

Operator spreading in random unitary circuits with orthogonal/symplectic-invariant gate distributions

Zhiyang Tan (谭志阳)  and Piet W. Brouwer

Dahlem Center for Complex Quantum Systems, Physics Department, Halle-Berlin-Regensburg Cluster of Excellence CCE, and Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany



(Received 2 June 2026; revised 4 August 2026; accepted 5 August 2026; published 18 August 2026)

We investigate operator spreading in random quantum circuits with gates drawn from orthogonal- or symplectic-invariant ensembles, revealing several key distinctions from the well-studied unitary-invariant case. We find that the ensemble-averaged Pauli-string weights relax to a ternary-valued structure, instead of the binary structure of unitary-invariant circuits. For orthogonal- or symplectic-invariant circuits, the domain wall separating trivial and scrambled regions has a finite width even for Haar-random gates, whereas domain walls are sharp for Haar-distributed random unitary circuits. We further find a fundamental dichotomy between random circuits with two-qubit gates from the two disconnected components of the orthogonal group: While the butterfly velocity for the special orthogonal ensemble lies between zero and the Haar value, the negative-determinant sector exhibits a nonzero lower bound for any gate distribution. Moreover, for qudit size $q = 2$, the butterfly velocity can exceed that of the Haar-random ensemble.

DOI: [10.1103/p5d4-dmj6](https://doi.org/10.1103/p5d4-dmj6)

I. INTRODUCTION

Random unitary circuits are minimal models for studying fundamental aspects of many-body quantum dynamics, including information spreading, quantum chaos, and thermalization [1–11]. These circuits consist of a sequence of gate operations acting on pairs of q -dimensional quantum degrees of freedom (“qudits”), whereby the gate operations are drawn from a statistical distribution [6]. Typically, the gate operations follow a brickwork structure, in which qudits are acted upon pairwise by two-qudit gates, with a pattern that alternates in space and time; see Fig. 1.

An initially local operator O acquires nonlocal correlations under the circuit evolution [12–14]. *Operator spreading* is the increase in the spatial support of the set of Pauli-string basis operators required to represent $O(t)$ with time t [14,15]. The decay of the four-point out-of-time-order correlator (OTOC) provides a direct probe of this spreading [6–9], and recent work has generalized this to higher-point OTOCs as measures of operator ergodicity in the full operator space [16,17].

For a one-dimensional random unitary circuit, operator spreading can be described in terms of a front propagating according to a drift–diffusion model [6–9]. The drift velocity, which is known as the “butterfly velocity”, and the diffusion constant were first calculated for the circuit with maximal randomness, which has its two-qudit gate operators drawn from a Haar distribution [6–9]. For Haar-distributed gates, the randomness of the local gate operation is maximal. This

raises the question of to what extent the properties of operator spreading found in this case are universal: which of them persist for a broader class of gate ensembles, and which new features emerge once the local randomness is reduced? Ensembles whose probability distribution is invariant under a group of basis transformations provide a natural setting in which this question can be addressed in a controlled manner.

In a recent publication, we addressed this question for random unitary circuits with an arbitrary basis-independent probability distribution for the two-qudit gates [18], i.e., probability distributions in which the two-qudit gate operator $\mathcal{U}$ satisfies the invariance property

$$P(\mathcal{U}) = P(\mathcal{V}\mathcal{U}\mathcal{V}^\dagger), \quad (1)$$

where $\mathcal{V}$ is an arbitrary unitary matrix. The Haar-random circuit is a special case of this larger class of unitary circuits, but the invariance property (1) also admits, e.g., circuits that continuously interpolate between the trivial circuit and the Haar-random one, such as continuous Brownian circuits [12,14]. Apart from a different drift velocity and diffusion constant, we found two qualitative differences between a fully random Haar-distributed circuit and a circuit with the more general unitary invariance property of Eq. (1): (i) whereas Haar-distributed unitary circuits have a sharp domain wall between the trivial and maximally random regions, general unitary-invariant circuits have a finite domain-wall width; (ii) for the Haar-random circuit, the probability of finding a given generalized Pauli operator σ_{p_x} in the Pauli-string expansion of $O(t)$ depends only on whether σ_{p_x} is the identity, whereas for a general unitary-invariant circuit, such a “binary” distribution emerges only after a finite “thermalization time”.

Antiunitary symmetries of the underlying Hamiltonian, such as time-reversal or particle-hole symmetry, endow the two-qudit gates with additional structure. In their presence,

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

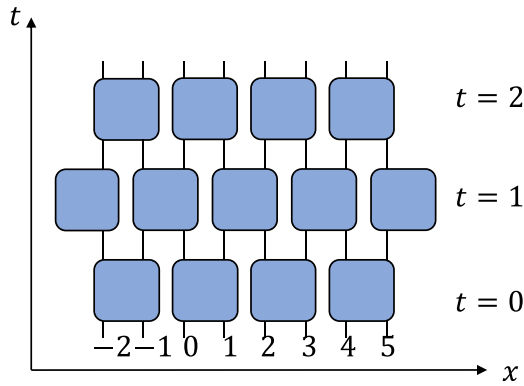


FIG. 1. Schematic of a random unitary circuit with a brickwork structure. The random circuit is composed of two-qudit gates (shown in blue) uncorrelated in space and time.

the natural basis transformations are no longer arbitrary unitary matrices, but orthogonal or symplectic ones. The orthogonal- and symplectic-Haar ensembles are therefore natural symmetry-constrained counterparts of the Haar-unitary circuit [19,20]. However, because the Haar measure combines the relevant basis invariance with maximal randomness, it does not allow one to distinguish which features of operator spreading originate from the symmetry constraint itself and which rely on the full Haar distribution. This motivates the study of the full class of orthogonal- and symplectic-invariant gate ensembles.

In this work, we consider operator spreading in random circuits in which the two-qudit gate operators satisfy an invariance property of the form (1), but with $\mathcal{V}$ an arbitrary orthogonal or symplectic matrix. Beyond this invariance condition, we impose no restriction on the probability distribution of the gates. This provides a setting in which the effects of the symmetry constraint can be disentangled from those of the gate randomness: properties shared by all gate distributions with a given invariance originate from the symmetry constraint alone, whereas properties that vary within the invariant class reflect the randomness of the gate distribution. Moreover, as in the unitary case, the invariance property (1) allows us to study continuous interpolations between trivial and Haar-random circuits, as well as continuous Brownian circuits.

An invariance property with respect to orthogonal or symplectic basis transformations is natural if the two-qudit gate operator $\mathcal{U}$ itself satisfies an additional symmetry, making it an orthogonal or symplectic matrix. However, there also exist random matrix ensembles in which $\mathcal{U}$ is unitary, but its probability distribution satisfies Eq. (1) with $\mathcal{V}$ restricted to the orthogonal or symplectic group [21,22]. Random circuits with orthogonal, unitary-symmetric, or symplectic two-qudit gates have recently been considered in Refs. [1,19,20,23–31]. Symplectic-invariant circuits require that the qudit array itself has an even-odd structure, as we discuss in further detail below.

As we show in this article, the drift velocity and the diffusion constant for random circuits with the orthogonal— or symplectic-invariance property differ quantitatively from their counterparts with unitary invariance. In addition, we also

find qualitative differences. For orthogonal- or symplectic-invariant circuits, we find that the domain-wall width remains finite even for Haar-distributed random circuits. We also find that the probability distribution of generalized Pauli operators approaches a “ternary” form, in which it depends not only on whether or not the generalized Pauli operator equals the identity operator, but also on whether it is symmetric or antisymmetric. Because the invariant gate distribution need not be Haar, the butterfly velocity can attain values inaccessible to the corresponding Haar ensemble: in the $\text{SO}^-(4)$ sector, its maximum exceeds the Haar-orthogonal value. Moreover, in this sector the minimum butterfly velocity is strictly positive for any gate distribution.

The remainder of this article is organized as follows: In Sec. II, we briefly review the formulation of random unitary circuits with a brickwork structure, closely following Ref. [6], and we review the expansion of an operator O in terms of Pauli strings. In Sec. III, we elaborate on the connection between the gate ensemble’s invariance property, Eq. (1), and the underlying symmetries of the Hamiltonian. We point out that a symplectic structure occurs naturally only if the qudits in the circuit have an even-odd structure, such that there is a natural involution squaring to -1 for every second qudit. We derive the governing equations for the time dependence of the Pauli-string density matrix in Sec. IV. The core of our analysis is presented in Sec. V, where we specialize to the case of qudit dimension $q = 2^m$. By mapping the quantum evolution to a classical stochastic growth model, we derive the butterfly velocity v_B and diffusion constant $\mathcal{D}$ for circuits with orthogonal and symplectic-invariant gate distributions. For circuits composed of gates that are themselves orthogonal or symplectic, we show that the Pauli-string weights relax to a ternary form after a finite thermalization time τ_t . In Sec. VI, we generalize these results from $q = 2^m$ to an arbitrary qudit dimension q . We illustrate our findings with three specific examples in Sec. VII. Finally, we conclude in Sec. VIII. Additional details of our calculations are provided in the Appendixes.

II. OPERATOR SPREADING IN RANDOM UNITARY CIRCUITS

The nonlocal correlations that an initially local operator O acquires under the circuit evolution can be described by expressing the Heisenberg-picture operator $O(t)$ [with $O(0) = O$] in terms of a set of Pauli-string basis operators $O_p = \otimes_x \sigma_{p_x}$, which are products of generalized Pauli operators σ_{p_x} associated with each qudit,

$$O(t) = \sum_p \gamma_p(t) O_p, \quad (2)$$

where the expansion coefficient $\gamma_p(t)$ denotes the amplitude of the basis operator O_p . The Pauli-string index $p = [\dots, p_{x-1}, p_x, p_{x+1}, \dots]$. Its support is the set of sites x for which σ_{p_x} is different from the identity operator. We define the front position of a Pauli string as the position of its rightmost nonidentity operator; all sites to the right of the front position therefore carry the identity operator. In Appendix A, we review the Weyl–Heisenberg choice for the generalized Pauli matrices, as well as a construction of generalized Pauli

matrices from tensor products of conventional 2×2 Pauli matrices for special values $q = 2^m$ of the qudit dimension. In either case, the generalized Pauli operators σ_{p_x} form a complete, orthonormal basis of operators for the qudit at the site x , satisfying

$$\frac{1}{q} \text{tr} \sigma_{p_x} \sigma_{q_x}^\dagger = \delta_{p_x, q_x}. \quad (3)$$

Correspondingly, the Pauli-string operators O_p satisfy the orthonormality relation

$$\frac{1}{q^L} \text{tr} O_p O_q^\dagger = \delta_{p, q}, \quad (4)$$

where L is the number of sites of the circuit. The Pauli-string amplitude $\gamma_p(t)$ satisfies the normalization condition

$$\sum_p |\gamma_p(t)|^2 = 1 \quad (5)$$

for all t .

The time evolution of an operator O in the Heisenberg picture is given by the relation

$$O(t) = U^\dagger(t, t') O(t') U(t, t'). \quad (6)$$

Following Refs. [6–8], the evolution matrix U is taken to be the product of two-qudit gates in a brickwork pattern,

$$U(t, t-1) = \begin{cases} \bigotimes_{\text{even } x} \mathcal{U}_{x, x+1}, & \text{if } t \text{ is even,} \\ \bigotimes_{\text{even } x} \mathcal{U}_{x-1, x}, & \text{if } t \text{ is odd,} \end{cases} \quad (7)$$

Here, $\mathcal{U}_{x,y}$ is a $q^2 \times q^2$ unitary matrix acting on the qudits at sites x and y . We take the two-qudit gate operators $\mathcal{U}_{x,y}$ from a probability distribution, which satisfies the invariance condition

$$P(\mathcal{U}_{x,y}) = P(\mathcal{V}_{x,y} \mathcal{U}_{x,y} \mathcal{V}_{x,y}^\dagger), \quad (8)$$

with $\mathcal{V}_{x,y}$ an arbitrary unitary matrix with the symmetry constraint

$$\mathcal{V}_{x,y}^\dagger = \Omega_{x,y} \mathcal{V}_{x,y}^T \Omega_{x,y}^\dagger, \quad (9)$$

where $\Omega_{x,y}$ is a symmetric or an antisymmetric unitary matrix. Since one may always choose the basis such that $\Omega_{x,y} = \mathbb{1}_{q^2}$ if $\Omega_{x,y}$ is symmetric and $\Omega_{x,y} = \mathbb{1}_{q^2/2} \otimes \sigma_2$ if $\Omega_{x,y}$ is antisymmetric, we refer to matrices $\mathcal{V}_{x,y}$ satisfying the constraint (9) as orthogonal and symplectic, respectively. The gate operator $\mathcal{U}_{x,y}$ may itself also be subject to additional symmetries, e.g., $\mathcal{U}_{x,y}$ may be orthogonal or symplectic, too, but this is not a necessary condition in our general analysis. We discuss random circuits for which the two-qudit gate operators $\mathcal{U}_{x,y}$ have an additional symmetry constraint in the next section.

The time evolution of the Pauli-string coefficients γ_p follows from that of the operator O , see Eq. (6). For a single time step, we have

$$\gamma_p(t) = \sum_a W_{ap}(t, t-1) \gamma_a(t-1) \quad (10)$$

where the summation variable a labels Pauli strings and

$$W_{ap}(t, t-1) = \frac{1}{q^L} \text{tr} U^\dagger(t, t-1) O_a U(t, t-1) O_p^\dagger.$$

For an evolution matrix of the form (7), this evaluates to

$$W_{ap} = \begin{cases} \prod_{x \text{ even}} \mathcal{W}_{ap; x, x+1} & t \text{ even,} \\ \prod_{x \text{ even}} \mathcal{W}_{ap; x-1, x} & t \text{ odd,} \end{cases} \quad (11)$$

Here, the factors $\mathcal{W}_{ap; x,y}$ do not depend on the full Pauli strings a and p , but only on the labels a_x, a_y, p_x , and p_y that belong to the sites x and y . Its explicit expression is given in Appendix B.

III. ORTHOGONAL AND SYMPLECTIC-INVARIANT ENSEMBLES

The invariance condition (1), with $\mathcal{V}_{x,y}$ orthogonal or symplectic, can be seen as a purely mathematical condition on the pair-qudit gate operation. Here, we consider symmetry conditions for the evolution of single qudits and of qudit pairs, for which such a symmetry restriction on the matrix $\mathcal{V}_{x,y}$ appears naturally. We consider qudits with q degrees of freedom and gate operations that arise from the operation of a Hamiltonian for a finite period of time, whereby the randomness appears because these Hamiltonians differ from time step to time step.

The restriction to invariance with orthogonal or symplectic basis transformations appears naturally if there exists an antiunitary involution

$$\psi_x \rightarrow Z_x \psi_x^* \quad (12)$$

on the q -dimensional single-qudit spinors, where Z_x is a $q \times q$ unitary matrix with $Z_x Z_x^* = \pm 1$, and the Hamiltonian that is applied to a single qudit or a pair of qudits is symmetric or antisymmetric under the involution. Specifically, we consider involutions corresponding to time-reversal symmetry and particle-hole symmetry. For time-reversal symmetry, the qudit Hamiltonian is symmetric under the involution, whereby Z_x is symmetric if the qudit has integer spin and antisymmetric if the qudit spin is half-integer. In the language of the tenfold-way classification [32–34], these two cases correspond to the Cartan classes AI and AII, respectively. For particle-hole symmetry, the qudit Hamiltonian is antisymmetric under the involution. The cases of symmetric and antisymmetric Z_x then correspond to Cartan classes D and C, respectively. For integer spin, we may choose the qudit basis such that

$$Z_x = \mathbb{1}. \quad (13)$$

For half-integer spin, q must be even, and we can choose the qudit basis such that

$$Z_x = \sigma_2 \otimes \mathbb{1}_{q/2}. \quad (14)$$

For a single-qudit gate operation described by the Hamiltonian H_x , one has the constraint

$$H_x = Z_x H_x^* Z_x^\dagger \quad \text{or} \quad H_x = -Z_x H_x^* Z_x^\dagger \quad (15)$$

for the case of time-reversal symmetry or particle-hole symmetry, respectively. Correspondingly, for the special choice of basis corresponding to Eqs. (13) and (14), the single-qudit gate operations are unitary-symmetric, unitary-antisymmetric, orthogonal, and symplectic for Cartan classes AI, AII, D, and C, respectively. Whereas these are well-defined symmetry properties for the evolution matrix of a single time step, the

properties of being unitary symmetric or unitary antisymmetric are not preserved if multiple time steps are taken and one ends up with evolution matrices that are simply unitary, without further symmetry constraints.

The generalized Pauli matrices σ_{p_x} form a basis for operators on the qudit x . If the qudit dimension $q = 2^m$, we can take tensor products of the standard 2×2 Pauli matrices for the generalized Pauli matrices. In this case, the generalized Pauli matrices σ_{p_x} can be chosen such that they have a well-defined parity $s_{p_x} \in \{1, -1\}$ under the involution (12),

$$\sigma_{p_x} = s_{p_x} Z_x \sigma_{p_x}^T Z_x^\dagger. \quad (16)$$

For general q , a generalization of Eq. (16) exists, which will be discussed in Sec. VI.

The invariance property (1) refers to two-qudit gates. The two-qudit Hamiltonian $H_{x,x+1}$ satisfies a constraint analogous to Eq. (15), but with

$$\Omega_{x,x+1} = Z_x \otimes Z_{x+1} \quad (17)$$

instead of Z_x . The corresponding two-qudit gate operator $\mathcal{U}_{x,x+1}$ then satisfies the symmetry relation

$$\mathcal{U}_{x,x+1} = \Omega_{x,x+1} \mathcal{U}_{x,x+1}^T \Omega_{x,x+1}^\dagger \quad (18)$$

for time-reversal symmetry and

$$\mathcal{U}_{x,x+1} = \Omega_{x,x+1} \mathcal{U}_{x,x+1}^* \Omega_{x,x+1}^\dagger \quad (19)$$

for particle-hole symmetry. Since $\Omega_{x,x+1}$ is symmetric if Z_x and Z_{x+1} are both symmetric as well as if they are both antisymmetric, the corresponding evolution operator $\mathcal{U}_{x,x+1}$ is unitary-symmetric or orthogonal [in the sense described below Eq. (9)], regardless of whether the spin of the individual qudits is half-integer or integer. This is to be expected, since the spin of two neighboring qudits is always an integer, regardless of whether the individual spins are of half-integer or integer type. Evolution operators of unitary-self-dual or symplectic type appear if the qudits in the circuit are alternately of half-integer and integer spin, e.g., if the Z_x are symmetric for even x and antisymmetric for odd x , so that $\Omega_{x,x+1} = Z_x \otimes Z_{x+1}$ is an antisymmetric matrix.

If the two-qudit gate evolution matrix $\mathcal{U}_{x,x+1}$ is orthogonal or symplectic, the transition factors $\mathcal{W}_{ap,x,x+1}$ of Eq. (11) are parity preserving for $q = 2^m$, i.e.,

$$\mathcal{W}_{ap,x,x+1} = 0 \quad \text{if } s_{p_x} s_{p_{x+1}} s_{a_x} s_{a_{x+1}} = -1. \quad (20)$$

A generalization to arbitrary q will be discussed in Sec. VI.

IV. TIME EVOLUTION OF PAULI-STRING DENSITY MATRIX

The Pauli-string coefficient $\gamma_p(t)$ contains the full information about the evolution of the operator O under the unitary dynamics. We introduce the second moment of the Pauli-string amplitudes,

$$\rho_{pq}(t) = \langle \gamma_p(t) \gamma_q^*(t) \rangle, \quad (21)$$

where the average is taken over the ensemble of two-qudit gates. The matrix $\rho(t)$ is Hermitian and positive semidefinite, and the normalization condition in Eq. (5) implies $\text{tr } \rho(t) = 1$. We therefore refer to it as the *Pauli-string density matrix*.

Its diagonal elements $\rho_{pp}(t)$ may be interpreted as a classical probability distribution over the set of Pauli strings p [6].

With the help of Eq. (10), the Pauli-string density matrix at time t can be expressed in terms of that at time $t - 1$ and the covariance of the transition amplitudes,

$$\rho_{pq}(t) = \sum_{a,b} \langle W_{ap} W_{bq}^* \rangle \rho_{ab}(t-1). \quad (22)$$

Using unitarity and the invariance of the identity operator under conjugation, one verifies that the evolution equation (22) admits the maximally random steady-state solution

$$\rho_{pq}^\infty(t) = q^{-2L} \prod_x \delta_{p_x, q_x} \quad (23)$$

as well as the trivial solution

$$\rho_{pq}^0(t) = \prod_x \delta_{p_x, 0} \delta_{q_x, 0}. \quad (24)$$

The butterfly velocity v_B corresponds to the speed at which a domain wall between these two solutions propagates through the qudit array. For definiteness, here we will consider a domain wall separating a maximally random region to the left (smaller x) from a trivial region to the right (larger x). Such a domain wall propagates to the right (in the positive x direction).

For circuits with orthogonal or symplectic gates, the parity-conservation condition (20) implies that the total parity of a Pauli string is preserved under time evolution. In principle, this allows for two maximally random steady-state solutions, each with well-defined global parity. A global parity constraint has no effect on how the steady-state solution looks locally. Hence, for the problem of operator spreading, it is sufficient to consider a domain wall between steady-state solutions that locally have the forms of Eqs. (23) and (24).

For a Haar-random unitary circuit, the expectation value

$$\langle W_{ap} W_{bq}^* \rangle = 0 \quad \text{if } a \neq b \text{ or } p \neq q. \quad (25)$$

Moreover, $\langle |W_{ap}|^2 \rangle$ only depends on whether the indices a and p are trivial or nontrivial. This means that the Pauli-string density matrix $\rho(t)$ acquires a “binary” form after the first time step: its only nonzero elements are the diagonal elements $\rho_{pp}(t)$, which depend only on whether p_x is trivial or nontrivial at each site x .

For a generic unitary-invariant circuit, Eq. (25) no longer holds, but there is a weaker constraint [18],

$$\langle W_{ap} W_{bq}^* \rangle = 0 \quad \text{if } a \neq b \text{ and } p = q \text{ or if } p \neq q \text{ and } a = b. \quad (26)$$

The constraint (26) is sufficient to show that the evolution of the diagonal elements ρ_{pp} can be solved separately from that of the off-diagonal elements, i.e., to find diagonal elements $\rho_{pp}(t)$ it is sufficient to know the diagonal elements $\rho_{aa}(t-1)$ at the preceding times,

$$\rho_{pp}(t) = \sum_a \rho_{aa}(t-1) \langle |W_{ap}|^2 \rangle. \quad (27)$$

In Ref. [18], we showed that the diagonal elements ρ_{pp} exponentially relax to the binary form after a finite relaxation time.

In an orthogonal-invariant or symplectic-invariant circuit, Eq. (26) holds only for a basis of generalized Pauli matrices

that have well-defined parity s_{p_x} , see Eq. (16). Such a basis exists for qudit size $q = 2^m$, but not for other qudit sizes. We therefore first consider the case $q = 2^m$ and postpone the discussion of general q to Sec. VI.

V. THE CASE $q = 2^m$

A. Mapping to a stochastic growth model

The transition probabilities $\langle |W_{ap}|^2 \rangle$ in the stochastic growth model (27) factorize in pair contributions from the two-qudit gates,

$$\langle |W_{ap}|^2 \rangle = \prod_{x \text{ even}} \begin{cases} \langle |\mathcal{W}_{ap;x,x+1}|^2 \rangle & \text{for } t \text{ even,} \\ \langle |\mathcal{W}_{ap;x-1,x}|^2 \rangle & \text{for } t \text{ odd.} \end{cases} \quad (28)$$

In Ref. [18], we showed that for a unitary-invariant circuit, the pair transition probability $\langle |\mathcal{W}_{ap;x,x+1}|^2 \rangle$ is of the form

$$\langle |\mathcal{W}_{ap;x,y}|^2 \rangle = \delta_{a,0} \delta_{p,0} + (1 - \delta_{a,0})(1 - \delta_{p,0}) \times [\mathcal{P}_1 + \mathcal{P}_2 \varphi(a, p) + \mathcal{P}_3 \delta_{a,p}], \quad (29)$$

where $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are coefficients that depend on moments of the pair-qudit evolution matrix $\mathcal{U}_{x,x+1}$ involving up to four factors of $\mathcal{U}_{x,x+1}$ or $\mathcal{U}_{x,x+1}^\dagger$ and the function $\varphi(a, p)$ is defined by

$$\varphi(a, p) = \begin{cases} 1 & \text{if } \sigma_{p_x} \otimes \sigma_{p_{x+1}} \text{ and } \sigma_{a_x} \otimes \sigma_{a_{x+1}} \text{ commute,} \\ -1 & \text{if } \sigma_{p_x} \otimes \sigma_{p_{x+1}} \text{ and } \sigma_{a_x} \otimes \sigma_{a_{x+1}} \text{ anticommute.} \end{cases} \quad (30)$$

For an orthogonal-invariant or symplectic-invariant circuit, $\langle |\mathcal{W}_{ap;x,y}|^2 \rangle$ is also of the form of Eq. (29), but the coefficients $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ additionally depend on the products $s_{a_x} s_{a_y}$ and $s_{p_x} s_{p_y}$ of the parities of the generalized Pauli matrices. Explicit expressions for the coefficients $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ are given in Appendix B.

For the Haar orthogonal and symplectic ensembles, the averages can be evaluated directly, without invoking the machinery required for the more general orthogonal-invariant and symplectic-invariant ensembles [35]. In this case, the coefficient $\mathcal{P}_1$ reads

$$\mathcal{P}_1 = \frac{2}{q^2 \mp 1} \begin{cases} \frac{1}{q^2 \pm 2}, & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = 1, \\ \frac{1}{q^2}, & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = -1, \\ 0, & \text{if } s_{p_x} s_{p_{x+1}} \neq s_{a_x} s_{a_{x+1}}. \end{cases} \quad (31)$$

(Recall that s_{p_x} , $s_{p_{x+1}}$, s_{a_x} , and $s_{a_{x+1}}$ are the parities of the generalized Pauli matrices.) In Eq. (31), the upper sign applies to the orthogonal case and the lower sign to the symplectic case. As in the Haar-random unitary case, one has

$$\mathcal{P}_2 = \mathcal{P}_3 = 0. \quad (32)$$

For the COE and CSE ensembles, standard Weingarten calculus gives

$$\begin{aligned} \mathcal{P}_1 &= \frac{q^4 \pm 4q^2 + 2}{q^4(q^2 \pm 1)(q^2 \pm 3)}, \\ \mathcal{P}_2 &= -\frac{2}{q^4(q^2 \pm 1)(q^2 \pm 3)}, \\ \mathcal{P}_3 &= \frac{2(q^2 \pm 2)}{q^2(q^2 \pm 1)(q^2 \pm 3)}. \end{aligned} \quad (33)$$

These coefficients are independent of the parities of the generalized Pauli matrices, and hence do not depend on the signs of $s_{p_x} s_{p_{x+1}}$ and $s_{a_x} s_{a_{x+1}}$. In these expressions, the upper sign corresponds to the COE and the lower sign to the CSE.

We give a simple derivation of Eq. (31) that does not use the full machinery for orthogonal- and symplectic-invariant circuits in Appendix B. For the corresponding derivation of Eq. (33), we refer to Ref. [35].

B. Projected ternary strings

Since the stochastic growth process defined by Eqs. (27) and (28) refers to the basis of qudit operators—the generalized Pauli matrices σ_{p_x} —, it has q^2 degrees of freedom per qudit. This obstructs an efficient simulation of the stochastic growth process for a large qudit dimension q . In Ref. [18], we showed that for a unitary-invariant circuit, by appropriate summation the growth process may be simplified to a growth process with only two degrees of freedom per qudit, specifying whether the index p_x is trivial ($p_x = 0$) or nontrivial ($p_x \neq 0$). Hereto, we defined the *binary Pauli string* $\bar{p}$ associated with p as the string $[\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{L-1}]$, where $\bar{p}_x = 0$ if $p_x = 0$ and $\bar{p}_x = 1$ if $p_x \neq 0$. Writing $p \rightarrow \bar{p}$ to denote the binary string $\bar{p}$ corresponding to the Pauli string p , the “projected binary-string distribution” $\bar{\rho}_{\bar{p}}$ corresponding to ρ_{pp} is then defined as

$$\bar{\rho}_{\bar{p}} = \sum_{p \rightarrow \bar{p}} \rho_{pp}, \quad (34)$$

i.e., $\bar{\rho}_{\bar{p}}$ is the sum of all diagonal elements ρ_{pp} with the same binary Pauli string $\bar{p}$. The evolution equation (22) closes when applied to the projected binary-string distribution, i.e., $\bar{\rho}_{\bar{p}}(t)$ depends only on $\bar{\rho}_{\bar{p}}(t-1)$ [18]. Since the projected binary-string distribution has only two degrees of freedom per qudit, its stochastic evolution equation can be easily simulated classically, independent of the actual qudit dimension q . Moreover, it is a good starting point for analytical approximation schemes.

A similar simplification is possible for an orthogonal-invariant or symplectic-invariant circuit, but with three degrees of freedom per qudit instead of two. In order to achieve this simplification, we consider a projection of the classical probabilities $\rho_p(t)$ onto a “ternary” probability string $\bar{\rho}_{\bar{p}}$, where the string index $\bar{p}$ is a list of ternaries $[\bar{p}_0, \bar{p}_1, \dots, \bar{p}_{L-1}]$ with $\bar{p}_x \in \{-1, 0, 1\}$, where $\bar{p}_x = 0$ if $\sigma_{p_x} = \mathbb{1}$ and $\bar{p}_x = s_{p_x}$ if $\sigma_{p_x} \neq \mathbb{1}$. [The parity s_{p_x} is defined in Eq. (16).] In analogy to Eq. (34), we write $p \rightarrow \bar{p}$ for the ternary string associated with the Pauli string p and define the *projected ternary-string distribution* $\bar{\rho}_{\bar{p}}$ from ρ_{pp} by the same equation.

In Appendix C, we show that there is a closed Markovian evolution equation for the projected ternary distribution function $\bar{\rho}_{\bar{p}}$,

$$\bar{\rho}_{\bar{p}}(t) = \sum_{\bar{a}} \bar{\rho}_{\bar{a}}(t-1) T_{\bar{a}\bar{p}}, \quad (35)$$

with transition probabilities $T_{\bar{a}\bar{p}}$ that are products of pair-transition probabilities,

$$T_{\bar{a}\bar{p}} = \prod_{x \text{ even}} \begin{cases} T_{\bar{a}_x \bar{a}_{x+1}; \bar{p}_x, \bar{p}_{x+1}} & \text{for } t \text{ even,} \\ T_{\bar{a}_{x-1} \bar{a}_x; \bar{p}_{x-1}, \bar{p}_x} & \text{for } t \text{ odd.} \end{cases} \quad (36)$$

We refer to Appendix C for explicit expressions.

The projected binary- or ternary-string distributions corresponding to the maximally random steady state (23) factorize site-wise,

$$\bar{\rho}_{\bar{p}}^{\infty}(t) = \prod_x \bar{r}_{\bar{p}_x}^{\infty}, \quad (37)$$

with

$$\bar{r}_{\bar{p}_x}^{\infty} = \frac{1}{q^2} [\delta_{\bar{p}_x,0} + (q^2 - 1) \delta_{\bar{p}_x,1}] \quad (38)$$

for the unitary case. For orthogonal- or symplectic-invariant random circuits, the maximally random steady state also has the form (37), but with

$$\bar{r}_{\bar{p}_x}^{\infty} = \frac{1}{q^2} \left\{ \delta_{\bar{p}_x,0} + \frac{q-1}{2} [(q+2) \delta_{\bar{p}_x,1} + q \delta_{\bar{p}_x,-1}] \right\}, \quad (39)$$

for qudits with integer spin [i.e., qudits that have an involution of the form (12) with symmetric Z_x] and

$$\bar{r}_{\bar{p}_x}^{\infty} = \frac{1}{q^2} \left\{ \delta_{\bar{p}_x,0} + \frac{q+1}{2} [(q-2) \delta_{\bar{p}_x,1} + q \delta_{\bar{p}_x,-1}] \right\}, \quad (40)$$

for qudits with half-integer spin (i.e., qudits that have an involution with antisymmetric Z_x). We recall that orthogonal-invariant two-gate distributions appear if all qudits have the same spin, whereas symplectic-invariant two-gate distributions

appear if the qudits alternate between integer and half-integer spin between even and odd x . The projected ternary-string distribution corresponding to the trivial solution (24) is

$$\bar{\rho}_{\bar{p}}^0(t) = \prod_x \delta_{\bar{p}_x,0} \quad (41)$$

for all cases.

C. Right-propagating n -point density

The next steps in the analysis of operator spreading in orthogonal-invariant or symplectic-invariant random circuits closely follow those of the analysis of operator spreading in unitary-invariant circuits in Ref. [18]. We say that the ternary string $\bar{p}$ has front position x , or equivalently “ends at x ” if x is the rightmost qudit position with $\bar{p}_x \neq 0$. (This implies that $\bar{p}_y = 0$ for all $y > x$.) Following Ref. [6], we define the “right-propagating density” $\bar{\rho}^{(0)}(\Delta x; t; \bar{a}_0)$ as the fraction of ternary Pauli strings whose front position is $x = t + \Delta x$ and whose last nontrivial element is equal to $\bar{a}_0 \in \{-1, 1\}$,

$$\bar{\rho}^{(0)}(\Delta x; t; \bar{a}_0) = \sum_{\bar{p} \text{ ends at } t+\Delta x} \bar{\rho}_{\bar{p}}(t) \delta_{\bar{p}_{t+\Delta x}, \bar{a}_0}. \quad (42)$$

(Note that the argument Δx is measured with respect to the ballistic propagation at unit velocity.) We also define the “right-propagating n -point density”

$$\bar{\rho}^{(n)}(\Delta x; t; \bar{a}_n, \dots, \bar{a}_0) = \sum_{\bar{p} \text{ ends at } t+\Delta x} \bar{\rho}_{\bar{p}}(t) \delta_{\bar{p}_{t+\Delta x}, \bar{a}_0} \delta_{\bar{p}_{t+\Delta x-1}, \bar{a}_1} \cdots \delta_{\bar{p}_{t+\Delta x-n}, \bar{a}_n}, \quad (43)$$

which is the fraction of ternary Pauli strings that end at $t + \Delta x$ and that have the sequence $\bar{a}_n, \dots, \bar{a}_0$ leading up to the end of the string at Δx .

From the stochastic evolution equation for the full ternary probability density $\bar{\rho}_{\bar{p}}(t)$, we may deduce evolution equations for the right-propagating density and for the right-propagating n -point densities. These are

$$\begin{aligned} \bar{\rho}^{(n)}(\Delta x; t; \bar{p}_n, \dots, \bar{p}_0) &= \sum_{\bar{a}_0, \dots, \bar{a}_n} T_{\bar{a}_n \bar{a}_{n-1}; \bar{p}_n \bar{p}_{n-1}} \cdots T_{\bar{a}_2 \bar{a}_1; \bar{p}_2 \bar{p}_1} T_{\bar{a}_0 0; \bar{p}_0 0} \bar{\rho}^{(n)}(\Delta x + 1; t - 1; \bar{a}_n, \dots, \bar{a}_0) \\ &+ \sum_{\bar{a}_0, \dots, \bar{a}_{n+1}} T_{\bar{a}_{n+1} \bar{a}_n; \bar{p}_n \bar{p}_{n-1}} \cdots T_{\bar{a}_3 \bar{a}_2; \bar{p}_2 \bar{p}_1} T_{\bar{a}_1 \bar{a}_0; \bar{p}_0 0} \bar{\rho}^{(n+1)}(\Delta x + 2; t - 1; \bar{a}_{n+1}, \bar{a}_n, \dots, \bar{a}_0) \end{aligned} \quad (44)$$

if n and Δx are both even. The summation variables $\bar{a}_j \in \{-1, 0, 1\}$ for $j = 1, \dots, n$; $\bar{a}_0$ takes the values -1 and 1 only. In the same manner, we find

$$\begin{aligned} \bar{\rho}^{(n)}(\Delta x; t; \bar{p}_n, \dots, \bar{p}_0) &= \sum_{\bar{a}_0, \dots, \bar{a}_n} \sum_{\bar{p}_{n+1}} T_{\bar{a}_n \bar{a}_{n-1}; \bar{p}_{n+1} \bar{p}_n} \cdots T_{\bar{a}_2 \bar{a}_1; \bar{p}_2 \bar{p}_1} T_{\bar{a}_0 0; \bar{p}_0 \bar{p}_0} \bar{\rho}^{(n)}(\Delta x; t - 1; \bar{a}_n, \dots, \bar{a}_0) \\ &+ \sum_{\bar{a}_0, \dots, \bar{a}_{n+1}} \sum_{\bar{p}_{n+1}} T_{\bar{a}_{n+1} \bar{a}_n; \bar{p}_{n+1} \bar{p}_n} \cdots T_{\bar{a}_3 \bar{a}_2; \bar{p}_2 \bar{p}_1} T_{\bar{a}_1 \bar{a}_0; \bar{p}_0 \bar{p}_0} \bar{\rho}^{(n+1)}(\Delta x + 1; t - 1; \bar{a}_{n+1}, \bar{a}_n, \dots, \bar{a}_0), \end{aligned} \quad (45)$$

if n is even and Δx is odd. The summation variables $\bar{a}_0, \dots, \bar{a}_n$ take the same values as for the case that n and Δx are both even; $\bar{p}_{n+1}$ takes the values $-1, 0$, and 1 . To avoid spurious even-odd effects, we will not consider the evolution equations for the right-propagating n -point density with n odd.

D. Mapping to a drift–diffusion process

The evolution equations (44) and (45) for the right-propagating n -point density both involve the $(n+1)$ -point

density. To obtain a closed set of equations, we follow Ref. [18] and truncate the evolution equations (44) and (45) at sufficiently high order n by replacing the $(n+1)$ -point density by its maximally random approximation,

$$\begin{aligned} \bar{\rho}^{(n+1)}(\Delta x; t; \bar{p}_{n+1}, \bar{p}_n, \dots, \bar{p}_0) \\ = \bar{\rho}^{(n)}(\Delta x; t; \bar{p}_n, \dots, \bar{p}_0) \bar{r}_{\bar{p}_{n+1}}^{\infty}, \end{aligned} \quad (46)$$

where $\bar{r}_{\bar{p}_x}^{\infty}$ was defined in Eqs. (38)–(40). The long-time solution of the truncated evolution equation is of the

form [18]

$$\begin{aligned} \bar{\rho}^{(n)}(\Delta x; t; \bar{p}_n, \dots, \bar{p}_0) \\ = R_1^{(n)}(\Delta x; t) V_1^{(n)}((-1)^{\Delta x}; \bar{p}_n, \dots, \bar{p}_0), \end{aligned} \quad (47)$$

where $V_1^{(n)}(\sigma; \bar{p}_n, \dots, \bar{p}_0)$ is independent of Δx and t (up to the parity of Δx) and $R^{(n)}(\Delta x; t)$ satisfies a drift-diffusion equation

$$\begin{aligned} \partial_t R_1^{(n)}(\Delta x, t) = (1 - v_B^{(n)}) \partial_{\Delta x} R_1^{(n)}(\Delta x, t) \\ + \frac{\mathcal{D}^{(n)}}{2} \partial_{\Delta x}^2 R_1^{(n)}(\Delta x, t). \end{aligned} \quad (48)$$

Explicit expressions for the drift velocity $v_B^{(n)}$ and the diffusion constant $\mathcal{D}^{(n)}$ in terms of the ternary-string transition matrices $T_{\sigma, \sigma'}$ of Eq. (59) are given in Appendix D. We verify that the results for v_B and $\mathcal{D}$ converge with respect to the truncation order n .

E. Approach to a binary or ternary distribution

In Sec. VB, we showed that a closed evolution equation can be obtained if instead of the full ρ_{pp} we consider a “projected ternary” weight $\bar{\rho}_{\bar{p}}$ that is obtained from ρ_{pp} by summing over all Pauli strings p with the same binary or ternary string $\bar{p}$. We say that the Pauli-string density matrix ρ is of “binary form” or “ternary form” if it is diagonal and all Pauli strings p that map to the same binary or ternary string $\bar{p}$ have the same probability ρ_{pp} . In Ref. [18], we showed that ρ_{pp} exponentially relaxes towards such a binary form for the unitary-invariant case and calculated the corresponding relaxation time. In fact, the exponential relaxation to the binary form applies not only to the diagonal elements ρ_{pp} , but to the full Pauli-string density matrix ρ , including its off-diagonal elements.

In analogy to the unitary-invariant case, the Pauli-string density matrix ρ of an orthogonal-invariant or symplectic-invariant random circuit relaxes towards a diagonal distribution of ternary form. In such a distribution, all Pauli strings p that have the same ternary string $\bar{p}$ have the same probability. Explicitly, we define the operator $\mathbb{B}$, which maps ρ_{pq} to its associated ternary distribution, as

$$\mathbb{B} \rho_{pq} = \delta_{pq} \left(\prod_x B_{\bar{p}_x} \right) \bar{\rho}_{\bar{p}}, \quad (49)$$

where $\bar{p}$ is the ternary string corresponding to p , and $\bar{\rho}_{\bar{p}}$ the projected ternary weight corresponding to ρ_{pp} . The factor $B_{\bar{p}_x}$ gives the uniform probability to find the generalized Pauli operator p_x given the ternary label $\bar{p}_x$.

In the orthogonal-invariant case, as well as at even sites in the symplectic-invariant case, the ternary projector is obtained by taking the upper signs in

$$B_{\bar{p}_x} = \delta_{\bar{p}_x, 0} + \frac{2}{(q \mp 1)(q \pm 2)} \delta_{\bar{p}_x, 1} + \frac{2}{q(q \mp 1)} \delta_{\bar{p}_x, -1}. \quad (50)$$

At odd sites in the symplectic-invariant case, the lower signs apply. For $q = 2$, however, the lower-sign expression in Eq. (50) is singular and therefore does not define the ternary projector. In this special case, the projector at odd sites is instead

$$B_{\bar{p}_x} = \delta_{\bar{p}_x, 0} + \frac{1}{3} \delta_{\bar{p}_x, -1}. \quad (51)$$

In Appendix E, we show that the two-norm $\|\rho_{pq} - \mathbb{B} \rho_{pq}\|_2 \rightarrow 0$ in the long-time limit.

VI. GENERAL QUDIT DIMENSION q

For an arbitrary qudit dimension q , it is not possible to find a set of generalized Pauli matrices that satisfy the orthonormality condition (3) and that have a well-defined parity under transposition or duality. As a consequence, it is not possible to construct a stochastic growth process involving only the diagonal elements ρ_{pp} of the Pauli-string density matrix. In both the orthogonal-invariant and the symplectic-invariant case, however, the Weyl–Heisenberg choice for the generalized Pauli matrices (see Appendix A) has the property that $Z_x \sigma_p^T Z_x^\dagger$ is again a generalized Pauli matrix, up to a phase factor. For each generalized Pauli matrix σ_{p_x} we therefore define the “transpose index” p_x^T and phase ϕ_{p_x} via the relation

$$Z_x \sigma_{p_x}^T Z_x^\dagger = \sigma_{p_x^T} e^{-i\phi_{p_x}}, \quad (52)$$

if Z_x is symmetric. For the Weyl–Heisenberg generalized Pauli matrices σ_{p_x} , the index p_x is represented by a pair of integers $p_x = (p'_x, p''_x)$ with $p'_x, p''_x \in \mathbb{Z}_q$, see Appendix A, and the phase shift ϕ_{p_x} is given by

$$\phi_{p_x} = \frac{2\pi}{q} p'_x p''_x. \quad (53)$$

Analogously, for antisymmetric Z_x , we define the “dual index” p_x^R and corresponding phase via the relation

$$Z_x \sigma_{p_x}^T Z_x^\dagger = \sigma_{p_x^R} e^{-i\phi_{p_x}}, \quad (54)$$

where $\phi_{p_x} = 2\pi p'_x p''_x / 2 + \pi(1 - \delta_{p''_x, 0})$. Here, the index p_x is represented by a triple of integers $p_x = (p'_x, p''_x, p'''_x)$ with $p'''_x \in \{0, 1, 2, 3\}$, see Appendix A.

For generalized Pauli matrices with this property, we generalize the definition (34) of the projected ternary weight as

$$\bar{\rho}_{\bar{p}} = \sum_{a,b} \rho_{ab} \prod_x \chi_{\bar{p}_x; a_x, b_x}, \quad (55)$$

where $\bar{p}_x \in \{-1, 0, 1\}$ and

$$\chi_{\bar{p}_x; a_x, b_x} = \begin{cases} \delta_{a_x, 0} \delta_{b_x, 0}, & \text{if } \bar{p}_x = 0, \\ \frac{1}{2} (1 - \delta_{a_x, 0}) (\delta_{a_x, b_x} + \bar{p}_x \delta_{a_x, b_x^T} e^{-i\phi_{a_x}}) & \text{else} \end{cases} \quad (56)$$

Here, b_x^T is used for orthogonal circuits and for even sites in symplectic circuits, whereas b_x^R is used for odd sites in symplectic circuits. For $q = 2$, $\bar{p}_x$ retains the same interpretation as defined in Sec. VB, and the projected ternary weight function constructed in Eq. (55) reduces to Eq. (34).

In Appendix C, we show that there is a well-defined stochastic growth model for the projected ternary weight $\bar{\rho}_{\bar{p}}$, which has the same structure as the stochastic growth process for the case $q = 2^m$ considered previously. As in the unitary-invariant case, under time evolution, the Pauli-string density matrix ρ relaxes to a “ternary-form” distribution, which is a distribution of the form

$$\rho_{ab} = \bar{\rho}_{\bar{p}} \prod_x \frac{\chi_{\bar{p}_x; a_x, b_x}^*}{q^2 \bar{r}_{\bar{p}_x}}, \quad (57)$$

with $\bar{\rho}_p$ the ternary distribution corresponding to ρ_{ab} . For more details, we refer to Appendix E.

VII. EXAMPLES

A. Haar-random ensembles

For the Haar-random ensembles, the evolution matrices $\mathcal{U}_{x,x+1}$ are randomly chosen from the unitary, orthogonal, or symplectic group. For the Haar-unitary random circuit, the pair transition probabilities $\langle |\mathcal{W}_{ap;xx+1}|^2 \rangle$ are of the form (29) with $\mathcal{P}_1 = 1/(q^4 - 1)$ and $\mathcal{P}_2 = \mathcal{P}_3 = 0$. The butterfly velocity and diffusion constant for this circuit were calculated in Ref. [6],

$$v_B = \frac{q^2 - 1}{q^2 + 1}, \quad \mathcal{D} = \frac{4q^2}{(q^2 + 1)^2}. \quad (58)$$

Haar-orthogonal random circuits with qudit dimension $q = 2^m$ were previously studied by Hunter-Jones in Ref. [35] using the standard orthogonal Haar integral. In our formulation, the transition probability matrix for this case takes the form of Eq. (29), with the coefficients $\mathcal{P}_j$ given by Eqs. (31) and (32). These results are algebraically equivalent to the transition probabilities obtained in Ref. [35], although the two derivations use different notation and organize the resulting equations differently. The corresponding formulation for arbitrary qudit dimension q is given in Appendix F.

The 9×9 pair transition probability matrices appearing on the right-hand side of Eq. (36) have the block structure

$$T = \begin{bmatrix} 1 & & \\ & T_{++} & T_{+-} \\ & T_{-+} & T_{--} \end{bmatrix}, \quad (59)$$

where the sign $\pm$ of the indices of the 4×4 blocks $T_{s_a s_p}$ refer to the combined parity of the indices a_x and a_{x+1} or p_x and p_{x+1} . For the Haar-orthogonal random circuit, the 4×4 transition matrices for the projected ternary weights are

$$T_{++} = \frac{1}{2(q^2 + 2)(q^2 - 1)} \begin{pmatrix} f^2 & g^2 & 2f & 2f \\ f^2 & g^2 & 2f & 2f \\ f^2 & g^2 & 2f & 2f \\ f^2 & g^2 & 2f & 2f \end{pmatrix}, \quad (60)$$

$$T_{--} = \frac{1}{2q^2(q^2 - 1)} \begin{pmatrix} fg & fg & 2g & 2g \\ fg & fg & 2g & 2g \\ fg & fg & 2g & 2g \\ fg & fg & 2g & 2g \end{pmatrix} \quad (61)$$

and $T_{+-} = T_{-+} = 0$, with $f = (q + 2)(q - 1)$ and $g = q(q - 1)$. Using the approximation scheme of Sec. V D, we then obtain for the zeroth order approximation for the butterfly velocity and diffusion constant

$$v_B^{(0)} = \frac{(q - 1)(q^3 + 2q^2 + 2q + 3)}{(q + 1)(q^3 + 2q + 1)}, \quad (62)$$

$$\mathcal{D}^{(0)} = \frac{4}{(q + 1)^2(q^3 + 2q + 1)^3} (q^9 + 2q^8 + 6q^7 + 10q^6 + 11q^5 + 19q^4 + 3q^3 + 15q^2 - q - 2). \quad (63)$$

The zeroth-order expressions given in Eqs. (62) and (63) coincide with those reported in Ref. [35]. We further evaluate the butterfly velocity and diffusion constant using truncation

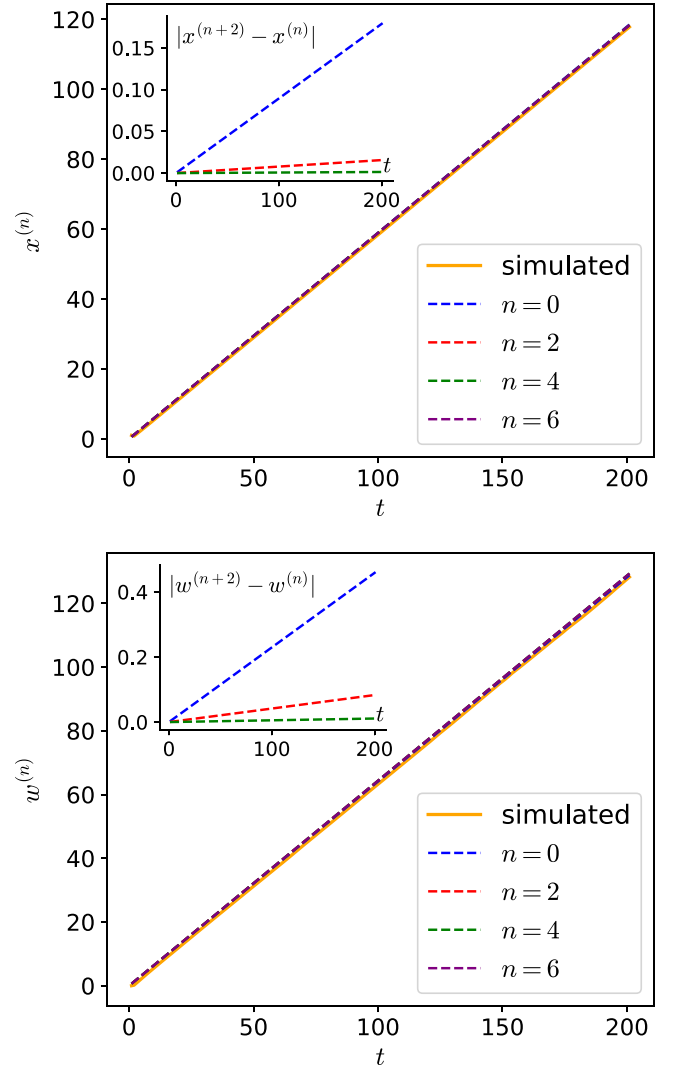


FIG. 2. Front position (top) and variance of front position (bottom) vs time t for a random unitary circuit with Haar-orthogonal two-qudit gates and qudit dimension $q = 2$. The solid lines in both panels represent results from direct numerical simulations of the classical stochastic growth process with system size $L = 200$, based on $N = 2 \times 10^5$ independent realizations. In the top panel, the slopes of the lines correspond to the butterfly velocity v_B , while in the bottom panel, the slopes yield the diffusion constant $\mathcal{D}$. The dashed curves indicate analytical results obtained from the truncation scheme at orders $n = 0, 2, 4$, and 6 . Because of the rapid convergence of the truncation, the analytical curves in the main panels are visually indistinguishable. To better resolve these details, the insets display the absolute differences between successive analytical approximations, $|x^{(n+2)} - x^{(n)}|$ (top) and $|w^{(n+2)} - w^{(n)}|$ (bottom), vs t . Here, $x^{(n)}$ and $w^{(n)}$ represent the n th order analytical approximations of the front position and variance of front position, respectively.

orders $n = 2, n = 4$, and $n = 6$. These results are presented in Fig. 2 for $q = 2$, alongside direct numerical simulations of the stochastic process defined by Eq. (35). The specific values for the butterfly velocity and diffusion constant at orders $n = 0, 2, 4$, and 6 are summarized in Table I. As shown, both quantities exhibit clear convergence as the truncation order increases. In contrast to the Haar-unitary circuit, Table I

TABLE I. Butterfly velocity $v_B^{(n)}$ and diffusion constant $\mathcal{D}^{(n)}$ for the Haar-orthogonal random circuit with $q = 2$, calculated analytically for orders $n = 0, 2, 4, 6$, respectively, and compared with results obtained from numerical simulations of the stochastic evolution of the projected ternary-string distribution $\bar{\rho}_{\bar{p}}$.

n	$v_B^{(n)}$	$\mathcal{D}^{(n)}$
0	0.5897435897	0.6408739190
2	0.5888499236	0.6431654164
4	0.5887719700	0.6435805364
6	0.5887652662	0.6436349980
sim.	0.5886992069	0.6413786349

demonstrates that the zeroth-order approximation is not exact for the Haar-orthogonal random circuit. Although the deviations between the higher-order and zeroth-order results are small, their existence implies that the domain-wall width does not strictly vanish in the Haar-orthogonal limit.

Expanding Eqs. (62) and (63) for large q yields $v_B^{(0)} = 1 - 2/q^2 + \mathcal{O}(q^{-4})$ and $\mathcal{D}^{(0)} = 4/q^2 + \mathcal{O}(q^{-4})$. Without the terms of $\mathcal{O}(q^{-2})$, this is the same result as that obtained from an exact treatment of the limit $q \rightarrow \infty$, see Eq. (B23). From this, we conclude that the zeroth-order approximation is exact in the limit $q \rightarrow \infty$.

B. Brownian special orthogonal circuit

In a Brownian circuit, the two-qudit operators $\mathcal{U}_{x,x+1}$ are obtained from the continuous time evolution with a random Hamiltonian with a Gaussian white-noise distribution [36,37]. Specifically,

$$\mathcal{U} = T_t e^{-i \int_0^t dt \mathcal{H}(t)}, \quad (64)$$

where T_t indicates the time-ordering prescription and $\mathcal{H}(t')$ is a $q^2 \times q^2$ random Hermitian matrix with zero mean and Gaussian delta-function correlations. For the Brownian special orthogonal circuit, $H(t)$ is antisymmetric. It has the two-time correlation function [38,39]

$$\langle H_{ij}(t) H_{kl}(t') \rangle = \frac{\lambda}{q^2} (\delta_{il} \delta_{jk} - \delta_{ik} \delta_{jl}) \delta(t - t'), \quad (65)$$

where λ is a parameter that describes the interpolation between the trivial and Haar-random-special-orthogonal two-qudit gate operators. This ensemble satisfies the orthogonal invariance property (8).

The moments of the distribution function for the Brownian special orthogonal circuit are given in Appendix G. The butterfly velocity and the diffusion constant for the Brownian special orthogonal circuit are shown in Figs. 3 and 4 for $q = 2$ and in the limit $q \rightarrow \infty$. In the large- q limit, the zeroth-order approximation is exact, and the butterfly velocity and the diffusion constant are

$$v_B^{(0)} = \frac{1 - e^{-2\lambda}}{1 + e^{-2\lambda}}, \quad (66)$$

$$\mathcal{D}^{(0)} = \frac{4e^{-2\lambda}(1 - e^{-2\lambda})}{(1 + e^{-2\lambda})^2}, \quad (67)$$

which is the same as for the unitary case.

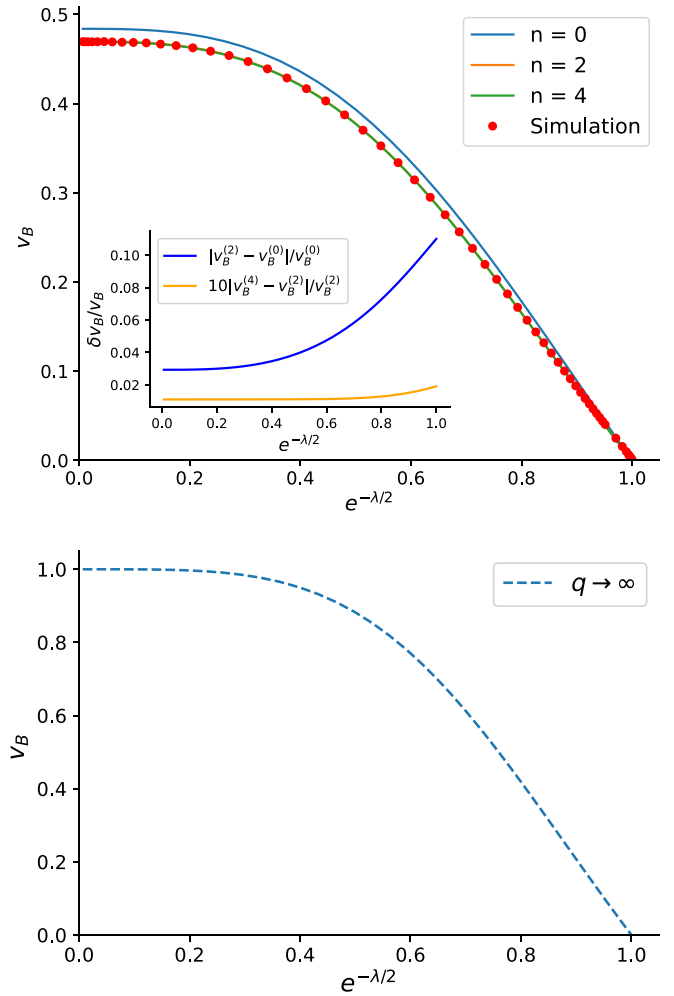


FIG. 3. Butterfly velocity for Brownian special orthogonal circuits with local dimension $q = 2$ and in the limit $q \rightarrow \infty$. The parameter λ interpolates between a trivial circuit and the Haar-random limit. (Top) Solid curves represent the analytical approximations $v_B^{(n)}$ for $q = 2$ at orders $n = 0, 2$, and 4 . Symbols denote data from direct numerical simulations (system size $L = 320$, evolved for 200 time steps, averaged over $N = 2 \times 10^5$ realizations). Because of the rapid convergence of the truncation scheme, the analytical results for all orders are virtually indistinguishable from the numerical data. (Inset) Relative differences between the approximations of order $n = 4$ and $n = 2$, and between $n = 2$ and $n = 0$. (Bottom) The dashed line shows the result for $q \rightarrow \infty$, where the approximation scheme is exact at $n = 0$.

The evolution coefficients for Brownian symplectic circuits can be obtained from those for orthogonal circuits by making the substitution $q^2 \rightarrow -q^2$ for all averages involving two-qudit operators. The resulting butterfly velocity and diffusion constant for $q = 2$ are shown in Figs. 9 and 10 below. (For $q \rightarrow \infty$, there is no difference between special orthogonal and symplectic circuits.)

C. SO^- ensembles

An instructive example is provided by SO^- ensembles, which consist of orthogonal matrices with determinant -1 . This set is topologically disconnected from the special

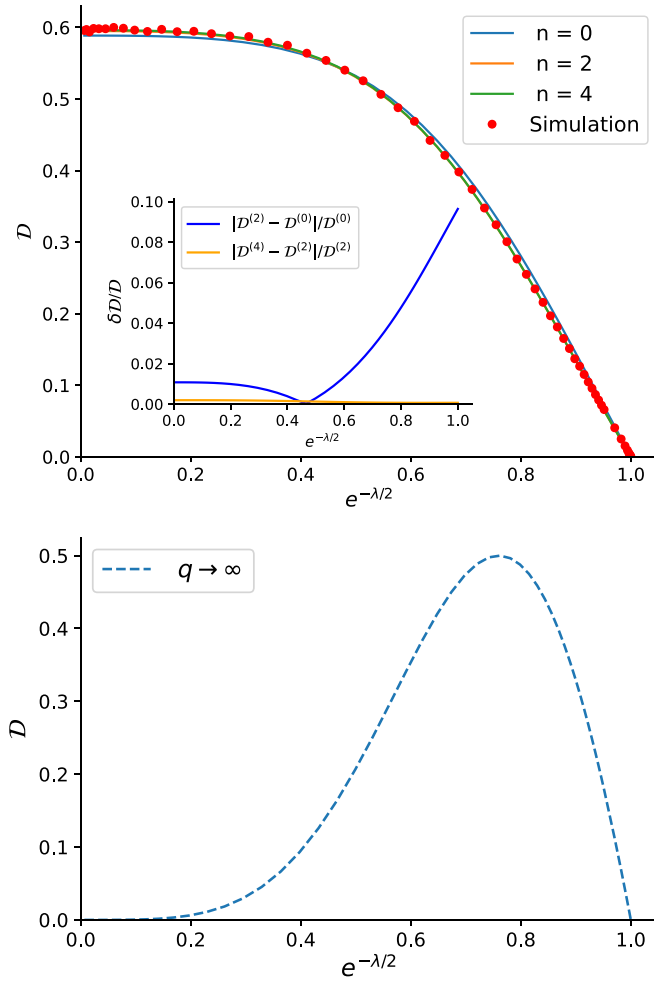


FIG. 4. Same as Fig. 3, but for the diffusion constant $\mathcal{D}$ for a Brownian special orthogonal circuit. The small fluctuations of the data points around the theoretical curve are attributed to statistical noise from the large, but finite, number of realizations N in the numerical simulations. Because of the rapid convergence of the truncation scheme, the analytical results for orders $n = 2$ and 4 are indistinguishable in the upper panel.

orthogonal group $\text{SO}(q^2)$ [40]. For odd qudit dimension q , $-\mathcal{U}_{x,x+1}$ has determinant -1 if $\mathcal{U}_{x,x+1}$ is an orthogonal matrix with determinant one. Since the operator spreading is unaffected by the simultaneous change $\mathcal{U}_{x,x+1} \rightarrow -\mathcal{U}_{x,x+1}$ for all x , the butterfly velocity and the diffusion constant in the SO^- circuit are the same as in the special orthogonal circuit obtained by replacing $\mathcal{U}_{x,x+1}$ by $-\mathcal{U}_{x,x+1}$ for all x . Such a relationship does not exist for even qudit dimension q . This can already be seen by noting that $\text{SO}^-(q^2)$ does not contain the identity matrix or a matrix proportional to it, so that orthogonal invariance rules out the possibility of purely diagonal evolutions, thereby ensuring a nonzero minimal spreading velocity. This qualitative distinction separates the two disconnected sectors of the orthogonal ensembles.

For $q = 2$, the pair transition probability $\langle |\mathcal{W}_{ap;x,x+1}|^2 \rangle$ takes the form given in Eq. (29). The parameters $\mathcal{P}_i$ for the

SO^- ensemble with $q = 2$ are

$$\begin{aligned} \mathcal{P}_1 &= \begin{cases} \frac{1}{9} - \frac{\mathcal{P}_3}{6} & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = 1, \\ \frac{1}{6} & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = -1, \\ 0 & \text{otherwise,} \end{cases} \\ \mathcal{P}_2 &= \begin{cases} \frac{\mathcal{P}_3}{2} & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = 1, \\ \frac{1}{6} & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = -1, \\ 0 & \text{otherwise,} \end{cases} \\ \mathcal{P}_3 &= \begin{cases} \mathcal{P}_3 & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = 1, \\ -\frac{1}{3} & \text{if } s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = -1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (68)$$

Here, the parameter $\mathcal{P}_3$ is

$$\mathcal{P}_3 = \frac{1 + 16\mathcal{R}_{1,3}}{75}, \quad (69)$$

where $\mathcal{R}_{1,3}$ is a fourth moment of the two-qudit evolution matrix $\mathcal{U}_{x,x+1}$,

$$\mathcal{R}_{1,3} = \frac{1}{q^4} \langle \text{tr } \mathcal{U}_{x,x+1} \text{tr } \mathcal{U}_{x,x+1}^3 \rangle. \quad (70)$$

From the expressions for $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$, it follows that the parameter space of the $\text{SO}^-(4)$ ensemble is effectively one dimensional: The distribution is characterized by the single real parameter $\mathcal{P}_3$. To find the range that $\mathcal{P}_3$ can take, we consider the structure of $q^2 \times q^2$ orthogonal matrices with $q = 2$ and determinant -1 : Every matrix $\mathcal{U}_{x,x+1}$ in $\text{SO}^-(4)$ possesses the eigenvalues $\{-1, 1, e^{i\theta}, e^{-i\theta}\}$ for some $\theta \in [0, \pi]$. The bounds of $\mathcal{P}_3$ are therefore determined by the minimum and maximum values of $\text{tr } \mathcal{U} \text{tr } \mathcal{U}^3 = 4 \cos \theta \cos 3\theta$, which are $-\frac{9}{4}$ and 4 , respectively. This constrains $\mathcal{P}_3$ to the interval

$$\mathcal{P}_3 \in \left[-\frac{1}{60}, \frac{1}{15}\right]. \quad (71)$$

Because of the restricted range of $\mathcal{P}_3$, the accessible butterfly velocities also lie within a finite interval. The maximum is attained for $\theta_* = (\pi \pm \arctan \sqrt{15})/2$, whereas the minimum occurs at $\theta = 0$ or $\theta = \pi$, corresponding to gates with eigenvalues $(1, 1, 1, -1)$ or $(-1, -1, -1, 1)$. Since these two spectra differ only by an overall factor of -1 , which acts trivially under conjugation, they generate the same operator dynamics. These involutions are the elements of $\text{SO}^-(4)$ closest to the trivial gate and play, within this disconnected sector, a role analogous to that of the identity in $\text{SO}(4)$.

For a fixed value of θ , orthogonal invariance requires uniform sampling over the corresponding conjugacy class, while the probability distribution of θ itself is not fixed by Eq. (1). The ensemble saturating the upper bound is $V = U \text{diag}(-1, 1, e^{i\theta_*}, e^{-i\theta_*})U^T$, where U is Haar distributed over the orthogonal group, while the ensemble saturating the lower bound is $V = U \text{diag}(-1, 1, 1, 1)U^T$. Neither extremal ensemble is Haar distributed over the $\text{SO}^-(4)$ sector.

Remarkably, the upper bound of the butterfly velocity in the SO^- ensemble for $q = 2$ exceeds the corresponding Haar-orthogonal value. Intuitively, the Haar-orthogonal measure samples the entire orthogonal group, including gates close to the trivial one, whereas an ensemble concentrated on the most strongly scrambling conjugacy classes of SO^- avoids this contribution. More concretely, the condition $\mathcal{P}_3 = -1/3$

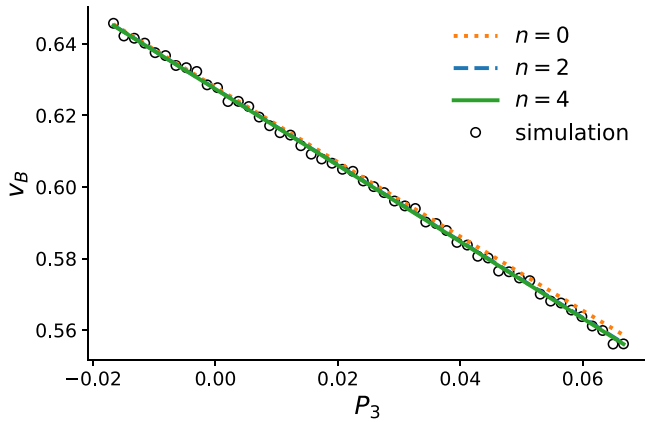


FIG. 5. Butterfly velocity for a SO^- circuit at $q = 2$. The parameter space of the circuit is one-dimensional and can be parameterized by P_3 , which in turn determines other parameters P_1 and P_2 . The data points represent the results of a direct numerical simulation conducted on a system of size $L = 320$ for 200 time steps, with averaging over $N = 5000$ independent realizations. The curves indicate analytical predictions derived from a truncation scheme at successive orders $n = 0, 2$, and 4 . Because of the rapid convergence of the truncation scheme, the analytical results for orders $n = 2$ and 4 are indistinguishable.

for $s_{p_x}s_{p_{x+1}} = s_{a_x}s_{a_{x+1}} = -1$ suppresses the probability that odd-parity operator strings retain their original form. Their resulting tendency to delocalize enhances the propagation speed beyond the Haar-orthogonal value. Numerical and analytical results for the SO^- ensembles are shown in Fig. 5.

VIII. DISCUSSION AND CONCLUSIONS

In this article, we extend the results of Ref. [18] from circuits composed of unitary-invariant two-qudit gates to those composed of orthogonal- or symplectic-invariant gates. As in the unitary case, the operator-spreading dynamics map at late times onto a classical stochastic growth model governed by a drift-diffusion equation, characterized by a butterfly velocity v_B and a diffusion constant $\mathcal{D}$. We derive explicit expressions for v_B and $\mathcal{D}$ in both the orthogonal- and symplectic-invariant ensembles.

There are four key differences from the random circuits with unitary invariance: (i) In the unitary-invariant case, averaging directly yields a closed Markov process on the space of Pauli strings; with orthogonal or symplectic symmetry, one must first project onto the invariant ternary subspace (in which a summation over degrees of freedom with the same symmetry has taken place) before obtaining a description in terms of a classical Markovian stochastic process for a generic dimension q . (ii) Instead of approaching a binary (trivial vs nontrivial) distribution, the system relaxes to a three-valued (“trivial,” “even,” “odd”) distribution. As in the unitary-invariant case, the relaxation time only depends on the parameters of the local two-qudit gate distribution. (iii) The domain wall between the trivial and maximally mixed regions is perfectly sharp in Haar-unitary circuits but has a finite width in orthogonal- or symplectic-invariant circuits, even when these circuits are composed of Haar-distributed

two-qudit gates. (iv) The bounds on the butterfly velocity depend qualitatively on the determinant sector of the orthogonal gate ensemble. In particular, the determinant -1 sector, denoted by SO^- , has a strictly positive lower bound for v_B and, for $q = 2$, can support a butterfly velocity exceeding the Haar-orthogonal value. Remarkably, neither extremal ensemble in the SO^- sector is Haar distributed, emphasizing the relevance of studying random gate ensembles that go beyond the Haar-distributed case.

Using the criterion introduced above, these differences can be separated into effects of symmetry and of gate randomness. The ternary structure of the Pauli-string weights and the need, for generic qudit dimension q , to project onto the invariant ternary subspace before a closed Markovian description is obtained are shared by all orthogonal- or symplectic-invariant gate distributions and thus originate from the symmetry alone; the relaxation time toward the ternary form, by contrast, is set by the gate distribution. The finite domain-wall width has both origins: reduced gate randomness widens the domain wall already in the unitary-invariant class [18], while the orthogonal or symplectic constraint renders the width finite even for Haar-distributed gates. The bounds of the butterfly velocity for even q , finally, fall outside this dichotomy: the strictly positive lower bound in the SO^- sector holds for any gate distribution, yet the SO^- ensembles share the same orthogonal invariance as the special orthogonal ones. This bound is instead set by the topology of the gate manifold—the connected component of the orthogonal group on which the distribution is supported—which thus constitutes a third factor controlling operator spreading, independent of both symmetry and randomness.

The physical significance of this enhanced butterfly velocity is clarified by comparison with dual-unitary circuits. Local operators propagate in these circuits at the maximal light-cone velocity, $v_B = 1$ [41], but this propagation need not be correlated with entanglement generation. The SWAP circuit, for example, transports a local operator along the light cone without growth of its support, while mapping every product state to another product state and hence having zero entangling power [42,43]. By contrast, the conjugation-invariant random circuits studied here belong to the generic setting in which operator spreading and entanglement growth are correlated [4,6,44]. Within this setting, the enhanced butterfly velocity found here corresponds to faster entanglement spreading and a greater effective entangling power of the local dynamics. It further suggests a larger spectral gap of the second-moment operator, consistent with the relation between local entangling power and the second-moment gap established for solvable unitary brickwork circuits in Ref. [45]. Such an enhancement would imply faster convergence toward the corresponding unitary, orthogonal, or symplectic 2-design.

We illustrated these results using two representative examples: the Haar-orthogonal circuit and the Brownian special orthogonal circuit, which interpolates between the trivial and Haar limits. In both cases, the domain-wall width remains finite. From the fast convergence of our calculations of the butterfly velocity with the approximation order n , we estimate a domain-wall width of $n_{\text{DW}} \sim 4$. Numerical simulations of the corresponding classical stochastic growth model—without assuming a maximally random distribution far from the end of

the Pauli string—show excellent agreement with our analytical predictions. Furthermore, we find that the results in the large- q limit coincide with those of the unitary case.

We identify three distinct temporal regimes for random quantum circuits with orthogonal- or symplectic-invariant gate distributions. *Short-time binary/ternary relaxation*: At early times, independent of the system size, the weight $\rho_{pp}(t)$ relaxes to a form that depends solely on whether each local generalized Pauli operator is trivial, even, or odd. *Intermediate thermalization*: A localized Pauli string spreads, with a finite-width domain wall separating high-entropy and trivial regions, propagating at velocity v_B . The crossover time for a site at position x scales as n_{DW}/v_B . *Long-time saturation*: At times of order L/v_B (for system size L), the distribution has extended across the entire system, completing the scrambling process [12–14].

The binary or ternary relaxation time is a circuit-level measure of Haar-like scrambling, rather than a measure of how closely the distribution of the local gates approximates the corresponding Haar distribution. Rather, it quantifies how rapidly the many-gate dynamics erases distinctions between Pauli strings within the same binary or ternary class. The Haar-special-orthogonal ensemble for $q = 2$ illustrates this distinction: a single local transition matrix has an additional eigenvalue-1 state outside the ternary subspace and therefore cannot produce ternary relaxation when repeatedly applied. This state is not preserved by the alternating even–odd brickwork structure, however, so the full circuit can still relax to ternary form.

More generally, convergence to a binary or ternary form from arbitrary initial conditions implies that no additional steady states retain information beyond the corresponding binary or ternary variables. Conversely, if this relaxation fails at the level of the full circuit, additional steady-state sectors are present. Continuous symmetries provide a natural setting in which such additional structure arises. In a $U(1)$ -symmetric circuit, for example, the conserved charge distinguishes charge-density operators, such as local Pauli- Z operators, from generic nonidentity operators [8,9]. The resulting deviations from the usual binary form are particularly apparent near the operator front, even when the bulk behind it already appears locally random-matrix-like. We leave the incorporation of continuous symmetries, including $U(1)$ and $SU(2)$, into the present framework for future investigation [46].

ACKNOWLEDGMENTS

We thank Adam Chaou, Vatsal Dwivedi, Jens Eisert, Ming Huang, Adam Nahum, Ryotaro Suzuki, and Guoyi Zhu for valuable discussions. Special thanks go to Yiping Deng for suggestions on figure editing. Financial support was provided by the Einstein Stiftung Berlin (Einstein Research Unit on Quantum Devices) and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Project No. 277101999 - CRC TR 183 (Project No. A03).

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: GENERALIZED PAULI MATRICES

1. Orthogonal case

The generalized Pauli matrices σ_a , $a = 0, 1, \dots, q^2 - 1$, form a basis for operators acting on a qudit, a quantum system with q degrees of freedom $|m\rangle$, $m = 0, 1, \dots, q - 1$. We represent the index $a = (a', a'')$ as a pair of two integers $a', a'' \in \mathbb{Z}_q$. The Weyl–Heisenberg choice for the complete family of q^2 -independent generalized Pauli matrices is [47]

$$\sigma_a = \sum_{m=0}^{q-1} \omega^{ma''} |m + a'\rangle \langle m|, \quad (A1)$$

where $\omega = e^{\frac{2\pi i}{q}}$ and $m + a'$ is taken mod q .

Without loss of generality, we may choose the qudit basis, such that the involution of Eq. (12) has $Z_x = \mathbb{1}$. In that case, $Z_x \sigma_a^\dagger Z_x^\dagger$ [see Eq. (16)] is simply the transpose, σ_a^T . The transpose of the generalized Pauli matrices (A1) is

$$(\sigma_a)^T = \sigma_{a^T} e^{-i\phi_a}, \quad (A2)$$

with $a^T = (-a', a'')$ and $\phi_a = 2\pi a' a'' / q$.

The generalized Pauli matrices satisfy the orthonormality relation

$$\text{tr} \sigma_a \sigma_b^\dagger = q \delta_{a,b} \quad (A3)$$

and the commutation relation

$$\sigma_a \sigma_b = \omega^{b'a'' - a'b''} \sigma_b \sigma_a. \quad (A4)$$

They also satisfy the completeness relation

$$\sum_a \text{tr} A \sigma_a^\dagger \text{tr} B^\dagger \sigma_a = q \text{tr} AB^\dagger \quad (A5)$$

for arbitrary $q \times q$ matrices A and B .

Since the evolution operators act on pairs of qudits, we will often consider the tensor product $\sigma_{a_x} \otimes \sigma_{a_y}$ of generalized Pauli matrices for two qudits at positions x and y . With the shorthand notations $a = (a_x, a_y) = (a'_x, a''_x, a'_y, a''_y)$ and $\Sigma_a = \sigma_{a_x} \otimes \sigma_{a_y}$, we observe that the $q^2 \times q^2$ matrices Σ_a satisfy the orthonormality relation

$$\text{tr} \Sigma_a \Sigma_b^\dagger = q^2 \delta_{a,b}, \quad (A6)$$

whereas their commutation relation is

$$\Sigma_a \Sigma_b = \varphi(a, b) \Sigma_b \Sigma_a, \quad (A7)$$

with

$$\varphi(a, b) = \omega^{b'_x a''_x - a'_x b''_x + b'_y a''_y - a'_y b''_y}. \quad (A8)$$

The function $\varphi(a, b)$ satisfies

$$\varphi(0, 0, b_x, b_y) = 1 \quad (A9)$$

and

$$\varphi(a, b) = \varphi(b, a)^*, \quad (A10)$$

$$\varphi(-a, b) = \varphi(a, b)^*. \quad (A11)$$

TABLE II. Numbers of generalized Pauli operators σ for $q = 2^m$ ($q > 2$) that commute or anticommute with a fixed Pauli matrix $\sigma' \neq \mathbb{1}$, for fixed parities of σ and σ' . A generalized Pauli operator σ is called even (e) or odd (o) if $\sigma = \sigma^T$ or $\sigma = -\sigma^T$, respectively.

parity σ	parity σ'	count $\sigma : [\sigma, \sigma'] = 0$	count $\sigma : \{\sigma, \sigma'\} = 0$
e	e'	$q(q+2)/4$	$q^2/4$
e	o'	$q^2/4$	$q(q+2)/4$
o	e'	$q(q-2)/4$	$q^2/4$
o	o'	$q^2/4$	$q(q-2)/4$

The sum rules for a generic dimension q^2 are

$$\begin{aligned} \sum_{a_x \neq 0} \varphi(a_x, 0, b_x, b_y) &= q^2 \delta_{b_x, 0} - 1, \\ \sum_{a_y \neq 0} \varphi(0, a_y, b_x, b_y) &= q^2 \delta_{b_y, 0} - 1, \\ \sum_{a_x \neq 0} \sum_{a_y \neq 0} \varphi(a_x, a_y, b_x, b_y) &= 1 - q^2 (\delta_{b_x, 0} + \delta_{b_y, 0}) + q^4 \delta_{b_x, 0} \delta_{b_y, 0}. \end{aligned} \quad (\text{A12})$$

In addition, there are sum rules that also involve the phase ϕ_a of Eq. (A2). For arbitrary $\bar{a}_x, \bar{a}_y \in \{-1, 1\}$ one has

$$\begin{aligned} \sum_{a_x \neq 0} \frac{1 + \bar{a}_x e^{-i\phi_{a_x}}}{2} \varphi(a_x, 0, b_x, b_y) &= F(b_x, \bar{a}_x), \\ \sum_{a_y \neq 0} \frac{1 + \bar{a}_y e^{-i\phi_{a_y}}}{2} \varphi(0, a_y, b_x, b_y) &= F(b_y, \bar{a}_y), \\ \sum_{a_x \neq 0} \sum_{a_y \neq 0} \frac{1 + \bar{a}_x e^{-i\phi_{a_x}}}{2} \frac{1 + \bar{a}_y e^{-i\phi_{a_y}}}{2} \varphi(a_x, a_y, b_x, b_y) \\ &= F(b_x, \bar{a}_x) F(b_y, \bar{a}_y), \end{aligned} \quad (\text{A13})$$

where

$$F(b, \bar{a}) = \frac{q^2 \delta_{b, 0} + q \bar{a} e^{-i\phi_b} - 1 - \bar{a}}{2}. \quad (\text{A14})$$

The three relations in Eq. (A13) are essential for deriving the stochastic equations governing the projected ternary weight $\rho_{\bar{p}}$, as defined in Sec. VI. For $q = 2$, these relations can be interpreted as sum rules for Pauli matrices with even and odd parities under transposition.

For dimensions $q = 2^m$, it is convenient to adopt an alternative operator basis, which is constructed from tensor products of the standard 2×2 Pauli matrices. On this basis, all generalized Pauli operators have a well-defined parity under transposition. The projections onto the ternary weights depend on the numbers of basis operators of each parity that commute or anticommute with a fixed basis operator. These numbers are listed in Table II.

2. Symplectic case

Symplectic invariance applies to qudits of half-integer spin. In this case, the qudit dimension q must be even. Without loss of generality, we may choose the qudit basis such that

the involution matrix Z_x of Eq. (12) takes the form

$$Z_x = \mathbb{1}_{q/2} \otimes \sigma_2 \quad (\text{A15})$$

where σ_2 is a standard 2×2 Pauli matrix. In this case, $Z_x \sigma_{a_x}^T Z_x^\dagger = \sigma_{a_x}^R$ is the dual of the generalized Pauli matrix σ_{a_x} [22].

Consistent with Eq. (A15), for the generalized Pauli matrices for a qudit of half-integer spin, we choose tensor products of the Weyl–Heisenberg generalized Pauli matrices and standard 2×2 Pauli matrices. Hence, we use generalized Pauli matrices of the form

$$\sigma_a = \sum_{m=0}^{q-1} \omega^{ma''} |m + a'\rangle \langle m| \otimes \sigma_{a'''}, \quad (\text{A16})$$

where $a = (a', a'', a''')$ and $\sigma_{a'''}$ is a standard 2×2 Pauli matrix with $a''' \in \{0, 1, 2, 3\}$. The dual matrix of such a generalized Pauli matrix is given by

$$(\sigma_a)^R = \sigma_{a^R} e^{-i\phi_a}, \quad (\text{A17})$$

where $a^R = (-a', a'', a''')$ and $\phi_a = 2\pi a' a'' / 2 + \pi(1 - \delta_{a''', 0})$.

The matrix $\Omega_{x,y} = Z_x \otimes Z_y$ that describes the involution operation for a two-qudit gate $\mathcal{U}_{x,y}$ is antisymmetric only if Z_x is symmetric and Z_y is antisymmetric or vice versa. For definiteness, we here focus on the case that Z_x is symmetric (i.e., the qudit x has integer spin) and Z_y is antisymmetric (i.e., the qudit y has half-integer spin), so that $Z_x = \mathbb{1}_q$ and $Z_y = \mathbb{1}_{q/2} \otimes \sigma_2$ and, hence, $\Omega_{x,y} = \mathbb{1}_{q^2/2} \otimes \sigma_2$. The tensor products Σ_a with $a = (a'_x, a''_x; a'_y, a''_y, a'''_y)$ that span the two-qudit operators are of the form $\Sigma_a = \sigma_{a_x} \otimes \sigma_{a_y}$, with σ_{a_x} a generalized Pauli matrix of the form (A1) and σ_{a_y} a generalized Pauli matrix of the form (A16). The two-qudit matrices Σ_a again satisfy the orthonormality relation (A6). They also satisfy the commutation relation

$$\Sigma_a \Sigma_b = \varphi(a, b) \Sigma_b \Sigma_a \quad (\text{A18})$$

with

$$\begin{aligned} \varphi(a, b) &= \omega^{b'_x a''_x - a'_x b''_x + b'_y a''_y - a'_y b''_y} \\ &\times e^{i\pi(1 - \delta_{a'''_y, 0})(1 - \delta_{b'''_y, 0})(1 - \delta_{a'''_y, b'''_y})}. \end{aligned} \quad (\text{A19})$$

The combined phase factors of both the generalized Pauli and standard Pauli matrix components determine the commutation relation in the symplectic case. For an arbitrary local dimension q , the phases defined in (A19) satisfy sum rules identical to those presented in Eqs. (A12). Furthermore, they obey additional sum rules analogous to the relations in Eq. (A13),

$$\begin{aligned} \sum_{a_x \neq 0} \frac{1 + \bar{a}_x e^{-i\phi_{a_x}}}{2} \varphi(a_x, 0, b_x, b_y) &= F(b_x, \bar{a}_x), \\ \sum_{a_y \neq 0} \frac{1 + \bar{a}_y e^{-i\phi_{a_y}}}{2} \varphi(0, a_y, b_x, b_y) &= F'(b_y, \bar{a}_y), \\ \sum_{a_x \neq 0} \sum_{a_y \neq 0} \frac{1 + \bar{a}_x e^{-i\phi_{a_x}}}{2} \frac{1 + \bar{a}_y e^{-i\phi_{a_y}}}{2} \varphi(a_x, a_y, b_x, b_y) \\ &= F(b_x, \bar{a}_x) F'(b_y, \bar{a}_y). \end{aligned} \quad (\text{A20})$$

TABLE III. Numbers of generalized Pauli operators σ for $q = 2^m$ ($q > 2$) that commute or anticommute with a fixed Pauli matrix $\sigma' \neq \mathbb{1}$, for fixed parities of σ and σ' . A generalized Pauli operator σ is called even (e) or odd (o) if $\sigma = \sigma^R$ or $\sigma = -\sigma^R$, respectively.

parity σ	parity σ'	count $\sigma : [\sigma, \sigma'] = 0$	count $\sigma : \{\sigma, \sigma'\} = 0$
e	e'	$q(q-2)/4$	$q^2/4$
e	o'	$q^2/4$	$q(q-2)/4$
o	e'	$q(q+2)/4$	$q^2/4$
o	o'	$q^2/4$	$q(q+2)/4$

Here, the function $F(b, \bar{a})$ is given in (A14), while

$$F'(b, \bar{a}) = \frac{q^2 \delta_{b,0} - q \bar{a} e^{-i\phi_b} - 1 - \bar{a}}{2}. \quad (\text{A21})$$

For $q = 2$, the relations in Eq. (A20) can be interpreted as sum rules for Pauli matrices of even and odd parities within the context of the duality operation.

For qudit dimension $q = 2^m$, it is again advantageous to employ an operator basis consisting of tensor products of Pauli matrices, so that all generalized Pauli operators have a well-defined parity under the duality operation. The numbers of basis matrices that commute or anticommute with a given basis matrix of even or odd parity are summarized in Table III.

APPENDIX B: CALCULATION OF $\langle W_{ap} W_{bq}^* \rangle$

The time evolution of $\rho_{pq}(t) = \langle \gamma_p(t) \gamma_q^*(t) \rangle$ involves the average $\langle W_{ap} W_{bq}^* \rangle$, see Eq. (22). The ensemble average $\langle W_{ap} W_{bq}^* \rangle$ factorizes as a product of two-qudit averages,

$$\langle W_{ap} W_{bq}^* \rangle = \prod_{x \text{ even}} \left\{ \langle \mathcal{W}_{ap;x,x+1} \mathcal{W}_{bq;x,x+1}^* \rangle \quad t \text{ even}, \right. \\ \left. \langle \mathcal{W}_{ap;x-1,x} \mathcal{W}_{bq;x-1,x}^* \rangle \quad t \text{ odd}, \right. \quad (\text{B1})$$

where the matrices $\mathcal{W}_{ap;x,y}$ and $\mathcal{W}_{bq;x,y}$ (with $y = x+1$ or $y = x-1$) are expressed in terms of the pair-qudit evolution matrices $\mathcal{U}_{x,y}$,

$$\mathcal{W}_{ap;x,y} = \frac{1}{q^2} \text{tr} \mathcal{U}_{x,y}^\dagger \Sigma_a \mathcal{U}_{x,y} \Sigma_p^\dagger. \quad (\text{B2})$$

Here, $\Sigma_a = \sigma_{a_x} \otimes \sigma_{a_y}$ and $\Sigma_p = \sigma_{p_x} \otimes \sigma_{p_y}$, see Appendix A.

In Sec. V A, we presented the two-qudit transition matrices for the CUE, COE, CSE, and Haar orthogonal and symplectic ensembles without giving their derivations. Since these calculations follow the same general procedure, we use the Haar orthogonal ensemble as a representative example to illustrate the diagrammatic evaluation. This example also provides a reference point for the more involved calculation of orthogonally or symplectically invariant gate distributions presented below.

Both the Haar-orthogonal example and the subsequent calculation for orthogonally or symplectically invariant gate distributions use a generalization of the diagrammatic method developed by Beenakker and one of the authors for averages over the unitary group [48] (following a structure originally developed by Weingarten [49] and Samuel [50] for large q and finite q , respectively) to integrations over the orthogonal and symplectic groups [51,52]. In this diagrammatic approach, elements of the two-qudit evolution matrix $\mathcal{U}_{x,y}$ and its complex

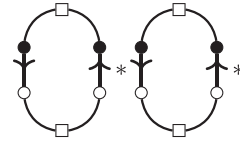


FIG. 6. Diagrammatic representation of the product $W_{ap} W_{bq}^*$ before taking the ensemble average.

conjugate are represented by single directed lines,

$$\mathcal{U} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \mathcal{U}^* = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}^*. \quad (\text{B3})$$

The generalized Pauli matrices Σ_a , Σ_b , Σ_p , and Σ_q are represented by solid lines terminating in open squares. Spatial indices are omitted from the diagrams for simplicity. The product $W_{ap} W_{bq}^*$ is thus represented by diagrams with four directed lines and four lines with open squares, whereas the joining of lines indicates matrix multiplication, see Fig. 6.

1. Special case: Haar-orthogonal and Haar-symplectic ensembles

We first consider the Haar orthogonal ensemble, for which $\mathcal{U}_{x,y}$ itself is a $q^2 \times q^2$ Haar-distributed orthogonal matrix. The average over the orthogonal group is evaluated by pairing the first and second indices of the matrix elements of $\mathcal{U}$ and/or $\mathcal{U}^*$. Diagrammatically, these pairings are represented by dashed contraction lines joining the corresponding filled dots, which denote first indices, and open dots, which denote second indices. When evaluating a diagram, each closed loop consisting of solid lines with squares and dashed contraction lines contributes a trace of the corresponding generalized Pauli matrices or a product of them. The loop structure consisting of solid lines with arrows and dashed contraction lines determines the Weingarten coefficient: V_2 if there is one loop involving all four dashed contraction lines, $V_{1,1}$ if there are two loops involving two dashed lines each [48].

The diagrams contributing to $\langle \mathcal{W}_{ap;x,x+1} \mathcal{W}_{bq;x,x+1}^* \rangle$ are shown in Fig. 7, as well as the evaluation of each diagram following the rules outlined above. Since an orthogonal matrix is real, $\mathcal{U}_{x,y} = \mathcal{U}_{x,y}^*$, and the asterisks may be omitted in these diagrams. The two Weingarten numbers entering this calculation are

$$V_{1,1} = \frac{q^2 + 1}{q^2(q^2 - 1)(q^2 + 2)}, \\ V_2 = -\frac{1}{q^2(q^2 - 1)(q^2 + 2)}. \quad (\text{B4})$$

Summing the contributions of all diagrams and evaluating the traces of products of generalized Pauli matrices, we verify that the average $\langle \mathcal{W}_{ap;x,x+1} \mathcal{W}_{bq;x,x+1}^* \rangle$ for qudit dimensions $q = 2^m$ has the property of Eq. (25). The resulting transition probability matrix $\langle |\mathcal{W}_{ap;x,y}|^2 \rangle$ is then given by Eqs. (31) and (32). For the symplectic case, the diagrams are the same as those shown in Fig. 7 and one arrives at the same result as for the orthogonal case up to the substitution $q^2 \rightarrow -q^2$. The transition matrices for the COE and CSE ensembles [see Eq. (33)] can be obtained by applying the corresponding integration rules in the same manner [35].

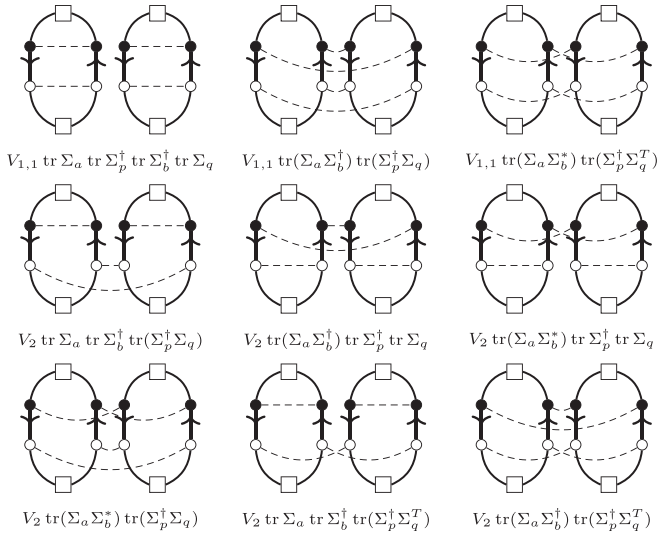


FIG. 7. Diagrams contributing to $\langle \mathcal{W}_{ap;x,x+1} \mathcal{W}_{bq;x,x+1}^* \rangle$ for Haar-distributed orthogonal gates. The Weingarten numbers $V_{1,1}$ and V_2 are given in Eq. (B4).

2. General case: orthogonal- and symplectic-invariant gate distribution

We now turn to the general case of a two-qudit evolution matrix $\mathcal{U}_{x,y}$ whose distribution is invariant under orthogonal or symplectic changes of basis. In contrast to the Haar case, $\mathcal{U}_{x,y}$ is not itself averaged over a Haar ensemble. Instead, its basis invariance allows the replacement

$$\mathcal{U}_{x,y} \rightarrow \mathcal{V}_{x,y} \mathcal{U}_{x,y} \mathcal{V}_{x,y}^\dagger, \quad (\text{B5})$$

where $\mathcal{V}_{x,y}$ is a statistically independent $q^2 \times q^2$ Haar-distributed orthogonal or symplectic matrix, followed by averaging over $\mathcal{V}_{x,y}$. The resulting second moment contains more Haar-distributed matrix factors than in the preceding Haar-orthogonal example, leading to a substantially larger number of possible contraction patterns.

We retain the single-line convention introduced above for $\mathcal{U}$ and $\mathcal{U}^*$ and represent $\mathcal{V}$ and $\mathcal{V}^*$ by double lines,

$$\mathcal{V} = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \quad \mathcal{V}^* = \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}^*. \quad (\text{B6})$$

where the spatial indices x and y are omitted for simplicity.

Only terms in which the first and second indices of all factors $\mathcal{V}$ and $\mathcal{V}^*$ coincide pairwise contribute to $\langle \mathcal{W}_{ap;x,y} \mathcal{W}_{bq;x,y}^* \rangle$. As in the Haar-orthogonal calculation, these pairings are represented diagrammatically by dashed contraction lines joining the corresponding filled dots, which denote first indices, and open dots, which denote second indices.

Following the approach in Ref. [18], we consider the two loop structures in the diagrams separately: (i) loops containing only dashed and doubled lines, and (ii) loops composed of dashed and solid lines. Loops of type (i) determine the Weingarten function. These are closed loops formed by alternating

dashed lines (contractions) and double lines (representing the Haar matrices $\mathcal{V}$ and $\mathcal{V}^*$). Defining the “length” c_i of each loop as the number of double lines associated with $\mathcal{V}$, with $i = 1, \dots, k$ and k the total number of loops, the corresponding Weingarten number is denoted by $V_{c_1, \dots, c_k}$. The five Weingarten numbers that appear in our calculation for the orthogonal-invariant case are

$$\begin{aligned} V_{1,1,2} &= \frac{-q^6 - 6q^4 - 3q^2 + 6}{D(q)}, \\ V_{2,2} &= \frac{q^4 + 5q^2 + 18}{D(q)}, \\ V_{1,3} &= \frac{2q^4 + 8q^2}{D(q)}, \\ V_{1,1,1,1} &= \frac{q^8 + 7q^6 + q^4 - 35q^2 - 6}{D(q)}, \\ V_4 &= \frac{-5q^2 - 6}{D(q)}, \end{aligned} \quad (\text{B7})$$

where

$$\begin{aligned} D(q) &= (q^2 - 3)(q^2 - 2)(q^2 - 1)q^2(q^2 + 1) \\ &\quad \times (q^2 + 2)(q^2 + 4)(q^2 + 6). \end{aligned} \quad (\text{B8})$$

Weingarten numbers for the symplectic-invariant case are obtained from Eqs. (B7) and (B8) by the substitution $q^2 \rightarrow -q^2$ [51,52]. (Recall that the integration is over orthogonal or symplectic matrices $\mathcal{V}_{x,y}$ of size $q^2 \times q^2$.)

Loops of type (ii) are built from alternating single directed lines (for $\mathcal{U}$ and $\mathcal{U}^*$) and dashed lines connecting open dots, as well as loops consisting of alternating single lines ending in open squares (for Σ_a , Σ_b , Σ_p , and Σ_q) and dashed lines connecting open dots. Each such loop contributes a trace over the product of matrices $\mathcal{U}$ and $\mathcal{U}^*$ or over generalized Pauli matrices Σ_a , Σ_b , Σ_p , and Σ_q , following their order within the loop.

In the diagrammatic evaluation of the second moment $\langle \mathcal{W}_{ap;x,y} \mathcal{W}_{bq;x,y}^* \rangle$, there are 105 possible contractions for the open dots and 105 for the closed dots, yielding a total of $105^2 = 11\,025$ diagrams. The diagrams are the same for the orthogonal and symplectic cases, but the outcome of the calculation of the diagrams is not, because of the different Weingarten numbers. Each set of 105 contraction patterns corresponds to the diagrams of the Brauer algebra [53]. We organize the diagrams according to the contractions of the closed dots, which determines the loops of type (ii) and, hence, the appearance of Kronecker deltas involving the indices a , b , p , and q . The 105 possible contraction patterns and their corresponding contributions are shown in Fig. 8. Each contribution C_j contains a factor B_j ($j = 1, 2, \dots, 105$), representing the contractions of the open dots, along with a product of traces over the generalized Pauli matrices Σ_a , Σ_b , Σ_p , and Σ_q . The expressions C_j of those diagrams are

$$\begin{aligned} C_1 &= B_1 \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_q & C_2 &= B_2 \text{tr} \Sigma_a \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \Sigma_q & C_3 &= B_3 \text{tr} \Sigma_a \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \Sigma_q \\ C_4 &= B_4 \text{tr} \Sigma_a \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_q & C_5 &= B_5 \text{tr} \Sigma_a \Sigma_b^* \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_q & C_6 &= B_6 \text{tr} \Sigma_a \Sigma_b^* \text{tr} \Sigma_p^\dagger \Sigma_q \end{aligned}$$

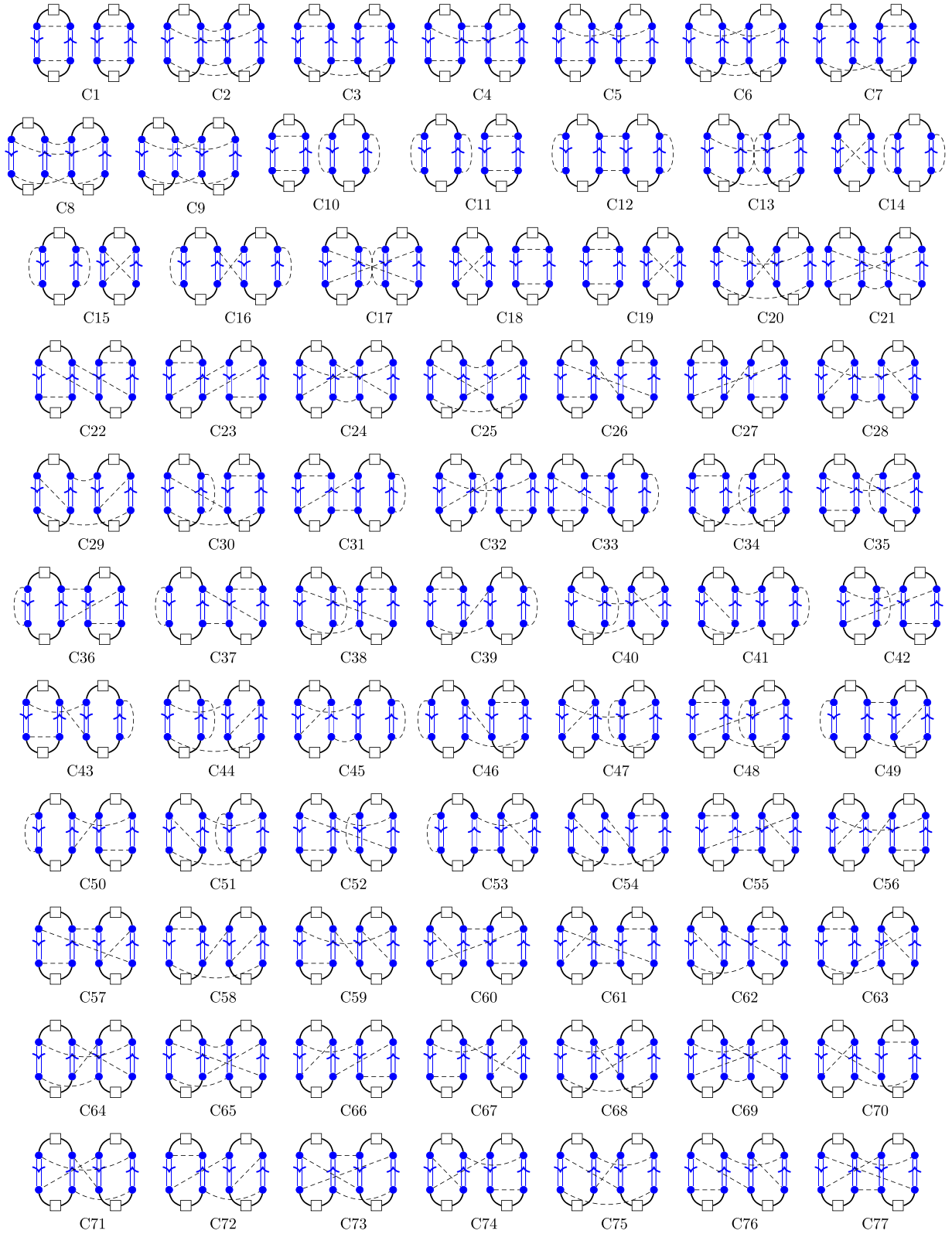


FIG. 8. Contractions C_j of closed dots (corresponding to the generalized Pauli matrices) and their contributions to $\langle \mathcal{W}_{apx,y} \mathcal{W}_{bqz,y} \rangle$. The coefficients B_j ($j = 1, 2, \dots, 105$) represent the contributions arising from the contractions of open dots, each of which involves a sum over 105 distinct terms. In the figure, each double line with two closed dots represents the product $\mathcal{V}_{x,y} \mathcal{U}_{x,y} \mathcal{V}_{x,y}^\dagger$, cf. Eqs. (B3) and (B6).

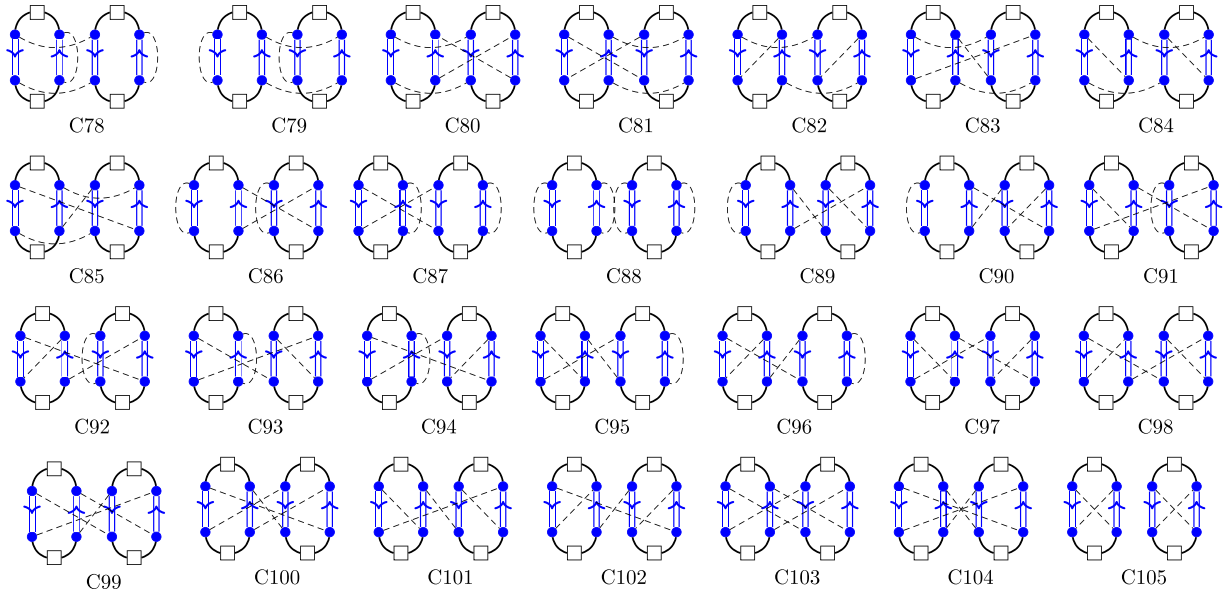


FIG. 8. (Continued.)

$$\begin{aligned}
 C_7 &= B_7 \text{tr} \Sigma_a \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \Sigma_q^T & C_8 &= B_8 \text{tr} \Sigma_a \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \Sigma_q^T & C_9 &= B_9 \text{tr} \Sigma_a \Sigma_b^* \text{tr} \Sigma_p^\dagger \Sigma_q^T \\
 C_{10} &= B_{10} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \Sigma_q & C_{11} &= B_{11} \text{tr} \Sigma_a \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_q & C_{12} &= B_{12} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q \Sigma_b^\dagger \\
 C_{13} &= B_{13} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q \Sigma_p^\dagger & C_{14} &= B_{14} \text{tr} \Sigma_a \Sigma_p^* \text{tr} \Sigma_b^\dagger \Sigma_q & C_{15} &= B_{15} \text{tr} \Sigma_a \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \Sigma_q^T \\
 C_{16} &= B_{16} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^* \Sigma_q^T & C_{17} &= B_{17} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^* \Sigma_p^\dagger & C_{18} &= B_{18} \text{tr} \Sigma_a \Sigma_p^* \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_q \\
 C_{19} &= B_{19} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \Sigma_q^T & C_{20} &= B_{20} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^* \Sigma_q^T & C_{21} &= B_{21} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^* \Sigma_b^\dagger \\
 C_{22} &= B_{22} \text{tr} \Sigma_a \Sigma_q \text{tr} \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger & C_{23} &= B_{23} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger \text{tr} \Sigma_q & C_{24} &= B_{24} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^\dagger \Sigma_q \\
 C_{25} &= B_{25} \text{tr} \Sigma_a \Sigma_q \Sigma_p^\dagger \Sigma_b^\dagger & C_{26} &= B_{26} \text{tr} \Sigma_a \Sigma_q^T \text{tr} \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger & C_{27} &= B_{27} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger \text{tr} \Sigma_q \\
 C_{28} &= B_{28} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q^T \Sigma_p^* & C_{29} &= B_{29} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q^T \Sigma_b^\dagger & C_{30} &= B_{30} \text{tr} \Sigma_a \Sigma_q \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \\
 C_{31} &= B_{31} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_q \Sigma_b^\dagger & C_{32} &= B_{32} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^\dagger \text{tr} \Sigma_q & C_{33} &= B_{33} \text{tr} \Sigma_a \Sigma_q \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \\
 C_{34} &= B_{34} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger \Sigma_q & C_{35} &= B_{35} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q \text{tr} \Sigma_p^\dagger & C_{36} &= B_{36} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^\dagger \text{tr} \Sigma_q \\
 C_{37} &= B_{37} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q \text{tr} \Sigma_b^\dagger & C_{38} &= B_{38} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger & C_{39} &= B_{39} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^* \Sigma_q^T \\
 C_{40} &= B_{40} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q^T \Sigma_p^\dagger & C_{41} &= B_{41} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q \Sigma_b^\dagger & C_{42} &= B_{42} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^\dagger \text{tr} \Sigma_q \\
 C_{43} &= B_{43} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q^T \text{tr} \Sigma_p^\dagger & C_{44} &= B_{44} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q \Sigma_p^\dagger & C_{45} &= B_{45} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q^T \Sigma_p^* \\
 C_{46} &= B_{46} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q^T \text{tr} \Sigma_b^\dagger & C_{47} &= B_{47} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q \Sigma_p^* & C_{48} &= B_{48} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_q^T \Sigma_b^* \\
 C_{49} &= B_{49} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q^T \Sigma_b^\dagger & C_{50} &= B_{50} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^* \text{tr} \Sigma_q & C_{51} &= B_{51} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q^T \Sigma_b^* \\
 C_{52} &= B_{52} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^* \text{tr} \Sigma_p^\dagger & C_{53} &= B_{53} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q \Sigma_b^* & C_{54} &= B_{54} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q^T \text{tr} \Sigma_b^\dagger \\
 C_{55} &= B_{55} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_q \Sigma_b^* & C_{56} &= B_{56} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^* \text{tr} \Sigma_q & C_{57} &= B_{57} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^\dagger \text{tr} \Sigma_p^\dagger \\
 C_{58} &= B_{58} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^* \Sigma_q & C_{59} &= B_{59} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_q^T \text{tr} \Sigma_p^\dagger & C_{60} &= B_{60} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^\dagger \text{tr} \Sigma_q \\
 C_{61} &= B_{61} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger & C_{62} &= B_{62} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q \text{tr} \Sigma_b^\dagger & C_{63} &= B_{63} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger \Sigma_q^T \\
 C_{64} &= B_{64} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^* \Sigma_q & C_{65} &= B_{65} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^\dagger \Sigma_b^\dagger & C_{66} &= B_{66} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^* \text{tr} \Sigma_q \\
 C_{67} &= B_{67} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q \text{tr} \Sigma_p^\dagger & C_{68} &= B_{68} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^* \Sigma_q^T & C_{69} &= B_{69} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^\dagger \Sigma_q \\
 C_{70} &= B_{70} \text{tr} \Sigma_a \Sigma_q \Sigma_p^* \text{tr} \Sigma_b^\dagger & C_{71} &= B_{71} \text{tr} \Sigma_a \Sigma_b^\dagger \Sigma_p^\dagger \Sigma_q^T & C_{72} &= B_{72} \text{tr} \Sigma_a \text{tr} \Sigma_p^\dagger \Sigma_q^T \Sigma_b^\dagger \\
 C_{73} &= B_{73} \text{tr} \Sigma_a \Sigma_q \Sigma_p^* \Sigma_b^\dagger & C_{74} &= B_{74} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^* \text{tr} \Sigma_q & C_{75} &= B_{75} \text{tr} \Sigma_a \Sigma_q \Sigma_p^\dagger \Sigma_b^*
 \end{aligned}$$

$$\begin{aligned}
C_{76} &= B_{76} \text{tr} \Sigma_a \Sigma_q \Sigma_b^* \text{tr} \Sigma_p^\dagger & C_{77} &= B_{77} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^* \Sigma_b^* & C_{78} &= B_{78} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q^T \Sigma_p^\dagger \\
C_{79} &= B_{79} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_q^T \Sigma_b^* & C_{80} &= B_{80} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^* \Sigma_q & C_{81} &= B_{81} \text{tr} \Sigma_a \Sigma_q \Sigma_p^* \Sigma_b^* \\
C_{82} &= B_{82} \text{tr} \Sigma_a \Sigma_b^* \Sigma_q \Sigma_p^* & C_{83} &= B_{83} \text{tr} \Sigma_a \Sigma_b^* \Sigma_p^\dagger \Sigma_q^T & C_{84} &= B_{84} \text{tr} \Sigma_a \Sigma_p^* \Sigma_q \Sigma_b^* \\
C_{85} &= B_{85} \text{tr} \Sigma_a \Sigma_q^T \Sigma_p^\dagger \Sigma_b^* & C_{86} &= B_{86} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^\dagger \Sigma_q & C_{87} &= B_{87} \text{tr} \Sigma_a \Sigma_q \Sigma_b^\dagger \Sigma_p^\dagger \\
C_{88} &= B_{88} \text{tr} \Sigma_a \Sigma_p^\dagger \text{tr} \Sigma_b^\dagger \Sigma_q & C_{89} &= B_{89} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^\dagger \Sigma_q^T & C_{90} &= B_{90} \text{tr} \Sigma_a \Sigma_p^\dagger \Sigma_b^* \Sigma_q \\
C_{91} &= B_{91} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^\dagger \Sigma_q & C_{92} &= B_{92} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^* \Sigma_p^* & C_{93} &= B_{93} \text{tr} \Sigma_a \Sigma_q \Sigma_b^* \Sigma_p^\dagger \\
C_{94} &= B_{94} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^\dagger \Sigma_p^\dagger & C_{95} &= B_{95} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^* \Sigma_q^T & C_{96} &= B_{96} \text{tr} \Sigma_a \Sigma_q \Sigma_b^\dagger \Sigma_p^* \\
C_{97} &= B_{97} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^* \Sigma_q & C_{98} &= B_{98} \text{tr} \Sigma_a \Sigma_q \Sigma_b^* \Sigma_p^* & C_{99} &= B_{99} \text{tr} \Sigma_a \Sigma_q \text{tr} \Sigma_p^\dagger \Sigma_b^* \\
C_{100} &= B_{100} \text{tr} \Sigma_a \Sigma_q^T \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger & C_{101} &= B_{101} \text{tr} \Sigma_a \Sigma_p^* \Sigma_b^\dagger \Sigma_q^T & C_{102} &= B_{102} \text{tr} \Sigma_a \Sigma_q^T \Sigma_b^\dagger \Sigma_p^* \\
C_{103} &= B_{103} \text{tr} \Sigma_a \Sigma_q \text{tr} \Sigma_p^\dagger \Sigma_b^\dagger & C_{104} &= B_{104} \text{tr} \Sigma_a \Sigma_q^T \text{tr} \Sigma_p^\dagger \Sigma_b^* & C_{105} &= B_{105} \text{tr} \Sigma_a \Sigma_p^* \text{tr} \Sigma_b^\dagger \Sigma_q^T
\end{aligned}$$

(For the symplectic case, the transpose T should be replaced by the dual R , complex conjugation is considered in the quaternion sense, and tr should be replaced by $-\text{tr}$, see Refs. [39,48].)

It remains to calculate the coefficients B_j and the products of traces of generalized Pauli matrices. In principle, the calculation of each coefficient B_j again involves the evaluation of 105 separate diagrams. To avoid having to perform this calculation for every coefficient B_j separately, we make use of the fact that many diagrams in Fig. 8 have the same form, implying that their coefficients B_j are identical. This gives the relations

$$\begin{aligned}
B_1 &= B_2; & B_3 &= B_4; \\
B_5 &= B_6 = B_7 = B_8; & B_{10} &= B_{11} = B_{12} = B_{13}; \\
B_{14} &= B_{15} = B_{16} = B_{17}; & B_{18} &= B_{19} = B_{20} = B_{21}; \\
B_{22} &= B_{23} = B_{24} = B_{25}; & B_{26} &= B_{27} = B_{28} = B_{29}; \\
B_{30} &= \dots = B_{33} = B_{34}^* = \dots = B_{37}^*; & B_{54} &= \dots = B_{57} = B_{58}^* = \dots = B_{61}^*; \\
B_{38} &= \dots = B_{45} = B_{46}^* = \dots = B_{53}^*; & B_{62} &= \dots = B_{69} = B_{70}^* = \dots = B_{77}^*; \\
B_{78} &= B_{79}^*; & B_{80} &= B_{81}^*; \\
B_{82} &= B_{83} = B_{84} = B_{85}; & B_{86} &= B_{87}^*; \\
B_{89} &= \dots = B_{92} = B_{93}^* = \dots = B_{96}^*; & B_{97} &= B_{98} = B_{99} = B_{100}; \\
B_{101} &= B_{102}; & B_{104} &= B_{105}.
\end{aligned} \tag{B9}$$

In addition, we use that $\langle \mathcal{W}_{ap;x,y} \mathcal{W}_{bq;x,y}^* \rangle = 0$ if $\Sigma_a = \mathbb{1}$ and $\Sigma_p \neq \mathbb{1}$. For the orthogonal case, this gives the constraints

$$\begin{aligned}
B_1 + q^2 B_3 + B_5 + 2 \text{Re } B_{30} + 2 \text{Re } B_{54} &= 0; \\
(1 + q^2) B_5 + B_9 + 2 \text{Re } B_{40} + 2 \text{Re } B_{70} &= 0; \\
B_{14} + (1 + q^2) B_{40}^* + B_{70} + B_{78}^* + B_{84} + B_{90} &= 0; \\
B_{18} + B_{28} + B_{40} + q^2 B_{54}^* + B_{70} + B_{90}^* + B_{101} &= 0; \\
B_{10} + B_{24} + q^2 B_{30}^* + B_{40}^* + B_{70}^* + B_{86} + B_{90} &= 0; \\
B_{40} + (1 + q^2) B_{70}^* + B_{80} + B_{84} + B_{90} + B_{100} &= 0; \\
q^2 B_{24} + 2 \text{Re } B_{30} + 2 \text{Re } B_{70} + B_{100} + B_{103} &= 0; \\
q^2 B_{28} + 2 \text{Re } B_{40} + 2 \text{Re } B_{54} + B_{100} + B_{105} &= 0.
\end{aligned} \tag{B10}$$

Similarly, from $\langle \mathcal{W}_{ap;x,y} \mathcal{W}_{bp';x,y}^* \rangle = 1$ if $a = p = b = q = 0$ we find

$$q^4 B_1 + q^2 (B_3 + B_5 + 2B_{10} + 2B_{18} + B_{24} + B_{28}) + 2(B_{14} + \text{Re } B_{30} - \text{Re } B_{40} + \text{Re } B_{54} + \text{Re } B_{70}) + B_{88} + B_{105} = 1. \tag{B11}$$

for the orthogonal case. To obtain the corresponding constraints for the symplectic case, one has to replace q^2 by $-q^2$ in Eqs. (B10) and (B11).

For qudit size $q = 2^m$, the pair-transition probability retains the form given in Eq. (29), with the coefficients $\mathcal{P}_i$ defined piecewise,

$$\begin{aligned} \mathcal{P}_1 &= \begin{cases} P_1 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = 1, \\ P'_1 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = -1, \\ P''_1 & \text{if } s_{p_x} s_{p_{x+1}} \neq s_{a_x} s_{a_{x+1}} \end{cases} \\ \mathcal{P}_2 &= \begin{cases} P_2 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = 1, \\ P'_2 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = -1, \\ P''_2 & \text{if } s_{p_x} s_{p_{x+1}} \neq s_{a_x} s_{a_{x+1}} \end{cases} \\ \mathcal{P}_3 &= \begin{cases} P_3 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = 1, \\ P'_3 & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = -1, \\ 0 & \text{if } s_{p_x} s_{p_{x+1}} \neq s_{a_x} s_{a_{x+1}} \end{cases} \end{aligned} \quad (\text{B12})$$

The parameters P_i, P'_i, P''_i can be expressed in terms of the coefficients B_j as

$$\begin{aligned} P_1 &= B_1 + B_9 + 2B_5 + q^{-2}(2B_{10} + 2B_{18} + 2B_{24} \\ &\quad + 2B_{28} + 8 \operatorname{Re} B_{40} + 8 \operatorname{Re} B_{70} + 2 \operatorname{Re} B_{78} \\ &\quad + 2 \operatorname{Re} B_{80} + 4B_{84}), \\ P_2 &= 2q^{-2}(B_{14} + \operatorname{Re} B_{86} + B_{100} + B_{101} + 4 \operatorname{Re} B_{90}), \\ P_3 &= 2B_{14} + 2B_{100} + B_{88} + B_{103} + 2B_{105}, \\ P'_1 &= B_1 + B_9 - 2B_5 + q^{-2}(2B_{10} + 2B_{18} + 2B_{24} \\ &\quad + 2B_{28} - 8 \operatorname{Re} B_{40} - 8 \operatorname{Re} B_{70} + 2 \operatorname{Re} B_{78} \\ &\quad + 2 \operatorname{Re} B_{80} + 4B_{84}), \\ P'_2 &= 2q^{-2}(B_{14} + \operatorname{Re} B_{86} + B_{100} + B_{101} - 4 \operatorname{Re} B_{90}), \\ P'_3 &= -2B_{14} - 2B_{100} + B_{88} + B_{103} + 2B_{105}, \\ P''_1 &= B_1 - B_9 + q^{-2}(2B_{10} + 2B_{18} + 2B_{24} + 2B_{28} \\ &\quad - 2 \operatorname{Re} B_{78} - 2 \operatorname{Re} B_{80} - 4B_{84}), \\ P''_2 &= 2q^{-2}(-B_{14} + \operatorname{Re} B_{86} - B_{100} + B_{101}). \end{aligned} \quad (\text{B13})$$

For these coefficients, the normalization condition $\sum_{p_x, p_y} |\mathcal{W}_{apx, y}|^2 = 1$ implies the constraints

$$\begin{aligned} \frac{q^4 + q^2 - 2}{2} P_1 + \frac{q^2 - 2}{2} P_2 + P_3 \\ + \frac{q^4 - q^2}{2} P'_1 - \frac{q^2}{2} P'_2 = 1, \\ \frac{q^4 - q^2}{2} P'_1 + \frac{q^2}{2} P'_2 + P'_3 \\ + \frac{q^4 + q^2 - 2}{2} P''_1 - \frac{q^2 + 2}{2} P''_2 = 1. \end{aligned} \quad (\text{B14})$$

Equations (B13) and (B14) are for the orthogonal case. For the symplectic case, one again has to replace $q^2 \rightarrow -q^2$.

Although these relations significantly reduce the number of coefficients B_j that have to be calculated separately, the number of remaining coefficients is still very large. A further simplification can be obtained if the two-qudit gate matrices $\mathcal{U}$ satisfy a further symmetry. Below, we consider the cases that $\mathcal{U}$ is orthogonal, symplectic, symmetric, and self-dual in detail.

3. Orthogonal and symplectic ensembles

When the two-qudit gate operators $\mathcal{U}$ satisfy $\mathcal{U}^\dagger = \mathcal{U}^T$ (orthogonal case) or $\mathcal{U}^\dagger = \mathcal{U}^R$ (symplectic case), i.e., $\mathcal{U}$ is an orthogonal or symplectic $q^2 \times q^2$ matrix, respectively, the symmetry constraints on the coefficients B_j simplify significantly. In both cases, we find that additional equalities hold for the coefficients B_j ,

$$\begin{aligned} B_1 &= B_2 = B_9, \\ B_3 &= B_4 = B_5 = B_6 = B_7 = B_8, \\ B_{10} &= B_{11} = B_{12} = B_{13} = B_{78} = B_{79}, \\ B_{14} &= B_{15} = B_{16} = B_{17} = B_{86} = B_{87}, \\ B_{18} &= B_{19} = B_{20} = B_{21} = B_{80} = B_{81}, \\ B_{22} &= \dots = B_{29} = B_{82} = \dots = B_{85}, \\ B_{30} &= \dots = B_{53}, \\ B_{54} &= \dots = B_{77}, \\ B_{89} &= \dots = B_{96}, \\ B_{97} &= \dots = B_{102}, \\ B_{103} &= B_{104} = B_{105}. \end{aligned} \quad (\text{B15})$$

These relations reduce the constraints of Eqs. (B10) to

$$\begin{aligned} 2B_{30} + 2B_{54} + B_1 + (1 + q^2)B_3 &= 0, \\ B_{24} + B_{10} + B_{14} + (1 + q^2)B_{30} + B_{54} + B_{90} &= 0, \\ B_{24} + B_{18} + B_{30} + (1 + q^2)B_{54} + B_{90} + B_{101} &= 0, \\ q^2 B_{24} + 2B_{30} + 2B_{54} + B_{101} + B_{103} &= 0. \end{aligned} \quad (\text{B16})$$

The normality condition (B11) simplifies to

$$\begin{aligned} q^4 B_1 + 2q^2(B_3 + B_{10} + B_{18} + B_{24}) \\ + 2B_{14} + 4(B_{30} + B_{54}) + B_{88} + B_{103} &= 1. \end{aligned} \quad (\text{B17})$$

Consequently, the parameters P_i, P'_i , and P''_i defined in Eq. (B13) become

$$\begin{aligned} P_1 &= 2B_1 + 2B_3 + q^{-2}(4B_{10} + 4B_{18} + 8B_{24} + 8B_{30} + 8B_{54}), \\ P_2 &= q^{-2}(4B_{14} + 4B_{101} + 8B_{90}), \\ P_3 &= 2B_{14} + 2B_{101} + B_{88} + 3B_{103}, \\ P'_1 &= 2B_1 - 2B_3 + q^{-2}(4B_{10} + 4B_{18} + 8B_{24} - 8B_{30} - 8B_{54}), \\ P'_2 &= q^{-2}(4B_{14} + 4B_{101} - 8B_{90}), \\ P'_3 &= B_{88} + 3B_{103} - 2B_{14} - 2B_{101}, \\ P''_1 &= 0, \\ P''_2 &= 0, \end{aligned} \quad (\text{B18})$$

and instead of the normalization constraints of Eq. (B14) one has

$$\begin{aligned} -\frac{q^2 + 2}{2} P_1 + \frac{q^2}{2} P'_1 - P_2 &= 0, \\ \frac{q^4 + q^2 - 2}{2} P_1 + \frac{q^2 - 2}{2} P_2 + P_3 &= 1, \\ \frac{q^4 - q^2}{2} P'_1 + \frac{q^2}{2} P'_2 + P'_3 &= 1. \end{aligned} \quad (\text{B19})$$

Equations (B16)–(B19) are for the orthogonal case; for the symplectic case, one has to replace q^2 by $-q^2$.

This system leaves three free parameters: P_3 , P'_3 , and P'_2 . The remaining coefficients B_j can be expressed in terms of the moments $\mathcal{R}$ of the two-qudit gate operator by straightforward evaluation of the corresponding diagrams,

$$\begin{aligned}
B_{14} &= q^2[(q^2 + 2)V_{2,2} + 4V_{1,3} + (12 + 2q^2)V_4] + q^4\mathcal{R}_{1;1}[(4 + q^2)V_{1,1,2} + q^2V_{2,2} + (8 + 4q^2)V_{1,3} + 12V_4] \\
&\quad + q^2\mathcal{R}_2[q^2V_{1,1,2} + (4 + q^2)V_{2,2} + 8V_{1,3} + (12 + 4q^2)V_4] + q^2\mathcal{R}_4(2V_{2,2} + 4V_{1,3}) \\
&\quad + q^6\mathcal{R}_{1,1,2}(V_{1,1,1,1} + V_{2,2} + 4V_{1,3}) + q^4\mathcal{R}_{1,3}(4V_{1,1,2} + 4V_4) + q^4\mathcal{R}_{2,2}(V_{1,1,2} + 2V_4) + q^8\mathcal{R}_{1,1,1,1}V_{1,1,2}, \\
B_{88} &= q^2[3q^2V_{2,2} + 18V_4] + q^4\mathcal{R}_{1;1}(6q^2V_{1,1,2} + 24V_{1,3}) + q^2\mathcal{R}_2(6q^2V_{2,2} + 24V_4) + 6q^2\mathcal{R}_4V_4 \\
&\quad + 6q^6\mathcal{R}_{1,1,2}V_{1,1,2} + 8q^4\mathcal{R}_{1,3}V_{1,3} + 3q^4\mathcal{R}_{2,2}V_{2,2} + q^8\mathcal{R}_{1,1,1,1}V_{1,1,1,1}, \\
B_{90} &= q^2[3V_{2,2} + 6V_{1,3} + (9 + 3q^2)V_4] + q^4\mathcal{R}_{1;1}[3V_{1,1,2} + 3V_{2,2} + (6 + 3q^2)V_{1,3} + (12 + 3q^2)V_4] \\
&\quad + q^2\mathcal{R}_2[3V_{1,1,2} + 3V_{2,2} + (6 + 3q^2)V_{1,3} + (12 + 3q^2)V_4] + q^2\mathcal{R}_4(3V_{1,1,2} + 3V_4) \\
&\quad + q^6\mathcal{R}_{1,1,2}(3V_{1,1,2} + 3V_4) + q^4\mathcal{R}_{1,3}(V_{1,1,1,1} + 3V_{2,2} + 4V_{1,3}) + 3q^4\mathcal{R}_{2,2}V_{1,3} + q^8\mathcal{R}_{1,1,1,1}V_{1,3}, \\
B_{101} &= q^2[2V_{1,1,2} + (q^2 + 4)V_{2,2} + 4V_{1,3} + (8 + 2q^2)V_4] + q^4\mathcal{R}_{1;1}[4V_{2,2} + 8V_{1,3} + (12 + 6q^2)V_4] \\
&\quad + q^2\mathcal{R}_2[4V_{1,1,2} + (8 + 4q^2)V_{1,3} + (12 + 2q^2)V_4] + q^2\mathcal{R}_4(V_{1,1,1,1} + V_{2,2} + 4V_{1,3}) \\
&\quad + q^6\mathcal{R}_{1,1,2}(2V_{2,2} + 4V_{1,3}) + q^4\mathcal{R}_{1,3}(4V_{1,1,2} + 4V_4) + q^4\mathcal{R}_{2,2}(2V_{1,1,2} + V_4) + q^8\mathcal{R}_{1,1,1,1}V_4, \\
B_{103} &= q^2[4V_{1,1,2} + (3q^2 + 4)V_{2,2} + 10V_4] + q^4\mathcal{R}_{1;1}[2q^2V_{2,2} + 8V_{1,3} + (16 + 4q^2)V_4] \\
&\quad + q^2\mathcal{R}_2[2q^2V_{1,1,2} + 16V_{1,3} + (8 + 4q^2)V_4] + q^2\mathcal{R}_4(4V_{1,1,2} + 2V_4) + q^6\mathcal{R}_{1,1,2}(2V_{1,1,2} + 4V_4) \\
&\quad + 8q^4\mathcal{R}_{1,3}V_{1,3} + q^4\mathcal{R}_{2,2}(V_{1,1,1,1} + 2V_{2,2}) + q^8\mathcal{R}_{1,1,1,1}V_{2,2}, \tag{B20}
\end{aligned}$$

where, again, for the symplectic case, one has to replace q^2 by $-q^2$. The moments $\mathcal{R}$ of the two-qudit gate operators are defined as

$$\begin{aligned}
\mathcal{R}_{1;1} &= q^{-4}\langle(\text{tr}\mathcal{U}_{x,x+1})^4\rangle, \\
\mathcal{R}_2 &= q^{-2}\langle\text{tr}\mathcal{U}_{x,x+1}^2\rangle, \\
\mathcal{R}_4 &= q^{-2}\langle\text{tr}\mathcal{U}_{x,x+1}^4\rangle, \\
\mathcal{R}_{1,3} &= q^{-4}\langle\text{tr}\mathcal{U}_{x,x+1}\text{tr}\mathcal{U}_{x,x+1}^3\rangle, \\
\mathcal{R}_{2,2} &= q^{-4}\langle(\text{tr}\mathcal{U}_{x,x+1}^2)^2\rangle, \\
\mathcal{R}_{1,1,1,1} &= q^{-8}\langle(\text{tr}\mathcal{U}_{x,x+1})^4\rangle, \\
\mathcal{R}_{1,1,2} &= q^{-6}\langle(\text{tr}\mathcal{U}_{x,x+1})^2\text{tr}\mathcal{U}_{x,x+1}^2\rangle. \tag{B21}
\end{aligned}$$

The definitions (B21) apply both to the orthogonal and the symplectic case.

In the large- q limit, the expressions for the average $\langle\mathcal{W}_{ap,x,y}\mathcal{W}_{bq,x,y}^*\rangle$ can be further simplified. We find that $B_{88} \sim \mathcal{O}(1)$, $B_{14} \sim \mathcal{O}(q^{-2})$, $B_{90} \sim B_{103} \sim \mathcal{O}(q^{-4})$, and $B_{101} \sim \mathcal{O}(q^{-6})$. As a consequence, of the eight coefficients P_j and P'_j , one has $P_3 = P'_3 = B_{88} + \mathcal{O}(q^{-2})$, whereas $P_j \sim \mathcal{O}(q^{-4})$ for $j = 1, 2$, and $(P_j - P'_j)/P_j \sim \mathcal{O}(q^{-2})$ for all $j = 1, 2, 3$. This results in a simple closed form for the zeroth-order truncated evolution equations in the limit of large qudit size q , which contains the coefficient P_3 only,

$$\begin{aligned}
\bar{\rho}^{(0)}(\Delta x; t) &= P_3 \bar{\rho}^{(0)}(\Delta x + 1; t - 1) \\
\bar{\rho}^{(0)}(\Delta x + 1; t) &= \bar{\rho}^{(0)}(\Delta x + 2; t - 1) + (1 - P_3) \bar{\rho}^{(0)}(\Delta x + 1; t - 1). \tag{B22}
\end{aligned}$$

[Corrections to these equations appear to order $\mathcal{O}(q^{-2})$.] The butterfly velocity and diffusion constant are easily found from Eqs. (B22),

$$\lim_{q \rightarrow \infty} v_B = \frac{1 - P_3}{1 + P_3}, \quad \lim_{q \rightarrow \infty} \mathcal{D} = \frac{4P_3(1 - P_3)}{(1 + P_3)^2}. \tag{B23}$$

The fact that for the limit of large q , Eqs. (B22) are closed without having to make further approximations implies that the truncation procedure of Sec. V D is exact in the limit $q \rightarrow \infty$. In the large- q limit, we obtain that

$$\lim_{q \rightarrow \infty} P_3 = \lim_{q \rightarrow \infty} \mathcal{R}_{1,1,1,1}. \tag{B24}$$

4. Symmetric and self-dual ensembles

The symmetry constraints on the coefficients B_j also simplify significantly if the two-qudit gate operators $\mathcal{U}$ are symmetric or self-dual matrices, $\mathcal{U} = \mathcal{U}^T$ or $\mathcal{U} = \mathcal{U}^R$ for the orthogonal or symplectic case, respectively. The circular orthogonal and symplectic ensembles (COE and CSE, respectively) from random matrix theory are examples of ensembles with such symmetry [39]. For symmetric or self-dual $\mathcal{U}$, the relations between the coefficients B_j are

$$\begin{aligned} B_1 &= B_2 = B_{18} = \cdots = B_{21} = B_{104} = B_{105}, \\ B_3 &= B_4 = B_{26} = \cdots = B_{29} = B_{54} = \cdots = B_{61} = B_{101} = B_{102}, \\ B_5 &= B_6 = \cdots = B_8 = B_{22} = \cdots = B_{25} = B_{62} = \cdots = B_{77} = B_{82} = \cdots = B_{85} = B_{97} = \cdots = B_{100}, \\ B_9 &= B_{80} = B_{81} = B_{103}, \\ B_{10} &= B_{11} = \cdots = B_{17}, \\ B_{78} &= B_{79}^* = B_{86}^* = B_{87}, \\ B_{30} &= B_{31} = \cdots = B_{33} = B_{34}^* = \cdots = B_{37}^* = B_{38} = \cdots = B_{45} = B_{46}^* = \cdots = B_{53}^* = B_{89}^* = \cdots = B_{92}^* = B_{93} = \cdots = B_{96}. \end{aligned} \quad (\text{B25})$$

These relations simplify the expressions in (B10) to a set of three linear constraints,

$$\begin{aligned} 2 \operatorname{Re} B_{30} + B_5 + B_1 + (q^2 + 2)B_3 &= 0, \\ (q^2 + 3)B_5 + B_9 + 2 \operatorname{Re} B_{30} &= 0, \\ B_{10} + (q^2 + 2)B_{30}^* + 2B_5 + B_{86} &= 0. \end{aligned} \quad (\text{B26})$$

Similarly, the normality condition (B11) simplifies to

$$(q^2 + 1)^2 B_1 + 2(q^2 + 1)(B_3 + B_5 + B_{10}) + 4 \operatorname{Re} B_{30} + B_{88} = 1. \quad (\text{B27})$$

Equations (B26) and (B27) are for the orthogonal case; for the symplectic case, one has to replace q^2 by $-q^2$. Because of the constraints of Eqs. (B25)–(B27), Eq. (B13) becomes

$$\begin{aligned} P_1 &= \frac{B_9(q^4 + 4q^2 - 2) - 2 \operatorname{Re} B_{30}(2q^4 + 7q^2 + 8)}{q^2(q^2 + 3)} - \frac{B_3(q^6 + 7q^4 + 14q^2 + 6)}{q^2(q^2 + 3)}, \\ P_2 &= \frac{2[B_9 + B_3(q^2 + 3) - \operatorname{Re} B_{30}(q^4 + q^2 - 8)]}{q^2(q^2 + 3)}, \\ P_3 &= B_3(q^6 + 4q^4 + q^2 - 4) + B_9 \left(4 - q^2 - \frac{8}{q^2 + 3} \right) + 2 \operatorname{Re} B_{30} \left(q^4 + q^2 - \frac{16}{q^2 + 3} \right) - 2q^2 B_{10} + 1, \\ P'_1 &= \frac{B_9(q^4 + 8q^2 + 14) - 2 \operatorname{Re} B_{30}(2q^4 + 11q^2 + 16)}{q^2(q^2 + 3)} - \frac{B_3(q^6 + 7q^4 + 14q^2 + 6)}{q^2(q^2 + 3)}, \\ P'_2 &= \frac{2[B_9 + B_3(q^2 + 3) - \operatorname{Re} B_{30}(q^4 + 9q^2 + 16)]}{q^2(q^2 + 3)}, \\ P'_3 &= 1 + B_3(q^6 + 4q^4 + q^2 - 4) + B_9 \left(4 - q^2 - \frac{4}{q^2 + 3} \right) + 2 \operatorname{Re} B_{30} \left(q^4 + 2q^2 - \frac{8}{q^2 + 3} \right) - 2B_{10}(q^2 + 2), \\ P''_1 &= - \frac{B_9 q^4 + (4B_9 - 2 \operatorname{Re} B_{30})q^2 + 6B_9 - 4B_{10}(q^2 + 3)}{q^2(q^2 + 3)} - \frac{B_3(q^6 + 7q^4 + 14q^2 + 6)}{q^2(q^2 + 3)}, \\ P''_2 &= \frac{2B_3}{q^2} - \frac{2[\operatorname{Re} B_{30}(q^4 + 5q^2) - 3B_9 + 2B_{10}(q^2 + 3)]}{q^2(q^2 + 3)}. \end{aligned} \quad (\text{B28})$$

The remaining coefficients B_j can be expressed in terms of the moments $\mathcal{R}$ of the two-qudit gate operator,

$$\begin{aligned} B_3 &= q^2[V_{1,1,1,1} + (2q^2 + 6)V_{1,1,2} + 5V_{2,2} + (4q^2 + 16)V_{1,3} + (2q^2 + 20)V_4] + q^4 \mathcal{R}_{2,2}(V_{2,2} + 3V_4) \\ &\quad + q^4 \mathcal{R}_{1,1}[4V_{1,1,2} + 4V_{2,2} + (4q^2 + 8)V_{1,3} + (4q^2 + 16)V_4] + q^6 \operatorname{Re} \mathcal{R}_{1,1,2}(2V_{2,2} + 2V_4) + q^8 \mathcal{R}_{1,1,1,1}V_4, \\ B_9 &= q^2[8V_{1,1,2} + 4(q^2 + 1)V_{2,2} + 16V_{1,3} + 4(q^2 + 5)V_4] + q^4 \mathcal{R}_{1,1}[16V_{1,3} + (16 + 8q^2)V_4] \\ &\quad + q^4 \mathcal{R}_{2,2}(V_{1,1,1,1} + 2V_{1,1,2} + V_{2,2}) + q^6 \operatorname{Re} \mathcal{R}_{1,1,2}(2V_{1,1,2} + 2V_{2,2}) + q^8 \mathcal{R}_{1,1,1,1}V_{2,2}, \end{aligned}$$

$$\begin{aligned}
B_{10} &= q^2[2q^2V_{1,1,2} + (2q^2 + 8)V_{2,2} + 16V_{1,3} + (4q^2 + 24)V_4] + 4q^4\mathcal{R}_{2,2}V_4 + 4q^6\text{Re}\mathcal{R}_{1,1;2}V_{1,3} + q^8\mathcal{R}_{1,1;1,1}V_{1,1,2} \\
&\quad + q^4\mathcal{R}_{1,1}[q^2V_{1,1,1,1} + (q^2 + 8)V_{1,1,2} + 2q^2V_{2,2} + (4q^2 + 8)V_{1,3} + 16V_4], \\
\text{Re}B_{30} &= q^2[6V_{1,1,2} + 6V_{2,2} + (4q^2 + 12)V_{1,3} + (4q^2 + 24)V_4] + q^4\mathcal{R}_{2,2}(2V_{1,3} + 2V_4) + q^6\text{Re}\mathcal{R}_{1,1;2}(V_{1,1,2} + V_{1,3} + 2V_4) \\
&\quad + q^4\mathcal{R}_{1,1}[V_{1,1,1,1} + (2q^2 + 3)V_{1,1,2} + 6V_{2,2} + (2q^2 + 10)V_{1,3} + (4q^2 + 12)V_4] + q^8\mathcal{R}_{1,1;1,1}V_{1,3}. \quad (\text{B29})
\end{aligned}$$

The relevant moments $\mathcal{R}$ required to calculate these coefficients are defined as

$$\begin{aligned}
\mathcal{R}_{1,1} &= q^{-4}\langle \text{tr } \mathcal{U}_{x,x+1} \text{ tr } \mathcal{U}_{x,x+1}^\dagger \rangle, \\
\mathcal{R}_{2,2} &= q^{-4}\langle \text{tr } \mathcal{U}_{x,x+1}^2 \text{ tr } \mathcal{U}_{x,x+1}^{\dagger 2} \rangle, \\
\mathcal{R}_{1,1;1,1} &= q^{-8}\langle (\text{tr } \mathcal{U}_{x,x+1})^2 (\text{tr } \mathcal{U}_{x,x+1}^\dagger)^2 \rangle, \\
\mathcal{R}_{1,1;2} &= q^{-6}\langle (\text{tr } \mathcal{U}_{x,x+1})^2 \text{ tr } \mathcal{U}_{x,x+1}^{\dagger 2} \rangle. \quad (\text{B30})
\end{aligned}$$

Following an argument analogous to that in Appendix B3, we find that, in the limit $q \rightarrow \infty$, the butterfly velocity and diffusion constant are again given by Eq. (B23), with P_3 evaluated for the symmetric or self-dual ensemble. In the large- q limit, we obtain the same form of P_3 as shown in Eq. (B24).

5. Combined symmetries

Finally, we consider the most constrained ensembles, where the matrix $\mathcal{U}$ possesses both symmetries. Specifically, we examine the Symmetric Orthogonal case ($\mathcal{U} = \mathcal{U}^* = \mathcal{U}^T$) and the Self-dual Symplectic case ($\mathcal{U} = \mathcal{U}^R = \mathcal{U}^\dagger$).

The combined symmetries impose constraints on the coefficients B_j ,

$$\begin{aligned}
B_1 &= B_2 = B_9 = B_{18} = \cdots = B_{21} = B_{80} = B_{81} \\
&= B_{103} = B_{104} = B_{105}; \\
B_3 &= B_4 = \cdots = B_8 = B_{22} = \cdots = B_{29} \\
&= B_{54} = \cdots = B_{77} = B_{82} \cdots = B_{85} \\
&= B_{97} = \cdots = B_{102}; \\
B_{10} &= B_{11} = \cdots = B_{17} = B_{78} = B_{79} = B_{86} = B_{87}; \\
B_{30} &= B_{31} = \cdots = B_{53} = B_{89} = \cdots = B_{96}. \quad (\text{B31})
\end{aligned}$$

The constraints of Eqs. (B10) and (B11) simplify to

$$\begin{aligned}
2B_{30} + B_1 + (3 + q^2)B_3 &= 0, \\
2B_{10} + (2 + q^2)B_{30} + 2B_3 &= 0 \quad (\text{B32})
\end{aligned}$$

and

$$\begin{aligned}
(1 + q^2)^2 B_1 + 4(1 + q^2)B_3 \\
+ 2(1 + q^2)B_{10} + 4B_{30} + B_{88} &= 1. \quad (\text{B33})
\end{aligned}$$

These relations are for the orthogonal case; for the symplectic case, one has to replace $q^2 \rightarrow -q^2$. Because of the constraints of Eqs. (B31)–(B33), the parameters defined in Eq. (B13) become

$$\begin{aligned}
P_1 &= B_{30} \left(-6 - \frac{4}{q^2} \right) - 2B_3(q^2 + 4), \\
P_2 &= B_{30} \left(-2 + \frac{4}{q^2} \right),
\end{aligned}$$

$$\begin{aligned}
P_3 &= B_3(q^6 + 5q^4 + 2q^2 - 8) + B_{30}(3q^4 + 6q^2 - 8) + 1, \\
P'_1 &= -\frac{2[B_{30}(3q^2 + 10) + B_3(q^4 + 6q^2 + 8)]}{q^2}, \\
P'_2 &= -\frac{2B_{30}(q^2 + 6)}{q^2}, \\
P'_3 &= B_3(q^6 + 5q^4 + 2q^2 - 8) + B_{30}(3q^4 + 8q^2 - 4) + 1, \\
P''_1 &= 0, \\
P''_2 &= 0.
\end{aligned}$$

Because of the high degree of symmetry, only two independent moments are required to determine the coefficients B_j ,

$$\begin{aligned}
B_3 &= q^2[V_{1,1,1,1} + (2q^2 + 6)V_{1,1,2} + (4q^2 + 16)V_{1,3} \\
&\quad + 6V_{2,2} + (5q^2 + 20)V_4] + q^4\mathcal{R}_{1,1}[4V_{1,1,2} \\
&\quad + (2q^2 + 4)V_{2,2} + (4q^2 + 8)V_{1,3} + (6q^2 + 16)V_4] \\
&\quad + q^8\mathcal{R}_{1,1;1,1}V_4, \\
B_{30} &= q^2[6V_{1,1,2} + 6V_{2,2} + (6q^2 + 12)V_{1,3} \\
&\quad + (6q^2 + 24)V_4] + q^4\mathcal{R}_{1,1}[V_{1,1,1,1} + 6V_{2,2} \\
&\quad + (3q^2 + 3)V_{1,1,2} + (3q^2 + 10)V_{1,3} \\
&\quad + (6q^2 + 12)V_4] + q^8\mathcal{R}_{1,1;1,1}V_{1,3}, \quad (\text{B34})
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{R}_{1,1} &= q^{-4}\langle \text{tr } \mathcal{U}_{x,x+1} \text{ tr } \mathcal{U}_{x,x+1} \rangle, \\
\mathcal{R}_{1,1;1,1} &= q^{-8}\langle (\text{tr } \mathcal{U}_{x,x+1})^2 (\text{tr } \mathcal{U}_{x,x+1})^2 \rangle. \quad (\text{B35})
\end{aligned}$$

An analogous large- q argument shows that the butterfly velocity and diffusion constant retain the form given in Eq. (B23), with P_3 evaluated for the ensembles considered in this subsection. As a concrete example, Ref. [54] considers the choice $V = O \text{diag}(-1, 1, \dots, 1) O^T$, where O is Haar distributed over the orthogonal group. In the large- q limit, we obtain the same form of P_3 as shown in Eq. (B24).

APPENDIX C: PROJECTED TRANSITION MATRIX

1. The case $q = 2^m$

For $q = 2^m$, to show that the projected binary-string distribution $\bar{\rho}_{\bar{p}}$ satisfies a closed evolution equation, it is sufficient to consider the adjacent qudits x and $x + 1$, which are subject to a two-qudit gate. From Eqs. (27) and (34), we then have

$$\begin{aligned}
\bar{\rho}_{\bar{p}_x, \bar{p}_{x+1}}(t) &= \sum_{a_x, a_{x+1}} \rho_{a_x, a_{x+1}; a_x, a_{x+1}}(t-1) \\
&\quad \times \sum_{p_x \rightarrow \bar{p}_x} \sum_{p_{x+1} \rightarrow \bar{p}_{x+1}} \langle |W_{a_x, a_{x+1}; p_x, p_{x+1}}|^2 \rangle. \quad (\text{C1})
\end{aligned}$$

In this expression, depending on $\bar{p}_x$ and $\bar{p}_{x+1}$, the summations over p_x and p_{x+1} are either restricted to $p_x = 0$ or $\bar{p}_{x+1} = 0$ or extend over all $p_x \neq 0$ or $p_{x+1} \neq 0$. We may then use the general form (29) of $\langle |W_{a_x, a_{x+1}; p_x, p_{x+1}}|^2 \rangle$, Eq. (A9), and the sum rules (A12) to see that $\sum_{p_x \rightarrow \bar{p}_x} \sum_{p_{x+1} \rightarrow \bar{p}_{x+1}} \langle |W_{a_x, a_{x+1}; p_x, p_{x+1}}|^2 \rangle$ depends only on whether or not a_x and a_{x+1} are equal to 0, i.e., they only depend on the projected binary indices $\bar{a}_x$ and $\bar{a}_{x+1}$ associated with a_x and a_{x+1} . This means that the right-hand side of Eq. (C1) can be expressed in terms of the projected binary string distribution $\bar{p}_{\bar{a}_x, \bar{a}_{x+1}}(t-1)$, which is what we set out to prove. Therefore, we can obtain a closed

Markovian evolution equation of the form of Eq. (35). Since the transition matrix $T_{\bar{a}\bar{p}}$ is constructed from a sum of the nonnegative pair probabilities $\langle |W_{a_x, a_{x+1}; p_x, p_{x+1}}|^2 \rangle$, its elements are also nonnegative. Similarly, for both the orthogonal- and symplectic-invariant cases, one can employ the sum rules in Eqs. (A13) and (A20) to obtain the projected pair transition probability matrix $T_{\bar{a}\bar{p}}$, which exhibits the block structure shown in Eq. (59).

For the orthogonal and symplectic invariant circuit, the blocks of the transition matrix have the structure

$$\begin{aligned} T_{++} &= \begin{pmatrix} T_{ee;ee} & T_{ee;eo} & T_{ee;e\mathbb{I}} & T_{ee;\mathbb{I}e} \\ T_{oo;ee} & T_{oo;eo} & T_{oo;e\mathbb{I}} & T_{oo;\mathbb{I}e} \\ T_{e\mathbb{I};ee} & T_{e\mathbb{I};eo} & T_{e\mathbb{I};e\mathbb{I}} & T_{e\mathbb{I};\mathbb{I}e} \\ T_{\mathbb{I}e;ee} & T_{\mathbb{I}e;eo} & T_{\mathbb{I}e;e\mathbb{I}} & T_{\mathbb{I}e;\mathbb{I}e} \end{pmatrix}, & T_{+-} &= \begin{pmatrix} T_{ee;eo} & T_{ee;oe} & T_{ee;o\mathbb{I}} & T_{ee;\mathbb{I}o} \\ T_{oo;eo} & T_{oo;oe} & T_{oo;o\mathbb{I}} & T_{oo;\mathbb{I}o} \\ T_{e\mathbb{I};eo} & T_{e\mathbb{I};oe} & T_{e\mathbb{I};o\mathbb{I}} & T_{e\mathbb{I};\mathbb{I}o} \\ T_{\mathbb{I}e;eo} & T_{\mathbb{I}e;oe} & T_{\mathbb{I}e;o\mathbb{I}} & T_{\mathbb{I}e;\mathbb{I}o} \end{pmatrix}, \\ T_{-+} &= \begin{pmatrix} T_{eo;ee} & T_{eo;eo} & T_{eo;e\mathbb{I}} & T_{eo;\mathbb{I}e} \\ T_{oe;ee} & T_{oe;eo} & T_{oe;e\mathbb{I}} & T_{oe;\mathbb{I}e} \\ T_{o\mathbb{I};ee} & T_{o\mathbb{I};eo} & T_{o\mathbb{I};e\mathbb{I}} & T_{o\mathbb{I};\mathbb{I}e} \\ T_{\mathbb{I}o;ee} & T_{\mathbb{I}o;eo} & T_{\mathbb{I}o;e\mathbb{I}} & T_{\mathbb{I}o;\mathbb{I}e} \end{pmatrix}, & T_{--} &= \begin{pmatrix} T_{eo;eo} & T_{eo;oe} & T_{eo;o\mathbb{I}} & T_{eo;\mathbb{I}o} \\ T_{oe;eo} & T_{oe;oe} & T_{oe;o\mathbb{I}} & T_{oe;\mathbb{I}o} \\ T_{o\mathbb{I};eo} & T_{o\mathbb{I};oe} & T_{o\mathbb{I};o\mathbb{I}} & T_{o\mathbb{I};\mathbb{I}o} \\ T_{\mathbb{I}o;eo} & T_{\mathbb{I}o;oe} & T_{\mathbb{I}o;o\mathbb{I}} & T_{\mathbb{I}o;\mathbb{I}o} \end{pmatrix}. \end{aligned} \quad (\text{C2})$$

Here, the indices $\mathbb{I}$, e , and o refer to the ternary indices $\bar{p} = 0, 1$, and -1 , corresponding to the identity, symmetric, and antisymmetric nonidentity channels, respectively. The nonvanishing elements of the projected transition matrix $T_{\bar{a}\bar{p}}$ are

$$\begin{aligned} T_{ee;ee} &= P_1 \left(\frac{q^2 + q - 2}{2} \right) \frac{q^2 + \sigma q - 2}{2} + P_2 \left(\frac{q - 2}{2} \right) \frac{\sigma q - 2}{2} + P_3, \\ T_{oo;ee} &= \frac{(q - 2)(\sigma q - 2)}{\sigma q^2} T_{ee;oo} = P_1 \left(\frac{q^2 + q - 2}{2} \right) \frac{q^2 + \sigma q - 2}{2} + P_2 \left(\frac{q + 2}{2} \right) \frac{\sigma q + 2}{2}, \\ T_{e\mathbb{I};ee} &= \frac{q^2 + \sigma q - 2}{2} T_{ee;e\mathbb{I}} = P_1 \left(\frac{q^2 + q - 2}{2} \right) \frac{q^2 + \sigma q - 2}{2} + P_2 \left(\frac{q - 2}{2} \right) \left(\frac{q^2 + \sigma q - 2}{2} \right), \\ T_{\mathbb{I}e;ee} &= \frac{q^2 + q - 2}{2} T_{ee;\mathbb{I}e} = P_1 \left(\frac{q^2 + q - 2}{2} \right) \frac{q^2 + \sigma q - 2}{2} + P_2 \left(\frac{\sigma q - 2}{2} \right) \left(\frac{q^2 + q - 2}{2} \right), \\ T_{eo;eo} &= P'_1 \left(\frac{q^2 - \sigma q}{2} \right) \left(\frac{q^2 + q - 2}{2} \right) + P'_2 \left(\frac{q - 2}{2} \right) \left(\frac{\sigma q}{2} \right) + P'_3, \\ T_{oe;oe} &= P'_1 \left(\frac{q^2 - q}{2} \right) \left(\frac{q^2 + \sigma q - 2}{2} \right) + P'_2 \left(\frac{\sigma q - 2}{2} \right) \left(\frac{q}{2} \right) + P'_3, \\ T_{eo;oe} &= \frac{q + 2\sigma}{q + 2} T_{oe;eo} = P'_1 \left(\frac{q^2 - q}{2} \right) \left(\frac{q^2 + \sigma q - 2}{2} \right) + P'_2 \left(\frac{\sigma q + 2}{2} \right) \left(\frac{q}{2} \right), \\ T_{o\mathbb{I};eo} &= \frac{(q + 2)(q - \sigma)}{2} T_{eo;o\mathbb{I}} = P'_1 \left(\frac{q^2 - \sigma q}{2} \right) \left(\frac{q^2 + q - 2}{2} \right) - P'_2 \left(\frac{q + 2}{2} \right) \left(\frac{q^2 - \sigma q}{2} \right), \\ T_{\mathbb{I}o;oe} &= \frac{(q + 2\sigma)(q - 1)}{2} T_{oe;\mathbb{I}o} = P'_1 \left(\frac{q^2 - q}{2} \right) \left(\frac{q^2 + \sigma q - 2}{2} \right) - P'_2 \left(\frac{\sigma q + 2}{2} \right) \left(\frac{q^2 - q}{2} \right), \\ T_{\mathbb{I}o;eo} &= \frac{q^2 + q - 2}{2} T_{eo;\mathbb{I}o} = P'_1 \left(\frac{q^2 - \sigma q}{2} \right) \left(\frac{q^2 + q - 2}{2} \right) + P'_2 \left(\frac{q^2 + q - 2}{2} \right) \left(\frac{\sigma q}{2} \right), \\ T_{o\mathbb{I};oe} &= \frac{q^2 + \sigma q - 2}{2} T_{oe;o\mathbb{I}} = P'_1 \left(\frac{q^2 - q}{2} \right) \left(\frac{q^2 + \sigma q - 2}{2} \right) + P'_2 \left(\frac{q^2 + \sigma q - 2}{2} \right) \left(\frac{q}{2} \right), \\ T_{oo;oo} &= P_1 \left(\frac{q^2 - q}{2} \right) \frac{q^2 - \sigma q}{2} + P_2 \left(\frac{q}{2} \right) \frac{\sigma q}{2} + P_3, \\ T_{e\mathbb{I};oo} &= \frac{q^2(q - \sigma)}{2(q + 2)} T_{oo;e\mathbb{I}} = P_1 \left(\frac{q^2 - q}{2} \right) \frac{q^2 - \sigma q}{2} - P_2 \left(\frac{q}{2} \right) \left(\frac{q^2 - \sigma q}{2} \right), \end{aligned}$$

$$\begin{aligned}
T_{\mathbb{I}e;oo} &= \frac{q^2(q-1)}{2(q+2\sigma)} T_{oo;\mathbb{I}e} = P_1 \left(\frac{q^2-q}{2} \right) \frac{q^2-\sigma q}{2} - P_2 \left(\frac{\sigma q}{2} \right) \left(\frac{q^2-q}{2} \right), \\
T_{e\mathbb{I};e\mathbb{I}} &= P_1 \left(\frac{q^2+q-2}{2} \right) + P_2 \left(\frac{q-2}{2} \right) + P_3, \\
T_{\mathbb{I}e;\mathbb{I}e} &= P_1 \left(\frac{q^2+\sigma q-2}{2} \right) + P_2 \left(\frac{\sigma q-2}{2} \right) + P_3, \\
T_{e\mathbb{I};\mathbb{I}e} &= \frac{q^2+\sigma q-2}{q^2+q-2} T_{\mathbb{I}e;e\mathbb{I}} = P_1 \left(\frac{q^2+\sigma q-2}{2} \right) + P_2 \left(\frac{q^2+\sigma q-2}{2} \right), \\
T_{\mathbb{I}o;o\mathbb{I}} &= \frac{q+\sigma}{q+1} T_{o\mathbb{I};\mathbb{I}o} = P'_1 \left(\frac{q^2-q}{2} \right) + P'_2 \left(\frac{q^2-q}{2} \right), \\
T_{o\mathbb{I};o\mathbb{I}} &= P'_1 \left(\frac{q^2-q}{2} \right) + P'_2 \left(\frac{q}{2} \right) + P'_3, \\
T_{\mathbb{I}o;\mathbb{I}o} &= P'_1 \left(\frac{q^2-\sigma q}{2} \right) + P'_2 \left(\frac{\sigma q}{2} \right) + P'_3, \\
T_{ee;eo} &= \frac{q}{q+2\sigma} T_{eo;ee} = P''_1 \left(\frac{q^2-\sigma q}{2} \right) \frac{q^2+q-2}{2} - P''_2 \left(\frac{q-2}{2} \right) \frac{\sigma q}{2}, \\
T_{ee;oe} &= \frac{q}{q+2} T_{oe;ee} = P''_1 \left(\frac{q^2-q}{2} \right) \frac{q^2+\sigma q-2}{2} - P''_2 \left(\frac{\sigma q-2}{2} \right) \frac{q}{2}, \\
T_{oo;eo} &= \frac{q-2\sigma}{q-2} T_{oe;oo} = \frac{q+2}{q} T_{eo;oo} = \frac{q+2}{q} T_{oe;oo} = P''_1 \left(\frac{q^2-\sigma q}{2} \right) \frac{q^2+q-2}{2} - P''_2 \left(\frac{q+2}{2} \right) \frac{\sigma q}{2}, \\
T_{e\mathbb{I};eo} &= \frac{q^2-\sigma q}{2} T_{eo;e\mathbb{I}} = P''_1 \left(\frac{q^2-\sigma q}{2} \right) \frac{q^2+q-2}{2} + P''_2 \left(\frac{q-2}{2} \right) \left(\frac{q^2-\sigma q}{2} \right), \\
T_{\mathbb{I}e;oe} &= \frac{q^2-q}{2} T_{oe;\mathbb{I}e} = P''_1 \left(\frac{q^2-q}{2} \right) \frac{q^2+\sigma q-2}{2} + P''_2 \left(\frac{\sigma q-2}{2} \right) \left(\frac{q^2-q}{2} \right), \\
T_{o\mathbb{I};ee} &= \frac{q^2+\sigma q-2}{2} T_{oe;e\mathbb{I}} = \frac{q^2+\sigma q-2}{2} T_{o\mathbb{I};e\mathbb{I}} = \frac{(q+2)(q^2+\sigma q-2)}{2q} T_{ee;o\mathbb{I}} = \frac{q+2}{q} T_{e\mathbb{I};oe}, \\
&= \frac{(q+2)(q^2+\sigma q-2)}{2q} T_{e\mathbb{I};o\mathbb{I}} = P''_1 \left(\frac{q^2+q-2}{2} \right) \frac{q^2+\sigma q-2}{2} - P''_2 \left(\frac{q+2}{2} \right) \frac{q^2+\sigma q-2}{2}, \\
T_{\mathbb{I}o;ee} &= \frac{q^2+q-2}{2} T_{eo;\mathbb{I}e} = \frac{q^2+q-2}{2} T_{\mathbb{I}o;\mathbb{I}e} = \frac{(q+2\sigma)(q^2+q-2)}{2q} T_{ee;\mathbb{I}o} = \frac{q+2\sigma}{q} T_{\mathbb{I}e;eo}, \\
&= \frac{(q+2\sigma)(q^2+q-2)}{2q} T_{\mathbb{I}e;\mathbb{I}o} = P''_1 \left(\frac{q^2+q-2}{2} \right) \frac{q^2+\sigma q-2}{2} - P''_2 \left(\frac{\sigma q+2}{2} \right) \frac{q^2+q-2}{2}, \\
T_{o\mathbb{I};oo} &= \frac{q^2-\sigma q}{2} T_{oo;o\mathbb{I}} = P''_1 \left(\frac{q^2-q}{2} \right) \frac{q^2-\sigma q}{2} + P''_2 \frac{q}{2} \left(\frac{q^2-\sigma q}{2} \right), \\
T_{\mathbb{I}o;oo} &= \frac{q^2-q}{2} T_{oo;\mathbb{I}o} = P''_1 \left(\frac{q^2-q}{2} \right) \frac{q^2-\sigma q}{2} + P''_2 \frac{\sigma q}{2} \left(\frac{q^2-q}{2} \right), \\
T_{o\mathbb{I};\mathbb{I}e} &= \frac{q^2+\sigma q-2}{q^2+q-2} T_{\mathbb{I}o;e\mathbb{I}} = \frac{q^2+\sigma q-2}{q^2-q} T_{\mathbb{I}e;o\mathbb{I}} = \frac{q+2\sigma}{q} T_{e\mathbb{I};\mathbb{I}o} = P''_1 \left(\frac{q^2+\sigma q-2}{2} \right) + P''_2 \left(\frac{q^2+\sigma q-2}{2} \right), \quad (C3)
\end{aligned}$$

where $\sigma = 1$ is for the orthogonal-invariant ensembles, and $\sigma = -1$ is for symplectic-invariant ensembles. This general formulation (C3) holds for all $q = 2^m$ with $m > 0$ in the orthogonal-invariant case, and with $m > 1$ in the symplectic-invariant case.

For the symplectic case, at odd time steps, the second and fourth indices have to be swapped with the first and third

indices. (Recall that for the symplectic ensemble we take the qudits to have integer spin for even x and half-integer spin and odd x .)

For a half-integer-spin qudit of size $q = 2$, the label “e” (corresponding to ternary index 1) does not occur, because all single-qudit operators are either equal to the unit operator or they are antisymmetric under the dual operation. Hence, for

even time steps, the second and fourth indices can take the values “o” (corresponding to ternary index -1) and “l” only. For the symplectic-invariant ensemble with $q = 2$, the nonvanishing elements of the ternary-projected transition matrix $T_{\bar{a}\bar{p}}$ at even time steps are

$$\begin{aligned}
 T_{\text{lo};\text{lo}} &= 3P'_1 - P'_2 + P'_3, \\
 T_{\text{ol};\text{lo}} &= 3(P'_1 + P'_2), \\
 T_{\text{eo};\text{lo}} &= 3P'_1 - P'_2, \\
 T_{\text{lo};\text{ol}} &= P'_1 + P'_2, \\
 T_{\text{ol};\text{ol}} &= P'_1 + P'_2 + P'_3, \\
 T_{\text{eo};\text{ol}} &= P'_1 - P'_2, \\
 T_{\text{oo};\text{oo}} &= 3P_1 - P_2 + P_3, \\
 T_{\text{el};\text{oo}} &= 3P_1 - 3P_2, \\
 T_{\text{oo};\text{el}} &= 2P_1 - 2P_2, \\
 T_{\text{el};\text{el}} &= 2P_1 + P_3, \\
 T_{\text{lo};\text{eo}} &= 6P'_1 - 2P'_2, \\
 T_{\text{ol};\text{eo}} &= 6P'_1 - 6P'_2, \\
 T_{\text{eo};\text{eo}} &= 6P'_1 + P'_3, \\
 T_{\text{oo};\text{lo}} &= 3P''_1 - P''_2, \\
 T_{\text{el};\text{lo}} &= 3P''_1 + 3P''_2, \\
 T_{\text{oo};\text{ol}} &= P''_1 + P''_2, \\
 T_{\text{el};\text{ol}} &= P''_1 - P''_2, \\
 T_{\text{lo};\text{oo}} &= 3P''_1 - P''_2, \\
 T_{\text{ol};\text{oo}} &= 3P''_1 + 3P''_2, \\
 T_{\text{eo};\text{oo}} &= 3P''_1 + P''_2, \\
 T_{\text{lo};\text{el}} &= 2P''_1 + 2P''_2, \\
 T_{\text{ol};\text{el}} &= 2P''_1 - 2P''_2, \\
 T_{\text{eo};\text{el}} &= 2P''_1, \\
 T_{\text{oo};\text{eo}} &= 6P''_1 + 2P''_2, \\
 T_{\text{el};\text{eo}} &= 6P''_1.
 \end{aligned} \tag{C4}$$

2. General qudit size q

For a general qudit size q , which is not a power of 2, we can also obtain a closed Markovian evolution equation of the form of Eq. (35), where the transition matrix $T_{\bar{a}\bar{p}}$ has a block structure as in Eq. (59).

This matrix is constructed by summing the transition probabilities $W_{ab;pq}$ over all Pauli strings p and q with the weight factors $\chi_{\bar{p};p,q}$ defined in Eq. (56), depending on whether the qudit has integer or half-integer spin. We find that this summation depends only on the ternary structure of the input indices, denoted $\bar{a}$. Consequently, the elements of $T_{\bar{a}\bar{p}}$ are built from sums over terms of the form

$$\frac{1}{q^4} \langle |\text{tr} \mathcal{U}_{x,y} \Sigma_{a;s} \mathcal{U}_{x,y}^\dagger \Sigma_{p;s}^\dagger|^2 \rangle,$$

where the matrices $\Sigma_{a;s}$ and $\Sigma_{p;s}$ are defined as

$$\begin{aligned}
 \Sigma_{a;s} &= \frac{1}{4} (\sigma_{a_x} + s_{a_x} \sigma_{a_x}^T) \otimes (\sigma_{a_{x+1}} + s_{a_{x+1}} \sigma_{a_{x+1}}^T), \\
 \Sigma_{p;s} &= \frac{1}{4} (\sigma_{p_x} + s_{p_x} \sigma_{p_x}^T) \otimes (\sigma_{p_{x+1}} + s_{p_{x+1}} \sigma_{p_{x+1}}^T)
 \end{aligned} \tag{C5}$$

and s_{a_x} , $s_{a_{x+1}}$, s_{p_x} , and $s_{p_{x+1}}$ take the value -1 and 1 .

For any $q > 2$, the elements of the projected transition matrix $T_{\bar{a}\bar{p}}$ retain the form given in Eq. (C3).

APPENDIX D: DRIFT-DIFFUSION EQUATION

To derive the drift-diffusion equation (48), we follow Ref. [18] and combine $\bar{\rho}^{(n)}(x; t)$ and $\bar{\rho}^{(n)}(x + 1, t)$ into a two-component spinor,

$$R^{(n)}(\Delta x; t) = \begin{pmatrix} \bar{\rho}^{(n)}(\Delta x; t) \\ \bar{\rho}^{(n)}(\Delta x + 1; t) \end{pmatrix}, \tag{D1}$$

where now Δx is always even. The evolution equations (44) and (45) with the truncation prescription (46), may then be represented as

$$\begin{aligned}
 R^{(n)}(\Delta x; t) &= D^{(n)} R^{(n)}(\Delta x; t - 1) \\
 &\quad + D'^{(n)} R^{(n)}(\Delta x + 2; t - 1),
 \end{aligned} \tag{D2}$$

where $D^{(n)}$ and $D'^{(n)}$ are $(4 \times 3^n) \times (4 \times 3^n)$ matrices. The transition matrices $D^{(n)}$ and $D'^{(n)}$ satisfy the normalization rule

$$\sum_i (D_{ij}^{(n)} + D'_{ij}{}^{(n)}) = 1 \tag{D3}$$

for each $j = 1, \dots, 4 \times 3^n$. We denote the left-eigenvectors, right-eigenvectors, and eigenvalues of $D^{(n)} + D'^{(n)}$ by $\tilde{V}_j^{(n)}$, $V_j^{(n)}$, and $d_j^{(n)}$, respectively. Equation (D3) ensures that the largest eigenvalue $d_1^{(n)} = 1$, with left-eigenvector $\tilde{V}_1^{(n)} = (1, 1, \dots, 1)^T$. In the two-component spinor notation of Eq. (D1), the right-eigenvectors $V_1^{(n)}$ are written

$$V_1^{(n)} = \begin{pmatrix} V_1^{(n)}(1) \\ V_1^{(n)}(-1) \end{pmatrix}. \tag{D4}$$

The right-eigenvector $V_1^{(n)}$ also appears in Eq. (47) of the main text, where we write its dependence on the ternary labels $\bar{p}_0, \dots, \bar{p}_n$ explicitly.

In the long-time limit, the solution of Eq. (D2) is of the form

$$R^{(n)}(\Delta x, t) = R_1^{(n)}(\Delta x, t) V_1^{(n)}, \tag{D5}$$

where $R_1^{(n)}(\Delta x, t)$ satisfies the drift-diffusion equation (48), with

$$v_B^{(n)} = 1 - 2d_{11}^{(n)}, \tag{D6}$$

the butterfly velocity and

$$\mathcal{D}^{(n)} = 4d_{11}^{(n)} [1 - d_{11}^{(n)}] + 8 \sum_{j \neq 1} \frac{d_{1j}^{(n)} d_{j1}^{(n)}}{1 - d_j^{(n)}}, \tag{D7}$$

the diffusion constant governing the diffusive spreading of the front. (Recall that Δx is measured with respect to a ballistic propagation at unit velocity so that the drift velocity measured

with respect to Δx is $v_B - 1$.) In Eqs. (D6) and (D7), we abbreviated

$$d_{ij}^{(n)} = \tilde{V}_i^{(n)T} D^{(n)} V_j^{(n)}. \quad (\text{D8})$$

In Eq. (D5), the function $R_1^{(n)}(\Delta x, t)$ describes the propagation of the ends of the Pauli strings, whereas the right-eigenvector $V_1^{(n)}$ contains information on the structure of the Pauli strings immediately behind the end.

The drift-diffusion equation can also be derived for the case that a complete basis of left- and right-eigenvectors of $D^{(n)} + D'^{(n)}$ does not exist. We refer to Ref. [18] for details.

In the case of symplectic-invariant circuits, a closed drift-diffusion equation can only be derived by considering two consecutive time steps as a single unit. We denote the truncated projected transition matrix for even time steps as $D^{(n)} + D'^{(n)}$, and for odd time steps as $\tilde{D}^{(n)} + \tilde{D}'^{(n)}$. This coarse-graining yields a drift-diffusion equation identical in form to Eq. (48), but with a modified butterfly velocity and diffusion constant given by

$$v_B^{(n)} = 1 - \tilde{d}_{11}^{(n)} \quad (\text{D9})$$

where

$$\tilde{d}_{11}^{(n)} = \tilde{V}_1^{(n)T} [\tilde{D}^{(n)} + D'^{(n)}(\tilde{D}^{(n)} + \tilde{D}'^{(n)})] V_1^{(n)}. \quad (\text{D10})$$

The diffusion constant is defined as

$$\begin{aligned} \mathcal{D}^{(n)} = & 2\tilde{d}_{11}^{(n)}(1 - \tilde{d}_{11}^{(n)}) + 4\tilde{V}_1^{(n)T} [\tilde{D}^{(n)} + D'^{(n)}(\tilde{D}^{(n)} + \tilde{D}'^{(n)})] \tilde{G}^{(n)} \\ & \times [(D^{(n)} + D'^{(n)})\tilde{D}^{(n)} + D'^{(n)}(\tilde{D}^{(n)} + \tilde{D}'^{(n)})] \\ & \times V_1^{(n)} + 4\tilde{V}_1^{(n)T} D'^{(n)} \tilde{D}'^{(n)} V_1^{(n)} \end{aligned} \quad (\text{D11})$$

with

$$\begin{aligned} \tilde{G}^{(n)} = & [\mathbb{1} - (D^{(n)} + D'^{(n)})(\tilde{D}^{(n)} + \tilde{D}'^{(n)}) \\ & + V_1^{(n)} \tilde{V}_1^{(n)T}]^{-1} - V_1^{(n)} \tilde{V}_1^{(n)T}. \end{aligned} \quad (\text{D12})$$

Here, $\tilde{V}_1^{(n)}$ and $V_1^{(n)}$ are the left and right eigenvectors, respectively, of the two-step propagator $(D^{(n)} + D'^{(n)})(\tilde{D}^{(n)} + \tilde{D}'^{(n)})$ corresponding to the eigenvalue 1. One can verify that the butterfly velocity and diffusion constant recover the original forms in Eqs. (D6) and (D7) when $D^{(n)} = \tilde{D}^{(n)}$ and $D'^{(n)} = \tilde{D}'^{(n)}$.

APPENDIX E: CONVERGENCE TO THE TERNARY FORM

We first give an outline of the arguments that show that the two-norm $\|\rho - \mathbb{B}\rho\|_2 \rightarrow 0$ in the long-time limit. In this Appendix, we use the notation

$$\mathbb{W}_{ab,pq} = W_{ap} W_{bq}^* \quad (\text{E1})$$

Both $\mathbb{W}$ and its ensemble average $\langle \mathbb{W} \rangle$ are operators on the Pauli-string density matrix ρ . Because the amplitudes W_{ap} factorize into pair contributions, Eq. (B1), the layer operator $\mathbb{W}$ is a tensor product of pair transition matrices, one for each two-qudit gate in the layer. For a single two-qudit gate U , the pair transition matrix is defined as

$$W(U)_{ab,pq} = \mathcal{W}_{ap,x,y} \mathcal{W}_{bq,x,y}^* |_{\mathcal{U}_{x,y}=U}, \quad (\text{E2})$$

with the pair amplitudes $\mathcal{W}_{ap,x,y}$ given in Eq. (B2).

For qudit size $q = 2^m$, the ternary-state projector $\mathbb{B}$ is defined in Eq. (50); for general qudit dimension q it is defined as

$$\begin{aligned} \mathbb{B}_{p_x, q_x, \bar{p}_x} = & \frac{2(1 - \delta_{p_x, 0}) \chi_{+1}^*(p_x, q_x) \delta_{\bar{p}_x, +1}}{q^2 + q - 2} \\ & + \frac{2(1 - \delta_{p_x, 0}) \chi_{-1}^*(p_x, q_x) \delta_{\bar{p}_x, -1}}{q^2 - q} \\ & + \delta_{p_x, q_x} \delta_{p_x, 0} \delta_{\bar{p}_x, 0}, \end{aligned} \quad (\text{E3})$$

with $\chi_{\bar{p}, a, b}$ defined in Eq. (56) for qudits. If $q = 2^m$, we can choose the tensor products of Pauli matrices as the operator basis, in which case Eq. (E3) simplifies to Eq. (50).

Using the fact that the ternary-state projector $\mathbb{B}$ and the evolution operator $\langle \mathbb{W} \rangle$ for ρ commute,

$$\mathbb{B} \rho_{pq}(t) = \sum_{a,b} \langle \mathbb{W}_{ab,pq} \rangle \mathbb{B} \rho_{ab}(t-1), \quad (\text{E4})$$

we find that for a succession of two time steps the two-norm

$$\|\rho(t) - \mathbb{B}\rho(t)\|_2 = \sqrt{\sum_{pq} |\rho_{pq}(t) - \mathbb{B}\rho_{pq}(t)|^2}, \quad (\text{E5})$$

satisfies the bound

$$\begin{aligned} & \|\rho(t+2) - \mathbb{B}\rho(t+2)\|_2 \\ & \leq \|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|_\infty' \|\rho(t) - \mathbb{B}\rho(t)\|_2, \end{aligned} \quad (\text{E6})$$

where $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|_\infty'$ denotes the largest singular value of the product of the two single-layer averaged transition matrices, evaluated after excluding the subspace of density matrices with $\rho - \mathbb{B}\rho = 0$. (Since the gates of the two layers are drawn independently, this product equals the average $\langle \mathbb{W}_2 \mathbb{W}_1 \rangle$ of the two-layer transition matrix.) Per qudit pair, this subspace is three dimensional, spanned by the trivial state and the maximally random even-parity and odd-parity states. The largest singular value of the pair transition matrix $\langle W(U) \rangle$ for a pair of neighboring qudits is 1. Its multiplicity is generically two, corresponding to the trivial state and the maximally random nontrivial state. Exceptions are the case of the trivial circuit, for which all singular values of the transition matrix are 1, and the case where the transition matrices are orthogonal or symplectic. In the latter cases, the multiplicity is three for $q > 2$, because there are separate maximally random even-parity and odd-parity states, and four for $q = 2$ in the orthogonal case, as discussed further below.

For the operator norm $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|_\infty'$, we find the bound

$$\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|_\infty'^2 \leq \lambda_W^2 \quad (\text{E7})$$

with

$$\begin{aligned} \lambda_W^2 = & \sigma_2^2 [\sigma_2^2 + (1 - \sigma_2^2) \|\mathbb{P}_1\|_\infty'^2] \\ & + (1 - \sigma_2^2) [\chi + (1 - \chi) \sigma_2^2] \|\mathbb{P}_2\|_\infty'^2. \end{aligned} \quad (\text{E8})$$

Here, $\mathbb{P}_j$ is the projector onto the eigenvalue-1 eigenspace of $\langle \mathbb{W}_j \rangle$, $j = 1, 2$, associated with the two consecutive layers of the brickwork circuit, and $\sigma_2 \in [0, 1)$ is the largest singular value of the pair transition matrix $\langle W(U) \rangle$ other than 1. Since $\langle \mathbb{W} \rangle$ is a tensor product of pair transition matrices $\langle W(U) \rangle$, σ_2 is also the largest subunit singular value of a full layer. Finally, $\chi := \|\mathbb{P}_1 \mathbb{P}_2\|_\infty'^2$; because the two layers are offset by

one site in the brickwork pattern, their eigenvalue-1 subspaces do not coincide, and χ characterizes the largest singular value of the product $\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle$ in a Haar-distributed circuit, evaluated after projecting out vectors satisfying $\rho - \mathbb{B}\rho = 0$. For Haar-distributed ensembles, the second largest singular value vanishes, $\sigma_2 = 0$, which leads to

$$\lambda_W^2 = \chi \|\mathbb{P}_2\|_\infty^2. \quad (\text{E9})$$

For qudit dimension $q > 2$, the quantity $\|\mathbb{P}_2\|_\infty^2$ vanishes, implying $\lambda_W = 0$, in agreement with the behavior of fully unitary circuits. In contrast, for $q = 2$, if the local gate belongs to the $\text{SO}(4)$ or $\text{SO}^-(4)$ ensemble, one finds $\|\mathbb{P}_2\|_\infty^2 = 1$, because of the additional singular vector, beyond the ternary states, associated with the largest singular value of $\langle \mathbb{W} \rangle$. Consequently, in this case,

$$\lambda_W^2 = \chi. \quad (\text{E10})$$

The parameter χ is strictly smaller than 1 and is independent of the specific properties of the probability distribution of the gate ensemble. Its precise value can be obtained numerically. Since all contributing parameters are strictly less than one, it follows that $\lambda_W < 1$. Moreover, since these parameters are independent of the system size, λ_W remains strictly smaller than 1 as the system size increases.

We define the decay time for convergence to a ternary distribution τ_t as

$$\tau_t = \frac{-1}{\ln \lambda_W}. \quad (\text{E11})$$

For Haar-distributed ensembles other than $\text{SO}(4)$ and $\text{SO}^-(4)$, the convergence to the ternary distribution occurs already after the first time step.

1. Detailed derivation

Before giving the full derivation, we first outline the logical structure. We denote the ensemble of two-qudit gate operators by $\mathcal{E}$ and the transition matrix corresponding to the two-qudit operator U by $W(U)$, Eq. (E2). We call ρ a *unimodular eigenvector* of $\langle W \rangle$ if $\langle W \rangle \rho = c\rho$ with $|c| = 1$. This condition is *a priori* weaker than the steady-state condition of Sec. IV, which requires $c = 1$; in the following we determine all unimodular eigenvectors for unitary-, orthogonal-, and symplectic-invariant ensembles.

(1) We show that the largest singular value of the ensemble average $\langle W \rangle$ is 1.

(2) We show that every unimodular eigenvector of $\langle W \rangle$ satisfies $W(U)\rho = c\rho$ for all $U \in \mathcal{E}$.

(3) We show that if the ensemble is unitary-invariant and $\rho_{i\mu}$ are unimodular eigenvectors, then $W(V)\rho_{i\mu} = \rho_{i\mu}$ for all $q^2 \times q^2$ unitary matrices V .

(4) We determine the unimodular eigenvectors for unitary-, orthogonal-, and symplectic-invariant ensembles, and identify which of them are the steady states of Sec. IV.

(5) We derive the inequality (E6).

(6) We give an upper bound for $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty$.

We begin by bounding the singular values of the averaged pair transition matrix $\langle W(U) \rangle$. The positive semidefiniteness of the variance of W implies that for any normalized two-qudit

state $\|\tilde{\rho}\| = 1$, one has the inequality

$$\begin{aligned} \tilde{\rho}^\dagger \langle (W - \langle W \rangle)(W^\dagger - \langle W^\dagger \rangle) \rangle \tilde{\rho} &= \tilde{\rho}^\dagger (\langle WW^\dagger \rangle - \langle W \rangle \langle W^\dagger \rangle) \tilde{\rho} \\ &= 1 - \tilde{\rho}^\dagger \langle W \rangle \langle W^\dagger \rangle \tilde{\rho} \\ &\geq 0. \end{aligned} \quad (\text{E12})$$

Here, $\langle \dots \rangle$ denotes the ensemble average over $\mathcal{E}$, and we have used $\langle W \rangle^\dagger = \langle W^\dagger \rangle$, together with the unitarity of $W_{ab;pq}$, which is guaranteed by

$$\begin{aligned} (WW^\dagger)_{ab;a'b'} &= \sum_{pq} W_{ab;pq} W_{a'b';pq}^* \\ &= \frac{1}{q^8} \sum_{pq} \text{tr} U^\dagger \Sigma_a U \Sigma_p^\dagger \text{tr} U^\dagger \Sigma_b^\dagger U \Sigma_q \\ &\quad \times \text{tr} U^\dagger \Sigma_{a'}^\dagger U \Sigma_p \text{tr} U^\dagger \Sigma_{b'} U \Sigma_q^\dagger \\ &= \delta_{aa'} \delta_{bb'}. \end{aligned} \quad (\text{E13})$$

It follows that

$$1 \geq \max_{\|\tilde{\rho}\|_2=1} (\tilde{\rho}^\dagger \langle W \rangle \langle W^\dagger \rangle \tilde{\rho}) = \|\langle W \rangle\|_\infty^2, \quad (\text{E14})$$

so that the largest singular value of $\langle W \rangle$ is bounded from above by 1.

If ρ is a unimodular eigenvector, we obtain that

$$\langle W \rangle^\dagger \rho = c^* \rho. \quad (\text{E15})$$

This follows from the observation that

$$\begin{aligned} \|\langle W \rangle^\dagger \rho - c^* \rho\|_2^2 &= \|\langle W \rangle^\dagger \rho\|_2^2 - \|\rho\|_2^2 \\ &\leq \|\langle W \rangle^\dagger\|_\infty^2 \|\rho\|_2^2 - \|\rho\|_2^2 \\ &\leq 0, \end{aligned} \quad (\text{E16})$$

where we have used $\|\langle W \rangle^\dagger\|_\infty^2 = \|\langle W \rangle\|_\infty^2 \leq 1$, and nonnegativity of the norm. It follows that a unimodular eigenvector also satisfies

$$\langle W \rangle \langle W \rangle^\dagger \rho = \langle W \rangle^\dagger \langle W \rangle \rho = \rho. \quad (\text{E17})$$

Moreover, we have

$$\begin{aligned} \|\langle W \rangle \rho - c\rho\|_2^2 &= \langle \rho^\dagger W^\dagger W \rho + \rho^\dagger \rho - 2 \text{Re}(\rho^\dagger W \rho) \rangle \\ &= 2\rho^\dagger \rho - 2 \text{Re}(c^* \rho^\dagger \langle W \rangle \rho) \\ &= 0 \end{aligned} \quad (\text{E18})$$

if $\langle W \rangle \rho = c\rho$ with $|c| = 1$. Since the squared two-norm is nonnegative for any W , Eq. (E18) implies that $W\rho = c\rho$ must hold for all $W(U)$, without taking the ensemble average. Therefore, ρ is a simultaneous eigenvector with eigenvalue c for the entire ensemble of W .

One may argue that the unimodular eigenvectors ρ would depend on the structure of the ensemble. We now demonstrate that for a unitary-invariant ensemble, the only unimodular eigenvectors are the two steady states identified in Sec. IV.

2. Unitary invariant ensembles

Equivalently, vectorizing the two-qudit Pauli operators Σ_a , the pair transition matrix of Eq. (E2) can be written in terms of the Hilbert–Schmidt inner product,

$$W(U)_{ab;pq} = \langle ab^* | U^* \otimes U \otimes U \otimes U^* | pq^* \rangle, \quad (\text{E19})$$

or, in matrix notation,

$$W(U) = U^* \otimes U \otimes U \otimes U^*. \quad (\text{E20})$$

Here, we have used the identities [55]

$$\text{tr}(A^\dagger B) = \langle \text{vec}(A) | \text{vec}(B) \rangle, \quad (\text{E21})$$

$$|\text{vec}(ABC)\rangle = C^T \otimes A |\text{vec}(B)\rangle \quad (\text{E22})$$

and we defined

$$\begin{aligned} \langle a | &= (\text{vec}(\Sigma_a^\dagger))^\dagger, \\ \langle b^* | &= (\text{vec}(\Sigma_b^\dagger))^T, \\ |p\rangle &= (\text{vec}(\Sigma_p^\dagger)), \\ |q^*\rangle &= (\text{vec}^*(\Sigma_q^\dagger)). \end{aligned} \quad (\text{E23})$$

Due to the invariance of the ensemble-averaged operator $\langle W \rangle$ under unitary transformations V from the ensemble $\mathcal{E}$, we find

$$\begin{aligned} \langle W \rangle &= \langle W(VUV^\dagger) \rangle \\ &= W(V) \langle W(U) \rangle W^\dagger(V) \end{aligned} \quad (\text{E24})$$

for all $q^2 \times q^2$ unitary matrices V .

Now, suppose $\{\rho_{i\mu}\}$ is an orthonormal basis set of eigenvectors of $\langle W \rangle$ with eigenvalue c_i with $|c_i| = 1$,

$$\langle W \rangle \rho_{i\mu} = c_i \rho_{i\mu}. \quad (\text{E25})$$

Then,

$$\begin{aligned} W(V) \langle W \rangle W^\dagger(V) \rho_{i\mu} &= \langle W \rangle W(V) \rho_{i\mu} \\ &= c_i W(V) \rho_{i\mu} \end{aligned} \quad (\text{E26})$$

for all $q^2 \times q^2$ unitary matrices V , so that $W(V) \rho_{i\mu}$ is also an eigenvector of $\langle W \rangle$ at the same eigenvalue c_i . Hence, for an arbitrary unitary matrix V , $W(V) \rho_{i\mu}$ can be expressed as a linear combination of the basis vectors $\rho_{i\mu}$,

$$W(V) \rho_{i\mu} = \sum_v C_{\mu v}^i(V) \rho_{iv}, \quad (\text{E27})$$

where the coefficients $C_{\mu v}^i(V)$ satisfy the orthonormality constraint

$$\sum_v C_{\mu v}^i(V) C_{\mu v}^{i*}(V) = 1. \quad (\text{E28})$$

The matrix $C_{\mu v}^i(V)$ further satisfies

$$C_{\mu v}^i(V^\dagger) = C_{v\mu}^{i*}(V), \quad (\text{E29})$$

$$\sum_{v'} C_{\mu v'}^i(V) C_{v'v}^i(U) = C_{\mu v}^i(UV), \quad (\text{E30})$$

where we have used $W(UV) = W(U)W(V)$.

We first consider the case that the ensemble $\mathcal{E}$ of two-qudit operators U forms a group, and relax this assumption later. Because $\mathcal{E}$ is invariant under unitary conjugation, $VUV^\dagger \in \mathcal{E}$ for all unitary matrices V , so the group formed by $\mathcal{E}$ is a normal subgroup of the full unitary group $U(q^2)$ [56]. If $\mathcal{E}$ is a group, then $U^2 \in \mathcal{E}$, and Eqs. (E25) and (E18) give $W(U^2) \rho_{i\mu} = c_i \rho_{i\mu}$, while $W(U^2) = W^2(U)$ gives

$$W(U^2) \rho_{i\mu} = W^2(U) \rho_{i\mu} = c_i^2 \rho_{i\mu}. \quad (\text{E31})$$

Comparing the two gives $c_i = c_i^2$, hence $c_i = 1$. Any $q^2 \times q^2$ unitary matrix V can be written as

$$V = (\det V)^{1/q^2} V', \quad (\text{E32})$$

where $V' \in \text{SU}(q^2)$. Moreover, any special unitary matrix can be expressed as a commutator [57,58],

$$V' = ABA^{-1}B^{-1}, \quad (\text{E33})$$

with $A, B \in U(q^2)$. Therefore,

$$\begin{aligned} C_{\mu v}^i(V) &= C_{\mu v}^i((\det V)^{1/q^2} V') \\ &= \sum_{v'} C_{\mu v'}^i(V') C_{v'v}^i((\det V)^{1/q^2} \mathbb{1}) \\ &= C_{\mu v}^i(V'), \end{aligned} \quad (\text{E34})$$

where we used $C_{v'v}^i((\det V)^{1/q^2} \mathbb{1}) = C_{v'v}^i(\mathbb{1}) = \delta_{v'v}$. We also note that

$$W(V) = W(V'), \quad (\text{E35})$$

because V and V' only differ by a factor, see Eq. (E32), and V and V^* appear equally often (twice each) in the tensor product defining $W(V)$, see Eq. (E19).

Likewise, any matrix $U_i \in \mathcal{E}$ can be written as $U_i = (\det U_i)^{1/q^2} U'_i$ with $U'_i \in \text{SU}(q^2)$. If $\mathcal{E}$ is a group, then the set $\{U'_i\}$ also forms a group. Indeed, if $U_i U_j = U_k$, then

$$\begin{aligned} U_i U_j &= (\det(U_i U_j))^{1/q^2} U'_i U'_j \\ &= (\det U_k)^{1/q^2} U'_k, \end{aligned} \quad (\text{E36})$$

so that

$$U'_i U'_j = U'_k. \quad (\text{E37})$$

Since $\mathbb{1} \in \mathcal{E}$, it follows that $\mathbb{1} \in \{U'_i\}$. Associativity holds automatically. Hence $\{U'_i\}$ is a group whenever $\mathcal{E}$ is. It is then a normal subgroup of $\text{SU}(q^2)$.

We now make use of the fact that there are only two kinds of normal subgroups of $\text{SU}(q^2)$ [56]: (i) subgroups of its center $\{e^{2\pi i k/q^2} \mathbb{1} \mid k = 0, 1, \dots, q^2 - 1\}$; and (ii) the entire group $\text{SU}(q^2)$. The first case yields $W(U)$ behaving like $W(\mathbb{1})$, corresponding to the trivial circuit and producing no convergence to the binary or ternary state; thus, only the second case is nontrivial. Therefore, we must consider the case that

$$W(V) \rho_{i\mu} = \rho_{i\mu} \quad (\text{E38})$$

for all $q^2 \times q^2$ unitary matrices V , if $\langle W \rangle \rho_{i\mu} = c_i \rho_{i\mu}$ with $|c_i| = 1$ for the ensemble average $\langle W \rangle$ over all $U \in \mathcal{E}$.

If the ensemble $\mathcal{E}$ does not form a group, we can reach the same conclusion as follows: For any $U \in \mathcal{E}$ with $U = (\det U)^{1/q^2} U'$, define

$$\mathcal{S} = \{VU'^k V^\dagger \mid k \in \mathbb{Z}, V \in U(q^2)\}. \quad (\text{E39})$$

Let G be the set of all finite products of elements of $\mathcal{S}$. Then G is a group. Indeed, if $g_1, g_2 \in G$ with $g_1 = s_1 s_2 \dots$ and $g_2 = m_1 m_2 \dots$ for $s_i, m_i \in \mathcal{S}$, then

$$g_1 g_2 = s_1 s_2 \dots m_1 m_2 \dots, \quad (\text{E40})$$

which is again a finite product of elements of $\mathcal{S}$. Moreover, for $A = s_1 s_2 \cdots s_m \in G$,

$$A^{-1} = s_m^{-1} \cdots s_2^{-1} s_1^{-1} \in G. \quad (\text{E41})$$

Since $\mathbb{1} \in \mathcal{S}$, we also have $\mathbb{1} \in G$, and associativity holds as before. Hence, G is a (normal) subgroup of $\text{SU}(q^2)$. By the same reasoning as above, G is either central or all of $\text{SU}(q^2)$. For a nontrivial circuit, one must be in the noncentral case, i.e., $G = \text{SU}(q^2)$.

Furthermore, for any $A = VU'^k V^\dagger \in \mathcal{S}$, one has

$$\begin{aligned} W(VU'^k V^\dagger) \rho_{i\mu} &= \sum_v C_{\mu\nu}^i(VU'^k V^\dagger) \rho_{i\nu} \\ &= c_i^k \rho_{i\mu}, \end{aligned} \quad (\text{E42})$$

so $C_{\mu\nu}^i(A)$ is a scalar (proportional to $\delta_{\mu\nu}$) on the subspace spanned by the $\{\rho_{i\mu}\}$. For any $M \in G$, write $M = A_1 A_2 \cdots$ with $A_1, A_2, \dots \in \mathcal{S}$. Then,

$$\begin{aligned} W(M) \rho_{i\mu} &= \sum_v C_{\mu\nu}^i(M) \rho_{i\nu} \\ &= W(A_1 A_2 \cdots) \rho_{i\mu} \\ &= W(A_1) W(A_2) \cdots \rho_{i\mu} \\ &= c_i(M) \rho_{i\mu}, \end{aligned} \quad (\text{E43})$$

and thus $C_{\mu\nu}^i(M)$ is a scalar for every $M \in G$.

Now we can prove that the constant $c_i(V') = 1$ for all $V' \in \text{SU}(q^2)$ if $|c_i(V')| = 1$. Here, we notice that

$$c_i(V') c_i(U') = c_i(V' U') = c_i(U' V'), \quad (\text{E44})$$

which means that $c_i(V')$ is a one-dimensional character of the connected group $\text{SU}(q^2)$. Therefore, we obtain that

$$c_i(V') = c_i(A) c_i(B) c_i(A^{-1}) c_i(B^{-1}) = 1, \quad (\text{E45})$$

where we wrote the special unitary matrix V' as a commutator, see Eq. (E33). For the same reason we discussed before, we can conclude $C_{\mu\nu}^i(V) = \delta_{\mu\nu}$ for all $V \in \mathcal{U}(q^2)$.

The finding that $W(V) \rho_{i\mu} = \rho_{i\mu}$ for all $V \in \mathcal{U}(q^2)$ if $\rho_{i\mu}$ satisfies $\langle W \rangle \rho_{i\mu} = c_i \rho_{i\mu}$ implies that ρ transforms under the trivial representation of the group action defined by $W(V)$. Consequently, the dimension of the subspace spanned by such invariant states ρ is determined by the multiplicity of the trivial representation in the decomposition of $W(V)$. To analyze this, we write the representation $W(V)$ as

$$W(V) = T^*(V) \otimes T(V), \quad \text{where } T(V) = V \otimes V^*. \quad (\text{E46})$$

Under the adjoint action $A \mapsto VAV^\dagger$, the representation $T(V)$ acting on the space of linear operators $L(V)$ decomposes as [59]

$$T(V) = T_0 \oplus T_{\text{adj}}, \quad (\text{E47})$$

where T_0 is the trivial representation (spanned by the identity matrix), and T_{adj} corresponds to the adjoint subspace (traceless operators). Therefore, the full representation $W(V)$

decomposes as

$$\begin{aligned} (T_0^* \oplus T_{\text{adj}}^*) \otimes (T_0 \oplus T_{\text{adj}}) &= (T_0^* \otimes T_0) \oplus (T_0^* \otimes T_{\text{adj}}) \\ &\quad \oplus (T_{\text{adj}}^* \otimes T_0) \oplus (T_{\text{adj}}^* \otimes T_{\text{adj}}). \end{aligned} \quad (\text{E48})$$

Since only tensor products of a representation with its complex conjugate contain the trivial representation, the subrepresentations $T_0^* \otimes T_0$ and $T_{\text{adj}}^* \otimes T_{\text{adj}}$ each contain the trivial representation [59,60]. Hence, the dimension of the invariant subspace (i.e., the multiplicity of eigenvalue 1) is two, corresponding exactly to the steady states identified in Sec. IV. Therefore, we can conclude that if $\langle W \rangle \rho = c \rho$ with $|c| = 1$, then ρ must be a superposition of the maximum random state and the trivial state. Both of these states obviously have eigenvalue $c = 1$.

3. Orthogonal invariant ensembles

A parallel argument applies to orthogonal-invariant ensembles. Unlike the unitary case, the orthogonal group is not connected, so there are two disjoint components: the special orthogonal group $\text{SO}(q^2)$ and the determinant -1 component $\text{SO}^-(q^2)$ [60]. For any matrix V'' in $\text{SO}^-(q^2)$, we can write

$$V'' = R V', \quad (\text{E49})$$

where $R = \text{diag}(-1, 1, 1, \dots)$ is a reflection and $V' \in \text{SO}(q^2)$.

As in the unitary case, if the ensemble $\mathcal{E}$ is a group, then $C_{\mu\nu}^i(U) = \delta_{\mu\nu}$ for all $U \in \mathcal{E}$. Moreover, $\mathcal{E}$ is closed under conjugation: $VUV^\dagger \in \mathcal{E}$ for all $V \in \text{O}(q^2)$. If $\mathcal{E}$ is not a group, the conclusion must be adjusted slightly. Let $\{U_i\}$ denote the elements of $\mathcal{E}$ and define

$$\mathcal{S} = \{V U_i^k V^\dagger \mid k \in \mathbb{Z}, U_i \in \mathcal{E}, V \in \text{O}(q^2)\}. \quad (\text{E50})$$

Let G be the set of all finite products of elements of $\mathcal{S}$. The same closure, inverse, identity, and associativity arguments as before show that G is a group. For any $A \in G$ we then have $C_{\mu\nu}^i(A) = c_i(A) \delta_{\mu\nu}$ (a scalar on the $\{\rho_{i\mu}\}$ subspace). Furthermore, $G \subset \text{U}(q^2)$ is a subgroup that is preserved under orthogonal conjugation, i.e., normalized by $\text{O}(q^2)$. In the $W(V)$ representation, central phases act trivially. Up to this trivial central action, there are exactly three nontrivial $\text{O}(q^2)$ -invariant possibilities for the image of G : the special orthogonal group $\text{SO}(q^2)$, the full orthogonal group $\text{O}(q^2)$, and the image of $\text{SU}(q)$, which is isomorphic to the projective unitary group $\text{PU}(q)$ [56,61]. Moreover, if $\mathcal{E}$ consists only of elements whose action on W lies in $\text{O}(q^2)$ [equivalently, in $\text{U}(1) \cdot \text{O}(q^2)$], then G is a normal subgroup of $\text{O}(q^2)$ [56,61]; for $q \geq 2$, the only nontrivial normal subgroups of $\text{O}(q^2)$ (besides its center) are $\text{SO}(q^2)$ and $\text{O}(q^2)$. If $\mathcal{E}$ contains an element outside $\text{O}(q^2)$ [or outside $\text{U}(1) \cdot \text{O}(q^2)$], then G has the same image on W as $\text{U}(q)$ [equivalently, as $\text{SU}(q)$], which is isomorphic to $\text{PU}(q)$. In particular, when $G = \text{SO}(q^2)$, every element of G is a commutator [57,58], which implies

$$C_{\mu\nu}^i(V') = \delta_{\mu\nu} \quad (\text{E51})$$

for all $V' \in G$. If $G = \text{O}(q^2)$, then for $V'' = R V'$ with $V' \in \text{SO}(q^2)$ we obtain

$$C_{\mu\nu}^i(V'') = C_{\mu\nu}^i(R) = c_i(R), \quad (\text{E52})$$

and since

$$c_i(R) c_i(R) = c_i(\mathbb{1}) = 1, \quad (\text{E53})$$

it follows that $c_i(R) = \pm 1$.

For the full group $\text{SO}(q^2)$, the representation of $W(V')$ decomposes as [59,60]

$$\begin{aligned} & (T_0 \otimes T_0) \oplus (T_{\text{sym}} \otimes T_{\text{sym}}) \oplus (T_{\text{ant}} \otimes T_{\text{ant}}) \\ & \oplus (T_0 \otimes T_{\text{sym}}) \oplus (T_{\text{sym}} \otimes T_0) \oplus (T_0 \otimes T_{\text{ant}}) \\ & \oplus (T_{\text{ant}} \otimes T_0) \oplus (T_{\text{sym}} \otimes T_{\text{ant}}) \oplus (T_{\text{ant}} \otimes T_{\text{sym}}). \end{aligned}$$

Only tensor products of a representation with its complex conjugate contain the trivial representation; since T_0 , T_{sym} , and T_{ant} are self-conjugate, the subrepresentations $T_0 \otimes T_0$, $T_{\text{sym}} \otimes T_{\text{sym}}$, and $T_{\text{ant}} \otimes T_{\text{ant}}$ each contain a trivial component. Therefore, when $q > 2$, there exist three independent eigenvectors with

$$W(V') \rho_i = \rho_i$$

for all $V' \in \text{SO}(q^2)$, namely

$$\rho_1 = \delta_{ab}\delta_{cd}, \quad \rho_2 = \delta_{ad}\delta_{bc}, \quad \rho_3 = \delta_{ac}\delta_{bd}.$$

Explicitly,

$$\begin{aligned} W(V')_{a'b'c'd';abcd} \delta_{ab}\delta_{cd} &= \delta_{a'b'}\delta_{c'd'}, \\ W(V')_{a'b'c'd';abcd} \delta_{ad}\delta_{bc} &= \delta_{a'd'}\delta_{b'c'}, \\ W(V')_{a'b'c'd';abcd} \delta_{ac}\delta_{bd} &= \delta_{a'c'}\delta_{b'd'}. \end{aligned} \quad (\text{E54})$$

These three vectors are common eigenvectors of $W(V)$ with eigenvalue 1 for all $V \in \text{O}(q^2)$ and they satisfy

$$W(R) \rho_i = \rho_i. \quad (\text{E55})$$

Hence, $\langle W \rangle \rho_i = \rho_i$ for $i = 1, 2, 3$ when $q > 2$. These invariant eigenstates ρ_i coincide with those identified in Sec. IV.

For $q = 2$, there is an additional common eigenvector of $W(V')$ for $\text{SO}(4)$ with eigenvalue 1, namely $\rho_4 = \epsilon_{abcd}$. This eigenvector does not correspond to the binary or ternary sectors and is not stable under the even-odd structure. Moreover,

$$W(R) \rho_4 = -\rho_4. \quad (\text{E56})$$

If the ensemble $\mathcal{E}$ lies entirely within one of the components $\text{SO}(4)$ or $\text{SO}^-(4)$, then there are four independent eigenvectors of $\langle W \rangle$ with $|c| = 1$. Specifically, if $\mathcal{E} \subset \text{SO}(4)$, then $\langle W \rangle \rho_i = \rho_i$ for $i = 1, 2, 3, 4$; if $\mathcal{E} \subset \text{SO}^-(4)$, then $\langle W \rangle \rho_i = \rho_i$ for $i = 1, 2, 3$ and $\langle W \rangle \rho_4 = -\rho_4$. However, if $\mathcal{E}$ contains elements from both components $\text{SO}(4)$ and $\text{SO}^-(4)$, then only ρ_1, ρ_2, ρ_3 remain eigenvectors of $\langle W \rangle$ with $|c| = 1$. [Here, we used the fact that if $\langle W \rangle \rho_i = c_i \rho_i$, then $W(V) \rho_i = c_i \rho_i$ for all $V \in \mathcal{E}$.]

If $\mathcal{E}$ contains elements not in $U(1) \cdot \text{O}(q^2)$, we reach the same conclusion as the unitary case.

4. Symplectic-invariant ensembles

For the symplectic-invariant ensemble, one proceeds almost the same as the orthogonal case, with a few small differences. The constructed group G is invariant under symplectic transformations. In the $W(V)$ representation, up to the trivial action, there are only two nontrivial possibilities of the

image of G : $\text{Sp}(q^2/2)$, the projective unitary group $\text{PU}(q^2)$ [56,61].

If the ensemble $\mathcal{E}$ only contains elements in the symplectic group, then there are three independent eigenvalue-1 eigenvectors of $W(V)$,

$$\rho_1 = \delta_{ab}\delta_{cd}, \quad \rho_2 = \delta_{ad}\delta_{bc}, \quad \rho_3 = Z_{ac}Z_{bd},$$

where the involution matrix Z is defined in Eq. (14). Unlike the orthogonal case, there does not exist a fourth independent eigenvalue-1 eigenvector for $q = 2$, as $\text{Sp}(2)$ is not a real matrix group, but a subgroup of $\text{SU}(4)$ [60].

If the ensemble $\mathcal{E}$ contains elements outside $U(1) \cdot \text{Sp}(q^2/2)$, then there exist only two independent eigenvalue-1 eigenvectors of $W(V)$, namely, ρ_1, ρ_2 .

5. Relaxation time

One can verify that $\langle W \rangle$ commutes with $\mathbb{B}$, by using the Weingarten calculus result in Appendix B, which also guarantees the validation of the projection method.

As an example, consider the term $C_{16} = B_{16} \text{tr}(\Sigma_a \Sigma_p^\dagger \Sigma_b^* \Sigma_q^T)$ and investigate its action on a density matrix ρ_{pq} of ternary form. The nonzero elements of such density matrices have $p = q$ or $p = q^T$, see Eq. (56). If $p = q$, then

$$q^{-2} \delta_{a'+b', 2p'} \delta_{a''b''} e^{i(a'-p')(p''-a'')} B_{16}$$

is obtained. Summing over all $p = (p_x, p_{x+1})$ where both indices are nonidentity ($p_x, p_{x+1} \neq 0$) yields

$$\xi(a_x, b_x) \xi(a_{x+1}, b_{x+1}) B_{16}$$

with $\xi(a, b) = \delta_{a,b} - q^{-1} \delta_{a,b^T} e^{-i\phi_a}$. If, instead, $p = q^T$, we multiply by $e^{-i\phi_p}$ and sum over all nonidentity indices $p_x, p_{x+1} \neq 0$, we obtain

$$\begin{aligned} & q^{-2} \delta_{a,b^T} e^{-i\phi_a} B_{16} [(1 - \delta_{a_x,0})(1 - \delta_{a_{x+1},0}) \\ & - (q^2 - 1)(1 - \delta_{a_x,0})\delta_{a_{x+1},0} - (q^2 - 1)\delta_{a_x,0}(1 - \delta_{a_{x+1},0})]. \end{aligned}$$

Both contributions depend only on $\delta_{a,b}$, $\delta_{a,b^T} e^{-i\phi_a}$, and on whether the indices a_x and a_{x+1} are zero or nonzero. Hence, they exhibit a clear ternary structure. Repeating the same analysis for the remaining terms shows that each term likewise assumes a ternary form. Because a and p play symmetric roles, summing over a and b with a fixed ternary pattern induces a ternary pattern on p and q . Consequently, the pair transition matrix $\langle W(U) \rangle$ commutes with the pair factor of the ternary projection operator $\mathbb{B}$ and hence, by the factorization of a layer into pair transition matrices, $\langle W \rangle$ commutes with the ternary projection operator $\mathbb{B}$. As a result, once the Pauli strings are in the ternary or binary form, then they will be in the ternary or binary form forever.

One can also verify that $\langle W \rangle$ is normal based on the Weingarten calculus results in Appendix B. Then it follows that $\langle W \rangle$ and $\langle W \rangle \langle W \rangle^\dagger$ share eigenvectors. Consequently, the eigenspace of $\langle W \rangle \langle W \rangle^\dagger$ corresponding to the maximum singular value coincides with the eigenspace of $\langle W \rangle$ with eigenvalue c satisfying $|c| = 1$ (as established above). Therefore, the invariant subspace of $\langle W \rangle$ is precisely the top-singular-value eigenspace of $\langle W \rangle \langle W \rangle^\dagger$.

Since we have proved that the $\langle \mathbb{W} \rangle$ commutes with the ternary projection operator $\mathbb{B}$, the bound of (E5) over two time steps follows. It remains to construct an upper bound for the norm $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty$. Hereto, we may restrict to vectors with $\rho - \mathbb{B}\rho \neq 0$. We recall that the primed norm $\|\cdot\|'_2$ is the largest singular value of the matrix evaluated after excluding vectors with $\rho - \mathbb{B}\rho = 0$. We write $\sigma_2 \in [0, 1)$ for the largest singular value of the pair transition matrix $\langle W(U) \rangle$ smaller than 1. An upper bound for $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty$ follows from submultiplicativity,

$$\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty \leq \|\langle \mathbb{W}_2 \rangle\|'_\infty \|\langle \mathbb{W}_1 \rangle\|'_\infty. \quad (\text{E57})$$

In the unitary case, since the only eigenvalue-1 states are the binary steady states, $\|\langle \mathbb{W}_1 \rangle\|'_\infty$ and $\|\langle \mathbb{W}_2 \rangle\|'_\infty$ are strictly less than 1. Hence, the product can be bounded as

$$\|\langle \mathbb{W}_2 \rangle\|'_\infty \|\langle \mathbb{W}_1 \rangle\|'_\infty \leq \sigma_2^2. \quad (\text{E58})$$

For the orthogonal-invariant ensemble with $q \geq 3$ and for the symplectic-invariant ensemble, the same conclusion as in the unitary case holds. Since $0 \leq \sigma_2 < 1$, it follows that in the long-time limit, the state ρ approaches the binary/ternary form $\mathbb{B}\rho$.

For the special case $q = 2$ in the orthogonal-invariant ensemble, $\mathbb{W}$ possesses additional eigenvalue-1 eigenstates, as discussed above. In this case, showing that $\rho(t)$ acquires a ternary form for large t requires extra care, because Eq. (E58) does not apply. We recall that $\mathbb{P}_j$ is the projector onto the eigenvalue-1 eigenspace of $\langle \mathbb{W}_j \rangle$, $j = 1, 2$, and set $\mathbb{Q}_j = \mathbb{1} - \mathbb{P}_j$. Since $\langle \mathbb{W} \rangle$ is normal, $\mathbb{P}_j^\dagger = \mathbb{P}_j$, $\mathbb{P}_j \langle \mathbb{W}_j \rangle = \mathbb{P}_j$, and $\|\mathbb{Q}_j \langle \mathbb{W}_j \rangle \rho\|_2 \leq \sigma_2 \|\rho\|_2$. For the overlap parameter, we have

$$\chi = \|\mathbb{P}_1 \mathbb{P}_2\|'_\infty = \sup_{\substack{\phi \in \text{Ran}(\mathbb{P}_2) \\ \|\phi\|_2=1 \text{ \& } \phi - \mathbb{B}\phi \neq 0}} \langle \phi | \mathbb{P}_1 | \phi \rangle. \quad (\text{E59})$$

For any ρ we set $\alpha = \|\mathbb{P}_1(\rho - \mathbb{B}\rho)\|_2$ and $|x\rangle = \langle \mathbb{W}_1 \rangle(\rho - \mathbb{B}\rho)$. Splitting off the $\mathbb{P}_2$ sector at the second layer and the $\mathbb{P}_1$ sector at the first, one finds

$$\begin{aligned} \|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty &\leq \sigma_2^2 [\sigma_2^2 + (1 - \sigma_2^2) \|\mathbb{P}_1\|'_\infty] \\ &\quad + (1 - \sigma_2^2) \sup_{\substack{\|\rho\|_2=1 \\ \rho - \mathbb{B}\rho \neq 0}} \langle x | \mathbb{P}_2 | x \rangle, \end{aligned} \quad (\text{E60})$$

where $\|\mathbb{P}_1\|'_\infty = \|\mathbb{P}_2\|'_\infty = 0$ if the only eigenvalue-1 vectors in $\text{Ran}(\mathbb{P}_1)$ and $\text{Ran}(\mathbb{P}_2)$ are ternary states, and $\|\mathbb{P}_1\|'_\infty = \|\mathbb{P}_2\|'_\infty = 1$ otherwise.

It remains to bound $\langle x | \mathbb{P}_2 | x \rangle$. We decompose $|x\rangle = a \hat{x}_1 + b \hat{x}'_1$ into the orthogonal ranges of $\mathbb{P}_1$ and $\mathbb{Q}_1$, with $\hat{x}_1, \hat{x}'_1$ normalized and $a, b \geq 0$ without loss of generality. The identities above give $a = \alpha$ and $b \leq \sigma_2 \sqrt{1 - \alpha^2}$.

Since $|x\rangle$ lies in the nonternary sector,

$$\langle x | \mathbb{P}_2 | x \rangle = \sup_{\substack{\phi \in \text{Ran}(\mathbb{P}_2) \\ \|\phi\|_2=1 \text{ \& } \phi - \mathbb{B}\phi \neq 0}} |\langle \phi | x \rangle|^2 \|\mathbb{P}_2\|'_\infty. \quad (\text{E61})$$

For any admissible ϕ , define $c = \langle \hat{x}_1 | \phi \rangle$ and $d = \langle \hat{x}'_1 | \phi \rangle$. Since $\hat{x}_1 \in \text{Ran}(\mathbb{P}_1)$,

$$|c|^2 \leq \langle \phi | \mathbb{P}_1 | \phi \rangle \leq \chi, \quad (\text{E62})$$

while normalization gives $|c|^2 + |d|^2 \leq 1$. Hence,

$$|\langle \phi | x \rangle| \leq a|c| + b|d| \leq a|c| + b\sqrt{1 - |c|^2}. \quad (\text{E63})$$

Writing $|c| = \cos \theta$, we therefore obtain

$$\langle x | \mathbb{P}_2 | x \rangle \leq \max_{\cos^2 \theta \leq \chi} (a \cos \theta + b \sin \theta)^2 \|\mathbb{P}_2\|'_\infty. \quad (\text{E64})$$

The maximum occurs either at the unconstrained extremum or at the boundary, hence

$$\langle x | \mathbb{P}_2 | x \rangle \leq \begin{cases} (a^2 + b^2) \|\mathbb{P}_2\|'_\infty, & \text{if } \frac{a^2}{a^2 + b^2} \leq \chi, \\ (a\sqrt{\chi} + b\sqrt{1 - \chi})^2 \|\mathbb{P}_2\|'_\infty, & \text{if } \frac{a^2}{a^2 + b^2} > \chi. \end{cases} \quad (\text{E65})$$

Inserting $a = \alpha$ and $b^2 \leq \sigma_2^2(1 - \alpha^2)$, and writing

$$\Xi = \alpha\sqrt{\chi} + \sigma_2\sqrt{1 - \alpha^2}\sqrt{1 - \chi}, \quad (\text{E66})$$

$$\begin{aligned} \eta &= \frac{a^2}{a^2 + b^2} \\ &= \frac{\|\mathbb{P}_1(\rho - \mathbb{B}\rho)\|_2^2}{\|\mathbb{P}_1(\rho - \mathbb{B}\rho)\|_2^2 + \|\mathbb{Q}_1 \langle \mathbb{W}_1 \rangle(\rho - \mathbb{B}\rho)\|_2^2}, \end{aligned} \quad (\text{E67})$$

we obtain the inequalities

$$\begin{aligned} \|\mathbb{P}_2 \langle \mathbb{W}_1 \rangle(\rho - \mathbb{B}\rho)\|_2^2 &\leq \begin{cases} (\alpha^2 + \sigma_2^2(1 - \alpha^2)) \|\mathbb{P}_2\|'_\infty, & \text{if } \eta \leq \chi, \\ \Xi^2 \|\mathbb{P}_2\|'_\infty, & \text{if } \eta > \chi. \end{cases} \end{aligned} \quad (\text{E68})$$

The two cases are bounded separately. In the first case, $\eta \leq \chi$ is equivalent to $b^2 \geq \frac{1 - \chi}{\chi} \alpha^2$, which together with $b^2 \leq \sigma_2^2(1 - \alpha^2)$ restricts α to

$$\alpha^2 \leq \frac{\sigma_2^2 \chi}{1 - \chi + \sigma_2^2 \chi}, \quad (\text{E69})$$

so that $\alpha^2 + \sigma_2^2(1 - \alpha^2) \leq \sigma_2^2/(1 - \chi + \sigma_2^2 \chi)$. In the second case, Ξ is maximal at

$$\alpha^2 = \frac{\chi}{\chi + \sigma_2^2(1 - \chi)} > \frac{\sigma_2^2 \chi}{1 - \chi + \sigma_2^2 \chi}, \quad (\text{E70})$$

where $\Xi^2 = \chi + (1 - \chi)\sigma_2^2$. Hence, the inequality simplifies to

$$\begin{aligned} \|\mathbb{P}_2 \langle \mathbb{W}_1 \rangle(\rho - \mathbb{B}\rho)\|_2^2 &\leq \max \left\{ \frac{\sigma_2^2}{1 - \chi + \sigma_2^2 \chi} \|\mathbb{P}_2\|'_\infty, \right. \\ &\quad \left. \times (\chi + (1 - \chi)\sigma_2^2) \|\mathbb{P}_2\|'_\infty \right\} \\ &\leq (\chi + (1 - \chi)\sigma_2^2) \|\mathbb{P}_2\|'_\infty, \end{aligned} \quad (\text{E71})$$

where the last step uses $(\chi + (1 - \chi)\sigma_2^2)(1 - \chi + \sigma_2^2 \chi) - \sigma_2^2 = \chi(1 - \chi)(1 - \sigma_2^2)^2 \geq 0$.

Combining these results gives Eq. (E8).

The parameter χ can be calculated numerically and is found to be less than one. This ensures that $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty < 1$, a result that is independent of the system size. While $\|\mathbb{P}_1\|'_\infty = \|\mathbb{P}_2\|'_\infty = 1$ for any $\chi > 0$, both norms vanish in the limit $\chi = 0$. Consequently, at $\chi = 0$, the vanishing of $\|\mathbb{P}_2\|'_\infty$ simplifies the bound to $\|\langle \mathbb{W}_2 \rangle \langle \mathbb{W}_1 \rangle\|'_\infty \leq \sigma_2^4$, thereby recovering the unitary limit.

APPENDIX F: HAAR ORTHOGONAL CIRCUIT FOR GENERAL QUDIT SIZE q

For a Haar orthogonal circuit of a general qudit size q , the steady state is defined in terms of the function

$$\rho_{p,s_p}^\infty(t) = \sum_q \gamma_p(t) \gamma_q^*(t) \prod_x \chi_{s_{p_x}}(p_x, q_x), \quad (\text{F1})$$

where each $s_{p_x} \in \{+1, -1\}$ and $\chi_{s_{p_x}}(p_x, q_x)$ is defined in Eq. (56). The function $\rho_{p,s_p}(t)$ follows a closed Markovian evolution equation,

$$\rho_{p,s_p}(t) = \sum_{a,s_a} \rho_{a,s_a}(t-1) |W_{ap,s_a s_p}|^2. \quad (\text{F2})$$

The transition probabilities $\langle |W_{ap}|^2 \rangle$ in this stochastic model factorize into pair contributions from the two-qudit gates, with the specific layer of gates depending on the parity of the timestep t ,

$$\langle |W_{ap,s_a s_p}|^2 \rangle = \prod_{x \text{ even}} \begin{cases} \langle |W_{ap,s_a s_p;x,x+1}|^2 \rangle & \text{for } t \text{ even,} \\ \langle |W_{ap,s_a s_p;x-1,x}|^2 \rangle & \text{for } t \text{ odd.} \end{cases} \quad (\text{F3})$$

The pair-transition probability $\langle |W_{ap;s_a s_p;x,y}|^2 \rangle$ for the Haar orthogonal circuit at a generic dimension q can be expressed in

the form of Eq. (29). For the Haar-random case, the coefficients are $\mathcal{P}_2 = \mathcal{P}_3 = 0$, and

$$\mathcal{P}_1 = \frac{\eta_{s_p}(p)}{q^2 - 1} \times \begin{cases} \frac{2}{q^2 + 2} & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = 1, \\ \frac{2}{q^2} & \text{if } s_{p_x} s_{p_{x+1}} = s_{a_x} s_{a_{x+1}} = -1, \\ 0 & \text{if } s_{p_x} s_{p_{x+1}} \neq s_{a_x} s_{a_{x+1}}, \end{cases} \quad (\text{F4})$$

where the function $\eta_{s_p}(p)$ is defined as

$$\eta_{s_p}(p) = \frac{(1 + s_{p_x} \Delta(p_x))(1 + s_{p_{x+1}} \Delta(p_{x+1}))}{4}, \quad (\text{F5})$$

with

$$\Delta(p_x) = \begin{cases} \delta_{p'_x,0}, & q \text{ odd,} \\ \delta_{p'_x,0} + \delta_{p'_x,\frac{q}{2}} (-1)^{p'_x}, & q \text{ even.} \end{cases} \quad (\text{F6})$$

APPENDIX G: BROWNIAN-MOTION CIRCUITS

To illustrate our findings, we use the Brownian-motion special orthogonal ensemble as an example, see Eqs. (64) and (65). In this ensemble λ is a parameter that interpolates between the trivial circuit ($\lambda = 0$) and the Haar-random circuit ($\lambda \rightarrow \infty$).

Based on the result of Ref. [54], we then find the following expressions for the moments of the orthogonal Brownian-motion two-qudit evolution operators:

$$\begin{aligned} \mathcal{R}_{1,1;1,1} &= \frac{3}{q^8} - 3 \frac{q^2 - 1}{q^8} e^{-\lambda} [e^{\frac{2\lambda}{q^2}} q^2 - (q^2 + 2)] + \frac{1}{24q^8} e^{-2\lambda} [e^{\frac{8\lambda}{q^2}} q^2 (q^2 - 1)(q^2 - 2)(q^2 - 3) \\ &\quad + 9e^{\frac{4\lambda}{q^2}} q^2 (q^2 - 1)(q^2 + 2)(q^2 - 3) + 4e^{\frac{2\lambda}{q^2}} q^2 (q^2 + 2)(q^2 + 1)(q^2 - 3) \\ &\quad + 9(q^2 - 1)(q^2 - 2)(q^2 + 1)(q^2 + 4) + e^{-\frac{4\lambda}{q^2}} q^2 (q^2 - 1)(q^2 + 1)(q^2 + 6)], \end{aligned} \quad (\text{G1})$$

$$\begin{aligned} \mathcal{R}_{1,1;2} &= \frac{(q^2 - 1)(q^2 + 2)}{q^6} e^{-\lambda} - \frac{q^2 - 1}{24q^6} e^{-2\lambda} [e^{\frac{8\lambda}{q^2}} q^2 (q^2 - 2)(q^2 - 3) \\ &\quad + 3e^{\frac{4\lambda}{q^2}} q^2 (q^2 + 2)(q^2 - 3) - 3(q^2 - 2)(q^2 + 1)(q^2 + 4) - e^{-\frac{4\lambda}{q^2}} q^2 (q^2 + 1)(q^2 + 6)], \end{aligned} \quad (\text{G2})$$

$$\begin{aligned} \mathcal{R}_{2,2} &= \frac{3}{q^4} - \frac{1}{q^4} e^{-\lambda} [e^{\frac{2\lambda}{q^2}} q^2 (q^2 - 1) - (q^2 - 2)(q^2 + 1)] + \frac{1}{24q^4} e^{-2\lambda} [e^{\frac{8\lambda}{q^2}} q^2 (q^2 - 1)(q^2 - 2)(q^2 - 3) \\ &\quad - 3e^{\frac{4\lambda}{q^2}} q^2 (q^2 - 1)(q^2 + 2)(q^2 - 3) + 4e^{\frac{2\lambda}{q^2}} q^2 (q^2 + 1)(q^2 + 2)(q^2 - 3) \\ &\quad - 3(q^2 - 1)(q^2 + 1)(q^2 - 4)(q^2 + 4) + e^{-\frac{4\lambda}{q^2}} q^2 (q^2 - 1)(q^2 + 1)(q^2 + 6)], \end{aligned} \quad (\text{G3})$$

$$\mathcal{R}_{1,3} = -\frac{1}{24q^2} e^{-2\lambda} [e^{\frac{8\lambda}{q^2}} (q^2 - 1)(q^2 - 2)(q^2 - 3) - e^{\frac{2\lambda}{q^2}} (q^2 + 1)(q^2 + 2)(q^2 - 3) + e^{-\frac{4\lambda}{q^2}} (q^2 - 1)(q^2 + 1)(q^2 + 6)], \quad (\text{G4})$$

$$\begin{aligned} \mathcal{R}_4 &= -\frac{q^2 - 1}{24q^2} e^{-2\lambda} [e^{\frac{8\lambda}{q^2}} q^2 (q^2 - 2)(q^2 - 3) - 3e^{\frac{4\lambda}{q^2}} q^2 (q^2 + 2)(q^2 - 3) \\ &\quad + 3(q^2 + 1)(q^2 + 4)(q^2 - 2) - e^{-\frac{4\lambda}{q^2}} q^2 (q^2 + 1)(q^2 + 6)], \end{aligned} \quad (\text{G5})$$

$$\mathcal{R}_{1,1} = \frac{1}{q^4} + \frac{q^2 - 1}{2q^4} e^{-\lambda} (q^2 e^{\frac{2\lambda}{q^2}} + 2 + q^2), \quad (\text{G6})$$

$$\mathcal{R}_2 = \frac{1}{q^2} - \frac{q^2 - 1}{2q^2} e^{-\lambda} (q^2 e^{\frac{2\lambda}{q^2}} - (q^2 + 2)). \quad (\text{G7})$$

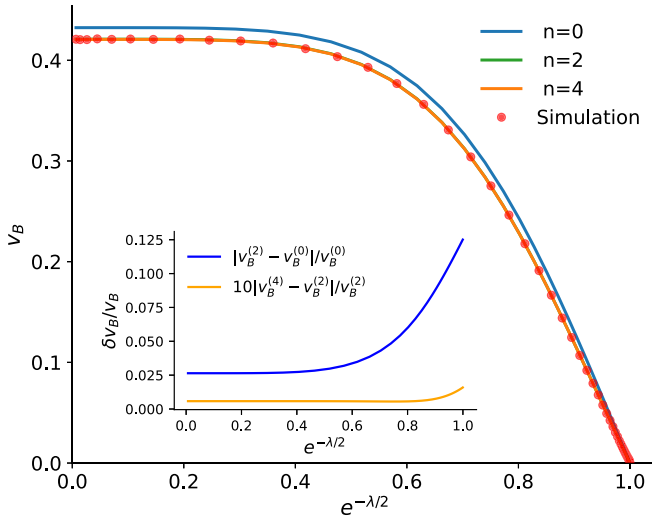


FIG. 9. Butterfly velocity for Brownian symplectic circuits ($q = 2$), plotted analogously to Fig. 3. The simulation parameters ($L = 320$, $t = 200$) are the same as in Fig. 3, while the ensemble average is taken over $N = 2 \times 10^4$ realizations. Note that the analytical results for $n = 2$ and 4 rapidly converge to the numerical data. (Inset) Relative deviations between approximations of orders $n = 4, 2$ and $n = 0$.

For the moments of the symplectic Brownian-motion two-qudit operators, for which Eq. (65) is replaced by

$$\langle H_{ij}(t)H_{kl}(t') \rangle = \frac{\lambda}{q^2}(\delta_{il}\delta_{jk} - \Omega_{ik}\Omega_{jl})\delta(t - t'), \quad (\text{G8})$$

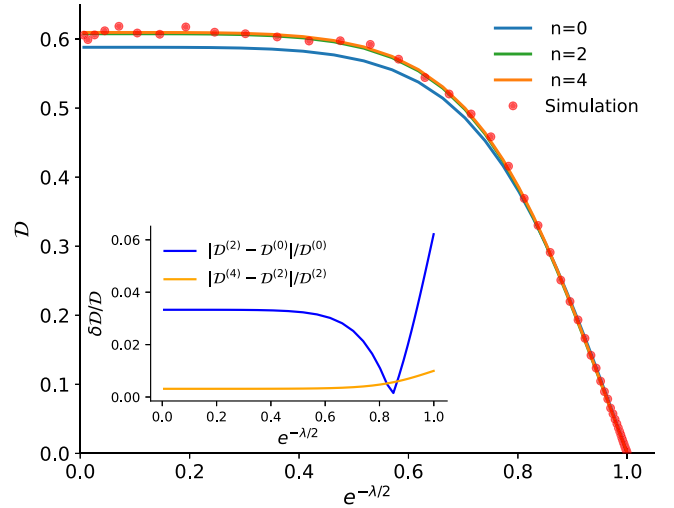


FIG. 10. Analogous to Fig. 9, but for the diffusion constant D in the Brownian symplectic ensemble. Deviations in the numerical data stem from finite-sample statistical noise. The analytical predictions for $n = 2$ and 4 coincide, demonstrating the rapid convergence of the expansion.

where the antisymmetric $q^2 \times q^2$ matrix Ω is defined in Eq. (17). In this case, one has to replace q^2 by $-q^2$ in Eqs. (G1)–(G7). Results for the butterfly velocity and diffusion constant in the Brownian symplectic circuit are shown in Figs. 9 and 10.

- [1] L. Arnaud and D. Braun, Distribution of interference in random quantum algorithms, *Phys. Rev. A* **75**, 062314 (2007).
- [2] L. Arnaud and D. Braun, Efficiency of producing random unitary matrices with quantum circuits, *Phys. Rev. A* **78**, 062329 (2008).
- [3] W. Brown and O. Fawzi, Scrambling speed of random quantum circuits, [arXiv:1210.6644](#).
- [4] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Quantum entanglement growth under random unitary dynamics, *Phys. Rev. X* **7**, 031016 (2017).
- [5] A. Chan, A. De Luca, and J. T. Chalker, Solution of a minimal model for many-body quantum chaos, *Phys. Rev. X* **8**, 041019 (2018).
- [6] A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, *Phys. Rev. X* **8**, 021014 (2018).
- [7] C. W. von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, OTOCs, and entanglement growth in systems without conservation laws, *Phys. Rev. X* **8**, 021013 (2018).
- [8] T. Rakovszky, F. Pollmann, and C. von Keyserlingk, Diffusive hydrodynamics of out-of-time-ordered correlators with charge conservation, *Phys. Rev. X* **8**, 031058 (2018).
- [9] V. Khemani, A. Vishwanath, and D. A. Huse, Operator spreading and the emergence of dissipative hydrodynamics under unitary evolution with conservation laws, *Phys. Rev. X* **8**, 031057 (2018).
- [10] M. P. Fisher, V. Khemani, A. Nahum, and S. Vijay, Random quantum circuits, *Annu. Rev. Condens. Matter Phys.* **14**, 335 (2023).
- [11] B. Skinner, Lecture notes: Introduction to random unitary circuits and the measurement-induced entanglement phase transition, [arXiv:2307.02986](#).
- [12] S. Xu and B. Swingle, Locality, quantum fluctuations, and scrambling, *Phys. Rev. X* **9**, 031048 (2019).
- [13] G. Styliaris, N. Anand, and P. Zanardi, Information scrambling over bipartitions: Equilibration, entropy production, and typicality, *Phys. Rev. Lett.* **126**, 030601 (2021).
- [14] S. Xu and B. Swingle, Scrambling dynamics and out-of-time-ordered correlators in quantum many-body systems, *PRX Quantum* **5**, 010201 (2024).
- [15] X. Mi, P. Roushan, C. Quintana, S. Mandrá, J. Marshall, C. Neill, F. Arute, K. Arya, J. Atalaya, R. Babbush, *et al.*, Information scrambling in quantum circuits, *Science* **374**, 1479 (2021).
- [16] S. Vardhan and J. Wang, Free mutual information and higher-point otc, [arXiv:2509.13406](#).
- [17] G. Q. AI, Observation of constructive interference at the edge of quantum ergodicity, *Nature (London)* **646**, 825 (2025).
- [18] Z. Tan and P. W. Brouwer, Operator spreading in random unitary circuits with unitary-invariant gate distributions, *Phys. Rev. B* **111**, 184301 (2025).
- [19] J. Haferkamp and N. Hunter-Jones, Improved spectral gaps for random quantum circuits: Large local dimen-

- sions and all-to-all interactions, *Phys. Rev. A* **104**, 022417 (2021).
- [20] O. Shaya, Z. Holmes, C. Hirche, and A. Angrisani, On the complexity of quantum states and circuits from the orthogonal and symplectic groups, *J. Phys. A: Math. Theor.* **59**, 205302 (2026).
- [21] P. J. Forrester, *Log-gases and Random Matrices* (Princeton University Press, Princeton, NJ, 2010).
- [22] M. L. Mehta, *Random Matrices* (Elsevier, Amsterdam, 2004).
- [23] A. K. Hashagen, S. T. Flammia, D. Gross, and J. J. Wallman, Real randomized benchmarking, *Quantum* **2**, 85 (2018).
- [24] M. West, A. A. Mele, M. Larocca, and M. Cerezo, Random ensembles of symplectic and unitary states are indistinguishable, *arXiv:2409.16500*.
- [25] T. Schuster, J. Haferkamp, and H.-Y. Huang, Random unitaries in extremely low depth, *Science* **389**, 92 (2025).
- [26] L. Grevink, J. Haferkamp, M. Heinrich, J. Helsen, M. Hinsche, T. Schuster, and Z. Zimborás, Will it glue? On short-depth designs beyond the unitary group, *arXiv:2506.23925*.
- [27] A. Sauliere, B. Magni, G. Lami, X. Turkeshi, and J. De Nardis, Universality in the anticoncentration of chaotic quantum circuits, *Phys. Rev. B* **112**, 134312 (2025).
- [28] B. Magni, M. Heinrich, L. Leone, and X. Turkeshi, Anticoncentration and state design of doped real Clifford circuits and tensor networks, *Phys. Rev. A*, 062446 (2026) **113**.
- [29] M. West, A. Anna Mele, M. Larocca, and M. Cerezo, Real classical shadows, *J. Phys. A: Math. Theor.* **58**, 245304 (2025).
- [30] K. Khanna, A. Kumar, R. Vasseur, and A. W. W. Ludwig, Random quantum circuits with time-reversal symmetry, *Phys. Rev. Res.* **8**, 013165 (2026).
- [31] N. Hunter-Jones and J. Haferkamp, Ideal random quantum circuits pass the LXEB test, *arXiv:2602.22692*.
- [32] A. Altland and M. R. Zirnbauer, Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, *Phys. Rev. B* **55**, 1142 (1997).
- [33] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Classification of topological insulators and superconductors in three spatial dimensions, *Phys. Rev. B* **78**, 195125 (2008).
- [34] A. Kitaev, Periodic table for topological insulators and superconductors, *AIP Conf. Proc.* **1134**, 22 (2009).
- [35] N. Hunter-Jones, Operator growth in random quantum circuits with symmetry, *arXiv:1812.08219*.
- [36] F. J. Dyson, A Brownian-Motion Model for the Eigenvalues of a Random Matrix, *J. Math. Phys.* **3**, 1191 (1962).
- [37] F. J. Dyson, A class of matrix ensembles, *J. Math. Phys.* **13**, 90 (1972).
- [38] M. L. Mehta and A. Pandey, On some Gaussian ensembles of Hermitian matrices, *J. Phys. A: Math. Gen.* **16**, 2655 (1983).
- [39] M. L. Mehta, Random matrices and matrix models: The JNU lectures, *Pramana - J. Phys.* **48**, 7 (1997).
- [40] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge university press, Cambridge, 2012).
- [41] B. Bertini, P. W. Claeys, and T. Prosen, Exactly solvable quantum many-body dynamics from space-time duality, *Rev. Mod. Phys.* **98**, 025001 (2026).
- [42] P. Zanardi, C. Zalka, and L. Faoro, Entangling power of quantum evolutions, *Phys. Rev. A* **62**, 030301(R) (2000).
- [43] X. Wang, B. C. Sanders, and D. W. Berry, Entangling power and operator entanglement in qudit systems, *Phys. Rev. A* **67**, 042323 (2003).
- [44] T. Zhou and A. Nahum, Emergent statistical mechanics of entanglement in random unitary circuits, *Phys. Rev. B* **99**, 174205 (2019).
- [45] R. Suzuki, H. Katsura, Y. Mitsuhashi, T. Soejima, J. Eisert, and N. Yoshioka, More global randomness from less random local gates, *arXiv:2410.24127*.
- [46] Y. Mitsuhashi, R. Suzuki, T. Soejima, and N. Yoshioka, Unitary designs of symmetric local random circuits, *Phys. Rev. Lett.* **134**, 180404 (2025).
- [47] J. Patera and H. Zassenhaus, The Pauli matrices in n dimensions and finest gradings of simple Lie algebras of type A_{n-1} , *J. Math. Phys.* **29**, 665 (1988).
- [48] P. W. Brouwer and C. W. J. Beenakker, Diagrammatic method of integration over the unitary group, with applications to quantum transport in mesoscopic systems, *J. Math. Phys.* **37**, 4904 (1996).
- [49] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19**, 999 (1978).
- [50] S. Samuel, $U(N)$ integrals, $1/N$, and the De Wit-'t Hooft anomalies, *J. Math. Phys.* **21**, 2695 (1980).
- [51] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic group, *Commun. Math. Phys.* **264**, 773 (2006).
- [52] S. Matsumoto, Weingarten calculus for matrix ensembles associated with compact symmetric spaces, *Random Matrices: Theory Appl.* **02**, 1350001 (2013).
- [53] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. Math.* **38**, 857 (1937).
- [54] Z. Tan and P. W. Brouwer, Brownian motion in orthogonal and symplectic groups, *arXiv:2607.05094*.
- [55] H. D. Macedo and J. N. Oliveira, Typing linear algebra: A biproduct-oriented approach, *Sci. Comput. Program.* **78**, 2160 (2013).
- [56] D. Dummit and R. Foote, *Abstract Algebra* (Wiley, Hoboken, NJ, 2003).
- [57] J. B. Fraleigh, *A First Course in Abstract Algebra* (Pearson Education, Boston, MA, 2003).
- [58] M. R. Sepanski, *Compact Lie Groups* (Springer, New York, 2007).
- [59] W. Fulton and J. Harris, *Representation Theory*, Graduate Texts in Mathematics (Springer, New York, 2004), Vol. 129.
- [60] B. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Graduate Texts in Mathematics (Springer, New York, 2003).
- [61] L. C. Grove, *Classical Groups and Geometric Algebra*, Graduate Studies in Mathematics (American Mathematical Society, Providence, RI, 2002), Vol. 39.