

Microscopic theory of an atomic spin diode

William J. Huddie  and Rembert A. Duine 

Institute for Theoretical Physics, Utrecht University, Princetonplein 5 3584CE Utrecht, The Netherlands
and Department of Applied Physics, Eindhoven University of Technology, P.O. Box 513, 5600MB Eindhoven, The Netherlands



(Received 5 March 2026; revised 10 July 2026; accepted 11 September 2026; published 30 September 2026)

We present a microscopic theory of an atomic spin diode. Our proposed system consists of two magnetic adatoms deposited on the surface of a two-dimensional electron gas with Rashba spin-orbit coupling. A local s - d type coupling between the local spins and the spins of the electrons induces a nonlocal Ruderman-Kittel-Kasuya-Yosida type interaction and a Dzyaloshinskii-Moriya interaction, in addition to dissipative interactions, between the spins. We derive the effective action for the spins using the Keldysh formalism. From the effective action, we also derive equations of motion for the spins which are shown to be of Landau-Lifshitz-Gilbert type, and give expressions for the effective field and Gilbert damping which appear in this equation. From our microscopic theory, we find that for an in-plane magnetic field perpendicular to the vector connecting the two atoms, the magnitude of the field and the distance between the atoms can always be tuned to engender perfectly diodic coupling. Our findings may pave the way for experimental realization of atomic spin diodes.

DOI: [10.1103/physrevb.144.144428](https://doi.org/10.1103/physrevb.144.144428)

I. INTRODUCTION

The field of spintronics is concerned with the design and engineering of new devices which are based on magnetic materials, with the aim that these will eventually replace conventional electronics [1–3]. Such electronic (CMOS-based) devices rely on the transistor, which only allows transport of charge across it when a gate voltage is applied. The device which underlies the operation of the transistor is called a p - n junction, or semiconducting diode, which allows the flow of electric current across it in one direction, but strongly suppresses flow in the opposite direction—this is an example of a nonreciprocal phenomenon [4,5].

In spintronics, however, one is interested in the transport of information using the spin of the electron as opposed to its charge. Such devices may transport information either using spin-polarized electric currents or using magnons, the quanta of spin waves, which have the advantage that the propagation of such quasiparticles generates much less heat than is lost to Joule heating in ordinary electronics [6]. One may imagine, for example, a so-called magnon transistor, allowing the transmission of magnons across it only under certain conditions. The design of such a device would also necessarily have to include some diode component, facilitating this nonreciprocal behavior. Nonreciprocal spin waves in certain systems have been known to exist since the 1960s [7], while recent experimental and theoretical works have demonstrated a variety of ways in which one may construct such a magnon transistor [8–11].

Reference [10] demonstrates, phenomenologically at least, the existence of a perfectly diodic (unidirectional) coupling in an archetypal synthetic antiferromagnet structure, in which the two ferromagnetic (FM) layers are coupled via a coherent magnetic interaction [e.g., the Dzyaloshinskii-Moriya interaction (DMI)], and an incoherent interaction (e.g., spin pumping across the spacer layer). The authors show that, under certain assumptions, the magnetization dynamics of each magnetic layer are qualitatively different; that is, when one layer is driven with a microwave field, the other layer undergoes its dynamics in response, but not vice versa. This phenomenon has subsequently also been corroborated using the Lindblad master equation formalism [11]. The key ingredients for the perfectly diodic behavior emphasized in Refs. [10,11] are DMI and dissipative interactions, which precisely cancel one another for a specific field. This is an implementation in a magnetic system of the ideas put forward in Ref. [12] for coupling photonic cavities.

In this work, we derive a different version of the diode condition which was derived in [10,11]. In contrast to the phenomenological starting points of these previous works, we begin from a microscopic model. The setup we consider, as depicted in Fig. 1, consists of a system of two atomic spins (magnetic adatoms), deposited on the surface of a two-dimensional electron gas (2DEG) substrate. Rashba-type spin-orbit coupling (SOC) is present, and we assume a contact s - d coupling between the localized spins and the spins of the electron gas. This kind of system has found wide interest in recent years thanks to advancements in spin-polarized scanning-tunneling microscopy techniques, which make it feasible to not only construct such systems but also to measure their spin-wave dispersion and various other magnetic properties [13–16].

Starting from a microscopic model allows us to derive, rather than assume the presence of, the coherent and dissipative interactions whose necessity was pointed out by Yuan

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

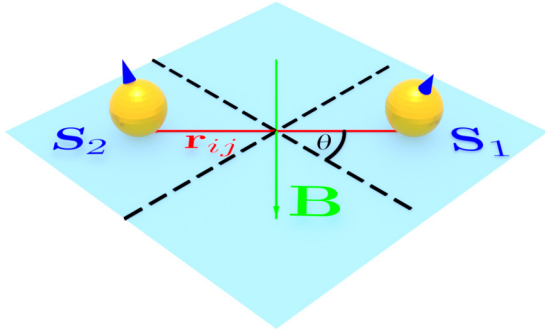


FIG. 1. Two atomic spins are deposited on the surface of a two-dimensional electron gas (2DEG). Local s - d type coupling to an electron bath induces coherent and dissipative interactions between the two spins.

et al. [10]. Moreover, this allows us to make contact with real material systems and assess the feasibility of the experimental realization of the diode effect.

Using a Keldysh approach to determine the effective dynamics of the atomic spins, we find that these dynamics are of Landau-Lifshitz-Gilbert (LLG) type. The effective equations of motion contain nonlocal coherent and dissipative interactions, as well as local Gilbert damping. Moreover, examining the magnetic susceptibility, we find that for an in-plane magnetic field perpendicular to the vector connecting the two atoms, the magnitude of the field and the distance between the atoms can always be tuned to engender unidirectional coupling. These findings may prove beneficial to future experimental studies of atomic spin diodes. The outline of this work is as follows. In Sec. II, we give a description of the model system and Hamiltonian and summarize our key results. In Sec. III, we develop the action of this system along the Keldysh contour. Using perturbation theory in the s - d coupling constant, we integrate out the electronic bath and derive an effective action on the spins. In Sec. IV, we expand the electronic response function to first order in frequency, allowing us to derive, under certain approximations, analytical expressions for the former. In Sec. V, we relate the real part of the retarded and advanced components of this response function to the nonlocal Ruderman-Kittel-Kasuya-Yosida (RKKY)-type coupling between the spins (including an antisymmetric component, which is identified as the DMI), while the imaginary part is related to the Gilbert damping. We show that the equations of motion of the spins are exactly in LLG form and study the linearized forms of these equations. From the latter, we derive a generalized version of the diode condition of Ref. [10] which applies when the nonlocal exchange and damping are tensorial in nature, as is found in our case. In Sec. VI, we numerically simulate the full nonlinear LLG and investigate the consequences of nonlinear effects for the quality of the atomic spin diode. We further clarify the role of finite temperature. We end with a short conclusion and discussion of our results and provide some outlook on possible continuations of our work.

II. MODEL AND SUMMARY OF RESULTS

In this section, we introduce the model considered in this work. We give the Hamiltonian of this model system, and

give a brief description of the methodology which is used to obtain the semiclassical equations of motion. Readers who are mainly interested in the primary results of our work may want to skip Secs. III and IV, and to this end we quote here the equations of motion and the conditions we derive on the parameters which appear in the latter such that the transmission of angular momentum between the two spins is unidirectional.

The setup we consider is depicted in Fig. 1. The Hamiltonian of this model system is given by

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_S + \hat{\mathcal{H}}_B + \hat{\mathcal{H}}_{SB}, \quad (1)$$

which we have split into system, bath, and system-bath interaction parts, in the context of an open quantum system [17]. This Hamiltonian consists of three parts. The first term, $\hat{\mathcal{H}}_S$ is the spin Hamiltonian of the localized spins; we write this as

$$\hat{\mathcal{H}}_S = - \sum_i \mathbf{B}_i \cdot \hat{\mathbf{S}}_i, \quad (2)$$

where $\mathbf{S}_i$ are the localized spins at sites 1 and 2. The sum is taken over spin sites $i = 1, 2$, and we have absorbed the units of $g\mu_B$ into the definition of $\mathbf{B}_i$.

The second term $\hat{\mathcal{H}}_B$ is the Hamiltonian of a two-dimensional electron gas with Rashba spin-orbit coupling; it is given by

$$\hat{\mathcal{H}}_B = \int d^2\mathbf{r} \, \hat{\psi}^\dagger(\mathbf{r}) \left(\frac{\hat{\mathbf{p}}^2}{2m} + \lambda_{SO}(\hat{\mathbf{z}} \times \hat{\mathbf{p}}) \cdot \boldsymbol{\sigma} \right) \hat{\psi}(\mathbf{r}), \quad (3)$$

where $\hat{\psi}^\dagger, \hat{\psi}$ are Weyl spinors in the standard σ^z eigenbasis $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\hat{\mathbf{p}} = -i\nabla$ is the momentum operator with m the effective electron mass, $\hat{\mathbf{z}}$ is the unit vector in the direction perpendicular to the plane of the 2DEG, and $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)^T$ is a vector of Pauli matrices. We shall hereafter refer to $\hat{\mathcal{H}}_B$ as the bath Hamiltonian, and to the 2DEG itself as the bath.

The last term, $\hat{\mathcal{H}}_{SB}$, is the system-bath interaction. This is assumed to be of s - d type and is of the general form [18]

$$\hat{\mathcal{H}}_{SB} = - \int d^2\mathbf{r} \, J_{sd}(\mathbf{r}) \mathbf{S}(\mathbf{r}) \cdot \mathbf{s}(\mathbf{r}), \quad (4)$$

where $J_{sd}(\mathbf{r})$ is the strength of the s - d coupling, $\mathbf{S}(\mathbf{r})$ is a localized spin at position $\mathbf{r}$, with $\mathbf{s}(\mathbf{r})$ the electronic spin density at that same point.

For simplicity, we assume a contact (δ -function type) interaction between localized spins and itinerant electrons, i.e., we write $J_{sd}(\mathbf{r}) = J_{sd} \sum_i \delta(\mathbf{r} - \mathbf{r}_i)$, which reduces Eq. (4) to

$$\hat{\mathcal{H}}_{SB} = -J_{sd} \sum_i \sum_\mu \hat{\mathbf{S}}_i^\mu \hat{s}_i^\mu. \quad (5)$$

Note that the dimensions of J_{sd} are [energy] $\times$ [area].

After performing the Fourier transform, we find

$$\hat{\mathcal{H}}_S = - \sum_{i=1,2} \mathbf{B}_i \cdot \mathbf{S}_i, \quad (6)$$

$$\hat{\mathcal{H}}_B = \sum_{\mathbf{k}s} \varepsilon_{\mathbf{k}s} \hat{c}_{\mathbf{k}s}^\dagger \hat{c}_{\mathbf{k}s}, \quad (7)$$

$$\hat{\mathcal{H}}_{SB} = -J_{sd} \sum_i \sum_\mu \hat{\mathbf{S}}_i^\mu \hat{s}_i^\mu. \quad (8)$$

Equations (7) and (8) are written in the Rashba basis $\{\hat{c}_{\mathbf{k}s}^\dagger, \hat{c}_{\mathbf{k}s}\}$, where $\mathbf{k} = (k_x, k_y)^T$, $s = \pm 1$ is the helicity index and ($\hbar = 1$)

$$\varepsilon_{\mathbf{k}s} = \frac{k^2}{2m} + s\lambda_{\text{SO}}k = \varepsilon_k + s\lambda_{\text{SO}}k \quad (9)$$

is the dispersion relation of electrons with momentum $\mathbf{k}$ in the Rashba band $s = \{+, -\}$, whereas the creation and annihilation operators in this basis are related to those in the spin- z eigenbasis $\sigma = \{\uparrow, \downarrow\}$ via

$$\begin{pmatrix} \hat{c}_{\mathbf{k}+} \\ \hat{c}_{\mathbf{k}-} \end{pmatrix} = U_{\mathbf{k}}^\dagger \begin{pmatrix} \hat{c}_{\mathbf{k}\uparrow} \\ \hat{c}_{\mathbf{k}\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} ie^{i\phi_k} & 1 \\ -ie^{i\phi_k} & 1 \end{pmatrix} \begin{pmatrix} \hat{c}_{\mathbf{k}\uparrow} \\ \hat{c}_{\mathbf{k}\downarrow} \end{pmatrix}. \quad (10)$$

The Rashba constant λ_{SO} appears as an overall prefactor to the simultaneous equations obtained from solving, in momentum space, for the eigenvectors of Eq. (3), so there is no dependence on this parameter in $U_{\mathbf{k}}$. Here, we define for conciseness

$$k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2}, \quad \phi_k = \arctan\left(\frac{k_y}{k_x}\right).$$

In the basis (10), the components of the electronic spin density $\hat{s}_i^\mu$ are given by

$$\hat{s}_i^\mu = \frac{1}{A} \sum_{\mathbf{k}\mathbf{k}', s s'} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_i} \langle s, \mathbf{k} | \sigma^\mu | s', \mathbf{k}' \rangle \hat{c}_{\mathbf{k}s}^\dagger \hat{c}_{\mathbf{k}'s'}. \quad (11)$$

Using the Keldysh path integral formalism [19], we develop the action of this model system. We integrate out the electron bath using second-order perturbation theory in the s - d coupling constant J_{sd} , thereby obtaining an effective action on the spins which is written in terms of the spin-density spin-density response function of the bath.

In the frequency domain, we expand the electronic response function to first order in frequency and show that the effective action yields the following equations of motion:

$$\dot{\mathbf{S}}_k(t) = \mathbf{S}_k(t) \times \left[\mathbf{B}^{\text{ext}} + \sum_j (\mathbf{\Gamma}_{ij} * \mathbf{S}_j)(t) + \boldsymbol{\xi}_j(t) \right], \quad (12)$$

where the friction kernel $\mathbf{\Gamma}_{ij}$ is obtained from the microscopic properties of the bath; in the low-frequency regime, Eq. (12) simplifies to

$$\dot{\mathbf{S}}_k(t) = \mathbf{S}_k(t) \times \left[\mathbf{B}^{\text{ext}} + \sum_j \mathbf{K}_{kj} \mathbf{S}_j(t) - \boldsymbol{\alpha}_{kj} \dot{\mathbf{S}}_j(t) + \boldsymbol{\xi}_j(t) \right], \quad (13)$$

which is in the form of a generalized LLG equation. The exchange tensor $\mathbf{K}_{ij}$ and the Gilbert damping tensor $\boldsymbol{\alpha}_{ij}$ contain both local ($i = j$) and nonlocal ($i \neq j$) terms. The relation of these terms to the microscopic properties of the bath is done in Sec. III, while explicit expressions for the bath response functions are given in Sec. IV. While the details of the derivation in Sec. IV are important for understanding which approximations are used to arrive at our result, and for where the structure of the exchange and damping tensors in Eq. (13) comes from, they do not have any impact on the conclusions made in Sec. V about the important bath-induced nonreciprocal physics.

Linearizing Eq. (13), we study the magnetic susceptibility and derive a condition on (a) the interatomic spacing $R =$

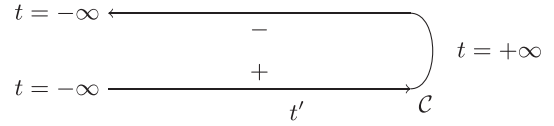


FIG. 2. Schematic representation of the time evolution along the Keldysh contour $\mathcal{C}$.

$|\mathbf{r}_{12}| = |\mathbf{r}_{21}|$ and (b) the applied magnetic field $B_0 = |\mathbf{B}^{\text{ext}}|$ such that the transmission of angular momentum between spin 1 and spin 2, or vice versa, is entirely unidirectional. In particular, we find that the transmission from spin 1 to spin 2 (or vice versa) vanishes at resonance, provided the magnetic parameters satisfy

$$D\alpha_1 + J_0\alpha_0 = 0, \quad (14)$$

where, for the specific geometry considered in Sec. V and as components of the exchange and damping tensors appearing in Eq. (13), $J_0 \equiv K_{12}^{xx} = K_{21}^{xx} = K_{12}^{zz} = K_{21}^{zz}$, $D \equiv K_{12}^{xz} = -K_{21}^{xz}$ is the DMI strength, $\alpha_0 \equiv \alpha_{12}^{xx} = \alpha_{21}^{xx} = \alpha_{12}^{zz} = \alpha_{21}^{zz}$ is the symmetric component of the damping, and $\alpha_1 \equiv \alpha_{12}^{xz} = -\alpha_{21}^{xz}$ is the chiral damping. In addition, we must also apply a specific magnetic field, whose strength is given by

$$B_{0,\eta_1\eta_2,\lambda}^\sigma = -\frac{D\sigma}{\alpha_0} - \frac{S}{2} (J_1(\eta_1 + \eta_2) + \lambda[(1 + \eta_1\eta_2)\Omega_0 + (1 - \eta_1\eta_2)\Omega_1]), \quad (15)$$

where $\sigma = +1$ denotes a diode biased in the $1 \rightarrow 2$ direction, $\sigma = -1$ the opposite case, $\lambda = \pm 1$ denotes the two resonance roots, $\eta_1, \eta_2 = \pm 1$ are used to denote whether the ground state of the spins is FM or antiferromagnetic (AFM), $\Omega_0 = \sqrt{D^2 + J_0^2}$, and $\Omega_1 = \sqrt{J_1^2 - \Omega_0^2}$, where $J_1 \equiv K_{12}^{yy} = K_{21}^{yy}$ is the other symmetric component of $\mathbf{K}_{ij}$. Details of this calculation are given in Sec. V.

III. EQUATIONS OF MOTION

In this section, we derive the equations of motion for the semiclassical spin dynamics of the system. We begin with a brief overview of the Keldysh path integral formalism [19–21], which is a widely used method for applying the techniques of quantum field theory to nonequilibrium systems, and finds diverse applications in a variety of areas of physics [22–24]. We then develop the combined action of the system and bath, integrate out the latter, and derive the equations of motion from the resulting effective action on the former. We find that these equations of motion are in the form of a generalized Landau-Lifshitz-Gilbert equation.

A. Keldysh path integral

To derive an expression for the real-time action, one considers the coherent-state path integral along the Keldysh time contour, depicted schematically in Fig. 2. At $t = -\infty$, the system and bath are initially in thermal equilibrium. An interaction between the system and the environment is then adiabatically switched on; this interaction reaches its full strength before the observation time t_0 . The interaction is then adiabatically switched off, and the system returns to

equilibrium along the negative branch of the contour. For works which apply similar techniques to study the dynamics of spin systems coupled to an environment, one may consult Refs. [25–28].

We develop the coherent path integral over fermionic (Grassmann) fields $\{\psi_{\mathbf{k}\alpha}^*, \psi_{\mathbf{k}\alpha}\}$ and coherent spin states $|g_i\rangle$ along the contour $\mathcal{C}$, and we find that the partition function is given by

$$Z = \int \mathcal{D}[g_1] \mathcal{D}[g_2] \int \mathcal{D}[\psi^*] \mathcal{D}[\psi] \exp(i\mathcal{S}[\mathbf{S}_1, \mathbf{S}_2, \psi^*, \psi]), \quad (16)$$

where $\mathbf{S}_i(t) = \langle g_i | \hat{\mathbf{S}}_i(t) | g_i \rangle$ is the expectation value of the spin operator $\hat{\mathbf{S}}_i$ in the spin-coherent state basis which is defined by the Euler angles $(\psi_i, \theta_i, \phi_i)$ as [29]

$$|g_i\rangle \equiv e^{-i\phi_i S_i^z} e^{-i\theta_i S_i^y} e^{-i\psi_i S_i^x} |S; S\rangle \\ = e^{-i\psi_i S} e^{-i\phi_i S_i^z} e^{-i\theta_i S_i^y} |S; S\rangle, \quad (17)$$

where $|S; S\rangle$ is the state with maximal spin, with S the spin quantum number. Note that the Euler angle ψ_i gives an overall phase and hence a gauge freedom to the spin coherent state. The choice of gauge is important, as certain choices of gauge will cause certain paths taken by the spins on the Keldysh contour to vanish pathologically [27]. We discuss this gauge freedom further in Sec. III B.

The action $\mathcal{S}[\mathbf{S}_1, \mathbf{S}_2, \psi^*, \psi]$ which weights the path integral in Eq. (16) is written as the sum of three parts:

$$\mathcal{S}[\mathbf{S}_1, \mathbf{S}_2, \psi^*, \psi] = \mathcal{S}_S[g_1, g_2] + \mathcal{S}_B[\psi^*, \psi] \\ + \mathcal{S}_{SB}[g_1, g_2, \psi^*, \psi], \quad (18)$$

where

$$\mathcal{S}_S = \sum_i \oint_{\mathcal{C}} dt [\langle g_i | i\partial_t | g_i \rangle - \mathbf{B}_i \cdot \mathbf{S}_i], \quad (19)$$

$$\mathcal{S}_B = \oint_{\mathcal{C}} dt \sum_{\mathbf{k}\alpha} \psi_{\mathbf{k}\alpha}^*(t) (i\partial_t - (\epsilon_{\mathbf{k}\alpha} - \mu)) \psi_{\mathbf{k}\alpha}(t), \quad (20)$$

$$\mathcal{S}_{SB} = J_{sd} \oint_{\mathcal{C}} dt \sum_i \sum_{\mu} S_i^{\mu}(t) s_i^{\mu}(t), \quad (21)$$

and the time-dependent electronic spin density is given by

$$s_i^{\mu}(t) = \frac{1}{A} \sum_{\mathbf{k}\mathbf{k}', ss'} e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}_i} \langle s, \mathbf{k} | \sigma^{\mu} | s', \mathbf{k}' \rangle \psi_{\mathbf{k}s}^*(t) \psi_{\mathbf{k}'s'}(t). \quad (22)$$

Here, the sums in Eq. (21) are taken over spatial indices (spin components) μ and site indices i . To obtain an effective action on the spins, we integrate out the fermionic fields. To do this, we assume that the s - d coupling constant J_{sd} can be treated perturbatively, such that we can expand the term $e^{i\mathcal{S}_{SB}}$ in powers of this quantity; this is a good approximation provided the spin-flip scattering amplitude density is small [30,31], related to the condition that the system is measured at a temperature much larger than the Kondo temperature [32–37]—i.e., we must be in a temperature regime where the RKKY interaction is dominant over the Kondo effect.

The validity of this expansion is not controlled by the parameter J_{sd} itself, but rather by the quantity $J_{sd}\rho$, where

ρ is the electronic density of states at the Fermi level; each power of the electronic spin density brings with it a factor of the (constant) density of states, and $J_{sd}\rho \ll 1$ is typically a very good approximation.

Expanding to second order in J_{sd} and performing the path integral over the fermionic fields, we find that the first-order term vanishes (physically, the Rashba 2DEG has no spontaneous spin-polarization), whereas the zeroth and second-order terms can be re-exponentiated to obtain an effective action on the spins of the form

$$i\mathcal{S}_{\text{eff}}[g_1, g_2] = i\mathcal{S}_0[g_1, g_2] + i\mathcal{S}_{\text{int}}[g_1, g_2], \quad (23)$$

where

$$i\mathcal{S}_{\text{int}} = i \oint_{\mathcal{C}} dt \oint_{\mathcal{C}} dt' \sum_{ij} \sum_{\mu\nu} S_i^{\mu}(t) \Pi_{ij}^{\mu\nu}(t, t') S_j^{\nu}(t'). \quad (24)$$

Here, we defined the electronic spin-density spin-density response function $\Pi_{ij}^{\mu\nu}(t, t')$ as

$$\Pi_{ij}^{\mu\nu}(t, t') = \frac{iJ_{sd}^2}{2} \langle s_i^{\mu}(t) s_j^{\nu}(t') \rangle_0 \\ = \frac{iJ_{sd}^2}{2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}') \cdot \mathbf{r}_{ij}} \\ \times \sum_{ss'} \mathcal{M}_{\mathbf{k}\mathbf{k}', ss'}^{\mu\nu} G_{ss}(\mathbf{k}, t, t') G_{s's'}(\mathbf{k}', t', t), \quad (25)$$

where G_{ss} is the electronic Green's function on the Keldysh contour, given by

$$G_{ss'}(\mathbf{k}, \mathbf{k}'; t, t') = \langle \psi_s(\mathbf{k}, t) \psi_{s'}^*(\mathbf{k}', t') \rangle_0 \delta_{ss'} \delta_{\mathbf{k}\mathbf{k}'}, \quad (26)$$

and we defined the matrix element $\mathcal{M}_{\mathbf{k}\mathbf{k}', ss'}^{\mu\nu}$ as

$$\mathcal{M}_{\mathbf{k}\mathbf{k}', ss'}^{\mu\nu} = \langle s, \mathbf{k} | \sigma^{\mu} | s', \mathbf{k}' \rangle \langle s', \mathbf{k}' | \sigma^{\nu} | s, \mathbf{k} \rangle.$$

The subscript 0 which appears in Eqs. (25) and (26) implies that the expectation values are taken with respect to the initial equilibrium state of the bath, i.e., without any system-bath interactions, such that time-reversal and inversion symmetry are preserved. We also assumed that the bath is in the thermodynamic limit, such that the momentum space sums can be replaced by integrals.

Having derived a formal expression for the effective action, the next step is to extract from this measurable real-time quantities. The prescription for converting quantities on the Keldysh contour to measurable observables is called the Keldysh rotation [19], which is the subject of the following section.

B. Keldysh rotation

We begin with some basic rules for calculus of observables which live on the Keldysh contour. Let $\mathcal{O}^+(t)$ denote an observable on the forward (+) branch of the contour, and $\mathcal{O}^-(t)$ the same observable on the backward (−) branch. The Keldysh rotation splits the observable into its so-called

classical and quantum parts:

$$\mathcal{O}_1^\pm(t) = \mathcal{O}_1^{\text{cl}}(t) \pm \frac{\mathcal{O}_1^{\text{q}}(t)}{2}, \quad (27)$$

which intuitively represent the average value of the observable and its fluctuating part, respectively. On the other hand, a two-time quantity transforms as

$$\mathcal{O}_2(t, t') = \Theta(t, t')\mathcal{O}_2^>(t, t') + \Theta(t', t)\mathcal{O}_2^<(t, t'), \quad (28)$$

where $\mathcal{O}_2^{\gtrless}(t, t')$ are the greater and lesser parts of the observable, respectively. Performing the Keldysh rotation as given by Eq. (27) leads one to define the retarded, advanced and Keldysh components of $\mathcal{O}_2$ as

$$\mathcal{O}_2^{R(A)}(t - t') = \pm \Theta(\pm(t - t'))(\mathcal{O}_2^>(t - t') - \mathcal{O}_2^<(t - t')) \quad (29)$$

$$\mathcal{O}_2^K(t - t') = \mathcal{O}_2^>(t - t') + \mathcal{O}_2^<(t - t'). \quad (30)$$

For example, the spins $\mathbf{S}_i$ are one-time quantities, and transform according to Eq. (27), whereas the Green's functions (26) are two-time quantities, and transform according to Eqs. (29) and (30), as we now discuss in more detail.

We first focus on the system part of the action, Eq. (19). Differentiation of Eq. (17) gives us

$$\langle g_i | i\partial_t | g_i \rangle = S[\dot{\psi}_i + \dot{\phi}_i \cos \theta_i], \quad (31)$$

where S is the spin quantum number. The magnetic field term in Eq. (19) becomes, if we assume $\mathbf{B}_i = (0, 0, B_i^z)^T$,

$$\langle g_i | \mathbf{B}_i \cdot \hat{\mathbf{S}}_i | g_i \rangle = SB_i^z \cos \theta_i, \quad (32)$$

from which we can see that $\mathcal{S}_S$ is

$$\mathcal{S}_S = S \sum_i \oint_C dt [\dot{\psi}_i + \dot{\phi}_i \cos \theta_i + B_i^z \cos \theta_i]. \quad (33)$$

Here is where the choice of gauge becomes important. We choose, following Refs. [27] and [38], $\psi = \chi - \phi$, and $\dot{\chi}^{\text{cl}} = \dot{\phi}^{\text{cl}}(1 - \cos \theta^{\text{cl}})$, $\chi^{\text{q}} = \phi^{\text{q}}(1 - \cos \theta^{\text{q}})$. Under this choice of gauge, Eq. (33) becomes, after Keldysh rotation,

$$\begin{aligned} \mathcal{S}_S = S \sum_i \int dt & \left[\phi_i^{\text{q}} \dot{\theta}_i^{\text{cl}} \sin \theta_i^{\text{cl}} \cos \frac{\theta_i^{\text{q}}}{2} \right. \\ & \left. + \left(\phi_i^{\text{q}} \frac{\dot{\theta}_i^{\text{q}}}{2} \cos \theta_i^{\text{cl}} - 2\dot{\phi}_i^{\text{cl}} \sin \theta_i^{\text{cl}} - 2B_i^z \sin \theta_i^{\text{cl}} \right) \sin \frac{\theta_i^{\text{q}}}{2} \right], \end{aligned} \quad (34)$$

where henceforth, a time integral without the contour subscript denotes an integral over the real line $t \in \mathbb{R}$.

We may make one further simplification to this action. Noting that our semiclassical Euler-Lagrange equations are obtained by variations with respect to quantum components

$\phi^{\text{q}}, \theta^{\text{q}}$, we may expand (34) to first order in these components:

$$\mathcal{S}_S = S \int dt [\phi_i^{\text{q}} \dot{\theta}_i^{\text{cl}} \sin \theta_i^{\text{cl}} - \theta_i^{\text{q}} (B_i + \dot{\phi}_i^{\text{cl}}) \sin \theta_i^{\text{cl}}]. \quad (35)$$

It is instructive to compute the classical and quantum components of $\langle g_i | \hat{\mathbf{S}}_i(t) | g_i \rangle \equiv \mathbf{S}_i(t)$. To first order in quantum components, we find

$$\mathbf{S}_i^{\text{cl}} = S \begin{pmatrix} \sin \theta_i^{\text{cl}} \cos \phi_i^{\text{cl}} \\ \sin \theta_i^{\text{cl}} \sin \phi_i^{\text{cl}} \\ \cos \theta_i^{\text{cl}} \end{pmatrix}, \quad (36)$$

$$\mathbf{S}_i^{\text{q}} = S \begin{pmatrix} \theta_i^{\text{q}} \cos \theta_i^{\text{cl}} \cos \phi_i^{\text{cl}} - \phi_i^{\text{q}} \sin \theta_i^{\text{cl}} \sin \phi_i^{\text{cl}} \\ \theta_i^{\text{q}} \sin \theta_i^{\text{cl}} \cos \phi_i^{\text{cl}} + \phi_i^{\text{q}} \cos \theta_i^{\text{cl}} \sin \phi_i^{\text{cl}} \\ -\theta_i^{\text{q}} \sin \theta_i^{\text{cl}} \end{pmatrix}, \quad (37)$$

giving us the useful relation

$$\mathbf{S}_i^{\text{q}} = \theta_i^{\text{q}} \frac{\partial \mathbf{S}_i^{\text{cl}}}{\partial \theta_i^{\text{cl}}} + \phi_i^{\text{q}} \frac{\partial \mathbf{S}_i^{\text{cl}}}{\partial \phi_i^{\text{cl}}}. \quad (38)$$

Performing the Keldysh rotation of $\mathcal{S}_{\text{int}}$ is no more difficult, though slightly more involved, as this term contains a double time integral along the Keldysh contour. One obtains

$$\begin{aligned} i\mathcal{S}_{\text{int}} = i \sum_{ij} \int dt \int dt' & \left[\mathbf{S}_i^{\text{q}}(t) \cdot [\mathbf{\Pi}_{ij}^R(t - t') \mathbf{S}_j^{\text{cl}}(t')] \right. \\ & + i \sum_{ij} \int dt \int dt' \mathbf{S}_j^{\text{cl}}(t) \cdot [\mathbf{\Pi}_{ij}^A(t - t') \mathbf{S}_i^{\text{q}}(t')] \\ & \left. + i \sum_{ij} \int dt \int dt' \frac{1}{2} \mathbf{S}_i^{\text{q}}(t) \mathbf{\Pi}_{ij}^K(t - t') \mathbf{S}_j^{\text{q}}(t') \right], \end{aligned} \quad (39)$$

where the retarded, advanced, and Keldysh components of the response function are given by

$$\mathbf{\Pi}^{R(A)}(t - t') = \pm \Theta(\pm(t - t'))(\mathbf{\Pi}^>(t - t') - \mathbf{\Pi}^<(t - t')); \quad (40)$$

$$\mathbf{\Pi}^K(t - t') = \mathbf{\Pi}^>(t - t') + \mathbf{\Pi}^<(t - t'). \quad (41)$$

Having converted the contour integrals into real-time integrals, we are in a position to study the dynamics of the spins under the influence of the coupling to the electron bath. The derivation of these equations is the subject of the following section.

C. Equations of motion

The third term in (39) is quadratic in quantum components of the spins. To remove this term, we perform a Hubbard-Stratonovich transformation [39,40], which exactly cancels this term at the cost of introducing a pair of auxiliary fields ξ_1 and ξ_2 . The partition function of the system becomes

$$\begin{aligned} Z = \int \mathcal{D}[\xi_1] \int \mathcal{D}[\xi_2] \int \mathcal{D}[g_1] \int \mathcal{D}[g_2] \\ \times \exp(i\mathcal{S}_{\text{eff}}[\hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2, \xi_1, \xi_2]), \end{aligned} \quad (42)$$

with the effective action

$$\begin{aligned} iS_{\text{eff}} = & iS \sum_i \int dt [\phi_i^q \dot{\theta}_i^{\text{cl}} \sin \theta_i^{\text{cl}} - \theta_i^q (B_i + \phi_i^{\text{cl}}) \sin \theta_i^{\text{cl}}] \\ & - \frac{i}{2} \sum_{ij} \int dt \xi_i(t) ((\Pi_{ij}^K)^{-1} * \xi_j)(t) \\ & + i \sum_{ij} \int dt S_i^q(t) \cdot [(\Pi_{ij}^R * S_j^{\text{cl}})(t)] \\ & + i \sum_{ij} \int dt S_j^{\text{cl}}(t) \cdot [(\Pi_{ij}^A * S_i^{\text{cl}})(-t)] \\ & + i \sum_{ij} \int dt S_i^q(t) \cdot \xi_j(t), \end{aligned} \quad (43)$$

where we rewrote the integral over t' as a convolution.

One observes from Eq. (43) that (i) the fluctuating magnetic fields have a noise spectrum defined by the Keldysh component of the magnetic field, namely,

$$\langle \xi_i(t) \xi_j(t') \rangle = -i \Pi_{ij}^K(t - t'), \quad (44)$$

and (ii) the Lagrangian of the system is given by

$$\begin{aligned} \mathcal{L}(t) = & S \sum_i [\phi_i^q \dot{\theta}_i^{\text{cl}} \sin \theta_i^{\text{cl}} - \theta_i^q (B_i + \phi_i^{\text{cl}}) \sin \theta_i^{\text{cl}}] \\ & + \sum_{ij} S_i^q(t) \cdot (\Pi_{ij}^R * S_j^{\text{cl}})(t) + S_j^q(t) \\ & \cdot (\Pi_{ij}^A * S_i^{\text{cl}})(-t) + \sum_{ij} S_i^q(t) \cdot \xi_j(t). \end{aligned} \quad (45)$$

From the Lagrangian, we obtain the equations of motion by solving the Euler Lagrange equations for the classical components of the angles ϕ_k^{cl} and θ_k^{cl} :

$$\frac{\partial \mathcal{L}}{\partial \phi_k^q} = \frac{\partial \mathcal{L}}{\partial \theta_k^q} = 0 \quad k = 1, 2. \quad (46)$$

Note that the right-hand side of the Euler-Lagrange equations, which depends on the time derivatives of the quantum components ϕ_k^q and θ_k^q , is zero by virtue of the semiclassical expansion of the action discussed above.

If we make use of Eq. (38) and rearrange a bit, we find that the equations of motion for ϕ_k^{cl} , θ_k^{cl} are

$$\dot{\theta}_k = -\frac{1}{S \sin \theta_k} \frac{\partial S_k(t)}{\partial \phi_k} \cdot \sum_j [(\Gamma_{kj} * S_j)(t) + \xi_j(t)], \quad (47)$$

$$\dot{\phi}_k = -B_k + \frac{1}{S \sin \theta_k} \frac{\partial S_k(t)}{\partial \theta_k} \cdot \sum_j [(\Gamma_{kj} * S_j)(t) + \xi_j(t)], \quad (48)$$

where we defined the friction kernel:

$$\Gamma_{ij}^{\mu\nu}(t - t') = (\Pi^R)_{ij}^{\mu\nu}(t - t') + (\Pi^R)_{ji}^{\nu\mu}(t' - t). \quad (49)$$

Henceforth, we omit the cl superscript for the sake of conciseness.

Following an approach similar to that which was used in Ref. [27], we show that these equations of motion are in fact exactly of the form of a generalized Landau-Lifshitz-Gilbert

equation, namely,

$$\dot{S}_k(t) = S_k(t) \times \left[\mathbf{B}^{\text{ext}} + \sum_j (\Gamma * S_j)(t) + \xi_j(t) \right]. \quad (50)$$

It is well-known that the retarded and advanced components of the response function Π are related as [19]

$$(\Pi^R)^\dagger = \Pi^A,$$

where $\dagger$ denotes complex conjugation and exchange of the time arguments, as well as the transpose with respect to both site and spin indices. One immediately concludes that the low-frequency expansion of Γ must give, in the frequency domain,

$$\Gamma(\omega) = \mathbf{K} + i\omega\boldsymbol{\alpha}, \quad (51)$$

where we defined the reactive and dissipative parts of the friction kernel as

$$\mathbf{K} \equiv 2 \lim_{\omega \rightarrow 0} \text{Re}[\Pi^R(\omega)]; \quad (52)$$

$$\boldsymbol{\alpha} \equiv 2 \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im}[\Pi^R(\omega)], \quad (53)$$

such that Eq. (12) reduces to Eq. (13), as claimed.

One now sees that to make further progress in understanding the bath-induced dynamics of the spins, we need to proceed with evaluating the real and imaginary parts of Π^R . This calculation forms the subject of the following section.

IV. CALCULATION OF THE RESPONSE FUNCTION

A. Response functions in the frequency domain

To make progress with the evaluation of the response function, we first extract its greater and lesser parts, and apply the following identities:

$$G^{\gtrless}(\mathbf{q}, \pm(t - t')) = \frac{1}{2\pi} \int d\omega e^{\mp i\omega(t - t')} G^{\gtrless}(\mathbf{q}, \omega); \quad (54)$$

$$iG_s^>(\mathbf{q}, \omega) = A_s(\mathbf{q}, \omega)(1 - N_F(\omega - \mu)); \quad (55)$$

$$iG_s^<(\mathbf{q}, \omega) = -A_s(\mathbf{q}, \omega)N_F(\omega - \mu), \quad (56)$$

where $N_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi-Dirac distribution function, and $A_s(\mathbf{q}, \omega)$ is the spectral function of electrons with momentum $\mathbf{q}$ and energy ω in the Rashba band s , defined by

$$A_s(\mathbf{q}, \omega) \equiv i[G_s^R(\mathbf{q}, \omega) - G_s^A(\mathbf{q}, \omega)], \quad (57)$$

which may be taken to have a δ -function form if the finite lifetime of electrons in the bath is neglected:

$$A_s(\mathbf{q}, \omega) = 2\pi\delta(\varepsilon_{\mathbf{q}s} - \omega). \quad (58)$$

Using the definitions from Eqs. (40) and (41), we obtain

$$\begin{aligned} (\Pi^{R(A)})_{ij}^{\mu\nu}(\tau) = & \pm \Theta(\pm\tau) \frac{iJ_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i(\omega' - \omega'')\tau} \\ & \times (N_F(\omega' - \mu) - N_F(\omega'' - \mu)) \\ & \times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_{ij}} \\ & \times \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega') A_{s'}(\mathbf{k}', \omega''), \end{aligned} \quad (59)$$

$$\begin{aligned}
(\mathbf{\Pi}^K)_{ij}^{\mu\nu}(\tau) &= \frac{iJ_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i(\omega' - \omega'')\tau} \\
&\times [N_F(\omega'' - \mu) + N_F(\omega' - \mu) \\
&- 2N_F(\omega'' - \mu)N_F(\omega' - \mu)] \\
&\times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_{ij}} \\
&\times \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'), \quad (60)
\end{aligned}$$

where we defined $\tau \equiv t - t'$ for brevity. The Fourier transform yields

$$\begin{aligned}
(\mathbf{\Pi}^{R(A)})_{ij}^{\mu\nu}(\omega) &= -\frac{J_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \frac{1}{\omega - (\omega' - \omega'') \pm i0} \\
&\times (N_F(\omega'' - \mu) - N_F(\omega' - \mu)) \\
&\times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_{ij}} \\
&\times \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'); \quad (61)
\end{aligned}$$

$$\begin{aligned}
(\mathbf{\Pi}^K)_{ij}^{\mu\nu}(\omega) &= \frac{iJ_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int d\omega'' \delta(\omega - (\omega' - \omega'')) \\
&\times [N_F(\omega'' - \mu) + N_F(\omega' - \mu) \\
&- 2N_F(\omega'' - \mu)N_F(\omega' - \mu)] \\
&\times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_{ij}} \\
&\times \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'). \quad (62)
\end{aligned}$$

Before we proceed any further, we note an important relation between the Keldysh component and the retarded/advanced components. Specifically, one has [19]

$$(\mathbf{\Pi}^K)_{ij}^{\mu\nu}(\omega) = \coth\left(\frac{\omega}{2T}\right) [(\mathbf{\Pi}^R)_{ij}^{\mu\nu}(\omega) - (\mathbf{\Pi}^A)_{ij}^{\mu\nu}(\omega)]. \quad (63)$$

This equation holds generally (in thermal equilibrium) whether one performs a low-frequency expansion of the response function or not, and constitutes a statement of the fluctuation-dissipation theorem (FDT): it relates the Keldysh component, determining the correlations of the fluctuating magnetic fields $\xi_i(t)$, to the imaginary part of $\mathbf{\Pi}^R$, which, as we shall see, determines the Gilbert damping, i.e., the dissipation.

We now continue with the calculation. First, we evaluate the momentum space integrals which appear in Eqs. (61) and (62). As shown in Appendix A, these integrals have the following matrix structure:

$$\begin{aligned}
&\sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') \\
&\times A_{s'}(\mathbf{k}', \omega') \quad (64)
\end{aligned}$$

$$\begin{aligned}
&= \frac{m^2}{2} \sum_{ss'} \frac{k_{0s}(\omega'') k_{0s'}(\omega')}{k_0(\omega'') k_0(\omega')} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij}) \\
&\times \Theta(\omega'' - \varepsilon_{0s}) \Theta(\omega' - \varepsilon_{0s'}), \quad (65)
\end{aligned}$$

where the matrix $\mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij})$ has components

$$\begin{aligned}
\mathbb{J}_{ss'}^{xx}(\omega', \omega''; \mathbf{r}_{ij}) &= J_0(k_{0s'}(\omega')R) J_0(k_{0s}(\omega'')R) \\
&- ss' \cos(2\theta) J_1(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R); \\
\mathbb{J}_{ss'}^{yy}(\omega', \omega''; \mathbf{r}_{ij}) &= J_0(k_{0s'}(\omega')R) J_0(k_{0s}(\omega'')R) \\
&+ ss' \cos(2\theta) J_1(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R); \\
\mathbb{J}_{ss'}^{zz}(\omega', \omega''; \mathbf{r}_{ij}) &= J_0(k_{0s'}(\omega')R) J_0(k_{0s}(\omega'')R) \\
&- ss' J_1(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R); \\
\mathbb{J}_{ss'}^{xy}(\omega', \omega''; \mathbf{r}_{ij}) &= \mathbb{J}_{ss'}^{yx}(\omega', \omega''; \mathbf{r}_{ij}) \\
&= -\frac{ss'}{2} \sin(2\theta) J_1(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R); \\
\mathbb{J}_{ss'}^{xz}(\omega', \omega''; \mathbf{r}_{ij}) &= -\mathbb{J}_{ss'}^{zx}(\omega', \omega''; \mathbf{r}_{ij}) \\
&= -(s' J_1(k_{0s'}(\omega')R) J_0(k_{0s}(\omega'')R) \\
&+ s J_0(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R)) \cos \theta; \\
\mathbb{J}_{ss'}^{yz}(\omega', \omega''; \mathbf{r}_{ij}) &= -\mathbb{J}_{ss'}^{zy}(\omega', \omega''; \mathbf{r}_{ij}) \\
&= (s' J_1(k_{0s'}(\omega')R) J_0(k_{0s}(\omega'')R) \\
&+ s J_0(k_{0s'}(\omega')R) J_1(k_{0s}(\omega'')R)) \sin \theta,
\end{aligned}$$

where $\mathbf{r}_{ij} = (R \cos \theta, R \sin \theta)$, $J_n(x)$ is the Bessel function of the first kind at order n and we defined

$$k_0(\varepsilon) = \sqrt{q_R^2 + 2m\varepsilon}, \quad k_{0s}(\varepsilon) = -sq_R + k_0(\varepsilon).$$

The step functions appearing in Eq. (65) are necessary to enforce the support of the spectral function $A_\alpha(\mathbf{k}, \omega)$ on the physical domain $k > 0$ for all values of ω .

B. Keldysh component

Our aim is to perform a low-frequency expansion of the expressions (61) and (62). For the retarded and advanced components, we expand to first order in ω , whereas for the Keldysh component it is sufficient to expand to zeroth order. Such an expansion guarantees that the response functions satisfy the low-frequency version of the FDT (63); we discuss this point in more detail later on.

We first evaluate the Keldysh component, which contains a δ function that sets $\omega'' \rightarrow \omega' - \omega$. Before proceeding further, we examine the effect that this has on the integrand. Focusing on the term containing Fermi functions, one can show that, after integrating over ω'' ,

$$\begin{aligned}
&[N_F(\omega'' - \mu) + N_F(\omega' - \mu) - 2N_F(\omega'' - \mu)N_F(\omega' - \mu)]_{\omega=0} \\
&\approx 2T \delta(\omega' - \mu), \quad (66)
\end{aligned}$$

in the limit that the derivative of the Fermi function with respect to the Fermi level μ can be approximated as a δ function; this approximation is excellent provided $T \ll T_F$, and is fulfilled experimentally in a variety of archetypal 2DEGs [41–51].

The Keldysh component becomes, to zeroth order in ω ,

$$\begin{aligned}
(\mathbf{\Pi}^K)_{ij}^{\mu\nu}(\omega) &\approx \frac{iT J_{sd}^2 m^2}{4\pi} \sum_{ss'} \left(1 - s \frac{sq_R}{k_F}\right) \left(1 - \frac{s' q_R}{k_F}\right) \\
&\times \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}), \quad (67)
\end{aligned}$$

where we defined the Fermi wave vector $k_F = \sqrt{2m\mu + q_R^2} = k_0(\mu)$ and $k_{Fs} = -sq_R + k_F$. We now proceed with the calculation of the retarded and advanced components.

C. Retarded and advanced components

We first split the energy denominator in Eq. (61) into its real and imaginary parts according to the Sokhotski–Plemelj theorem [52,53], and substitute Eq. (65) for the momentum space integrals, yielding

$$\begin{aligned} & (\Pi^{R(A)})_{ij}^{\mu\nu}(\omega) \\ &= -\frac{J_{sd}^2 m^2}{4} \sum_{ss'} \mathcal{P} \int_{\varepsilon_{0s'}}^{\infty} \frac{d\omega'}{2\pi} \int_{\varepsilon_{0s}}^{\infty} \frac{d\omega''}{2\pi} \\ & \quad \times \frac{(N_F(\omega'' - \mu) - N_F(\omega' - \mu))}{\omega - (\omega' - \omega'')} \\ & \quad \times \frac{k_{0s}(\omega'')k_{0s'}(\omega')}{k_0(\omega')k_0(\omega'')} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij}) \\ & \quad \pm \frac{i\pi J_{sd}^2 m^2}{4} \sum_{ss'} \int_{\varepsilon_{0s'}}^{\infty} \frac{d\omega'}{2\pi} \int_{\varepsilon_{0s}}^{\infty} \frac{d\omega''}{2\pi} \delta(\omega - (\omega' - \omega'')) \\ & \quad \times (N_F(\omega'' - \mu) - N_F(\omega' - \mu)) \frac{k_{0s}(\omega'')k_{0s'}(\omega')}{k_0(\omega')k_0(\omega'')} \\ & \quad \times \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij}), \end{aligned} \quad (68)$$

where $\mathcal{P}$ means that the result of the integral is to be understood in the Cauchy principal value sense, and we set the lower bounds of the integrals using the step functions in Eq. (65).

The frequency integrals in Eq. (69) are easily evaluated: We first use the δ function to evaluate the integral over ω'' , setting $\omega'' \rightarrow \omega' - \omega$ as before. This shows that the zeroth order term in the low frequency expansion of (69) vanishes when evaluated at $\omega = 0$. The first-order term is easily evaluated using integration by parts. We find that

$$\text{Im}[(\Pi^{R(A)})_{ij}^{\mu\nu}(\omega)] \approx \frac{\pm \omega J_{sd}^2 m^2}{16\pi} \sum_{ss'} \frac{k_{Fs} k_{Fs'}}{k_F^2} \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}). \quad (70)$$

It is this imaginary component of the retarded response function which enters into the Gilbert damping in the equation of motion (13). Applying the identity (53), we find for the Gilbert damping tensor:

$$\begin{aligned} \alpha_{ij}^{\mu\nu} &= 2 \lim_{\omega \rightarrow 0} \frac{1}{\omega} \text{Im}[(\Pi^R)_{ij}^{\mu\nu}(\omega)] \\ &= \frac{J_{sd}^2 m^2}{8\pi} \sum_{ss'} \frac{k_{Fs} k_{Fs'}}{k_F^2} \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}) \\ &= \frac{\alpha}{4} \sum_{ss'} \left(1 - \frac{sq_R}{k_F}\right) \left(1 - \frac{s'q_R}{k_F}\right) \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}), \end{aligned} \quad (71)$$

where the local damping α is the limit of Eq. (71) as $R \rightarrow 0$ and is given by

$$\alpha = \frac{J_{sd}^2 m^2}{2\pi}. \quad (72)$$

The structure of Eq. (71) reproduces the expression which was recently derived for the Gilbert damping at a heavy metal/ferromagnet interface [54].

Before proceeding with the principal value parts, we observe that

$$(\Pi^K)_{ij}^{\mu\nu}(\omega) = \frac{2T}{\omega} [(\Pi^R)_{ij}^{\mu\nu}(\omega) - (\Pi^A)_{ij}^{\mu\nu}(\omega)],$$

which is the low-frequency version of the FDT given in Eq. (63). The fact that these two forms agree in the $\omega \rightarrow 0$ limit justifies the different orders to which Π^K and Π^R are expanded (zeroth and first order, respectively).

Using this result, and recalling Eqs. (44) and (53), we find that, in the time domain,

$$\langle \xi_i(t) \xi_j(t') \rangle = T \alpha_{ij} \delta(t - t'). \quad (73)$$

For the principal value part (68), we may not trivially expand the denominator in powers of ω . Examining the singular denominator, one sees that the integral has a line singularity on $\omega' = \omega + \omega''$. In Appendix B, we discuss how this singularity can be removed. We then compute the long-range distance dependence of the exchange, finding agreement with previous works on the RKKY interaction in 2D [55,56], and also evaluate the integral numerically. These different methods are discussed in more detail in Appendix B, including detailed explanations of the approximations we use. To evaluate the integral numerically, we use

$$\begin{aligned} \mathbf{K}_{ij}^{\mu\nu} &= \frac{J_{sd}^2 m k_F^2}{8\pi^2} \sum_{nn'} \sum_{ss'} c_{nn',ss'}^{\mu\nu}(\theta_{ij}) \int_0^\infty d\rho \\ & \quad \times [\hat{P}_{ns}(\rho) \hat{H}_{n's'}(\rho) + \hat{H}_{ns}(\rho) \hat{P}_{n's'}(\rho)], \end{aligned} \quad (74)$$

where

$$\begin{aligned} \hat{P}_{ns}(\rho) &= \int_0^\infty dE e^{-\rho E} \left(1 - \frac{sy}{1+E}\right) J_n(x(1-sy+E)); \\ \hat{H}_{ns}(\rho) &= \int_0^{d_s} dh e^{-\rho h} \left(1 - \frac{sy}{1-h}\right) J_n(x(1-sy-h)). \end{aligned}$$

Here, $E = m\xi/k_F^2$, $h = m\zeta/k_F^2$, $y = q_R/k_F$, $x = k_F R$, and $d_s = m(\mu - \varepsilon_{0s})/k_F^2$. The coefficients $c_{nn',ss'}^{\mu\nu}$ can be deduced from Eq. (65); n, n' are the orders of the Bessel functions which appear therein.

For the case where we want to understand the long-range distance dependence, the integrals in Eq. (68) can, as mentioned above, be evaluated analytically. One obtains, using Eq. (52):

$$\begin{aligned} \mathbf{K}_{ij}^{\mu\nu} &= \frac{J_{sd}^2 m k_F^2}{8\pi^2} \sum_{ss'} \sum_{nn'} c_{nn',ss'}^{\mu\nu}(\theta_{ij}) F_{ss'}(y) \\ & \quad \times \frac{\cos(x(2 - (s + s')y) - (\phi_n + \phi_{n'}))}{x^2}. \end{aligned} \quad (75)$$

Here,

$$F_{ss'}(y) = \begin{cases} 1 & y \ll 1 \\ \sqrt{(1-sy)(1-s'y)} & \text{otherwise} \end{cases}$$

and $\phi_n + \phi_{n'} = (n + n' + 1)\pi/2$. For the sake of later conciseness, we here define the prefactor common to Eqs. (74)

and (75) as

$$K = \frac{J_{sd}^2 m k_F^2}{8\pi^2}. \quad (76)$$

Having derived expressions for the various components of the response function, we proceed to further analysis of the equations of motion (13).

V. GENERALIZED LANDAU-LIFSHITZ-GILBERT EQUATION

A. Equations of motion for a particular geometry

We begin by expanding the equation of motion (13) for each spin:

$$\dot{\mathbf{S}}_1 = \mathbf{S}_1 \times \mathbf{B}^{\text{ext}} - \alpha \mathbf{S}_1 \times \dot{\mathbf{S}}_1 + \mathbf{S}_1 \times [\mathbf{K}_{12} \mathbf{S}_2 - \alpha_{12} \dot{\mathbf{S}}_2], \quad (77)$$

$$\dot{\mathbf{S}}_2 = \mathbf{S}_2 \times \mathbf{B}^{\text{ext}} - \alpha \mathbf{S}_2 \times \dot{\mathbf{S}}_2 + \mathbf{S}_2 \times [\mathbf{K}_{21} \mathbf{S}_1 - \alpha_{21} \dot{\mathbf{S}}_1]. \quad (78)$$

To study the nonreciprocal physics of interest, we neglect in the first instance the stochastic field terms; this is exact at zero temperature. Thermal fluctuations will only play an important role when the thermal energy T is large compared to the other energy scales in the system, such as the strength of the external bias field, the magnetic exchange parameters, and the frequency ω_d of the applied drive.

The reader will also note that we have neglected other possible sources of damping; for example, due to the coupling of the spins to phonons in the substrate. Such intrinsic damping could, in principle, be either local or nonlocal. In the first case, this damping corresponds to a renormalization of the local damping parameter α : $\alpha \rightarrow \alpha + \alpha_{\text{intrinsic}}$. In the latter case, it corresponds to a renormalization of the components of the damping tensor in Eq. (80) as $\alpha_0 \rightarrow \alpha_0 + \alpha_{\text{intrinsic}}$, etc. In both cases, the existence of the conditions for unidirectionality that we derive are not affected.

Equations (77) and (78) contain no local exchange terms of the form $\mathbf{K}_{11}$ or $\mathbf{K}_{22}$. These are omitted because, as shown in Appendix A, they are proportional to the identity, and hence contribute no torque to the LLG. Physically, this is expected; a Rashba 2DEG has an isotropic Fermi surface, prohibiting local anisotropy in a spin polarization which is induced by a magnetic impurity.

Now, choosing a particular geometry such that $\mathbf{r}_{12} = R\hat{\mathbf{x}} = -\mathbf{r}_{21}$, we find that the DMI is parallel to $\hat{\mathbf{y}}$. In this geometry, the exchange tensors $\mathbf{K}_{12} = \mathbf{K}_{21}^T$ and $\alpha_{12} = \alpha_{21}^T$ have the following structure:

$$\mathbf{K}_{12} = \mathbf{K}_{21}^T = \begin{pmatrix} J_0 & 0 & D \\ 0 & J_1 & 0 \\ -D & 0 & J_0 \end{pmatrix}; \quad (79)$$

$$\alpha_{12} = \alpha_{21}^T = \begin{pmatrix} \alpha_0 & 0 & \alpha_1 \\ 0 & \alpha_2 & 0 \\ -\alpha_1 & 0 & \alpha_0 \end{pmatrix}. \quad (80)$$

Before proceeding, we briefly discuss the structure of Eqs. (79) and (80). One observes that these are (i) anisotropic in space and (ii) contain antisymmetric contributions. In the case of the exchange tensor, this antisymmetric contribution is the DMI, whereas in the case of the damping tensor it corresponds to chiral damping. The spin-orbit coupling of the bath, in combination with the broken inversion symmetry

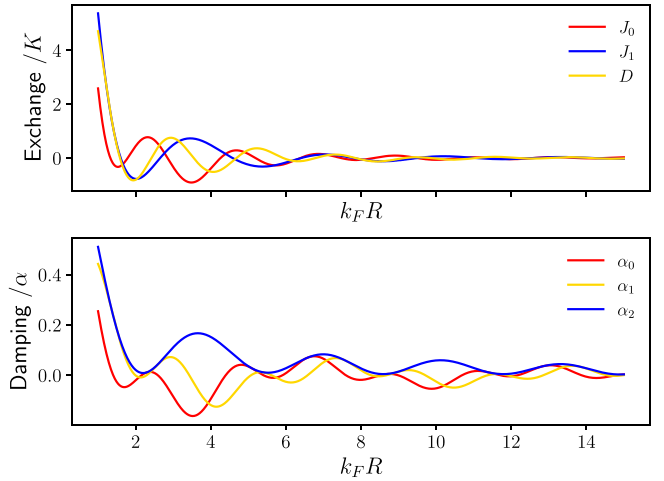


FIG. 3. Exchange and damping parameters in Eqs. (79) and (80), normalized by their respective prefactors in Eqs. (71), (74), and (75).

which is induced by the presence of the spins on the surface of the former, leads to this anisotropic, antisymmetric structure [57,58].

Figure 3 also shows that, for certain values of $k_F R$, the damping components α_0 and α_1 can be negative. However, this does not indicate the existence of antidamping or a spin-transfer-torque-like effect; one can demonstrate this by constructing the covariance matrix

$$\alpha_{\text{cov}} \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \quad (81)$$

and finding its eigenvalues. One finds that these eigenvalues are always greater than or equal to zero. In other words, α_{cov} is positive semidefinite, meaning that the nonlocal damping describes a purely dissipative process [17].

The quantities J_0 , J_1 , D and α_0 , α_1 , α_2 may be deduced from Eqs. (71) and (74) [or (75)], respectively. These quantities are plotted in Fig. 3 for the ratio $q_R/k_F = 0.5$. In the following subsection, we linearize the equations of motion (77) and (78) and examine the magnetic susceptibility.

B. Linearized dynamics and nonreciprocity

We now proceed to study the linearization of the equations of motion (77) and (78). Since $\hat{\mathbf{y}}$ is an eigenstate of the exchange tensor (79) with eigenvalue J_1 , if we choose $\mathbf{B}^{\text{ext}}$ parallel to this direction, then

$$\mathbf{S}_1^{(0)} = \eta_1 S \hat{\mathbf{y}}; \quad (82)$$

$$\mathbf{S}_2^{(0)} = \eta_2 S \hat{\mathbf{y}} \quad (83)$$

is a torque-free equilibrium of the equations of motion (77) and (78). Here, $\eta_1, \eta_2 = \pm 1$. To determine whether Eqs. (82) and (83) define the magnetic ground state, and let $\mathbf{B}^{\text{ext}} = B_0 \hat{\mathbf{y}}$. The ground state energies for each branch are then

$$E_{\eta_1 \eta_2} = -J_1 S^2 \eta_1 \eta_2 - B_0 S (\eta_1 + \eta_2). \quad (84)$$

To proceed, we linearize the equations of motion (77) and (78) around the equilibrium defined by Eqs. (82) and (83). For the sake of convention, we first rotate into a local equilibrium

frame for each spin, which can be accomplished by means of the following rotation:

$$\mathbf{R}_k^T \hat{\mathbf{z}} = \eta_k \hat{\mathbf{y}}, \quad (85)$$

whence we find

$$\mathbf{R}_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\eta_k \\ 0 & \eta_k & 0 \end{pmatrix}. \quad (86)$$

In this rotated frame, we then consider small transverse deviations u_{ix} , u_{iy} in the $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ plane. Projecting the equations of motion onto this plane, transforming to local circular coordinates

$$\psi_i = u_{ix} + i\eta_i u_{iy}, \quad (87)$$

and taking the Fourier transform, we find that Eqs. (77) and (78) reduce to

$$(\chi^{\eta_1 \eta_2}(\omega))^{-1} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \quad (88)$$

where the branch-dependent magnetic susceptibility is given by

$$\chi_{\eta_1 \eta_2}(\omega) = \frac{1}{\det((\chi^{\eta_1 \eta_2}(\omega))^{-1})} \begin{pmatrix} \chi_{11}^{\eta_1 \eta_2}(\omega) & \chi_{12}^{\eta_1 \eta_2}(\omega) \\ \chi_{21}^{\eta_1 \eta_2}(\omega) & \chi_{22}^{\eta_1 \eta_2}(\omega) \end{pmatrix}. \quad (89)$$

Here, the matrix elements are given by

$$\begin{aligned} \chi_{11}^{\eta_1 \eta_2}(\omega) &= i(\omega + B_0 + \eta_2 J_1 S + i\eta_1 S \alpha \omega); \\ \chi_{22}^{\eta_1 \eta_2}(\omega) &= i(\omega + B_0 + \eta_1 J_1 S + i\eta_2 S \alpha \omega); \\ \chi_{12}^{\eta_1 \eta_2}(\omega) &= \eta_2 S(D + iJ_0 + \alpha_0 \omega - i\alpha_1 \omega); \\ \chi_{21}^{\eta_1 \eta_2}(\omega) &= -\eta_1 S(D - iJ_0 - \alpha_0 \omega - i\alpha_1 \omega); \end{aligned}$$

$$\det((\chi^{\eta_1 \eta_2}(\omega))^{-1}) = \chi_{11}^{\eta_1 \eta_2}(\omega) \chi_{22}^{\eta_1 \eta_2}(\omega) - \chi_{12}^{\eta_1 \eta_2}(\omega) \chi_{21}^{\eta_1 \eta_2}(\omega).$$

Now, taking a weak-damping limit such that $\alpha \ll 1$ [which by virtue of Eq. (71) also implies α_0 and α_1 are small], we find that the resonance frequencies are given by

$$\omega_{\eta_1 \eta_2, \lambda}^r = \omega_{\eta_1 \eta_2, \lambda}^{(0)} [1 + i\kappa_{\eta_1 \eta_2, \lambda}], \quad (90)$$

where $\lambda = \pm 1$ labels the two resonance roots, and the undamped resonance frequencies and linewidths are given by

$$\begin{aligned} \omega_{\eta_1 \eta_2, \lambda}^{(0)} &= -B_0 - \frac{SJ_1(\eta_1 + \eta_2)}{2} \\ &\quad - \frac{\lambda S[\Omega_0(1 + \eta_1 \eta_2) + \Omega_1(1 - \eta_1 \eta_2)]}{2} \end{aligned} \quad (91)$$

and

$$\begin{aligned} \kappa_{\eta_1 \eta_2, \lambda} &= -\frac{\alpha S(\eta_1 + \eta_2)}{2} + \frac{\lambda S}{2} \left[\frac{(1 + \eta_1 \eta_2)\Gamma_1}{\Omega_0} \right. \\ &\quad \left. + \frac{(1 - \eta_1 \eta_2)\Gamma_2}{\Omega_1} \right], \end{aligned} \quad (92)$$

respectively. Here, $\Omega_0 = \sqrt{D^2 + J_0^2}$ is the FM resonance frequency, $\Omega_1 = \sqrt{J_1^2 - \Omega_0^2}$ is the AFM resonance frequency, $\Gamma_1 = D\alpha_1 + J_0\alpha_0$, and $\Gamma_2 = J_1\alpha - \Gamma_1$. Note that, depending on which method in Appendix B is used to compute the

exchange, Ω_1 may be imaginary, indicating that an AFM alignment of the spins is unstable.

Upon substituting these resonance frequencies in the off-diagonal elements χ_{12} and χ_{21} , and again dropping terms of second order or higher in the dampings, we find

$$\chi_{12}^{\eta_1 \eta_2, \lambda} = S\eta_2 [D - \eta_1 \alpha_0 \omega_{\eta_1 \eta_2, \lambda}^{(0)} - i(\eta_1 J_0 + \alpha_1 \omega_{\eta_1 \eta_2, \lambda}^{(0)})], \quad (93)$$

$$\chi_{21}^{\eta_1 \eta_2, \lambda} = -S\eta_1 [D + \eta_1 \alpha_0 \omega_{\eta_1 \eta_2, \lambda}^{(0)} + i(\eta_1 J_0 - \alpha_1 \omega_{\eta_1 \eta_2, \lambda}^{(0)})]. \quad (94)$$

Now, true unidirectional transmission of angular momentum from spin 1 to spin 2 (for example) corresponds to $\chi_{12}(\omega_{r\pm}) = 0$ while $\chi_{21}(\omega_{r\pm}) \neq 0$. The parameter which can be controlled to satisfy this condition is the strength of the external field B_0 . Hence, we solve for a condition on B_0 such that the real and imaginary parts are simultaneously zero. Our results can be parametrized by introducing the parameter σ , where $\sigma = +1$ corresponds to unidirectional transmission from spin 1 to spin 2 ($\chi_{12} = 0$; $\chi_{21} \neq 0$), and $\sigma = -1$ to the opposite case. We find

$$\begin{aligned} B_{0, \eta_1 \eta_2, \lambda}^\sigma &= -\frac{D\sigma}{\alpha_0} - \frac{S}{2} (J_1(\eta_1 + \eta_2) + \lambda[(1 + \eta_1 \eta_2)\Omega_0 \\ &\quad + (1 - \eta_1 \eta_2)\Omega_1]); \end{aligned} \quad (95)$$

where in all cases we must also impose an additional condition on the magnetic parameters:

$$D\alpha_1 + J_0\alpha_0 = 0, \quad (96)$$

for all cases.

In light of the discussion of the possibility of intrinsic damping given above, we note that a nonlocal damping will serve to decrease the (leading) first term in Eq. (95), thus decreasing the magnetic field required to realize the diode effect; this may be beneficial for applications.

In a generic system, there are always certain interatomic spacings R at which the condition(s) are satisfied. For the same ratio $q_R/k_F = 0.5$ as used in Fig. 3, we plot Eq. (96) in Fig. 4; one observes that there are indeed numerous crossings of the horizontal axis, confirming that the condition (96) can be satisfied.

Taking the first root of Eq. (96) in the range depicted in Figs. 3 and 4, which occurs at $k_F R \approx 1.605$, and reasonable values for the parameters K and α of $K \sim 10 \mu\text{eV}$, $\alpha \sim 0.005$, we compute the relevant magnetic parameters in Eqs. (79) and (80) and plot the resulting transmission, at the field given by Eq. (95) for $\sigma = -1$, in Fig. 5. One observes from this figure that the transmission from spin 2 to spin 1 has a peak at the driving frequency, whereas the transmission from spin 1 to spin 2 exhibits a Fano antiresonance at the same frequency, confirming that this channel is suppressed.

Having derived an exact diode condition in a linearized LLG formalism, we now proceed to analyze, by means of numerical simulations, the effect of nonlinearities on the size and existence of this diode effect.

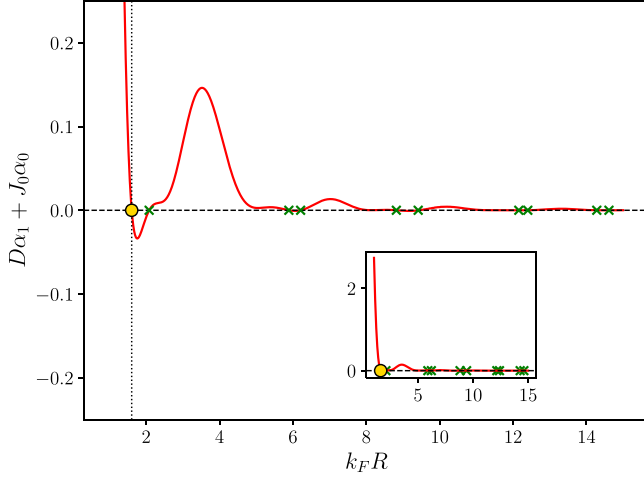


FIG. 4. Condition in Eq. (96) as a function of $k_F R$, with the selected root plotted as a gold circle, and other roots plotted with green crosses. The inset plot depicts the variation of this quantity across the same range of $k_F R$ as depicted in Fig. 3.

VI. NUMERICAL SIMULATION OF THE SPIN DIODE EFFECT

A. Nonlinearities and their effect on diode quality

The analysis presented in Sec. V relies on the assumption that the driving field is much smaller in magnitude than the bias field B_0 . To clarify the effect of small nonlinearities, we present here a simulation of the full LLG (77) and (78). Using a simple first-order Heun scheme, we study the dynamics under the influence of a magnetic field of the form

$$\mathbf{B}^{\text{ext}}(t) = B_0 \hat{\mathbf{y}} + B_1 r(t) [\cos(\omega_d t) \hat{\mathbf{x}} + h \sin(\omega_d t) \hat{\mathbf{z}}] \quad (97)$$

in the laboratory frame, where B_1 is the drive amplitude, which for linear dynamics must be much smaller than B_0 , $r(t)$ is the ramp function, defined as

$$r(t) = \begin{cases} 0 & t < 0 \\ 1 - \cos\left(\frac{\pi t}{\tau}\right) & 0 \leq t < \tau \\ 1 & t \geq \tau. \end{cases} \quad (98)$$

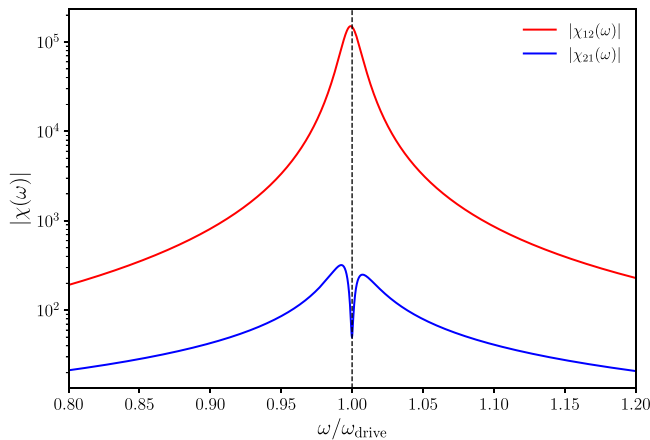


FIG. 5. Magnetic susceptibilities $|\chi_{12}|$ and $|\chi_{21}|$ with the conditions (95) and (96), plotted on a log scale.

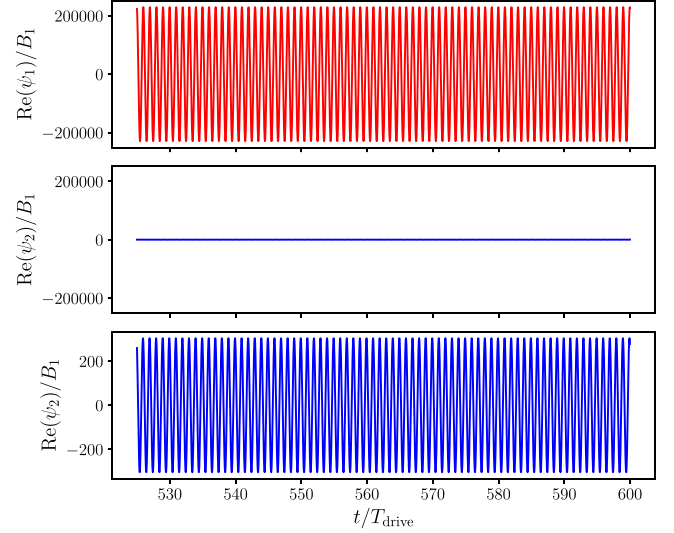


FIG. 6. Real-time dynamics of ψ_1 and ψ_2 , clearly showing the 3 orders of magnitude difference in the response of the allowed channel (drive spin 2, measure spin 1) versus the suppressed channel (drive spin 1, measure spin 2).

ω_d is the drive frequency (here, the resonance frequency computed from Eq. (91), and $h = \pm 1$ is the handedness, defining which circular mode is excited. Here, the ramp function represents the smooth turning-on of the ac field in a time τ .

Taking the same magnetic parameters as for the diode candidate in Sec. V, which was used to produce Figs. 4 and 5, we simulate the full LLG. From this simulation, we obtain the time series data for $S_{i\alpha}^x(t)$ and $S_{i\alpha}^z(t)$, where as before $i = 1, 2$ and where $\alpha = 1, 2$ denotes which spin is driven. From this, we rotate into the local frame for each spin and work in circular coordinates using Eqs. (85) and (87). We then compute the transmission amplitudes:

$$A_{i\alpha} \equiv \langle e^{-i\omega t} \psi_{i\alpha}(t) \rangle_t. \quad (99)$$

The quality of the diode is then characterized by the ratio

$$\mathcal{Z}_{2 \rightarrow 1} = \frac{A_{12}}{A_{21}} = \frac{1}{\mathcal{Z}_{1 \rightarrow 2}}; \quad (100)$$

if this is small, then the transmission from spin 2 to spin 1 when the former is driven is much smaller than the transmission from the latter to the former when the latter is driven.

In Fig. 6, we plot the real-time dynamics of ψ_1 and ψ_2 for suitable magnetic parameters. In this case, the diode is biased in the $2 \mapsto 1$ direction; the response of spin 2 to spin 1 when the latter is driven is seen to be 3 orders of magnitude smaller than in the opposite case, demonstrating the diode effect in operation. Note that the response of 2 to 1 is not nonzero, which is to be expected: In a full LLG simulation, small nonlinearities which were neglected in the analysis of Sec. V induce a comparatively small, but still nonzero response.

To investigate the effect of these nonlinearities, we measured the effect of increasing the amplitude B_1 of the ac driving field on the diode ratio. The results are depicted in Fig. 7, clearly showing that, as the driving field amplitude B_1 approaches and exceeds the same order of magnitude as the bias field B_0 , the perfect unidirectionality which is demonstrated by the linearized calculation of Sec. V disappears.

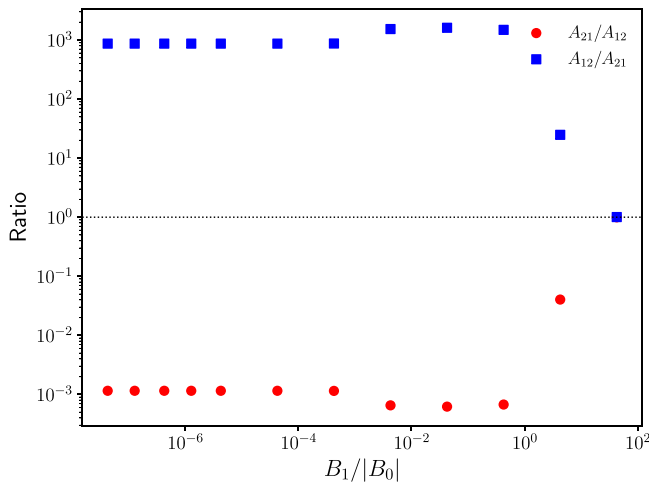


FIG. 7. Dependence of the diode ratios on the ac driving field amplitude.

To conclude this section, we compute the numerical susceptibility for each direction of the diode in frequency space by sweeping the driving frequency and using the fast Fourier transform (FFT), and plot these in Fig. 8. Comparing to Fig. 5, we see that the transmission is strongly suppressed in one direction, but it is still nonzero; this is also consistent with Fig. 6.

VII. CONCLUSION

In this work, we developed a model for an atomic spin diode, consisting of two localized spins (magnetic adatoms), deposited on the surface of a two-dimensional electron gas. We derived the semiclassical equations of motion for the steady-state spin dynamics using the Keldysh path integral formalism, and showed that these are very similar in form to the phenomenological equations which were the starting point of Ref. [10]. Our microscopic approach gives exact expressions for both the local and nonlocal magnetic exchange and Gilbert damping parameters which enter into these equations of motion. Under certain assumptions, these all have analytical expressions.

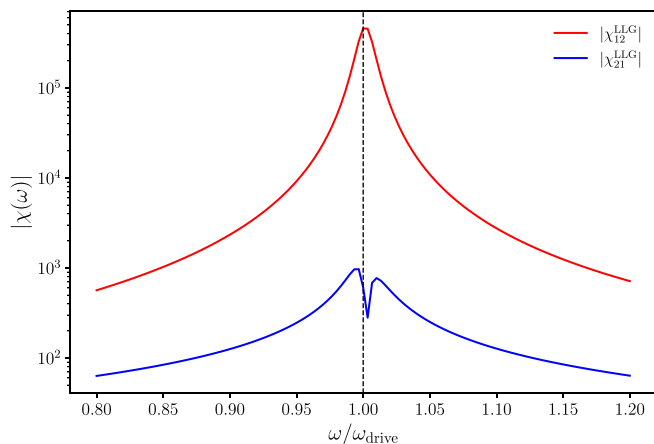


FIG. 8. Susceptibility computed from the time series data using the FFT.

The results we obtained rest on several key assumptions and approximations. First, we treated the s - d coupling constant perturbatively to integrate out the bath. Further, we assumed that the temperature of the system is much smaller than the Fermi temperature of the bath. These first two are the only two approximations that are necessary to arrive at an analytical expression for the damping; their validity is discussed in Secs. III and IV, respectively.

To obtain the quoted expression for the exchange tensor, we needed to make, depending on the method used, two or more further approximations to evaluate the frequency integrals in Eq. (68). The validity of these approximations is discussed in Appendix B.

As emphasized in Ref. [10], a system which possesses both DMI, long known to give nonreciprocal effects in spin-wave physics, and a nonlocal Gilbert damping, can exhibit unidirectional transmission of spin between the two magnetic subsystems, provided symmetric exchange is absent. To this end, we also studied the magnetic susceptibility, which is obtained by linearizing the equations of motion and taking the Fourier transform. By tuning of the exchange and damping parameters and carefully choosing the strength of the applied magnetic field, we find that the transmission of angular momentum between spin 1 and spin 2 (or vice versa) can always be made unidirectional.

In the work of Ref. [10], the authors made the initial assumption that the symmetric exchange (denoted as J_0 or J_1 in this work) was zero; when they allowed for this parameter to be nonzero, they found that their “magnon transmission” was strongly nonreciprocal but no longer unidirectional. In this work, we were able to show that this condition is not necessary to allow for exact unidirectional transmission of angular momentum between the two spins; our diode conditions (95) and (96) rather require a certain relation between the magnetic parameters whose realizability derives from the oscillatory nature of the latter, providing for high tunability. In particular, Eq. (96) can be satisfied independent of the underlying microscopic properties of the bath, such as the Fermi energy, Rashba splitting, and effective electron mass; these quantities only determine the relative magnitudes of the magnetic parameters.

Our work provides a derivation, starting from a microscopic model, of the conditions for unidirectional spin-wave propagation which were laid down phenomenologically in Ref. [10]. We emphasize that the conditions we derived are generic, though for the experimental parameters that we were able to infer, we found that the magnetic fields required to satisfy the condition (95) may be too large to realize experimentally [34,42–44,59–62]. Nevertheless, the field of magnetic adatoms on surfaces is a very active one, and we hope that our work motivates further study of experimental setups in which realizing these conditions is more feasible. Two possible directions are (i) substrate gating: the magnitude of B_0 in Eq. (95) is seen to be dominated by the term $D/\alpha_0 \propto K/\alpha \propto k_F^2/m$; gating could be used to reduce k_F and hence also B_0 , and (ii): replacing the 2DEG substrate with, for example, a ferromagnet, leveraging the resulting exchange bias effect to reduce the magnetic field required, or eliminate it completely.

We also foresee several other extensions to our work, which could include going beyond the semiclassical approximation, using a more realistic form of the s - d interaction, explicitly considering the crystalline structure of the electronic substrate, or considering a system of more than two spins.

ACKNOWLEDGMENTS

W.J.H. acknowledges the research program Materials for the Quantum Age (QuMat) for financial support. This

program (Registration No. 024.005.006) is part of the Gravitation program financed by the Dutch Ministry of Education, Culture and Science (OCW).

DATA AVAILABILITY

The code used to generate the plots used in this paper is available as a public repository at [63]. No data were generated in this study; the code only implements relevant equations.

APPENDIX A: COMPUTING THE DAMPING [EQ. (71)]

We begin by quoting Eq. (61) from the main text:

$$\begin{aligned} (\Pi^{R(A)})_{ij}^{\mu\nu}(\omega) &= -\frac{J_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \frac{1}{\omega - (\omega' - \omega'') \pm i0} (N_F(\omega'' - \mu) - N_F(\omega' - \mu)) \\ &\quad \times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'); \\ (\Pi^K)_{ij}^{\mu\nu}(\omega) &= \frac{iJ_{sd}^2}{2} \int \frac{d\omega'}{2\pi} \int d\omega'' \delta(\omega - (\omega' - \omega'')) [N_F(\omega'' - \mu) + N_F(\omega' - \mu) - 2N_F(\omega'' - \mu)N_F(\omega' - \mu)] \\ &\quad \times \sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'). \end{aligned}$$

1. Evaluating the momentum space integrals

The momentum space integrals are of the form

$$\sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega'). \quad (\text{A1})$$

To proceed, we first rewrite the spectral function. As defined in Eq. (57) of the main text, this is given by

$$A_s(\mathbf{k}, \omega) = 2\pi \delta(\varepsilon_{\mathbf{k}s} - \omega).$$

We wish to employ the distributional identity

$$\delta(f(k)) = \sum_{k_{0i}} \frac{\delta(k - k_{0i})}{f'(k_{0i})},$$

where $\{k_{0i}\}$ are the roots of $f(k) = 0$, to rewrite the spectral function in terms of k . Identifying $f(k) = \varepsilon_{\mathbf{k}s} - \hbar\omega$, and taking note of the domain $k \geq 0$, we find one real root:

$$k_{0\alpha}(\omega) = -sq_R + \sqrt{q_R^2 + 2m\omega},$$

whereas the derivative of $f(k)$ is

$$f'(k) = \frac{1}{m}(k + \alpha q_R),$$

which, evaluated at the roots $k_{0\alpha}(\omega)$, gives

$$f'(k_{0\alpha}(\omega)) = \frac{1}{m} \sqrt{q_R^2 + 2m\omega} \equiv \frac{k_0(\omega)}{m},$$

where $q_R = m\lambda_{\text{SO}}$, and we discarded the negative branch of the square root because the physical domain of k is $k \geq 0$. It is also important to consider the support of the spectral function. For this to be well-defined in the distributional sense, we must choose ω such that the function $f(k)$ has real roots on the whole domain of k . Differentiation of $\varepsilon_{\mathbf{k}\alpha}$ yields

$$\frac{\partial \varepsilon_{\mathbf{k}s}}{\partial k} = \frac{k}{m} + s\lambda_{\text{SO}} = 0.$$

For the $s = +1$ branch, we find no solutions for $k > 0$. For the $s = -1$ branch, we find a minimum at $k = m\lambda_{\text{SO}} = q_R$. Hence

$$\varepsilon_{0s} = \begin{cases} 0 & s = +1 \\ -\frac{q_R^2}{2m} & s = -1. \end{cases}$$

Therefore, in rewriting the spectral function, we must include a factor of $\Theta(\omega - \varepsilon_{0s})$. To summarize, we have

$$A_s(\mathbf{k}, \omega) = 2\pi m \frac{1}{k_0(\omega)} \delta(k - k_{0s}(\omega)) \Theta(\omega - \varepsilon_{0s}).$$

We now return to the matrix elements

$$\mathcal{M}_{\mathbf{k}\mathbf{k}',ss'}^{\mu\nu} = \langle s, \mathbf{k} | \sigma^\mu | s', \mathbf{k}' \rangle \langle s', \mathbf{k}' | \sigma^\nu | s, \mathbf{k} \rangle,$$

where the helicity eigenstates are given by

$$|s, \mathbf{k}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -is e^{i\phi_k} \end{pmatrix}.$$

In detail, we have

$$\begin{aligned} \langle s, \mathbf{k} | \sigma^x | s', \mathbf{k}' \rangle &= \frac{i}{2} [s e^{-i\phi} - s' e^{i\phi'}]; \\ \langle s', \mathbf{k}' | \sigma^x | s, \mathbf{k} \rangle &= \frac{i}{2} [s' e^{-i\phi'} - s e^{i\phi}]; \\ \langle s, \mathbf{k} | \sigma^y | s', \mathbf{k}' \rangle &= -\frac{1}{2} [s e^{-i\phi} + s' e^{i\phi'}]; \\ \langle s', \mathbf{k}' | \sigma^y | s, \mathbf{k} \rangle &= -\frac{1}{2} [s e^{i\phi} + s' e^{-i\phi'}]; \\ \langle s, \mathbf{k} | \sigma^z | s', \mathbf{k}' \rangle &= \frac{1}{2} [1 - s s' e^{i(\phi' - \phi)}]; \\ \langle s', \mathbf{k}' | \sigma^z | s, \mathbf{k} \rangle &= \frac{1}{2} [1 - s s' e^{i(\phi - \phi')}], \end{aligned}$$

where we defined $\phi = \phi_k$ and $\phi' = \phi_{k'}$. Therefore, the relevant products are

$$\begin{aligned} \mathcal{M}_{ss'}^{xx}(\mathbf{k}, \mathbf{k}') &= \frac{1}{2} [1 - s s' \cos(\phi + \phi')], \\ \mathcal{M}_{ss'}^{yy}(\mathbf{k}, \mathbf{k}') &= \frac{1}{2} [1 + s s' \cos(\phi + \phi')], \\ \mathcal{M}_{ss'}^{zz}(\mathbf{k}, \mathbf{k}') &= \frac{1}{2} [1 - s s' \cos(\phi - \phi')], \\ \mathcal{M}_{ss'}^{xz}(\mathbf{k}, \mathbf{k}') &= \frac{i}{2} [s' \cos \phi' - s \cos \phi] = -\mathcal{M}_{ss'}^{zx}(\mathbf{k}, \mathbf{k}'), \\ \mathcal{M}_{ss'}^{yz}(\mathbf{k}, \mathbf{k}') &= \frac{i}{2} [s \sin \phi - s' \sin \phi'] = -\mathcal{M}_{ss'}^{zy}(\mathbf{k}, \mathbf{k}'). \end{aligned}$$

We first consider the $i \neq j$ case, defining $\mathbf{r}_{ij} = -\mathbf{r}_{ji} = (R \cos \theta, R \sin \theta)$. The three types of integral which enter into the expression (A1) are

$$\begin{aligned} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{\pm i \mathbf{q} \cdot \mathbf{r}_{ij}} A_s(\mathbf{q}, \omega) &= \frac{m}{2\pi} \int_0^\infty dq q \delta(q - k_{0s}(\omega)) \Theta(\omega - \varepsilon_{0s}) \int_0^{2\pi} d\phi e^{\pm i q R \cos(\phi - \theta)} \\ &= m \int_0^\infty dq q \delta(q - k_{0s}(\omega)) \Theta(\omega - \varepsilon_{0s}) J_0(qR) \\ &= m J_0(k_{0s}(\omega) R) \Theta(\omega - \varepsilon_{0s}), \\ \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{\pm i \mathbf{q} \cdot \mathbf{r}_{ij}} A_s(\mathbf{q}, \omega) \cos \phi &= \pm i m J_1(k_{0s}(\omega) R) \Theta(\omega - \varepsilon_{0s}) \cos \theta, \\ \int \frac{d^2 \mathbf{q}}{(2\pi)^2} e^{\pm i \mathbf{q} \cdot \mathbf{r}_{ij}} A_s(\mathbf{q}, \omega) \sin \phi &= \pm i m J_1(k_{0s}(\omega) R) \Theta(\omega - \varepsilon_{0s}) \sin \theta, \end{aligned}$$

where J_n is the Bessel function of the first kind at order n . Using the results above, we find after some algebra that the momentum space integrals have the following matrix structure:

$$\sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega') = \frac{m^2}{2} \sum_{ss'} \frac{k_{0s}(\omega'') k_{0s'}(\omega')}{k_0(\omega') k_0(\omega'')} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij}) \Theta(\omega'' - \varepsilon_{0s}) \times \Theta(\omega' - \varepsilon_{0s'}),$$

which is Eq. (65) from the main text.

Now, let us consider the local case $i = j$. In this case, $\mathbf{r}_{ij} = \mathbf{0}$, which significantly simplifies the matrix structure of Eq. (65). We have

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} A_s(\mathbf{k}, \omega) = 2\pi m \int_0^\infty dk \frac{k}{k_0(\omega)} \delta(k - k_{0s}(\omega)) \int_0^{2\pi} d\phi = m \int_0^\infty dk \frac{k}{k_0(\omega)} \delta(k - k_{0s}(\omega)) = \frac{mk_{0s}(\omega)}{k_0(\omega)};$$

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2} A_s(\mathbf{k}, \omega) \cos \phi = \frac{m}{2\pi} \int_0^\infty dk \frac{k}{k_0(\omega)} \delta(k - k_{0s}(\omega)) \int_0^{2\pi} d\phi \cos \phi = 0 = \int \frac{d^2\mathbf{k}}{(2\pi)^2} A_s(\mathbf{k}, \omega) \sin \phi,$$

making the structure of Eq. (A1) much simpler. This is, namely,

$$\sum_{ss'} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \int \frac{d^2\mathbf{k}'}{(2\pi)^2} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}_{ij}} \mathcal{M}_{ss'}^{\mu\nu}(\mathbf{k}, \mathbf{k}') A_s(\mathbf{k}, \omega'') A_{s'}(\mathbf{k}', \omega') = \left[\frac{m^2}{k_0(\omega') k_0(\omega'')} \sum_{ss'} k_{0s'}(\omega') k_{0s}(\omega'') \right] \mathbb{1}.$$

2. Computing the damping [Eq. (71)]

a. Nonlocal damping

We may now proceed with the derivation of the nonlocal damping. We take Eq. (61) from the main text and split the complex denominator into its real and imaginary parts according to the Sokhotski-Plemelj theorem, which states that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{x \pm i\varepsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi \delta(x),$$

where $\mathcal{P}$ denotes the Cauchy principal value. For the delta function part, we therefore have

$$\begin{aligned} & \pm \frac{i\pi J_{sd}^2 m^2}{4} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty \frac{d\omega'}{2\pi} \int_{\varepsilon_{0s}}^\infty \frac{d\omega''}{2\pi} \delta(\omega - (\omega' - \omega'')) (N_F(\omega'' - \mu) - N_F(\omega' - \mu)) \frac{k_{0s}(\omega'') k_{0s'}(\omega')}{k_0(\omega') k_0(\omega'')} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega''; \mathbf{r}_{ij}) \\ &= \pm \frac{i\pi J_{sd}^2 m^2}{4} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty \frac{d\omega'}{(2\pi)^2} (N_F(\omega' - (\omega + \mu)) - N_F(\omega' - \mu)) \frac{k_{0s}(\omega' - \omega) k_{0s'}(\omega')}{k_0(\omega') k_0(\omega' - \omega)} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega' - \omega; \mathbf{r}_{ij}) \\ &= \pm \frac{i\pi J_{sd}^2 m^2}{4} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty \frac{d\omega'}{(2\pi)^2} (N_F(\omega' - (\omega + \mu)) - N_F(\omega' - \mu)) f(\omega, \omega') \quad \boxed{f(\omega, \omega') \equiv \frac{k_{0s}(\omega' - \omega) k_{0s'}(\omega')}{k_0(\omega') k_0(\omega' - \omega)} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega' - \omega; \mathbf{r}_{ij})} \\ &\approx \pm \frac{iJ_{sd}^2 m^2 \omega}{16\pi} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty d\omega' \frac{\partial}{\partial \omega} (N_F(\omega' - (\omega + \mu)) - N_F(\omega' - \mu)) f(\omega, \omega')|_{\omega=0} \\ &= \pm \frac{iJ_{sd}^2 m^2 \omega}{16\pi} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty d\omega' f(\omega = 0, \omega') \frac{\partial}{\partial \omega} (N_F(\omega' - (\omega + \mu)) - N_F(\omega' - \mu)) \\ &\approx \pm \frac{iJ_{sd}^2 m^2 \omega}{16\pi} \sum_{ss'} \int_{\varepsilon_{0s'}}^\infty d\omega' f(\omega = 0, \omega') \delta(\omega' - \mu) \\ &\approx \pm \frac{iJ_{sd}^2 m^2 \omega}{16\pi} \sum_{ss'} \frac{k_{0s}(\mu) k_{0s'}(\mu)}{k_0(\mu) k_0(\mu)} \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}), \end{aligned}$$

which is identical to Eq. (70). The use of the first delta function to collapse the ω'' integral is always valid, whereas the use of the second delta function (which comes from approximating the Fermi function's derivative as a delta function provided $T \ll T_F$)

to collapse the ω' integral is valid provided $\mu > 0$. So, the nonlocal damping is found from Eq. (53) to be

$$\alpha_{ij} = 2 \lim_{\omega \rightarrow 0} \frac{1}{\omega} \frac{J_{sd}^2 m^2 \omega}{16\pi} \sum_{ss'} \frac{k_{0s}(\mu) k_{0s'}(\mu)}{k_0(\mu) k_0(\mu)} \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}) = \frac{J_{sd}^2 m^2}{8\pi} \sum_{ss'} \frac{k_{0s}(\mu) k_{0s'}(\mu)}{k_0(\mu) k_0(\mu)} \mathbb{J}_{ss'}^{\mu\nu}(\mu, \mu; \mathbf{r}_{ij}).$$

To make connection with Eq. (80) from the main text, we consider the case where $\theta = 0$. One finds in this case:

$$\begin{aligned} \alpha_{ij}^{xx} &= \frac{\alpha}{4} \sum_{ss'} \left(1 - \frac{sq_R}{k_F R}\right) \left(1 - \frac{s'q_R}{k_F}\right) [J_0(k_{Fs}R)J_0(k_{Fs'}R) - ss'J_1(k_{Fs}R)J_1(k_{Fs'}R)] = \alpha_{ij}^{zz} \equiv \alpha_0, \\ \alpha_{ij}^{yy} &= \frac{\alpha}{4} \sum_{ss'} \left(1 - \frac{sq_R}{k_F R}\right) \left(1 - \frac{s'q_R}{k_F}\right) [J_0(k_{Fs}R)J_0(k_{Fs'}R) - ss'J_1(k_{Fs}R)J_1(k_{Fs'}R)] \equiv \alpha_2, \\ \alpha_{ij}^{xz} &= -\frac{\alpha}{4} \sum_{ss'} \left(1 - \frac{sq_R}{k_F R}\right) \left(1 - \frac{s'q_R}{k_F}\right) [s'J_1(k_{Fs'}R)J_0(k_{Fs}R) + sJ_0(k_{Fs}R)J_1(k_{Fs'}R)] \equiv \alpha_1 = -\alpha_{ij}^{zx}, \\ \alpha_{ij}^{yz} &= \alpha_{ij}^{zy} = \alpha_{ij}^{xy} = \alpha_{ij}^{yx} = 0. \end{aligned}$$

b. Local damping

We can follow an identical derivation as was used above to compute the local damping. The result is

$$\alpha = 2 \frac{J_{sd}^2 m^2}{16\pi} \sum_{ss'} \frac{k_{0s}(\mu) k_{0s'}(\mu)}{k_0(\mu) k_0(\mu)} = \frac{J_{sd}^2 m^2}{2\pi}.$$

APPENDIX B: COMPUTING THE EXCHANGE [EQS. (74) AND (75)]

Method 1: Numerical integration

a. Formulation in terms of integral transforms

We begin with Eq. (68):

$$-\frac{J_{sd}^2 m^2}{4} \sum_{ss'} \mathcal{P} \int_{\varepsilon_{0s'}}^{\infty} \frac{d\omega'}{2\pi} \int_{\varepsilon_{0s}}^{\infty} \frac{d\omega''}{2\pi} \frac{(N_F(\omega'' - \mu) - N_F(\omega' - \mu))}{\omega - (\omega' - \omega'')} \frac{k_{0s}(\omega'') k_{0s'}(\omega')}{k_0(\omega'') k_0(\omega')} \mathbb{J}_{ss'}^{\mu\nu}(\omega', \omega'', \mathbf{r}_{ij}). \quad (\text{B1})$$

This equation can also be written in the form

$$-C \sum_{ss'} \sum_{nn'} \mathcal{P} \int_{\varepsilon_{0s'}}^{\infty} d\omega' \int_{\varepsilon_{0s}}^{\infty} d\omega'' c_{nn',ss'}^{\mu\nu}(\theta) \frac{N_F(\omega'' - \mu) - N_F(\omega' - \mu)}{\omega - (\omega' - \omega'')} B_{ns}(\omega'') B_{n's'}(\omega'), \quad (\text{B2})$$

where $C = (J_{sd} m / 4\pi)^2$, $c_{nn',ss'}^{\mu\nu}(\theta)$ encodes the coefficients in Eq. (65), and

$$B_{ns}(\varepsilon) = \left(1 - \frac{sq_R}{\sqrt{2m\varepsilon + q_R^2}}\right) J_n(R(-sq_R + \sqrt{2m\varepsilon + q_R^2})). \quad (\text{B3})$$

So we see that our task is to evaluate an integral of the generic form

$$\mathcal{P} \int_{\varepsilon_{0s'}}^{\infty} d\omega' \int_{\varepsilon_{0s}}^{\infty} d\omega'' \frac{(N_F(\omega'' - \mu) - N_F(\omega' - \mu))}{\omega - (\omega' - \omega'')} B_{ns}(\omega'') B_{n's'}(\omega'). \quad (\text{B4})$$

To begin, we first write the Fermi functions as step functions; this approximation is exact at zero temperature but remains extremely good provided the temperature $T \ll T_F$, as discussed above. The weight $W = N_F(\omega'' - \mu) - N_F(\omega' - \mu) = \Theta(\mu - \omega'') - \Theta(\mu - \omega')$ is equal to -1 if $\mu > \omega' > \varepsilon_{0s'}$ and $\infty > \omega'' > \mu$, and $+1$ if $\infty > \omega' > \mu$ and $\mu > \omega'' > \varepsilon_{0s}$. Note that if $\mu > 0$, then for all $s, s' = \pm 1$, $\mu > \varepsilon_{0s}, \varepsilon_{0s'}$. Then, we get two particle-hole contributions to the integral:

$$-\mathcal{P} \int_{\varepsilon_{0s'}}^{\mu} d\omega' \int_{\mu}^{\infty} d\omega'' \frac{B_{ns}(\omega'') B_{n's'}(\omega')}{\omega - (\omega' - \omega'')}, \quad (\text{B5})$$

$$+\mathcal{P} \int_{\mu}^{\infty} d\omega' \int_{\varepsilon_{0s}}^{\mu} d\omega'' \frac{B_{ns}(\omega'') B_{n's'}(\omega')}{\omega - (\omega' - \omega'')}. \quad (\text{B6})$$

The integral still diverges on the singular line $\omega' = \omega'' + \omega$. To make progress, we first make the substitutions $\omega'' = \mu + \xi$, $\omega' = \mu - \zeta$ in Eq. (B5), and $\omega' = \mu + \xi$, $\omega'' = \mu - \zeta$ in Eq. (B6). Upon exchanging the bounds of integration (being careful

with signs), we obtain

$$-\mathcal{P} \int_0^\infty d\xi \int_0^{\mu-\varepsilon_{0s}} d\zeta \frac{B_{ns}(\mu+\xi)B_{n's'}(\mu-\zeta)}{\omega+(\xi+\zeta)}, +\mathcal{P} \int_0^{\mu-\varepsilon_{0s}} d\zeta \int_0^\infty d\xi \frac{B_{ns}(\mu-\zeta)B_{n's'}(\mu+\xi)}{\omega-(\xi+\zeta)}.$$

We now discuss how the singularity can be removed. Let $u = \xi + \zeta$ and $t = \xi/(\xi + \zeta)$. The Jacobian is $+u$, and the measure transforms to $d\xi d\zeta = u du dt$. In this form, we can show that exchanging the static limit $\omega \rightarrow 0$ with the integrals does not present a problem, as we have:

$$\frac{d\xi d\zeta}{\omega + \xi + \zeta} \mapsto \frac{u du dt}{\omega + u} \rightarrow 1 \text{ as } \omega \rightarrow 0, \quad \frac{d\xi d\zeta}{\omega - (\xi + \zeta)} \mapsto \frac{u du dt}{\omega - u} \rightarrow -1 \text{ as } \omega \rightarrow 0.$$

After taking the static limit, the integrand acquires the form

$$-\int_0^1 dt \int_0^{D_s/(1-t)} du B_{ns}(\mu + ut) B_{n's'}(\mu - u(1-t)) - \int_0^1 dt \int_0^{D_s/(1-t)} du B_{ns}(\mu - u(1-t)) B_{n's'}(\mu + ut), \quad (\text{B7})$$

where we defined the band bottom $D_s = \mu - \varepsilon_{0s}$. Equivalently,

$$-\int_0^\infty d\xi \int_0^{D_s} d\zeta \frac{B_{ns}(\mu+\xi)B_{n's'}(\mu-\zeta)}{\xi+\zeta} - \int_0^\infty d\xi \int_0^{D_s} d\zeta \frac{B_{ns}(\mu-\zeta)B_{n's'}(\mu+\xi)}{\xi+\zeta}. \quad (\text{B8})$$

We now introduce the transform

$$\frac{1}{\xi + \zeta} = \int_0^\infty d\tau e^{-\tau(\xi+\zeta)},$$

and also define

$$P_{ns}(\tau) = \int_0^\infty d\xi e^{-\tau\xi} B_{ns}(\mu + \xi), \quad (\text{B9})$$

$$H_{ns}(\tau) = \int_0^{E_s} d\zeta e^{-\tau\zeta} B_{ns}(\mu - \zeta) \quad (\text{B10})$$

as the particle-side and hole-side transforms, respectively. The integral is transformed to

$$-\int_0^\infty d\tau [P_{ns}(\tau)H_{n's'}(\tau) + H_{ns}(\tau)P_{n's'}(\tau)]. \quad (\text{B11})$$

We have reduced a double integral to an integral over a sum of products of one-dimensional transforms, which should be more stable numerically. In the following section, we discuss a further approximation that can be made to make this numerical integration even easier.

b. Linearizing the dispersion relation

Recall

$$k_{0s}(\varepsilon) = -sq_R + \sqrt{2m\varepsilon + q_R^2} \approx k_F - sq_R + \frac{(\varepsilon - \mu)}{v_F}, \quad (\text{B12})$$

where $v_F = k_F/m$ is the Fermi velocity. The function $B_{ns}(\varepsilon)$ becomes

$$B_{ns}(\varepsilon) \approx \left(1 - \frac{sq_R}{k_F + (\varepsilon - \mu)/v_F}\right) J_n\left(R\left(k_F - sq_R + \frac{\varepsilon - \mu}{v_F}\right)\right). \quad (\text{B13})$$

We now define a few variables. Let $x \equiv k_F R$; $y \equiv q_R/k_F$; $V \equiv v_F k_F$; $E \equiv \xi/V$; $\rho \equiv \tau V$. Then the particle-side transform becomes

$$P_{ns}(\tau) = V \hat{P}_{vs}(\rho); \quad (\text{B14})$$

$$\hat{P}_{vs}(\rho) = \int_0^\infty dE e^{-\rho E} \left(1 - \frac{sy}{1+E}\right) J_n(x(1-sy+E)) \quad (\text{B15})$$

$$= I_n(a_s, \rho; x) - sy Q_n(a_s, \rho; x), \quad (\text{B16})$$

where

$$I_v(a_s, \rho; x) = \int_0^\infty dE e^{-\rho E} J_n(x(a_s + E));$$

$$Q_v(a_s, \rho; x) = \int_0^\infty dE \frac{e^{-\rho E}}{1+E} J_n(x(a_s + E))$$

and we defined $a_s = 1 - sy$. Focusing on the first of these terms, we see that

$$\begin{aligned} I_v(a_s, \rho; x) &= e^{\rho a_s} \int_{a_s}^{\infty} dz e^{-\rho z} J_n(xz) = e^{\rho a_s} \left[\int_0^{\infty} dz e^{-\rho z} J_n(xz) - \int_0^{a_s} dz e^{-\rho z} J_n(xz) \right] = e^{\rho a_s} \left[\mathcal{L}\{J_n(xz); \rho\} - \int_0^{a_s} dz e^{-\rho z} J_n(xz) \right] \\ &= e^{\rho a_s} \left[\frac{(\sqrt{\rho^2 + x^2} - \rho)^n}{x^n \sqrt{\rho^2 + x^2}} - \int_0^{a_s} dz e^{-\rho z} J_n(xz) \right]. \end{aligned}$$

Thus, in the case of a linearized dispersion, we can reduce the improper integral of an oscillatory function to the sum of a known integral transform and a finite integral. We now focus on Q_n . This can be simplified by introducing

$$\frac{1}{1+E} = \int_0^{\infty} d\lambda e^{-\lambda(1+E)},$$

which gives

$$Q_n(a_s, \rho; x) = \int_0^{\infty} d\lambda e^{-\lambda} \int_0^{\infty} dE e^{-(\rho+\lambda)E} J_n(x(a_s + E)) = \int_0^{\infty} d\lambda e^{-\lambda} I_n(a_s, \rho + \lambda; x);$$

so, to summarize:

$$\hat{P}_{ns}(\rho) = I_n(a_s, \rho; x) - \int_0^{\infty} d\lambda e^{-\lambda} I_n(a_s, \rho + \lambda; x). \quad (\text{B17})$$

Analogously, we may rewrite the hole-side transform as

$$H_{ns}(\tau) = V \hat{H}_{ns}(\rho), \quad (\text{B18})$$

where

$$\hat{H}_{ns}(\rho) = \int_0^{d_s} \int dh e^{-\rho h} \left(1 - \frac{sy}{1-h} \right) J_n(x(1-sy-h)), \quad (\text{B19})$$

and we defined $d_s = D_s/V$. Equation (B11) becomes

$$-V \int_0^{\infty} d\rho [\hat{P}_{ns}(\rho) \hat{H}_{n's'}(\rho) + \hat{H}_{ns}(\rho) \hat{P}_{n's'}(\rho)]. \quad (\text{B20})$$

Before we give the numerical results which are obtained from these formulas, we first note that the expression for I_v given above is only numerically stable when ρa is not too large. For large ρa_s , the exponential prefactor is large, whereas the two terms in brackets will cancel one another; the latter two constitute the exponentially suppressed tail of the integral. For large ρa_s , we can go back to the original definition of I_n :

$$\begin{aligned} I_n(a_s, \rho; x) &= \int_0^{\infty} dE e^{-\rho E} J_n(x(a_s + E)) \quad \boxed{r \equiv \rho E} \\ &= \frac{1}{\rho} \int_0^{\infty} dr e^{-r} J_n\left(x\left(a_s + \frac{r}{\rho}\right)\right). \end{aligned}$$

In this version, the oscillatory behavior is exponentially suppressed by the prefactor e^{-r} , making the integral numerically well-behaved. We can summarize these results for I_n as follows:

$$I_n(a_s, \rho; x) = \begin{cases} e^{\rho a_s} \left[\frac{(\sqrt{\rho^2 + x^2} - \rho)^n}{x^n \sqrt{\rho^2 + x^2}} - \int_0^{a_s} dz e^{-\rho z} J_n(xz) \right] & \rho a_s \leq p^* \\ \frac{1}{\rho} \int_0^{\infty} dr e^{-r} J_n\left(x\left(a_s + \frac{r}{\rho}\right)\right) & \rho a_s > p^*, \end{cases}$$

where p^* is some threshold, above which we switch to the second method for computing I_n . Substitution of (B20) in (B4) yields

$$\lim_{\omega \rightarrow 0} \text{Re}[(\Pi^{R(A)})^{\mu\nu}_{ij}(\omega)] = \frac{J_{sd}^2 k_F^2 m}{16\pi^2} \sum_{nn'} \sum_{ss'} c_{nn',ss'}^{\mu\nu}(\theta) \int_0^{\infty} d\rho [\hat{P}_{ns}(\rho) \hat{H}_{n's'}(\rho) + \hat{H}_{ns}(\rho) \hat{P}_{n's'}(\rho)]. \quad (\text{B21})$$

2. Method 2: Analytic long-range dependence of the exchange

In this section, we compute the long-range dependence of the exchange interaction, obtaining a universal RKKY-type interaction kernel. Starting from Eqs. (B8) and (B13), we substitute the asymptotic form of the Bessel functions:

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi(2n+1)}{4}\right);$$

in this case, valid provided $(k_F - sq_R)R \gg 1$, to obtain

$$\begin{aligned} B_{ns}(\mu + \xi)B_{n's'}(\mu - \zeta) &\propto \cos(k_F R - sq_R R + \Gamma\xi - \phi_n) \cos(k_F R - s'q_R R - \Gamma\zeta - \phi_{n'}) \\ &\propto \frac{1}{2} [\cos(2k_F R - (s + s')q_R R + \Gamma(\xi - \zeta) - (\phi_n + \phi_{n'})) + \cos((s' - s)q_R R + \Gamma(\xi + \zeta) + \phi_{n'} - \phi_n)], \\ B_{ns}(\mu - \zeta)B_{n's'}(\mu + \xi) &\propto \cos(k_F R - sq_R R - \Gamma\zeta) \cos(k_F R - s'q_R R + \Gamma\xi) \\ &\propto \frac{1}{2} [\cos(2k_F R - (s + s')q_R R + \Gamma(\xi - \zeta) - (\phi_n + \phi_{n'})) + \cos((s' - s)q_R R - \Gamma(\xi + \zeta) + \phi_{n'} - \phi_n)]. \end{aligned}$$

Here, $\Gamma = R/v_F$ and $\phi_n = (2n + 1)\pi/4$. Making a further standard approximation, we replace the slowly varying (compared to the cosine term) prefactors by their values at the Fermi surface and find an overall Fermi-surface prefactor of

$$\frac{1}{\pi k_F R} \frac{(1 - sq_R/k_F)(1 - s'q_R/k_F)}{\sqrt{(1 - sq_R/k_F)(1 - s'q_R/k_F)}} = \frac{1}{\pi k_F R} \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)}.$$

Note that, in contrast to previous calculations of this kind, e.g., Ref. [56], we do not replace $q_R/k_F \rightarrow 1$. This would correspond to a weak SOC limit; for the nonreciprocal physics studied in the main text, it is important that full SOC dependence is retained, if it is desirable that the DMI be large.

Now, we are ready to proceed with evaluating Eq. (B8), using these approximate forms of the Bessel functions. Defining $\Phi_{nn',ss'}^+ = (2k_F - (s + s')q_R)R - (\phi_n + \phi_{n'})$; $\Phi_{nn',ss'}^- = (s - s')q_R R + \phi_{n'} - \phi_n$, we find that Eq. (B8) becomes, up to prefactors:

$$\begin{aligned} & - \int_0^\infty d\xi \int_0^{D_{s'}} d\zeta \frac{[\cos(\Gamma(\xi - \zeta) + \Phi_{nn',ss'}^+) + \cos(\Gamma(\xi + \zeta) + \Phi_{nn',ss'}^-)]}{\xi + \zeta}, \\ & - \int_0^\infty d\xi \int_0^{D_s} d\zeta \frac{[\cos(\Gamma(\xi - \zeta) + \Phi_{nn',ss'}^+) + \cos(\Gamma(\xi + \zeta) - \Phi_{nn',ss'}^-)]}{\xi + \zeta}. \end{aligned}$$

In the following, we only retain the sum-phase term in the above. In our logic, this term is the real-space version of the standard long-distance RKKY contribution; in momentum space, it arises from the nonanalyticity of the static susceptibility at the Fermi-surface spanning wave vector $q = 2k_F$; the Fourier transform of the latter gives the characteristic RKKY dependence [55,56,64]. We first focus on those terms containing the sum phase $\Phi_{nn',ss'}^+$. In the asymptotic expansion, the leading oscillatory contribution comes from energies on the scale $\xi, \zeta \sim v_F/R$. If $D_s \gg v_F/R$, then contributions at the valence band edge are asymptotically suppressed, and we may take the limit $D_s, D_{s'} \rightarrow \infty$. The sum-phase term becomes

$$\begin{aligned} \int_0^\infty d\xi \int_0^\infty d\zeta \frac{\cos(\Gamma(\xi - \zeta) + \Phi_{nn',ss'}^+)}{\xi + \zeta} v &= \xi + \zeta; u = \xi - \zeta = \frac{1}{2} \int_0^\infty \frac{dv}{v} \int_{-v}^v \cos(\Gamma u + \Phi_{nn',ss'}^+) \\ &= \frac{1}{2\Gamma} \int_0^\infty \frac{dv}{v} [\sin(\Gamma u + \Phi_{nn',ss'}^+)]_{-v}^v = \frac{\cos(\Phi_{nn',ss'}^+)}{\Gamma} \int_0^\infty dv \frac{\sin(\Gamma v)}{v} \\ &= \frac{\pi \cos(\Phi_{nn',ss'}^+)}{2\Gamma}. \end{aligned}$$

The difference phase term is slightly different. This term is not singular in momentum space; rather, it contains either smooth components of the response, or endpoint-dependent terms associated with a finite E_s . In the same Fermi-surface extension $E_s \rightarrow \infty$ as above, one can show that the leading $O(1/\Gamma)$ corrections from this term cancel between the two particle-hole sectors, and the leading finite-edge corrections are of order $O(1/\Gamma^2 E_s)$, so we can neglect them. Putting the two sum-phase terms together and collecting all prefactors, Eq. (B4) becomes

$$- \frac{1}{mR^2} \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \cos(2k_F R - (s + s')q_R R - (\phi_n + \phi_{n'})). \quad (\text{B22})$$

Substitution in Eq. (68) from the main text then yields

$$\lim_{\omega \rightarrow 0} \text{Re}[(\Pi^{R(A)})_{ij}^{\mu\nu}(\omega)] = \frac{J_{sd}^2 m k_F^2}{(4\pi)^2} \sum_{ss'} \sum_{nn'} c_{nn',ss'}^{\mu\nu}(\theta) \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \frac{\cos(2k_F R - (s + s')q_R R - (\phi_n + \phi_{n'}))}{(k_F R)^2}; \quad (\text{B23})$$

where, for the combinations of n and n' which appear in Eq. (65), the cosine term gives

n, n'	$\cos(\Phi_{nn',ss'}^+ R),$
$0, 0$	$\sin(2k_F R - (s + s')q_R R),$
$0, 1$	$-\cos(2k_F R - (s + s')q_R R),$
$1, 1$	$-\sin(2k_F R - (s + s')q_R R).$

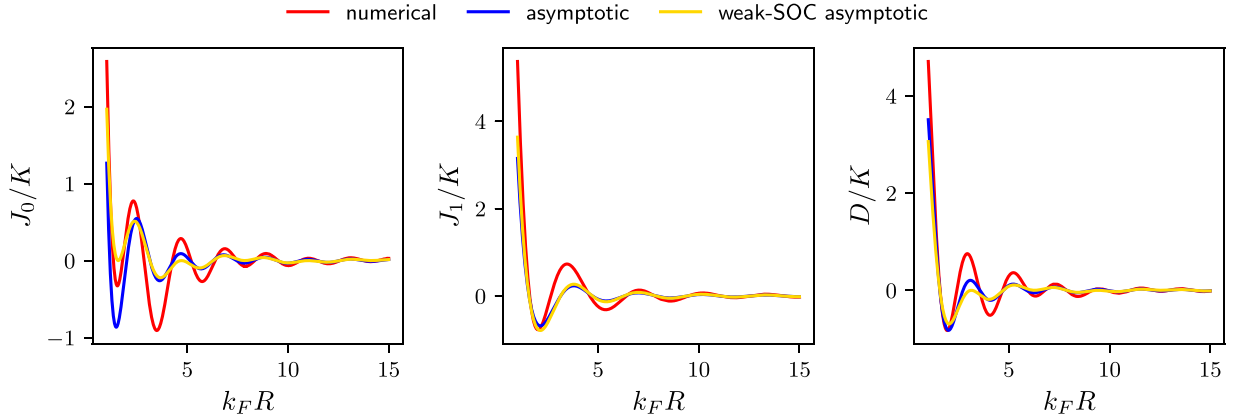


FIG. 9. The exchange constants J_0 , J_1 and the DMI strength D , calculated using Eqs. (B21)–(B23), for the geometry discussed in Sec. V.

For concreteness, we now quote the various components of the exchange tensor in the particular geometry $\mathbf{r}_{12} = -\mathbf{r}_{21} = R\hat{\mathbf{x}}$ studied in the main text, with (a) full SOC dependence retained:

$$\begin{aligned}
 J_{xx} \equiv J_0 &= \frac{K}{(k_F R)^2} \sum_{ss'} (1 + ss' \cos(2\theta)) \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \sin(2k_F R - (s + s')q_R R) \\
 &= \frac{2K}{(k_F R)^2} \left[\left(1 - \frac{q_R}{k_F}\right) \sin(2(k_F - q_R)R) + \left(1 + \frac{q_R}{k_F}\right) \sin(2(k_F + q_R)R) \right] = J_{zz}, \\
 J_{yy} \equiv J_1 &= \frac{K}{(k_F R)^2} \sum_{ss'} (1 - ss' \cos(2\theta)) \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \sin(2k_F R - (s + s')q_R R) = \frac{4K}{(k_F R)^2} \sqrt{1 - \frac{q_R^2}{k_F^2}} \sin(2k_F R), \\
 J_{xy} &= J_{yx} = 0, \\
 J_{xz} \equiv D &= \frac{K}{(k_F R)^2} \sum_{ss'} (s + s') \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \cos(2k_F R - (s + s')q_R R) \\
 &= \frac{2K}{(k_F R)^2} \left[\left(1 - \frac{q_R}{k_F}\right) \cos(2(k_F - q_R)R) - \left(1 + \frac{q_R}{k_F}\right) \cos(2(k_F + q_R)R) \right] = -J_{zx}, J_{yz} = 0,
 \end{aligned}$$

and (b) in a weak-SOC limit,

$$\begin{aligned}
 J_{xx} \equiv J_0 &= \frac{K}{(k_F R)^2} \sum_{ss'} (1 + ss' \cos(2\theta)) \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \sin(2k_F R - (s + s')q_R R) \\
 &= 4K \frac{\sin(2k_F R)}{(k_F R)^2} \cos(2q_R R) = J_{zz}, \\
 J_{yy} \equiv J_1 &= \frac{K}{(k_F R)^2} \sum_{ss'} (1 - ss' \cos(2\theta)) \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \sin(2k_F R - (s + s')q_R R) \\
 &= \frac{4K}{(k_F R)^2} \sin(2k_F R), \quad J_{xy} = J_{yx} = 0, \\
 J_{xz} \equiv D &= \frac{K}{(k_F R)^2} \sum_{ss'} (s + s') \sqrt{\left(1 - s \frac{q_R}{k_F}\right) \left(1 - s' \frac{q_R}{k_F}\right)} \cos(2k_F R - (s + s')q_R R) \\
 &= 4K \frac{\sin(2k_F R)}{(k_F R)^2} \sin(2q_R R) = -J_{zx}, \quad J_{yz} = 0,
 \end{aligned}$$

where $K = J_{sd}^2 m k_F^2 / 8\pi^2$. The components of the exchange tensor as given in Eq. (79), computed using the three different methods described above, are plotted in Fig. 9. One observes that, while the asymptotic expansion captures the phase of the Bessel functions quite well, it can wildly over- or underestimate the magnitude of these functions, leading to significant disagreement between the full and asymptotic forms, particularly at very small $k_F R$. In particular, the DMI is seen from Fig. 9 to decay much more slowly than the asymptotic expansion predicts.

- [1] A. Barman, G. Gubbiotti, S. Ladak, A. O. Adeyeye, M. Krawczyk, J. Gräfe, C. Adelmann, S. Cotořana, A. Naeemi, V. I. Vasyuchka, B. Hillebrands, S. A. Nikitov, H. Yu, D. Grundler, A. V. Sadovnikov, A. A. Grachev, S. E. Sheshukova, J.-Y. Duquesne, M. Marangolo, G. Csaba, *et al.*, The 2021 magnonics roadmap, *J. Phys.: Condens. Matter* **33**, 413001 (2021).
- [2] B. Flebus, D. Grundler, B. Rana, Y. C. Otani, I. Barsukov, G. Gubbiotti, P. Landeros, Q. Wang, T. van der Sar, *et al.*, The 2024 magnonics roadmap, *J. Phys.: Condens. Matter* **36**, 363501 (2024).
- [3] B. Lenk, H. Ulrichs, F. Garbs, and M. Münzenberg, The building blocks of magnonics, *Phys. Rep.* **507**, 107 (2011).
- [4] W. Shockley, The theory of p - n junctions in semiconductors and p - n junction transistors, *Bell Syst. Tech. J.* **28**, 435 (1949).
- [5] S. M. Sze and K. K. Ng, *Physics of Semiconductor Devices*, 3rd ed. (Wiley-Interscience, Hoboken, NJ, 2006).
- [6] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, Magnon spintronics, *Nat. Phys.* **11**, 453 (2015).
- [7] R. W. Damon and J. R. Eshbach, Magnetostatic modes of a ferromagnet slab, *J. Phys. Chem. Solids* **19**, 308 (1961).
- [8] A. V. Chumak, A. A. Serga, and B. Hillebrands, Magnon transistor for all-magnon data processing, *Nat. Commun.* **5**, 4700 (2014).
- [9] J. Lan, W. Yu, R. Wu, and J. Xiao, Spin-wave diode, *Phys. Rev. X* **5**, 041049 (2015).
- [10] H. Y. Yuan, R. Lavrijsen, and R. A. Duine, Unidirectional magnetic coupling induced by chiral interaction and nonlocal damping, *Phys. Rev. B* **107**, 024418 (2023).
- [11] J. Zou, S. Bosco, E. Thingstad, J. Klinovaja, and D. Loss, Dissipative spin-wave diode and nonreciprocal magnonic amplifier, *Phys. Rev. Lett.* **132**, 036701 (2024).
- [12] A. Metelmann and A. A. Clerk, Nonreciprocal photon transmission and amplification via reservoir engineering, *Phys. Rev. X* **5**, 021025 (2015).
- [13] A. A. Khajetoorians, J. Wiebe, B. Chilian, S. Lounis, S. Blügel, and R. Wiesendanger, Atom-by-atom engineering and magnetometry of tailored nanomagnets, *Nat. Phys.* **8**, 497 (2012).
- [14] A. A. Khajetoorians, D. Wegner, A. F. Otte, and I. Swart, Creating designer quantum states of matter atom-by-atom, *Nat. Rev. Phys.* **1**, 703 (2019).
- [15] D.-J. Choi, N. Lorente, J. Wiebe, K. von Bergmann, A. F. Otte, and A. J. Heinrich, *Colloquium: Atomic spin chains on surfaces*, *Rev. Mod. Phys.* **91**, 041001 (2019).
- [16] A. Spinelli, B. Bryant, F. Delgado, J. Fernández-Rossier, and A. F. Otte, Imaging of spin waves in atomically designed nanomagnets, *Nat. Mater.* **13**, 782 (2014).
- [17] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- [18] K. Yosida, *Theory of Magnetism*, Springer Series in Solid-State Sciences Vol. 122 (Springer, Berlin, Heidelberg, 1996).
- [19] A. Kamenev, *Field Theory of Non-equilibrium Systems* (Cambridge University Press, Cambridge, 2011).
- [20] L. V. Keldysh, Diagram technique for nonequilibrium processes, *Sov. Phys. JETP* **20**, 1018 (1965).
- [21] J. Schwinger, Brownian motion of a quantum oscillator, *J. Math. Phys.* **2**, 407 (1961).
- [22] A. Kamenev and A. Levchenko, Keldysh technique and nonlinear sigma-model: Basic principles and applications, *Adv. Phys.* **58**, 197 (2009).
- [23] J. Rammer and H. Smith, Quantum field-theoretical methods in transport theory of metals, *Rev. Mod. Phys.* **58**, 323 (1986).
- [24] L. M. Sieberer, M. Buchhold, and S. Diehl, Keldysh field theory for driven open quantum systems, *Rep. Prog. Phys.* **79**, 096001 (2016).
- [25] R. A. Duine, A. S. Núñez, J. Sinova, and A. H. MacDonald, Functional Keldysh theory of spin torques, *Phys. Rev. B* **75**, 214420 (2007).
- [26] A. S. Núñez and R. A. Duine, Effective temperature and Gilbert damping of a current-driven localized spin, *Phys. Rev. B* **77**, 054401 (2008).
- [27] R. C. Verstraten, T. Ludwig, R. A. Duine, and C. M. Smith, Fractional Landau-Lifshitz-Gilbert equation, *Phys. Rev. Res.* **5**, 033128 (2023).
- [28] M. G. Quarenta, M. Tharmalingam, T. Ludwig, H. Y. Yuan, L. Karwacki, R. C. Verstraten, and R. A. Duine, Bath-induced spin inertia, *Phys. Rev. Lett.* **133**, 136701 (2024).
- [29] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, Cambridge, 2010).
- [30] J. Kondo, Resistance minimum in dilute magnetic alloys, *Prog. Theor. Phys.* **32**, 37 (1964).
- [31] P. W. Anderson, G. Yuval, and D. R. Hamann, Exact results in the Kondo problem. II. Scaling theory, qualitatively correct solution, and some new results on one-dimensional classical statistical models, *Phys. Rev. B* **1**, 4464 (1970).
- [32] E. Turco, M. Aapro, S. C. Ganguli, N. Krane, R. Drost, N. Sobrino, A. Bernhardt, M. Jurićek, R. Fasel, P. Ruffieux, P. Liljeroth, and D. Jacob, Demonstrating Kondo behavior by temperature-dependent scanning tunneling spectroscopy, *Phys. Rev. Res.* **6**, L022061 (2024).
- [33] Y.-h. Zhang, S. Kahle, T. Herden, C. Stroh, M. Mayor, U. Schlickum, M. Ternes, P. Wahl, and K. Kern, Temperature and magnetic field dependence of a Kondo system in the weak coupling regime, *Nat. Commun.* **4**, 2110 (2013).
- [34] V. Madhavan, W. Chen, T. Jamneala, M. F. Crommie, and N. S. Wingreen, Local spectroscopy of a Kondo impurity: Co on Au(111), *Phys. Rev. B* **64**, 165412 (2001).
- [35] R. Hiraoka, E. Minamitani, R. Arafune, N. Tsukahara, S. Watanabe, M. Kawai, and N. Takagi, Single-molecule quantum dot as a Kondo simulator, *Nat. Commun.* **8**, 16012 (2017).
- [36] P. Wahl, P. Simon, L. Diekhöner, V. S. Stepanyuk, P. Bruno, M. A. Schneider, and K. Kern, Exchange interaction between single magnetic adatoms, *Phys. Rev. Lett.* **98**, 056601 (2007).
- [37] W. Wan, R. Harsh, A. Meninno, P. Dreher, S. Sajan, H. Guo, I. Errea, F. de Juan, and M. M. Ugeda, Evidence for ground state coherence in a two-dimensional Kondo lattice, *Nat. Commun.* **14**, 7005 (2023).
- [38] A. Shnirman, Y. Gefen, A. Saha, I. S. Burmistrov, M. N. Kiselev, and A. Altland, Geometric quantum noise of spin, *Phys. Rev. Lett.* **114**, 176806 (2015).
- [39] J. Hubbard, Calculation of partition functions, *Phys. Rev. Lett.* **3**, 77 (1959).
- [40] H. T. C. Stoof, K. Gubbels, and D. Dickerscheid, *Ultracold Quantum Fields* (Springer, Dordrecht, 2009).
- [41] S. LaShell, B. A. McDougall, and E. Jensen, Spin splitting of an Au(111) surface state band observed with angle resolved photoelectron spectroscopy, *Phys. Rev. Lett.* **77**, 3419 (1996).
- [42] F. Reinert, G. Nicolay, S. Schmidt, D. Ehm, and S. Hüfner, Direct measurements of the I -gap surface states on the (111)

- face of noble metals by photoelectron spectroscopy, *Phys. Rev. B* **63**, 115415 (2001).
- [43] G. Nicolay, F. Reinert, S. Hüfner, and P. Blaha, Spin-orbit splitting of the Γ -gap surface state on Au(111) and Ag(111), *Phys. Rev. B* **65**, 033407 (2001).
- [44] M. Hoesch, M. Muntwiler, V. N. Petrov, M. Hengsberger, L. Patthey, M. Shi, M. Falub, T. Greber, and J. Osterwalder, Spin structure of the Shockley surface state on the (111) surface of gold, *Phys. Rev. B* **69**, 241401(R) (2004).
- [45] A. Tamai, W. Meevasana, P. D. C. King, C. W. Nicholson, A. de la Torre, E. Rozbicki, and F. Baumberger, Spin-orbit splitting of the Shockley surface state on Cu(111), *Phys. Rev. B* **87**, 075113 (2013).
- [46] S. Hüfner, F. Reinert, S. Schmidt, G. Nicolay, and F. Forster, Photoemission investigation of the L -Gap surface states on clean and rare gas-covered noble metal (111)-surfaces, *Z. Phys. Chem.* **222**, 407 (2008).
- [47] C. R. Ast, J. Henk, A. Ernst, L. Moreschini, M. C. Falub, D. Pacilé, P. Bruno, K. Kern, and M. Grioni, Giant spin splitting through surface alloying, *Phys. Rev. Lett.* **98**, 186807 (2007).
- [48] I. Gierz, B. Stadtmüller, J. Vuorinen, M. Lindroos, F. Meier, J. H. Dil, K. Kern, and C. R. Ast, The structural influence on the Rashba-type spin-splitting in surface alloys, *Phys. Rev. B* **81**, 245430 (2010).
- [49] F. Meier, V. Petrov, S. Guerrero, C. Mudry, L. Patthey, J. Osterwalder, and J. H. Dil, Unconventional Fermi surface spin textures in the $\text{Bi}_x\text{Pb}_{1-x}/\text{Ag}(111)$ surface alloy, *Phys. Rev. B* **79**, 241408(R) (2009).
- [50] D. Pacilé, C. R. Ast, M. Papagno, C. Dasilva, L. Moreschini, M. Falub, A. P. Seitsonen, and M. Grioni, Electronic structure of an ordered Pb/Ag(111) surface alloy: Theory and experiment, *Phys. Rev. B* **73**, 245429 (2006).
- [51] A. Varykhalov, D. Marchenko, M. R. Scholz, E. D. L. Rienks, T. K. Kim, G. Bihlmayer, J. Sánchez-Barriga, and O. Rader, Ir(111) surface state with giant Rashba splitting persists under graphene in air, *Phys. Rev. Lett.* **108**, 066804 (2012).
- [52] S. Weinberg, *The Quantum Theory of fields. Vol. I: Foundations* (Cambridge University Press, Cambridge, 1995).
- [53] J. Plemelj, *Problems in the Sense of Riemann and Klein*, Interscience Tracts in Pure and Applied Mathematics Vol. 16 (Interscience Publishers, John Wiley & Sons, New York, 1964).
- [54] F. Reyes-Osorio and B. K. Nikolić, Nonlocal damping of spin waves in a magnetic insulator induced by normal, heavy, or altermagnetic metallic overlayer: A Schwinger-Keldysh field theory approach, *Phys. Rev. B* **110**, 214432 (2024).
- [55] M. T. Béal-Monod, Ruderman-Kittel-Kasuya-Yosida indirect interaction in two dimensions, *Phys. Rev. B* **36**, 8835 (1987).
- [56] H. Imamura, P. Bruno, and Y. Utsumi, Twisted exchange interaction between localized spins embedded in a one- or two-dimensional electron gas with Rashba spin-orbit coupling, *Phys. Rev. B* **69**, 121303(R) (2004).
- [57] T. Moriya, Anisotropic superexchange interaction and weak ferromagnetism, *Phys. Rev.* **120**, 91 (1960).
- [58] I. Dzyaloshinsky, A thermodynamic theory of “weak” ferromagnetism of antiferromagnetics, *J. Phys. Chem. Solids* **4**, 241 (1958).
- [59] V. Madhavan, W. Chen, T. Jamneala, M. F. Crommie, and N. S. Wingreen, Tunneling into a single magnetic atom: Spectroscopic evidence of the Kondo resonance, *Science* **280**, 567 (1998).
- [60] N. Knorr, M. A. Schneider, L. Diekhöner, P. Wahl, and K. Kern, Kondo effect of single Co adatoms on Cu surfaces, *Phys. Rev. Lett.* **88**, 096804 (2002).
- [61] T. Jamneala, V. Madhavan, W. Chen, and M. F. Crommie, Scanning tunneling spectroscopy of transition-metal impurities at the surface of gold, *Phys. Rev. B* **61**, 9990 (2000).
- [62] L. Limot, E. Pehlke, J. Kröger, and R. Berndt, Surface-state localization at adatoms, *Phys. Rev. Lett.* **94**, 036805 (2005).
- [63] <https://github.com/toor/asd>.
- [64] E. H. Hwang and S. D. Sarma, Screening, Kohn anomaly, Friedel oscillation, and RKKY interaction in bilayer graphene, *Phys. Rev. Lett.* **101**, 156802 (2008).