

Translation symmetry restoration in integrable systems: The noninteracting case

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The study of symmetry restoration has recently emerged as a fruitful means to extract high-level information on the relaxation of quantum many-body systems. However, while the restoration of internal symmetries has been investigated intensively, that of spatial symmetries has hitherto only been considered in the context of random unitary circuits. Here, we present a complementary study of translation symmetry restoration in integrable systems. In particular, we consider a one-dimensional chain of spinless, noninteracting fermions quenched from a $\nu > 1$ shift invariant state, and follow the local restoration of one-site shift invariance using the Frobenius distance ΔF_A between the state on a subsystem and its symmetrized counterpart. Distinct from the case of random unitary circuits, where symmetry restoration occurs abruptly for times proportional to the subsystem size, we find that symmetry here is restored smoothly and over timescales of the order of the subsystem size squared. We also find that the so-called “quantum Mpemba effect” is readily observed. Most importantly, we show that—in contrast to the case of continuous internal symmetries—this discrete symmetry restoration is not qualitatively described by a quasiparticle picture for ΔF_A , and therefore goes beyond the hydrodynamic description. Our results can be directly extended to higher dimensions.

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Introduction. Studying the dynamics of quantum many-body systems is often complicated by the fact that local observables generate specific artifacts that are hard to filter out. Recent research has shown that a convenient way to circumvent this problem is to study the restoration of the symmetries of the dynamics that are broken by the initial state, as measured via the distance between the state of a local subsystem and its symmetrized counterpart [1]. This has revealed a surprising phenomenon connected with the equilibration process: sometimes the symmetry is restored faster when the initial state is less symmetric. This effect bears a conceptual similarity with the Mpemba effect [2], occurring when warmer water freezes faster than colder water, and has been accordingly dubbed the quantum Mpemba effect (QME) [1,3]. Since its first observation, QME has been theoretically predicted to occur in many different classes of systems ranging from integrable (free and interacting) [4–18] to strongly nonintegrable [19–22] systems and even observed in trapped ion quantum simulators [23].

Despite the intense research on the subject, most of the studies conducted so far have considered the restoration of internal symmetries— $U(1)$ symmetries in the vast majority of the cases. This restriction comes from a technical, rather than conceptual, reason: namely, these symmetries generate

particularly simple charges that simplify the analysis. Studying the restoration of other kinds of symmetries, however, can drastically widen the scope of this line of research. For instance, spatial symmetries are arguably a more generic feature of quantum many-body systems than internal ones. The study of spatial symmetry restoration has been initiated, and so far only considered, in Ref. [21], which focused on a class of ergodic (quantum chaotic) quantum systems—random unitary circuits—and demonstrated the occurrence of QME for intermediate sizes. Here, we complement this study by considering translation symmetry restoration in integrable systems under constant Hamiltonian evolution. For the sake of simplicity, we focus on a simple noninteracting case.

More specifically, we consider an infinite (or periodic) one-dimensional chain of spinless fermions on the lattice, whose dynamics are described by the tight binding Hamiltonian

$$H_{\text{TB}} = \sum_j (c_j^\dagger c_{j+1} + \text{H.c.}). \quad (1)$$

Although this Hamiltonian is invariant under single-site translations— $[H_{\text{TB}}, T] = 0$, where T is the translation operator $T c_j T^{-1} = c_{j+1}$ —we consider initial states that are only invariant under ν -site shifts. Specifically, we consider product states of the form $|\Psi\rangle = \otimes_j |\Psi_\nu\rangle$, where $|\Psi_\nu\rangle$ is a Gaussian state defined on ν sites. The dynamics of the tight-binding model from states of this kind were recently considered in Ref. [24], which studied lattices of arbitrary dimension $d \geq 1$. Our upcoming discussion can similarly be generalized to higher dimensions.

Following Ref. [21], to study the breaking and local restoration of the translation symmetry in a subsystem A , we use the *Frobenius distance* between the reduced density matrix $\rho_A(t)$ and its symmetrized counterpart. These matrices are defined as

$$\bar{\rho}_A(t) = \frac{1}{v} \sum_{j=0}^{v-1} \rho_A^{(j)}(t), \quad \rho_A^{(j)}(t) \equiv \text{tr}_A[T^j \rho(t) T^{-j}], \quad (2)$$

where $\rho(t) = |\Psi_t\rangle\langle\Psi_t|$ is the state of the system at time t . Consequently, the (normalized) Frobenius distance reads

$$\Delta F_A(t) \equiv \frac{\|\rho_A(t) - \bar{\rho}_A(t)\|}{\|\rho_A(t)\|}, \quad (3)$$

where $\|A\| \equiv \sqrt{\text{tr}(AA^\dagger)}$ is the Frobenius norm and $\rho_A = \text{tr}_{\bar{A}}(\rho)$ is the reduced density matrix. As discussed in Ref. [21], when studying the restoration of spatial symmetries, Eq. (3) is more convenient than the entanglement asymmetry introduced in Ref. [1] because the latter is much harder to access analytically. In our case, however, the two quantities become equivalent at leading order in t and $|A| = l$. Furthermore, and which will be explicitly justified in due course, we note that the following discussion on spatial symmetry breaking should be compatible with any measure of symmetry breaking $f_A(\rho, t)$ that satisfies both $f_A \geq 0 \forall t$ and $f_A = 0 \iff \bar{\rho}_A = \rho_A$ [25].

Exact dynamics. The dynamics of $\Delta F_A(t)$ can be computed exactly by exploiting the Gaussianity of both the initial state and the dynamics, which allows us to write the Frobenius distance in terms of determinants of $l \times l$ matrices in the presence of number conservation [25]. For instance, in Fig. 1 we present the translation symmetry restoration in two families of states,

$$\begin{aligned} |\psi_2(\lambda_2)\rangle &= \prod \frac{(c_{2j}^\dagger + \lambda_2 c_{1+2j}^\dagger)}{\sqrt{1 + \lambda_2^2}} |0\rangle, \\ |\psi_4(\lambda_4)\rangle &= \prod c_{4j}^\dagger \frac{(c_{2+4j}^\dagger + \lambda_4 c_{1+4j}^\dagger)}{\sqrt{1 + \lambda_4^2}} |0\rangle. \end{aligned} \quad (4)$$

Interestingly, we see that the Frobenius distance decays as a power law in time in the asymptotic limit. This is in contrast with the sharp, initial-state-independent restoration at $t \propto l$ observed in random unitary circuits [21]. Moreover, the figure displays two distinct cases where QME is observed, as indicated by the shaded orange boxes. The later crossing occurs between an instance of $|\psi_2(\lambda_2)\rangle$ and one of $|\psi_4(\lambda_4)\rangle$, explicitly involving two states with different degrees of initial symmetry breaking. The earlier crossing occurs between two instances of $|\psi_4(\lambda_4)\rangle$, and the amount of symmetry breaking is implicitly controlled by the superposition parameter λ_4 . However, this figure also demonstrates that QME does not always occur, as it is absent between $\lambda_2 = 0.15$ and $\lambda_4 = 0.1$.

Quasiparticle picture. In many cases, the asymptotic dynamics of free fermions at sufficiently large scales can be described by the quasiparticle picture *à la* Calabrese and Cardy [26]. The intuition behind this picture is that, in integrable systems, the large-scale dynamics can be described in terms of the motion of stable quasiparticles created by the initial state that move as free classical particles with

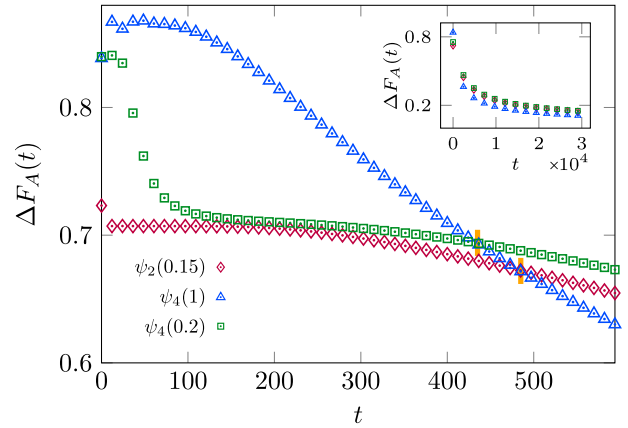


FIG. 1. The Frobenius distance $\Delta F_A(t)$, obtained via exact diagonalization using Gaussian methods for subsystem size $l = 100$ and different initial states. The states are of the form $|\psi_v(\lambda)\rangle$ where v is the translational invariance and λ is the superposition parameter, and their exact definitions are given in Eq. (4). Crossings between the Frobenius distance of different states, which correspond to instances of QME, are indicated by the shaded orange boxes. Inset: Main plot extended to late times, showing that no further crossings occur.

fixed velocity. The correlations among the quasiparticles are generated by the initial state but, crucially, *not by the dynamics*, which is able to reduce the calculation of how relevant quantities evolve in time to an exercise of simple kinematics [27]. Among other dynamical quantities, this picture is most celebrated for successfully capturing the entanglement growth [26, 28–30], as well as the restoration of internal symmetries [1, 14]. Here, we develop a similar approach to describe the asymptotic behavior of $\Delta F_A(t)$ in the case of translation symmetry restoration.

In our case, the quasiparticles are nothing but a semiclassical version of the momentum modes whose velocities are given by $v(k) = \epsilon'(k) = -2 \sin(k)$, where $\epsilon(k) = \cos(k)$ is the dispersion relation of the Hamiltonian in Eq. (1). In order to assign them well-defined positions, one must operate at an “intermediate resolution,” dividing space into unit cells of size δ that satisfy $a \ll \delta \ll l$ for lattice spacing a and performing a *partial Fourier transform* of the initial state within each cell [24, 31]. The resulting picture then reduces to simple kinematics when the initial correlations are confined to small, regular groups of quasiparticles, as is the case for us: one finds that the translational symmetry of the initial state confers a v -plet structure to these correlations [24]. More precisely, a given mode in the reduced Brillouin zone $k \in [0, 2\pi/v)$ is correlated only with those from the same cell with $k' \in \{k + 2\pi/v, k + 4\pi/v, \dots, k + 2(v-1)\pi/v\}$.

Armed with this fact, we assume that the quench generates v -plets in localized in cells of size $\Delta \ll l$ and their full evolution can therefore be classically determined. This allows us to write the semiclassical representation of the time-evolved state [24, 31] as $\rho_A(t) = \bigotimes_x \bigotimes_p \rho_A(x, p)$, where $x \in \mathbb{Z}_{L/\Delta}$, $p \in \frac{2\pi}{\Delta} \mathbb{Z}_{\Delta/v}$, and $\rho_A(x, p)$ is given in terms of the *semiclassical modes*

$$c_{x,p} = \frac{1}{\sqrt{\Delta}} \sum_{n \in \mathbb{Z}_{\Delta}^d} e^{ipn} c_{\Delta x + n}, \quad p \in \frac{2\pi}{\Delta} \mathbb{Z}_{\Delta}^d. \quad (5)$$

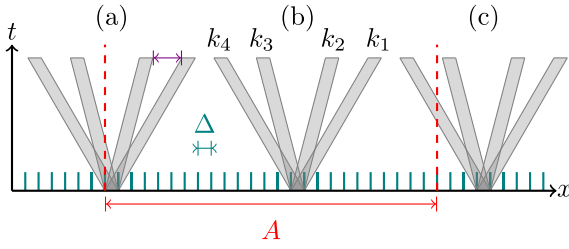


FIG. 2. Different bipartitions of a correlated multiplet for a $v = 4$ initial state with respect to subsystem A . The multiplet is produced in a unit cell of size Δ and each mode $k_n(p) = p + 2\pi n/v$ propagates with fixed velocity $v_n(p) = 2 \sin(p + 2\pi n/v)$. (a) A bipartition in which two modes are inside the subsystem. The purple dimension line indicates a *relative velocity* between adjacent modes, which will be important for later discussion. (b) The bipartition in which all modes are inside the subsystem. This contributes to the Frobenius distance but not to the entanglement entropy (see main text). (c) A bipartition in which a single mode is inside the subsystem. This contributes to the entanglement entropy but not to the Frobenius distance.

More precisely, $\rho_A(x, p)$ is formed by considering each v -plet of correlated modes produced by the initial state tracing out all those that are out of the subsystem A at time t [24]. Note that, because $|\Psi_0\rangle$ is Gaussian, the correlations involving $\rho_A(x, p)$ can be computed from the two-point *correlation matrix*

$$[C_A(x, p)]_{k,k'} \equiv \text{tr}(\rho_A(x, p) \hat{c}_{(x,p+k)}^\dagger \hat{c}_{(x,p+k')}). \quad (6)$$

To compute the Frobenius distance of Eq. (3), we recast the j -site shifted operators $\rho_A^{(j)}(t)$ of Eq. (2) in this semiclassical form, replacing $\rho_A(x, p)$ with $\rho_A^{(j)}(x, p) \equiv T_A^j \rho_A(x, p) T_A^{-j}$ where T_A the periodic one-site translation operator in the subsystem A [32]. This eventually gives us [25]

$$\Delta F_A(t)|_{\text{QP}} = \sqrt{\frac{(v-1)}{v} - \frac{1}{v} \sum_{j=1}^{v-1} r_A^{(j)}(t)}, \quad (7)$$

where we set

$$r_A^{(j)}(t) = \exp \left[\int_{A \times [0, \frac{2\pi}{v}]} \frac{dx dk}{2\pi} \ln \frac{\text{tr}[\rho_A(x, k) \rho_A^{(j)}(x, k)]}{\text{tr}[\rho_A^2(x, k)]} \right]. \quad (8)$$

Two interesting comments are in order concerning this expression. First, although Eqs. (7) and (8) allow one to compute the Frobenius distance by establishing how multiplets of correlated quasiparticles are split between A and $\bar{A}$, the contributions controlling symmetry restoration differ from those controlling the saturation of spatial entanglement [26]. Indeed, while bipartitions where one single quasiparticle is either in or out of the system, e.g., Fig. 2(a), contribute to the spreading of entanglement, they *do not* contribute to Eqs. (7) and (8). This is easily understood by viewing Eq. (8) together with Eq. (6): the translation operator is inert on a Gaussian single-particle density matrix, and so, for these bipartitions, the dynamical term in Eq. (7) must vanish. Instead, rather than the longest-surviving bipartitions, constituting a single mode inside the subsystem, it is the next longest-surviving bipartitions—those with exactly *two* modes inside

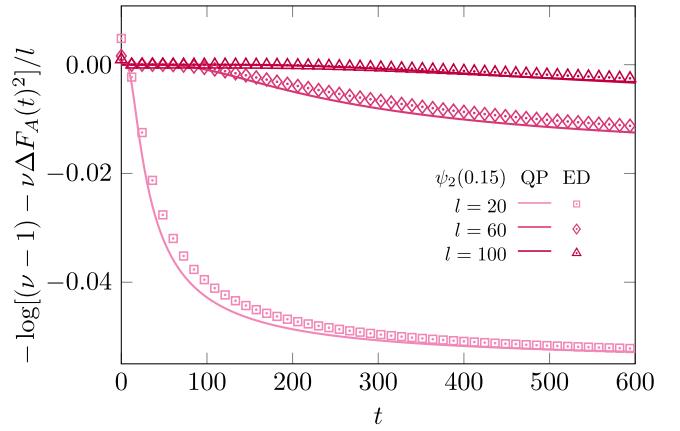


FIG. 3. (a) Comparison of exact diagonalization (data points) and quasiparticle method (solid lines) for $-\ln[(v-1) - v\Delta F_A(t)^2]/l$ as subsystem size is increased, for $\psi_2(0.15)$. For (b) [the $\psi_4(1)$ state], refer to the Supplemental Material [25].

the subsystem, e.g., Fig. 2(c)—which determine the asymptotic behavior of the Frobenius distance. On the other hand, it is longest-surviving bipartitions that characterize the hydrodynamic (fixed t/l) regime wherein the state of the subsystem is locally stationary [33–35]. That these bipartitions do not enter into the symmetry restoration is a deceptively simple observation that nonetheless anticipates much of what will become apparent in this Letter, namely, that—even when studied *within* the quasiparticle picture—the symmetry restoration goes beyond hydrodynamics. It is also clear that statement is independent of the chosen measure of symmetry breaking, since any such measure can be written in terms of the moments of Eq. (8) for which the argument holds.

Second, Eq. (7) involves two further approximations in addition to the basic quasiparticle picture: (i) In the definition of $\rho_A^{(j)}(x, k)$ we applied the translation only to the subsystem of interest. In other words, we assumed

$$\text{tr}[\rho_A^{(i)}(t) \rho_A^{(j)}(t)] \simeq \text{tr}[\rho_A(t) T_A^{j-i} \rho_A(t) T_A^{i-j}], \quad (9)$$

where $\simeq$ denotes equality at leading order in the subsystem size. This amounts to neglecting the boundary terms induced by a *full-system* translation of the *partial* Fourier creation operator. (ii) We computed the logarithm of the right-hand side of Eq. (9) using the quasiparticle picture. Although one can rigorously show that this correctly accounts for the leading order in the subsystem size [14,28], there are generically nonextensive corrections that we neglected. This step is inoffensive in the case of continuous symmetries—see, e.g., Refs. [3,4,11,14]—where the leading contribution to $\Delta F_A(t)$ comes from a window of the integral over continuous charge sectors, analogous to the discrete summation in Eq. (7), in which the integration variable α is infinitesimal. In our case, however, the analogous summation only involves terms with finite $i - j$, and so forces us to reconsider this point.

Concerning (i), we find that Eq. (9) does indeed hold, with our results reported in Fig. 3 demonstrating a rapid convergence of the shift-dependent quantity $-\ln[(v-1) - v\Delta F_A(t)^2]/l$ obtained via exact diagonalization (ED) to its quasiparticle picture (QP) prediction for increasing subsystem

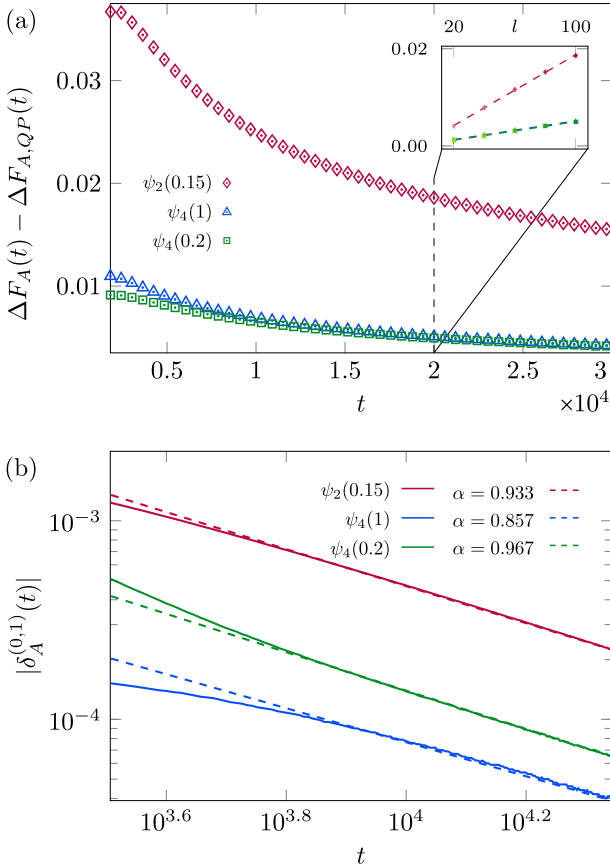


FIG. 4. (a) The difference between exact diagonalization and quasiparticle solutions to the Frobenius distance, $\Delta F_A(t) - \Delta F_{A,QP}(t)$, for different initial states and subsystem size $l = 100$. (b) The leading nonextensive correction $\delta_A^{(0,1)}(t)$, defined in Eq. (10), for each of these initial states and subsystem size $l = 200$. This panel is displayed on log-log scale at late times only, and the slopes of the dashed lines approximate the decay of the correction at late times as $t^{-\alpha}$. Inset (a): Example, shown for $t = 20\,000$, of the linear scaling of the difference $\Delta F_A(t) - \Delta F_{A,QP}(t)$ with system size.

size. The innocent-looking neglect of nonextensive terms (ii), however, leads to substantial disagreements between ED and QP that appear to worsen for increasing values of l , as shown in the inset of Fig. 4(a). This can be understood by noting that the corrections

$$\delta_A^{(i,j)}(t) = \ln \frac{\text{tr}[\rho_A^{(i)}(t)\rho_A^{(j)}(t)]}{\text{tr}[\rho_A^{(0)}(t)^2]} - \ln r_A^{(j-i)}(t), \quad (10)$$

generate nontrivial coefficients for the terms in the sum in Eq. (7). In fact, as demonstrated by the second panel of Fig. 4, for large times one has $\delta_A^{(i,j)}(t) \simeq t^{-\alpha}$ with $\alpha < 1$, while one can show that $\ln r_A^{(j-i)}(t) \simeq t^{-1}$ [25]. This means that, for large enough times and any fixed l , the leading contribution to the ratio of traces is given by $\delta_A^{(i,j)}(t)$, not $\ln r_A^{(j-i)}(t)$. Therefore, very surprisingly, we find that translation-symmetry restoration in noninteracting systems on the lattice, and the associated QME, are *not* described by the quasiparticle picture.

Scaling limit. Although we saw that the quasiparticle picture does not describe the large time limit of a fixed subsystem, one can still wonder whether Eq. (7) provides a

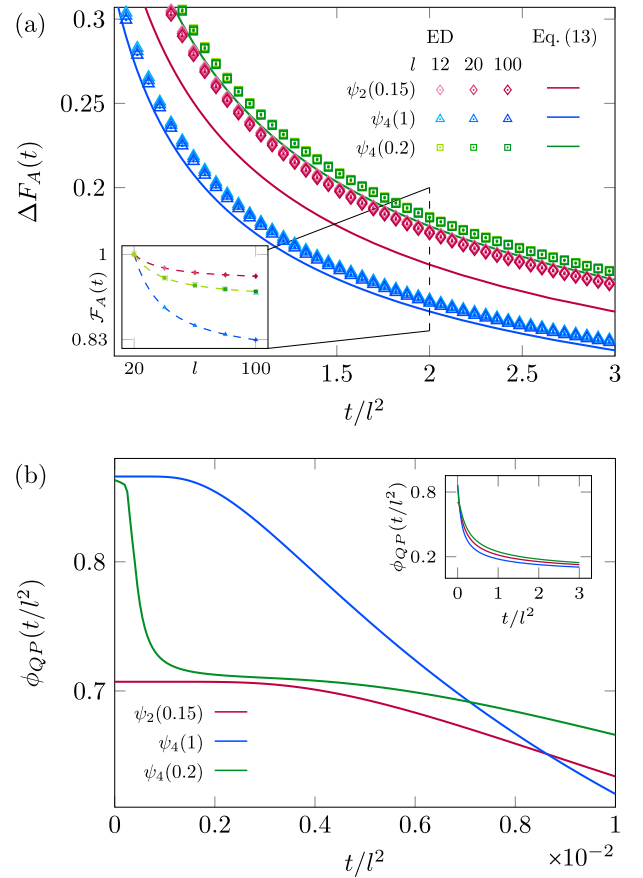


FIG. 5. The Frobenius distance in the scaling limit t/l^2 for different initial states and subsystem sizes. (a) compares the method of exact diagonalization (data points) at increasing subsystem size to the functional form $\phi_{QP}(\xi)$ given by Eq. (11) obtained in the $l \rightarrow \infty$ limit of the quasiparticle method (solid lines), while (b) shows $\phi_{QP}(\xi)$ at earlier times. Inset (a): Example, shown for $t/l^2 = 2$, of the scaling with system size compared to the prediction of $\phi_{QP}(\xi)$, as defined via the normalized function $\mathcal{F}_A(t, l) = [\Delta F_A(t, l) - \phi_{QP}(\xi)] / [\Delta F_A(t, l_{\min}) - \phi_{QP}(\xi)]$. The data are well described by a power-law fit of the form $al^{-b} + c$, where going from top to bottom we find $a = 1.13, 2.02, 3.98, b = 1.02, 0.97, 1.03$, and $c = 0.95, 0.91, 0.78$. Inset (b): The main plot of (b) extended to late times, showing that no further crossings occur.

quantitative description of symmetry restoration in a certain “scaling limit,” i.e., when t and l are simultaneously taken to infinity along a particular curve. The usual “ballistic” scaling limit of $t, l \rightarrow \infty$ with fixed t/l produces a trivial behavior of the Frobenius distance, which remains pinned to the initial value and shows no symmetry restoration. Remarkably, however, Fig. 5(a) shows that the Frobenius distance approaches a well-defined scaling form in the “diffusive” scaling limit, where both t and l are simultaneously taken to infinity while keeping fixed $\xi = t/l^2$. This scaling limit explicitly emerges from the expansion of Eq. (8) in the $t \rightarrow \infty$ limit [25], and provides an ideal setting for investigating the predictive power of the quasiparticle picture. In particular, noting that this limit takes the form

$$\lim_{l \rightarrow \infty} \ln r_A^{(j)}(\xi l^2) = -\frac{a_j}{\xi}, \quad a_j \geq 0, \quad (11)$$

we see that $\phi_{\text{QP}}(\xi) = \lim_{l \rightarrow \infty} \Delta F_A(\xi l^2)|_{\text{QP}}$ is a positive function of ξ that approaches 0 in the limit of infinite ξ , consistently describing a symmetry restoration process. This function bears explicit dependence on the initial state and, as shown in Fig. 5(b), is able to exhibit Mpemba-type crossings.

To ascertain whether $\phi_{\text{QP}}(\xi)$ gives an exact description of the symmetry restoration in the scaling limit, one should once again estimate the contribution of the corrections $\delta_A^{(i,j)}(t)$. Indeed, although the latter decay in time as $t^{-\alpha}$, they can in principle grow as $l^{\alpha+\beta}$, where $\beta < 1$ is given by the subextensive nature of $\delta_A^{(i,j)}(t)$ in the ballistic scaling limit. Our numerical analysis suggests that $\beta = \alpha$ and, accordingly, the corrections give a finite contribution in the scaling limit: the inset of Fig. 5(a) shows that the difference between the rescaled ED data and $\phi_{\text{QP}}(\xi)$ does not seem to approach 0 for large l . Nevertheless, $\phi_{\text{QP}}(\xi)$ is able to at least give a qualitative description of the symmetry restoration as it captures the correct timescale and remains close to the ED data for increasing l .

Discussion. Our results, which concern local restoration of translational symmetry in the tight-binding model, can be summarized as follows. First, we have seen that this symmetry restoration follows a polynomial law for large times—in contrast with the exponential one observed for random unitary circuits [21]—and can exhibit QME for arbitrary subsystem sizes. In fact, we observed that the Frobenius distance attains a nontrivial scaling form in the “diffusive” limit where time t and subsystem size l are simultaneously taken to infinity with fixed $\xi = t/l^2$. Second, we have found that—although the system is trivially integrable—this symmetry restoration goes beyond the quasiparticle picture and is described by the latter only qualitatively in the diffusive scaling limit. This means in particular that for any given l the asymptotic behavior in time is *not* described by the quasiparticle picture. We attributed this to the discrete nature of the symmetry that is restored.

Although the quasiparticle description is only qualitative, it still gives insight into the physical mechanism behind the occurrence of QME in integrable systems. The multiplets of quasiparticles contributing to the Frobenius distance at late times are generically those with the smallest *relative* velocities between modes. This matches the physical intuition: modes moving with very similar velocity spend more time close to each other and take longer to dephase [36,37]. The relaxation time is therefore determined by the density of modes with near-identical velocities, and by their contribution to the Frobenius distance. On the other hand, the initial value of the Frobenius distance depends on the full distribution of modes, thereby making it possible to manufacture initial states with greater initial Frobenius distance but shorter relaxation time, and vice versa, as required for QME. This mechanism generalizes the one identified in Ref. [11] beyond the case of pair production and makes it obvious that QME is not just a property of the *single-particle* distribution of quasiparticles, but rather of the *two-particle* one. This leads to the conclusion that QME is a phenomenon that fundamentally goes beyond the hydrodynamic description.

We expect the phenomenology described here to be a generic feature of discrete translation symmetry restoration in integrable systems. The latter should also survive small integrability breaking terms provided that the quasiparticle lifetimes extend beyond the onset of the late-time regime (cf. Ref. [38]).

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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