topology with a squashed metric and with $\theta = \pi$ at the massless point of QED₃, leading to the infrared effective action (1)

$$S_{M_i^j=0}^{\text{ir}} = \frac{f}{2} \int d^2x dt \delta^{ab} \text{Tr}(\partial_a g \partial_b g^{-1}) + \text{squashing} \\ + \frac{1}{24\pi} \int d^2x dt \epsilon^{abc} \text{Tr}(g^{-1} \partial_a g g^{-1} \partial_b g g^{-1} \partial_c g), \quad (3)$$

where g is an $SU(2)$ matrix and the squashing term breaks the symmetry of the nonlinear sigma model to $SU(2) \times U(1)$ by squashing the target space metric.

As we turn on masses M_i^j , the geometry of the S^3 slightly changes and the θ angles change as well. Indeed, the mass term $M_i^j \sim \delta_j^i$ preserves $SU(2) \times U(1)$ and is time-reversal odd. Hence the map between the operators at low energies near the massless point is [35]

$$\bar{\Psi}^i \Psi_i \longleftrightarrow \epsilon^{abc} \text{Tr}(g^{-1} \partial_a g g^{-1} \partial_b g g^{-1} \partial_c g). \quad (4)$$

Eventually as we further increase the masses and reach $\theta = 0 \bmod 2\pi$, we encounter $O(4)$ WF fixed points and hence an enhanced “custodial” symmetry. We can now identify $\bar{\Psi}_i \Psi^i$ with the unique relevant singlet of $O(4)$ WF [36], which has an accidental time-reversal symmetry in the IR that acts differently from that of massless QED₃. The summary of the phase diagram is in Fig. 1. While our discussion is somewhat tied to the concrete continuum theory (1), the central ideas of how the $O(4)$ WF model could feature in the dynamics and how the anomalies and SPT phases could match (and the role of the S^3 nonlinear sigma model at $\theta = \pi$) should be more general and apply to various other systems in the same universality class (in particular certain lattice systems).

For $N_f = 3$, the time reversal breaking theory with $|k| = 1/2$ may also have a symmetry breaking quantum phase similar to the $N_f = 2$ scenario we discuss here. For $N_f \geq 4$ and general k , however, it is expected to be just a standard second order phase transition [37].

It would be interesting to verify our scenario using a lattice simulation in the Villain formalism [38–40], which allows one to suppress monopoles and thereby simulate the $SU(2) \times U(1)$ invariant theory. It would also be interesting to study the theory deformed by various monopole operators.

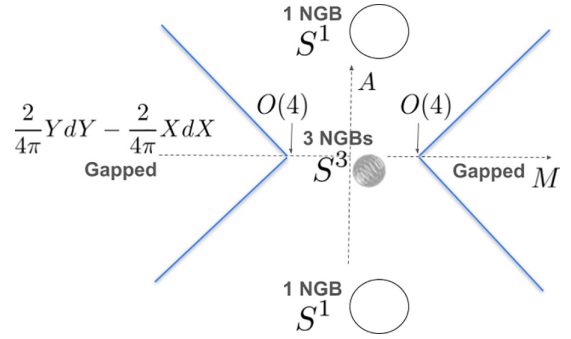


FIG. 1. The proposed phase diagram of QED₃. The horizontal axis corresponds to the $SU(2) \times U(1)$ preserving mass term while the vertical axis corresponds to the mass term in the adjoint representation of $SU(2) \times U(1)$. We denote these two types of mass terms by M and A , respectively. On the horizontal axis, in the symmetry broken phase, the theta angle of the sigma model varies continuously from 0 to 2π , with $\theta = \pi$ at $M = 0$. The $\pm 45^\circ$ lines correspond to $O(2)$ Wilson-Fisher transitions and these lines meet at two $O(4)$ Wilson-Fisher transitions.

While this paper was being finalized, we became aware of [41], where the same conclusion regarding the massless theory is reached via complementary arguments.

Weakly-coupled limits of QED₃. We denote the global symmetry gauge field which couples to the two fermions with opposite charge ± 1 by X and the gauge field that couples to the monopole symmetry by Y . The dynamical gauge field is denoted by a and it couples to the two Dirac fermions of charge 1. We take the Lagrangian to be

$$L_{\text{kinetic}} + M_1 \bar{\Psi}_1 \Psi_1 + M_2 \bar{\Psi}_2 \Psi_2 + \frac{1}{2\pi} a dY + L_{\text{ct.}}, \quad (5)$$

where L_{kinetic} was given in (1) and $L_{\text{ct.}} = \frac{1}{4\pi} Y dY - \frac{1}{4\pi} X dX$ represents the fact that we are adding some convenient counter-terms in the ultraviolet for the X, Y background gauge fields.

Let us explain the reason for adding these particular counter-terms. The global structure of the gauge group that X, Y couple to is in fact

$$\frac{U(1) \times U(1)}{\mathbb{Z}_2}. \quad (6)$$

The $\mathbb{Z}_2$ quotient is due to the fact that odd monopoles must be dressed by an odd number of fermions. That means that in gapped phases the following counter-terms correspond to well-defined SPT phases

$$\frac{4k_1}{4\pi} Y dY, \quad \frac{4k_2}{4\pi} X dX, \quad \frac{2k_3}{4\pi} (Y dY - X dX), \quad (7)$$

with $k_{1,2,3} \in \mathbb{Z}$. The reason that we have picked the specific counter-terms in (5) is so that in the gapped phases below we obtain well-defined SPT phases.

When the masses M_1, M_2 are both large compared to e^2 , the long distance limit can be easily understood by integrating out the fermions at one loop.

(1) When both M_1, M_2 are negative and large we can integrate both fermions out and we obtain the low-energy theory

$$-\frac{1}{4\pi}ada + \frac{1}{2\pi}adY - \frac{1}{4\pi}XdX + L_{ct.} \longrightarrow \frac{2}{4\pi}YdY - \frac{2}{4\pi}XdX, \quad (8)$$

where in the final step we have integrated out the dynamical gauge field a , which is at level 1 and hence the low-energy theory is invertible. We see that we have obtained a gapped phase with a well-defined SPT term, in accordance with (7).

(2) When both M_1, M_2 are positive and large we obtain the low-energy theory

$$\frac{1}{4\pi}ada + \frac{1}{2\pi}adY + \frac{1}{4\pi}XdX + L_{ct.} \longrightarrow 0, \quad (9)$$

where in the final step we have again integrated out the dynamical gauge field a . Again we have obtained a well-defined gapped phase.

(3) When M_1 is large and positive while M_2 is large and negative we find the low-energy theory

$$-\frac{1}{4e^2}F^2 + \frac{1}{2\pi}ad(X+Y) + \frac{1}{4\pi}YdY - \frac{1}{4\pi}XdX. \quad (10)$$

We have restored the kinetic term for the gauge field since there is no Chern-Simons term and thus the usual two-derivatives kinetic term dominates at long distances. This is a $U(1) \times U(1) \rightarrow U(1)$ symmetry breaking phase with a monopole condensate. The spontaneously broken monopole symmetry couples to $X+Y$. Similarly, when M_1 is large and negative while M_2 is large and positive, we find a symmetry breaking phase with a monopole condensate. Now the spontaneously broken symmetry couples to $X-Y$.

It looks alarming that the counter-term in (10) is improperly quantized. However, this is a symmetry breaking phase with a Nambu-Goldstone boson due to monopole condensation. We will explain why we are allowed to add *certain* improperly quantized terms in this phase.

The semiclassical limits can be seen in Fig. 1, where $M = M_1 + M_2$, $A = M_1 - M_2$. It is clear from the weakly-coupled limits that the physics at small M_1, M_2 must be nontrivial to accommodate these gapped SPT and gapless phases. It is also clear that it cannot be a single Landau-Ginzburg transition.

The physics on the $\pm 45^\circ$ lines that separate the four weakly-coupled quadrants can be readily understood. One Dirac fermion is massless on those lines. Since a $U(1)$ gauge field at level $\pm 1/2$ coupled to a single Dirac fermion is in the same universality class as the $O(2)$ Wilson-Fisher transition according to the dualities proposed in Refs. [42,43], we conclude that each of the $\pm 45^\circ$ lines represents a second-order phase transition in the $O(2)$ Wilson-Fisher universality class.

SPT absorption. We now address the Nambu-Goldstone phase (10) in more detail, and in particular, the question about the normalization of the counter-term $\frac{1}{4\pi}YdY - \frac{1}{4\pi}XdX$ which is clearly incompatible with (7).

We first prove that the partition functions of the $U(1)$ Nambu-Goldstone theory on manifolds with nontrivial $X+Y$ bundles vanish. For convenience denote $B = X+Y$ which is an ordinary, properly normalized $U(1)$ background gauge

field and consider the theory

$$-\frac{1}{4e^2}F^2 + \frac{1}{2\pi}adB + \dots, \quad (11)$$

where $\dots$ stand for possible higher derivative terms. The equations of motion in (11) dictate that $d \star F \sim dB$, which by the Gauss law implies that dB has to integrate to zero for the theory to have a nonzero partition function.

Going back to our case we see that our improperly quantized term $\frac{1}{4\pi}YdY - \frac{1}{4\pi}XdX = \frac{1}{4\pi}(Y+X)d(Y-X)$ is linear in $Y+X$, which we argued is a trivial gauge field and hence the improper quantization raises no difficulties.

The fact that the counter-terms do not have to be properly quantized in the NGB phase (11) does not mean that they are inessential. In the theory (11) there are defects which are electrically charged (we can represent them by Wilson lines). The counter-term $\frac{1}{4\pi}YdY - \frac{1}{4\pi}XdX$ leads to possibly fractional global charges of such electric defects. This is because in the presence of electric sources we can have nontrivial $B = X+Y$ bundles and the noninvariance of the partition function under gauge transformations of the background fields X and Y reflects the global symmetry charges of the electric sources.

We will later match these global charges as a consistency check on the phase diagram.

The whole discussion above can be phrased in the language of SPT absorption and symmetry fractionalization in a phase with one-form symmetry. This is carried out in the supplemental material.

The $O(4)$ transition. We now make some elementary comments about the $O(4)$ model deformed by quadratic terms that preserve $U(1) \times U(1)$. For convenience we assemble the four scalar fields $\phi_A = \phi_{1,2,3,4}$ into a matrix as

$$\mathcal{Z} = \begin{pmatrix} Z_1 & -Z_2^* \\ Z_2 & Z_1^* \end{pmatrix} \quad (12)$$

with $Z_1 = \phi_1 + i\phi_2$, $Z_2 = \phi_3 + i\phi_4$. We take the kinetic term to be as usual $|\partial Z_1 + iY Z_1 + iX Z_1|^2 + |\partial Z_2 + iY Z_2 - iX Z_2|^2$, with X, Y background gauge fields corresponding to a $U(1) \times U(1)$ symmetry. Note that the global structure is as before $\frac{U(1) \times U(1)}{\mathbb{Z}_2}$.

We take the potential to preserve $U(1) \times U(1)$:

$$V = \tilde{M}^2(|Z_1|^2 + |Z_2|^2) - A(|Z_1|^2 - |Z_2|^2) + \lambda(|Z_1|^2 + |Z_2|^2)^2.$$

The phase diagram can be readily obtained, see Fig. 2. On the $\pm 45^\circ$ lines in Fig. 2, we have one massless complex scalar and hence the $O(2)$ Wilson-Fisher transition. This is exactly what we have found on the $\pm 45^\circ$ lines in QED₃.

Note that for $A > \tilde{M}^2$ and $A > 0$, the broken symmetry couples to $X+Y$ since Z_1 condenses while for $A < -\tilde{M}^2$ and $A < 0$, the broken symmetry couples to $X-Y$ since Z_2 condenses. In between, for $A = 0$ there is $O(4)$ symmetry and the broken phase has 3 NGBs living on S^3 . This sigma model with S^3 can be written as

$$\frac{f}{2} \int d^2x dt \delta^{ab} \text{Tr}(\partial_a g \partial_b g^{-1}), \quad (13)$$

with g an $SU(2)$ group element. Since it is the standard $O(4)$ transition, there is no θ angle. A variant of the above

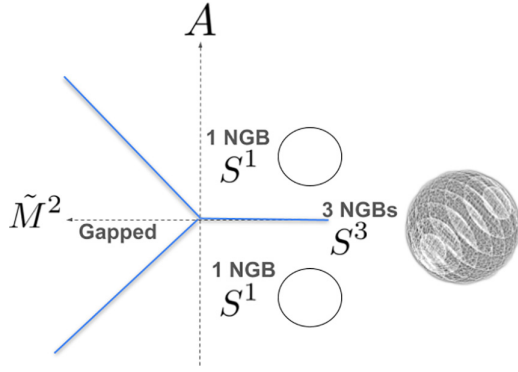


FIG. 2. The phase diagram of the $O(4)$ Wilson-Fisher model deformed by mass terms preserving $U(1) \times U(1)$.

discussion is to add the counter-term (properly quantized, compatible with (7)) $\frac{2}{4\pi}YdY - \frac{2}{4\pi}XdX$ in the ultraviolet. This counter-term then survives in the infrared of the gapped phase for large positive $\tilde{M}^2$.

Let us compare the broken phases we obtain here with those of QED_3 , (10). In the $O(4)$ model we obtain the broken phase (when Z_1 condenses)

$$-\frac{1}{4e^2}F^2 + \frac{1}{2\pi}adY. \quad (14)$$

On top of this, we may or may not add $\frac{2}{4\pi}YdY - \frac{2}{4\pi}XdX$, depending if it was added in the ultraviolet. [Here we wrote the low-energy theory of the NGB in the $O(4)$ model using a dual $U(1)$ gauge field to make the comparison with QED_3 easier.]

To make contact with QED_3 we need one last step, which is to match the fractionalization classes in the symmetry broken phases of the two models. In the $O(4)$ model in the phase with Z_1 condensed, the excitations on top of the vortex which carry charge under the unbroken symmetry are the Z_2, Z_2^* particles, whose charge under the $X - Y$ gauge field is ± 1 . Therefore whether or not we add the SPT term $\frac{2}{4\pi}YdY - \frac{2}{4\pi}XdX$ the vortices carry integer charge under the unbroken symmetry. Now consider the QED_3 model. In a background with unit background charge, we must excite, for instance, a fermion, which carries $X - Y$ charge $\pm 1/2$, however, the counterterm in (5) makes the total unbroken charge into an integer and hence the quantum numbers of the defects agree.

Enhanced symmetry in QED_3 . QED_3 with $M_1 = M_2$ in (5) preserves at short distances the continuous symmetry $SU(2) \times U(1)$. The phase diagram in Fig. 1, leads to symmetry enhancement to $O(4)$ at certain special values of the masses. We need to show that this enhancement is natural from the RG group point of view – i.e. does not require additional tuning.

We consider Landau-Ginzburg models for the matrix (12). We can act from the left and right by $SU(2)$ matrices $\mathcal{Z} \rightarrow A\mathcal{Z}B^\dagger$, which induce $SO(4) = \frac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$ transformations of the components $\phi_{1,2,3,4}$. The determinant of $\mathcal{Z}$ is $\sum_A \phi_A^2$ which is why the $SU(2)_L \times SU(2)_R$ action induces $SO(4)$ transformations. We would like to prove that any $SU(2) \times U(1)$ preserving potential respects the full $SU(2)_L \times SU(2)_R$ symmetry. (This is related to the “custodial” symmetry in

Higgs physics [44].) Indeed, to preserve $SU(2)_L$ we have to use the combination $\mathcal{Z}^\dagger \mathcal{Z}$. This a priori transforms in the adjoint+singlet of $SU(2)_R$, however, because of Bose symmetry only the singlet of $SU(2)_R$ is present. Therefore any $SU(2) \times U(1)$ invariant potential is fully $SU(2)_L \times SU(2)_R$ invariant. Therefore encountering the $O(4)$ Wilson-Fisher transition is entirely natural because all the $SU(2) \times U(1)$ invariant operators which break $SU(2)_L \times SU(2)_R$ are expected to be highly irrelevant. For instance, consider the matrix $\mathcal{Z}^\dagger \partial_\mu \mathcal{Z}$. All its elements are $SU(2)_L$ invariant. For instance, consider the combination $|Z_1^* \partial_\mu Z_1 + Z_2^* \partial_\mu Z_2|^2$. While it is irrelevant at the fixed point, it has a somewhat important effect in the symmetry broken phase. It modifies the metric on the S^3 target space. Indeed, one can put a left-invariant metric on S^3 which is not right-invariant (it is only invariant under the Cartan on the right). This is a “squashed” S^3 , which we anticipated by adding the squashing terms in (2)

Time reversal symmetry. Importantly, QED_3 at the massless point (1) is time reversal symmetric. The charge ± 1 monopoles are bosons in the doublet of $SU(2)$. Let us assemble the four monopole operators into a matrix, analogous to (12):

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & -\mathcal{M}_2^* \\ \mathcal{M}_2 & \mathcal{M}_1^* \end{pmatrix}. \quad (15)$$

The microscopic symmetries $\frac{SU(2)_L \times U(1)_R}{\mathbb{Z}_2}$ of QED_3 act on this quartet of monopoles by $\mathcal{M} \rightarrow U\mathcal{M}V$ with $U \in SU(2)_L$ and $V = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$.

There is also a microscopic charge conjugation symmetry which acts as $\mathcal{M} \rightarrow \mathcal{M}i\sigma_2$. This symmetry reverses the monopole number and extends the symmetry to $\frac{SU(2)_L \times \text{Pin}^-(2)}{\mathbb{Z}_2}$, because charge conjugation is an element of order 4 [45].

Finally, the massless QED_3 point respects time reversal symmetry which acts on the quartet of monopoles as

$$T_{UV} : \mathcal{M}(t) \rightarrow i\sigma_2 \mathcal{M}(-t). \quad (16)$$

This definition satisfies $T_{UV}^2 = -1$ on $\mathcal{M}$. More generally [45] $T_{UV}^2 = (-1)^q$, with q the monopole charge. A virtue of this definition is that it commutes with $SU(2)_L$ because of the property of $SU(2)$ matrices $-i\sigma_2 U i\sigma_2 = U^*$ for $U \in SU(2)$. The anomalies for such time reversal symmetry were discussed in Ref. [46].

In the ultraviolet theory, the mass deformation with $M_1 = M_2$ is time-reversal odd. From the phase diagram in Fig. 1, we see that the counter-term for the $\frac{U(1) \times U(1)}{\mathbb{Z}_2}$ symmetry jumps, which indicates a mixed anomaly between T_{UV} and $\frac{U(1) \times U(1)}{\mathbb{Z}_2}$. It is therefore necessary for our proposal to match this anomaly at the massless point. Since in our scenario we find a condensate (2) and since (16) satisfies $T_{UV}^2 = -1$ on the elementary monopole, it follows that it is not possible to have a time-reversal invariant condensate of the elementary monopole. In other words, the time reversal symmetry is broken spontaneously in the S^3 nonlinear sigma model phase.

Even though the time reversal symmetry is spontaneously broken, we still need to match the anomaly (it is matched by the physics of various defects [47]). A related problem is that our discussion of the SPT absorption mechanism pertained to the NGB phase, which had a one-form symmetry. There is no

one-form symmetry in the S^3 symmetry breaking phase since there are no vortices.

Therefore we have two remaining issues to address: How does the SPT phase jump as we move horizontally in the S^3 phase of Fig. 1? And how does the time reversal symmetry anomaly match at the middle point, $M = 0$? Since the microscopic theory breaks time reversal symmetry at the points of the $O(4)$ transition by irrelevant operators, it is expected that in the broken phase a θ angle can be present. The θ angle is associated to $\pi_3(S^3)$, and we can add it to the sigma model

$$\frac{f}{2} \int d^2x dt \delta^{ab} \text{Tr}(\partial_a g \partial_b g^{-1}) + \theta \Gamma, \quad (17)$$

with $\Gamma = \frac{1}{24\pi^2} \int d^2x dt \epsilon^{abc} \text{Tr}(g^{-1} \partial_a g g^{-1} \partial_b g g^{-1} \partial_c g) \in \mathbb{Z}$ is the winding number of space-time into the sigma model target space. An interesting fact is that as we change $\theta \rightarrow \theta + 2\pi$, nothing happens in the bulk physics, but when we couple the system to the background fields X, Y , the partition function jumps precisely by $\frac{2}{4\pi} X dX - \frac{2}{4\pi} Y dY$, as we show in Ref. [25]. Therefore we can account for the jump in the SPT phase by allowing θ to change continuously in the S^3 symmetry broken phase from 0 to 2π , where we encounter our two $O(4)$ WF transitions. At the massless point of QED₃, we find

precisely $\theta = \pi$ as required by time reversal symmetry. This reproduces the time reversal 't Hooft anomaly since $\theta = \pi$ is not quite time reversal invariant in the presence of background fields, analogously to the discussion in Ref. [48]. See also Refs. [49,50] where it was emphasized that $\theta \rightarrow \theta + 2\pi$ leads to edge modes, i.e., an SPT phase. In conclusion, the massless theory (1) flows to the $\pi = \theta$ sigma model with S^3 target space topology and as we deform by the mass term M in the ultraviolet, θ changes and at $\theta = 0, 2\pi$ we encounter two WF $O(4)$ transitions, and as the mass is further increased, we obtain a gapped phase or a gapped SPT phase, depending on the sign of M .

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