

Solvable embedding mechanism for one-dimensional spinless and Majorana fermions in higher-dimensional spin- $\frac{1}{2}$ magnets

Sumiran Pujari ^{*}

*Department of Physics, Indian Institute of Technology Bombay, Mumbai, Maharashtra 400076, India
and Max Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany*



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We write down a class of two-dimensional quantum spin- $\frac{1}{2}$ Hamiltonians whose eigenspectra are exactly solvable via the Jordan-Wigner transformation. The general structure corresponds to a suitable grid composed of XY or X-Ising spin chains and Z-Ising spin chains and is generalizable to higher dimensions. They can host stacks of one-dimensional spinless fermion liquids with gapless excitations and power-law correlations coexisting with ordered spin moments (localized spinless fermions). Bond-dependent couplings thus can be an alternate mechanism than geometric frustration of $SU(2)$ -symmetric couplings to obtain spinless fermionic excitations. Put in a different way, bond-dependent couplings allow for an embedding of one-dimensional spinless fermion (Tomonaga-Luttinger) liquids and solids and also Majorana excitations in higher dimensions. They can accommodate a simpler set of site-local conserved quantities apart from the more intricate, interlocked set of plaquette-local or bond-local conserved quantities in Kitaev's honeycomb model with Majorana excitations. The proposed grid structure may provide an architecture for quantum engineering with controllable qubits.

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I. INTRODUCTION

Emergent fractionalized excitations are a remarkable manifestation of strongly correlated many-body physics. A canonical example is the one-dimensional (1D) spin- $\frac{1}{2}$ antiferromagnetic chain with fermionic spinon excitations [1] which are colloquially said to be “half” of regular (bosonic) spin-wave excitations or magnons in presence of magnetic ordering. Magnons being close to product quantum states with low quantum entanglement allow one to form a classical image as quanta of smooth undulations of the ordered magnetic moments like phonons for lattice vibrations and photons for electromagnetic waves. Spinons on the other hand may not have simple classical images. They are generally speaking a complicated superposition of the underlying spin states with high quantum entanglement. In 1D one can have images of spinon excitations as dressed defects of some underlying order, e.g., 1D transverse field quantum Ising model or the valence bond solid phase of the 1D Ghosh-Majumdar model where the spinon excitations can be imagined as a dressed domain wall excitation. However, this image does not easily extend to other liquid-like situations especially in $d > 1$. Such emergent excitations are thus fascinating apart from the qualitative new effects they engender.

Finding solvable models where such physics is explicit is of importance as proofs-of-principle. They serve as counterparts to field theoretic arguments that rely on low-energy approximations, perturbative renormalization group treatments and universality [2]. The downside is that solvability is often restricted to points in the phase diagram [3]. The Kitaev honeycomb model is a famous solvable model of fractionalized (free fermionic) Majorana excitations [4–6] on top of a spin liquid (SL) ground state in 2D $S = \frac{1}{2}$ antiferromagnets. In this work, we will write down a set of exactly solvable spin- $\frac{1}{2}$ models with bond-dependent couplings which host spinless fermionic or Majorana excitations akin to the Kitaev model but with rather different physics. The technological application possibilities due to the new physics exist both for materials engineering and artificial quantum technologies. They are discussed in the final Sec. IV (in particular Sec. IV B).

To motivate our line of thought, take an infinite XY spin chain $H = J \sum_{\langle i,j \rangle} (S_i^x S_j^x + S_i^y S_j^y)$ as shown in the top panel of Fig. 1. Following standard notation, $i, j, \dots$ refer to lattice sites, S_i^μ refer to spin- $\frac{1}{2}$ angular momentum operators on site i and $c_i^\dagger, c_i$ refer to fermionic creation and destruction operators at site i . This is solved using the well-known Jordan-Wigner (JW) mapping [7]:

$$\begin{aligned} c_i^\dagger &= \prod_{j<i} (-1)^{S_j^z + 1/2} S_i^+ \\ c_i &= \prod_{j<i} (-1)^{S_j^z + 1/2} S_i^- \\ n_i &\equiv c_i^\dagger c_i = S_i^z + \frac{1}{2}, \end{aligned}$$

^{*}Contact author: sumiran.pujari@iitb.ac.in (on sabbatical leave)

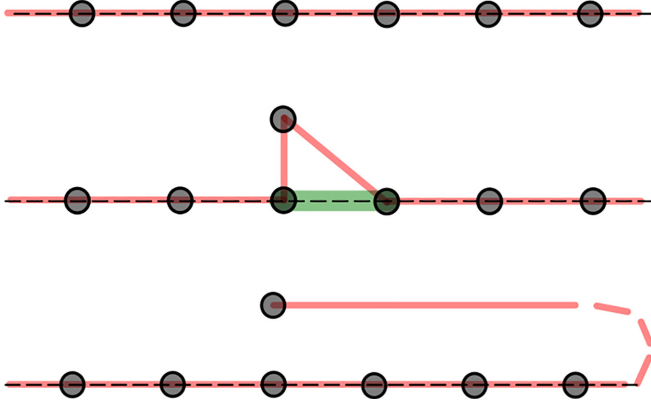


FIG. 1. Gray dots represent $S = \frac{1}{2}$ degrees of freedom, black dashed lines represent the XY couplings, and the light red line represents the Jordan-Wigner string, respectively. The top panel shows a part of an infinite XY chain. The middle and bottom panels show the same XY chain along with an additional single, disconnected spin for two different choices of the Jordan-Wigner string geometry.

where $S_i^\pm = \frac{S_i^x \pm S_i^y}{2}$ often represented by Pauli 2×2 matrices. This map respects the mutual consistency of fermion anticommutation algebra $c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij}$ and spin angular momentum algebra $S_i^\alpha S_j^\beta - S_j^\beta S_i^\alpha = i \delta_{ij} \epsilon_{\alpha\beta\gamma} S_i^\gamma$, where δ , ϵ are the Kronecker and Levi-Civita symbols, respectively. The fermionic vacuum is chosen here to be $\prod_i |\downarrow_i\rangle$. The geometry of “ $j < i$ ” in the above map defines the JW-string. After the mapping, the XY chain becomes the free fermions hopping chain $H = \frac{J}{2} \sum_{\langle i,j \rangle} c_i^\dagger c_j + \text{H.c.}$

Now consider an additional single, disconnected “off-chain” spin as in the middle panel of Fig. 1. The full solution is simply the product state of the disconnected spin and the JW-solved Fermi sea, i.e., $|\psi\rangle = |\sigma\rangle \otimes |\text{JWSL}_c\rangle$, $|\text{JWSL}_c\rangle = \prod_{\epsilon_p \leq 0} c_p^\dagger |\Omega\rangle$, and $\epsilon_p = J \cos(p)$ where p refers to reciprocal momentum. Consider the situation when the off-chain spin is included in the JW-string as in Fig. 1 middle panel. If the off-chain spin is in $|\downarrow\rangle$ state, then the XY chain maps to spinless fermions hopping as before with uniform hopping amplitudes on all bonds. If it is in $|\uparrow\rangle$ state (occupied state), then hopping amplitudes are not uniform. The green-highlighted bond in Fig. 1 middle panel has oppositely signed hopping amplitude compared to the rest of the bonds. However this change of sign in hopping amplitudes is a pure gauge artifact in 1D. One can remove it by the following transformation: $c_i \rightarrow d_i$ on sites left of the green bond, and $c_i \rightarrow -d_i$ to the right. This is special to 1D and hopping amplitude signs can not be “gauged away” in $d > 1$. They rather specify fluxes in the elementary plaquette units in higher dimensions. Thus for any $|\sigma\rangle = a |\uparrow\rangle + b |\downarrow\rangle$ for the disconnected spin, the solution is $(a |\uparrow\rangle \otimes |\text{JWSL}_c\rangle + b |\downarrow\rangle \otimes |\text{JWSL}_d\rangle)$ since $|\text{JWSL}_c\rangle$ and $|\text{JWSL}_d\rangle$ are degenerate.

Of course, one may just not include the disconnected spin in the JW-string, i.e., put it at the “very end” as shown in the bottom panel of Fig. 1 to not affect the JW-hopping amplitudes, which gives $|\sigma\rangle \otimes |\text{JWSL}_c\rangle$. We are not bothering with formal issues akin to the Hilbert’s paradox of the Grand Hotel, i.e., finite size chains can be handled without any essential difference to the argument as the results will show. The above

argument can be applied to any number of disconnected off-chain spins. Thus off-chain spins can bring in only pure gauge artifacts if they are themselves not involved kinetically with the spin chain. Once off-chain spins are involved kinetically, then JW-solvability generally breaks down.

The rest of the paper is organized as follows: In Sec. II, the basic design principle behind the solvable models is given. Section III describes the results obtained for different categories of model constructions as outlined in Sec. II. Section IV summarizes the various magnetic states found in the previous sections including discussions on their stability when going beyond the solvable parameter regime (Sec. IV A), possibilities for physical realization and some imagined technological applications (Sec. IV B).

II. MODEL CONSTRUCTION

The class of models that we will write down now avoid the “off-chain kinetics” as in the gedanken discussed in the previous Sec. I. The model construction will consist of collection of spin chains with on-chain kinetic terms and off-chain spins that are restricted to have (“interaction”) Z-Ising couplings with the on-chain spins. For such a bond-dependent scenario, the JW-transformation can be used fruitfully and gauge artifacts of the above sort are removed by a judicious “snake-like” choice for the JW-string [Fig. 2(a)]. This leads to a 2D grid of XY/X-Ising and Z-Ising chains that may be written in general as

$$H = \sum_{\langle i,j \rangle_\alpha} J_{\langle i,j \rangle_\alpha} (S_i^x S_j^x + \alpha S_i^y S_j^y) + \sum_{\langle i,j \rangle_{Z-1}} J_{\langle i,j \rangle_{Z-1}} S_i^z S_j^z, \quad (1)$$

where $\alpha = 1, \alpha = 0$ correspond to XY and X-Ising couplings, respectively. The set of $\{\langle i, j \rangle_\alpha\}$ and $\{\langle i, j \rangle_{Z-1}\}$ bonds will be indicated by black and orange solid lines respectively in the subsequent diagrams. The above models belong to the more general class of JW-solvable models as laid out in Refs. [8,9] and related papers [10–15]. This embedding can also be extended to $d > 2$ if the restriction on kinetic terms to on-chains is respected. We will restrict ourselves to uniform couplings, i.e., $J_{\langle i,j \rangle_\alpha} = J_\alpha$, $J_{\langle i,j \rangle_{Z-1}} = J_{Z-1}$. Under the JW map we get

$$H = \frac{J_\alpha}{2} \sum_{\langle i,j \rangle_\alpha} (c_i^\dagger c_j + (1 - \alpha) c_i^\dagger c_j^\dagger + \text{H.c.}) + J_{Z-1} \sum_{\langle i,j \rangle_{Z-1}} \left(n_i - \frac{1}{2} \right) \left(n_j - \frac{1}{2} \right). \quad (2)$$

Through the above construction, S_k^z or n_k on an off-chain site k become conserved c -numbers, i.e., effectively classical (Ising or Z_2) variables, that converts H into a quadratic form thereby leading to the solution of the full eigenspectrum.

Bond-dependent couplings in spin- $\frac{1}{2}$ Hamiltonians have assumed importance since Jackeli and Khaliullin’s seminal work [16]. The proposed class of bond-dependent models can thus form a new material class apart from so-called Kitaev materials [17–22]. The models naturally divide into geometrically bipartite lattices containing no triangular motifs, and geometrically nonbipartite lattices containing triangular motifs. The Hamiltonians are however frustrated in all cases due to the competition of the on-chain XY/X-Ising and Z-Ising terms similar to the Kitaev model. A further subdivision

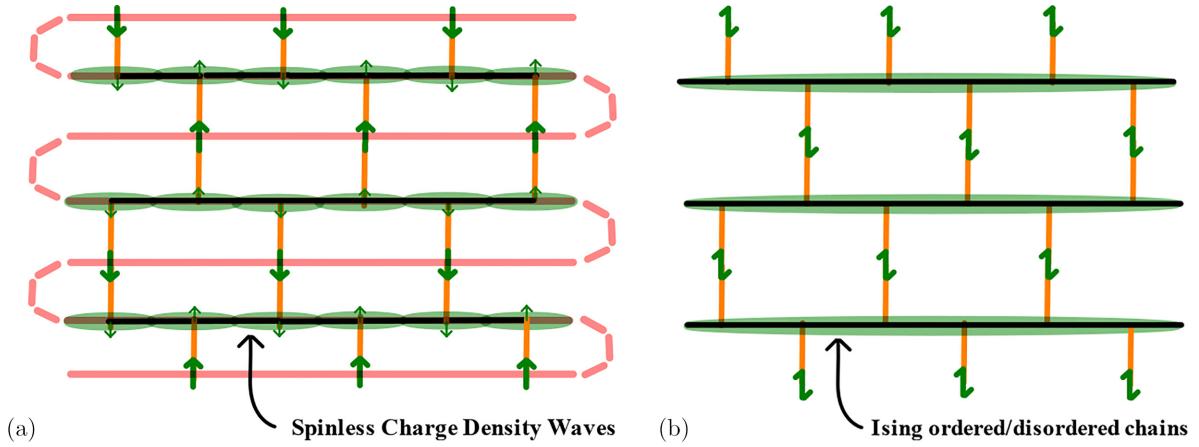


FIG. 2. Example of a lattice without triangular motifs. Black lines represent XY and X-Ising couplings in panels (a) and (b), respectively. Orange lines represent Z-Ising couplings. The light red line is the snake-like choice for the JW-string throughout this work. The ground state sector corresponds to an antiferromagnetic [(a); see Sec. III B for further details], paramagnetic [(b); see Sec. III E for further details] arrangement of the conserved off-chain spins with gapped on-chain spinon excitations. The difference in the size of the off-chain spins and on-chain spins in panel (a) illustrates the following: since off-chain spins are conserved, the Z-moment size is maximal, while the on-chain Z-spins fluctuate due to the XY terms thus leading to a reduced moment size compared to the off-chain spins.

concerns if the on-chain terms are of XY or X-Ising type. In the following, we will consider ferromagnetic signs for the couplings without any loss of generality. There are local unitaries that can give all possible sign combinations and resultant ground states.

III. RESULTS

This section is organized as follows: Section III A gives a concise description of the numerical methods used to obtain the results in the following subsections. Section III B discusses the case of bipartite XY model construction, followed by in Sec. III C the cases of nonbipartite XY and X-Ising model constructions. Then a subsection is devoted to analytical arguments for perturbative stability of the magnetic ground states found in the previous subsections. The climax of this section is the bipartite X-Ising model construction in Sec. III E which hosts an intriguing liquid-like ground state.

A. Numerical computations

We first briefly discuss the details of the numerical computations that are presented in the following sections. For numerical purposes, we can focus on just one set of on-chain spin chain and its corresponding off-chain spins and transfer the result to the rest of the spin chains. In other words, the exact solution applies to infinite extent in the perpendicular direction to the spin chains. Mathematica codes based on them are available at this link [23]. The codes also have further annotations that will be of help to the interested reader.

Starting from Eq. (3) after JW-transforming Eq. (1), we fix the values of the conserved off-chain spins' S^z quantum numbers to either $+\frac{1}{2}$ or $-\frac{1}{2}$, or equivalently the conserved occupation number of the off-chain JW-fermions' n to 0 or 1. For an off-chain of length L , there are 2^L different configurations for the conserved off-chain spins. Each off-chain

configuration effectively provides a background on-site potential to the on-chain JW-fermions, i.e.,

$$H = J_\alpha \sum_{\langle i,j \rangle_\alpha} (c_i^\dagger c_j + (1 - \alpha) c_i^\dagger c_j^\dagger + \text{H.c.}) + J_{Z-1} \sum_{\langle i,j \rangle_{Z-1}} \left(n_i - \frac{1}{2} \right) \left(n_j - \frac{1}{2} \right), \quad (3)$$

where it is understood that depending on the lattice geometry $\langle n_j \rangle$ in the second term above corresponds to the conserved off-chain degrees of freedom with value either 0 or 1.

For the XY cases, the above quadratic fermionic Hamiltonian for the on-chain spin chain of length L can now be easily diagonalized in the single-particle sector for each conserved off-chain sector. This is computationally just a $L \times L$ matrix diagonalization. The lowest energy state in each sector is found by summing all the negative single-particle eigenvalues in that sector. To find the overall many-body ground state, one needs to find that sector which hosts highest magnitude of the above negative single-particle sum. For small chain lengths, this can be done exhaustively as in the provided codes. For chains of bigger lengths, the exponential growth of 2^L makes it difficult to an exhaustive search. One can take advantage of translation symmetry here for homogeneous couplings to restrict the set of conserved off-chain configurations to those ones that are not related by translation symmetry. This has also been shown in the provided codes.

For the X-Ising cases which lack $U(1)$ symmetry or total $\sum_i S_i^z$ conservation (equivalently $\sum_i n_i$), one needs to follow the standard Bogoliubov-de Gennes setup [24] to find the single-particle states in this “superconducting” situation with no particle conservation but only fermion parity conservation. Computationally, this only leads to a doubling of the size of the single-particle Hamiltonian since now we have to handle both the particle and hole channels together. The additional cost of diagonalizing a $2L \times 2L$ is negligible and does not become a bottleneck. Through these computations, one can verify the various ground states discussed in the main text

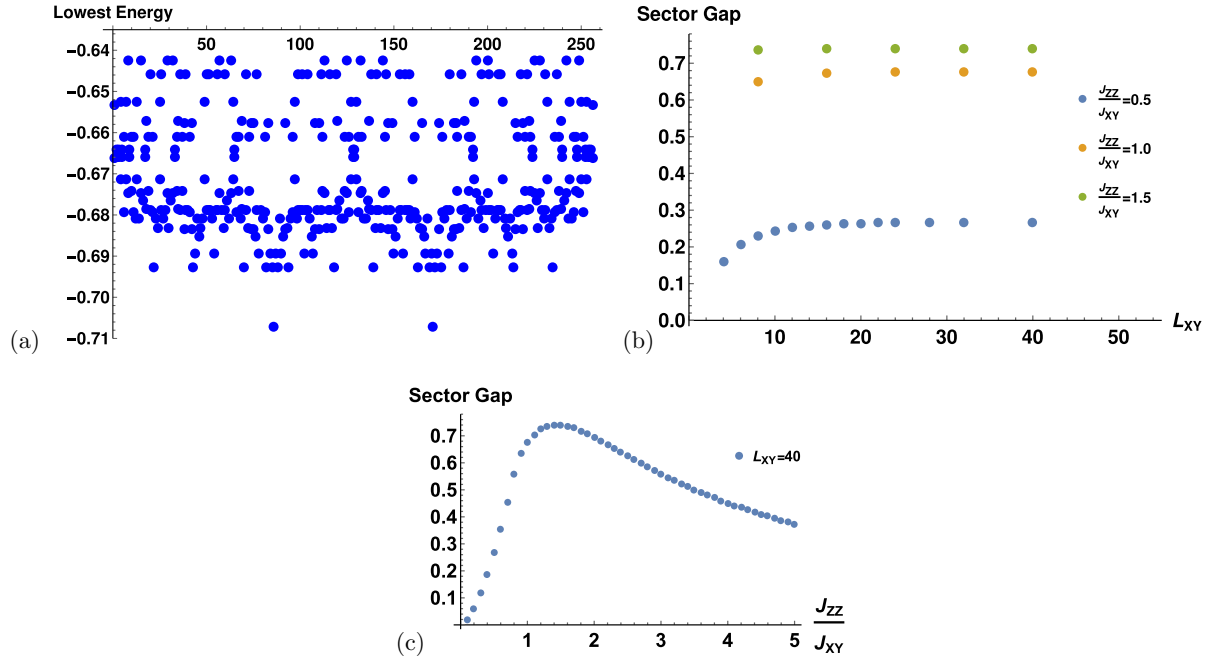


FIG. 3. Panel (a) shows the lowest energies (per site and in units of $J_{XY} = 1$) from each conserved off-chain spin sector for an $L = 8 + 8$ chain in Sec. III A for the bipartite XY model with $\frac{J_{ZZ}}{J_{XY}} = 0.5$ [Fig. 2(a); see Sec. III B for further details]. The X-axis indexes the different sectors through the decimal equivalent of the binary-valued off-chain spin configuration. One can verify that the two lowest energies corresponds to the two antiferromagnetic configurations for the off-chain spins ($85_{10} \equiv 0101010_2$ and $170_{10} \equiv 10101010_2$). Panel (b) shows the finite gap between the lowest energies from other sectors and the ground state energies for different system sizes. Panel (c) shows how this sector gap evolves with $\frac{J_{ZZ}}{J_{XY}}$ for $L = 40$.

apart from the perturbative arguments for their selection discussed later.

B. Bipartite XY

An example of this construction is shown in the panel (a) of Fig. 2. The ground state is found to be an antiferromagnet in the Z-direction. The JW-spectrum on the chains is gapped implying exponentially decaying transverse XY correlations as well. The ground state can also be thought of as a spinless fermion charge density wave. In contrast to the Kitaev model where one takes recourse to Lieb's theorem [25] to rigorously pin down the ground state sector, here we are not aware of any rigorous way of proving the ground state sector selection. As shown in Fig. 3(a), one numerically finds that the ground state sector indeed corresponds to an antiferromagnetic conserved off-chain spins as schematically indicated in Fig. 2(a). Such an arrangement corresponds to on-site potentials with alternating sign, i.e., modulating at wave vector π , for the on-chain JW-fermions which leads to gap opening near the Fermi level (at $\pm\pi/2$) along with charge density wave ordering at the same wave vector. Ground state sector selection can be argued for perturbatively in the $\frac{J_{XY}}{J_{Z1}} \gg 1$ and $\frac{J_{XY}}{J_{Z1}} \ll 1$ limits. For $\frac{J_{XY}}{J_{Z1}} \ll 1$, the on-chain spin aligns with its partner off-chain spin to satisfy the ferromagnetic Z-Ising coupling. The neighboring on-chain spins now prefer to antialign to gain energy through virtual hops. A different argument can also be made for this sector selection in the other $\frac{J_{XY}}{J_{Z1}} \gg 1$ limit [26]. These perturbative stability arguments in the solvable regime are further discussed in Sec. III D. The ground state

interpolates smoothly between the two limits with the ratio of gap opening size to spinless Fermi sea bandwidth being controlled by $\frac{J_{XY}}{J_{Z1}}$. Upon monitoring the lowest energy states in other conserved off-chain spin sectors, one finds a finite energy gap as well as shown in Fig. 3(b). It is also numerically observed that the off-chain spin configuration corresponding to the lowest excited sector changes once as we increase $\frac{J_{ZZ}}{J_{XY}}$ from domain wall of size one to domain wall of size two in the off-chain antiferromagnetic configuration as in Fig. 3(c). It is not clear if this is just energetics or something deeper. Thus this construction leads to gapped antiferromagnetic ground states. A rigorous proof of this fact is desirable. The X-Ising variant on the right of Fig. 2 of this construction will be addressed later in Sec. III E.

C. Nonbipartite XY and X-Ising

An example of this construction is shown in Fig. 4(a). Its ground state turns out to be a ferromagnet in the Z-direction as evidenced in Fig. 5(a). These constructions are in fact solvable points in the general Kagome antiferromagnet phase diagram! The JW-fermion spectrum on the chains now remains gapless implying algebraically decaying transverse XY correlations. This can be thought of as a quasi-long-range-ordered form of the so-called $U(1)$ -symmetry breaking spin superfluid order in higher dimensions in the literature [27]. Again the above is determined numerically. Here too, we can make similar perturbative arguments in $\frac{J_{XY}}{J_{Z1}} \gg 1$ and $\frac{J_{XY}}{J_{Z1}} \ll 1$ limits for the ground state sector selection. There is an interpolation between the fully polarized state for $\frac{J_{Z1}}{J_{XY}} > 1$ and a

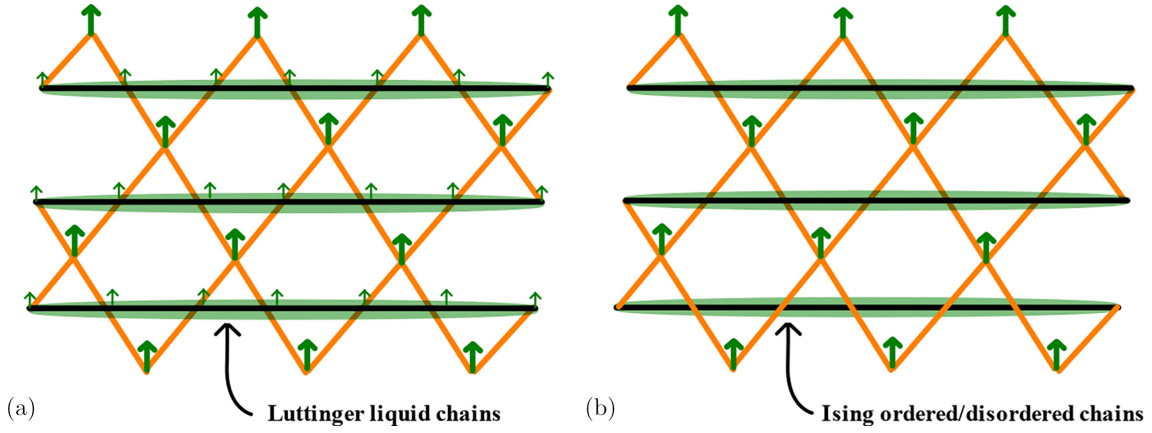


FIG. 4. Example of a lattice with triangular motifs. Black lines represent XY and X-Ising couplings in panels (a) and (b) respectively. Orange lines represent Z-Ising couplings. The ground state sector corresponds to a ferromagnetic arrangement of the conserved off-chain spins for both cases, with gapless on-chain spinon excitations in (a), and gapped on-chain Majorana excitations in (b). See Sec. III C for further details. The difference in the size of the off-chain spins and on-chain spins again illustrates the difference in the Z-moment size of the off-chain and on-chain spins as discussed in the caption of Fig. 3.

half-filled spinless Fermi sea as $\frac{J_{Z1}}{J_{XY}} \rightarrow 0$. Upon monitoring the lowest energy states in other conserved off-chain spin sectors, one again finds a finite energy gap as shown in Fig. 5(a). It is also numerically observed that the off-chain spin configuration corresponding to the lowest excited sector keeps changing as we increase $\frac{J_{Z1}}{J_{XY}}$ till $\frac{J_{Z1}}{J_{XY}} = 1$ as in

Fig. 5(c). It is not clear if this is just energetics or something deeper. Gaplessness and finite sizes further complicate matters for smaller values of $\frac{J_{Z1}}{J_{XY}}$ in this nonbipartite XY case. Such complications were absent in the bipartite XY case in Sec. III B. Thus we find that this construction leads to ferromagnetic ground states with gapless on-chain

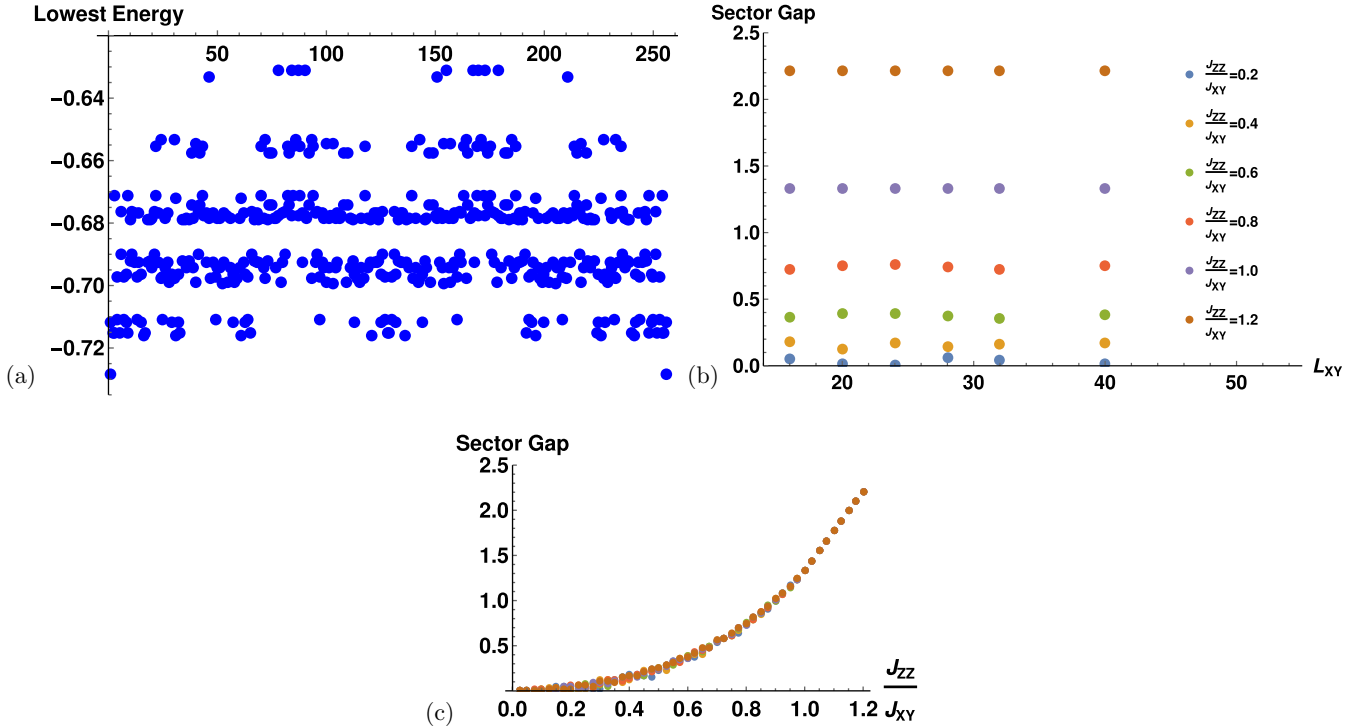


FIG. 5. Panel (a) shows the lowest energies (per site and in units of $J_{XY} = 1$) from each conserved off-chain spin sector for an $L = 8 + 8$ chain in Sec. III A for the nonbipartite XY construction with $\frac{J_{Z1}}{J_{XY}} = 0.5$ [Fig. 4(a); see Sec. III C for further details]. The X-axis indexes the different sectors through the decimal equivalent of the binary-valued off-chain spin configuration. One can verify that the two lowest energies corresponds to the two ferromagnetic configurations for the off-chain spins ($0_{10} = 00000000_2$ and $255_{10} = 11111111_2$). Panel (b) shows the finite gap between the lowest energies from other sectors and the ground state energies for different system sizes. Panel (c) shows how this sector gap evolves with $\frac{J_{Z1}}{J_{XY}}$ for these system sizes.

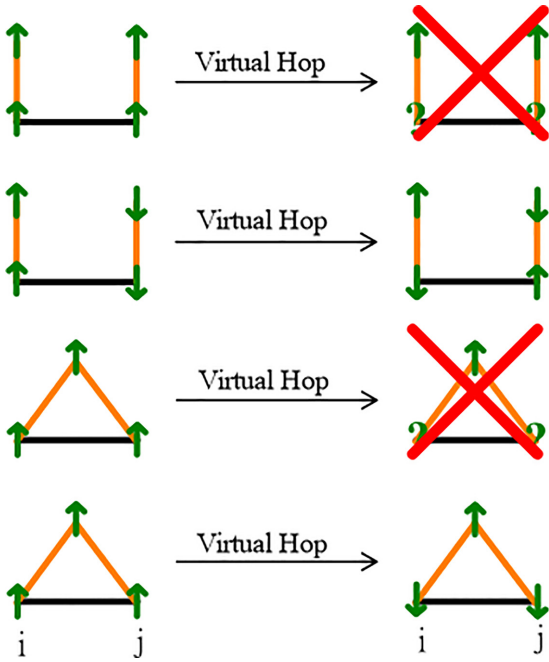


FIG. 6. Schematic of the virtual hopping processes in the $\frac{J_{XY}}{J_{Z1}} \ll 1$ limit for bipartite XY, X-Ising, and nonbipartite XY, X-Ising cases from top to bottom. See the beginning of Sec. III D for the details.

excitations. A rigorous proof of the ground state sector selection is again desirable. The X-Ising variant of this construction has related physics [Fig. 4(b)]. The ground state corresponds to a ferromagnetic configuration of the off-chain spins which effectively provides a transverse field to the on-chain spins. Thus depending on $\frac{J_{Z1}}{J_{X1}}$, an Ising ordered or disordered state obtains for the on-chain spins which in terms of the JW fermions would be a topologically nontrivial or trivial superconducting state respectively with gapped Majorana excitations and the transition at $\frac{J_{Z1}}{J_{X1}} = 0.5$ on this (kagome)

lattice. The sector gaps also remain finite like in the nonbipartite XY variant.

D. Perturbative arguments for ground state sector selection

The $\frac{J_{XY}}{J_{Z1}} \ll 1$ or Ising limit for the bipartite XY case was alluded to earlier in Sec. III B. The top two panels of Fig. 6 summarize this argument. An analogous argument can be made for the nonbipartite XY and X-Ising cases as shown in the lower panels of Fig. 6. The virtual hops are caused by the XY or X-Ising on-chain couplings. Note for the nonbipartite XY case, there is no virtual hopping energy gain to be had due to the triangular nature of the elementary motifs as in the third panel of Fig. 6. For the nonbipartite X-Ising case, virtual hopping energy can be gained from the superconducting part of the X-Ising couplings as in the fourth panel of Fig. 6 while keeping the on-chain short-range Ising correlations intact.

In the opposite XY limit for $\frac{J_{Z1}}{J_{XY}} \ll 1$, one can argue by considering one on-chain spin chain and its partner off-chain spins. Let us start with the bipartite XY case. The lattice Hamiltonian is

$$H = \frac{J_{XY}}{2} \sum_{\langle i,j \rangle} c_i^\dagger c_j + \text{H.c.} + J_{Z1} \sum_i \left(n_i - \frac{1}{2} \right) \langle S_i^z \rangle, \quad (4)$$

where $\langle S_i^z \rangle$ are the conserved off-chain partners of on-chain JW-fermions for the lattice site i . The first term equals $\sum_p \epsilon(p) c_p^\dagger c_p$ with $p \in (-\pi, \pi]$. In the XY limit, we can focus our attention to the modes near the Fermi points $p_F = \pm \frac{\pi}{2}$ of the unperturbed many-body ground state which is the half-filled spinless Fermi sea, i.e.,

$$c_i \sim \sum_{-\Lambda < \delta k < \Lambda} \frac{1}{\sqrt{L}} \left(e^{i(\frac{\pi}{2} + \delta k)r_i} c_{\delta k, r} + e^{i(-\frac{\pi}{2} + \delta k)r_i} c_{\delta k, l} \right). \quad (5)$$

where $\Lambda \ll \pi$ is a cutoff scale much smaller than the lattice scale. l and r stand for left-propagating and right-propagating modes or “helicities”, respectively, i.e., their corresponding group velocities are $\propto -|\delta k|$ and $\propto +|\delta k|$, respectively. With the above, the perturbation term can be evaluated to

$$\begin{aligned} \sum_i \langle S_i^z \rangle \left(n_i - \frac{1}{2} \right) &= \sum_{-\Lambda < \delta k, \delta k' < \Lambda} \frac{1}{L} \left(\langle \widetilde{S^z} \rangle (\delta k' - \delta k) c_{\delta k, r}^\dagger c_{\delta k', r} + \langle \widetilde{S^z} \rangle (\delta k' - \delta k) c_{\delta k, l}^\dagger c_{\delta k', l} \right. \\ &\quad \left. + \langle \widetilde{S^z} \rangle (\pi + (\delta k' - \delta k)) c_{\delta k, r}^\dagger c_{\delta k', l} + \langle \widetilde{S^z} \rangle (\pi + (\delta k' - \delta k)) c_{\delta k, l}^\dagger c_{\delta k', r} \right) - \sum_i \frac{\langle S_i^z \rangle}{2} \end{aligned} \quad (6)$$

where $\langle \widetilde{S^z} \rangle(p) \equiv \sum_i e^{ipr_i} \langle S_i^z \rangle$ is the (discrete) Fourier transform of the off-chain spin configuration. r_i is understood to be a set of integers say from 0 to $L-1$, etc. Also $\langle \widetilde{S^z} \rangle(p) = \langle \widetilde{S^z} \rangle(-p)$ since $\langle S^z \rangle \in \mathbb{R}$. Now we ask for which conserved off-chain spin configuration do we gain the most energy upon including the perturbation. The first two terms in Eq. (6) connect modes with same helicities, whereas the last two terms in Eq. (6) connect modes with different helicities. It is intuitively clear that the most energy to be gained is when the perturbation term from the off-chain configuration connects modes that are closest in energy

(ideally degenerate energy modes either with same or different helicity). The best that we can do from this point of view is to either have a ferromagnetic off-chain configuration for which $\langle \widetilde{S^z} \rangle(p=0) \neq 0$ ($= \frac{L}{2}$) and all other $\langle \widetilde{S^z} \rangle(p) = 0$, or an antiferromagnetic off-chain configuration for which $\langle \widetilde{S^z} \rangle(p=\pi) \neq 0$ ($= \frac{L}{2}$) and all other $\langle \widetilde{S^z} \rangle(p) = 0$. If we take the ferromagnetic case with $\langle S_i^z \rangle = \frac{1}{2}$, then the net many-body energy gain due to shifts in the one-particle energies is $J_{Z1} \left(\sum_{\epsilon(p) + \frac{J_{Z1}}{2} < 0} \frac{1}{2} - \sum_{-\Lambda < \delta k < \Lambda} \sum_{\alpha=\{l, r\}} \frac{1}{4} \right)$ which is lower bounded by $-J_{Z1} \sum_{-\Lambda < \delta k < \Lambda} \sum_{\alpha=\{l, r\}} \frac{1}{4} = -\sum_{-\Lambda < \delta k < \Lambda} \frac{J_{Z1}}{2}$. For this we keep in mind that the counting of modes should

remain the same, i.e., $\sum_{\delta k} \sum_{\alpha=\{l,r\}} = \sum_i$ which leads to $\sum_i \frac{\langle S_i^z \rangle}{2} = \sum_{\delta k} \sum_{\alpha=\{l,r\}} \frac{1}{4}$.

If we take the antiferromagnetic case, the last real-space sum term in Eq. (6) is now equal to $\sum_i \langle S_i^z \rangle = 0$. The effect of the perturbation is now to open a gap at the Fermi points in the single-particle spectrum which “pushes down” all energy eigenvalues below the Fermi energy (by different amounts), and “pushes up” all energy eigenvalues above the Fermi energy. This leads to a decrease in the many-body ground state energy which is again the half-filled spinless Fermi sea but now with a gap to (single-particle and multi-particle) excitations. The many-body energy gain will generically be of the form $-\sum_{\Lambda < \delta k < \Lambda} \sqrt{\frac{J_{Z1}^2}{2} + (\dots)^2}$ where $\dots$ represent the energy difference of the close-in-energy modes of different helicities separated by a momentum of π which are now hybridized by the perturbation term in Eq. (6). This

is a greater many-body energy gain than the ferromagnetic off-chain configuration. This is how the antiferromagnetic off-chain spin configuration gets selected in the XY limit. The above can be made rigorous by making rigorous the intuitive step taken above of “connecting modes closest in energy through the perturbation to gain the most reduction in the (unperturbed) many-body ground state energy”. This is an open and a mathematical physics question even though the physical reason for the ground state sector selection is clear.

For the nonbipartite XY case, the lattice Hamiltonian is

$$H = J_{XY} \sum_{\langle i,j \rangle} c_i^\dagger c_j + \text{H.c.} + J_{Z1} \sum_i (n_i + n_{i+1} - 1) \langle S_i^z \rangle. \quad (7)$$

Upon redoing the steps above, we get

$$\begin{aligned} \sum_i \langle S_i^z \rangle (n_i + n_{i+1} - 1) &= \sum_{\delta k, \delta k' < \Lambda} ((1 + e^{i(\delta k' - \delta k)}) \widetilde{\langle S^z \rangle} (\delta k' - \delta k) c_{\delta k, r}^\dagger c_{\delta k', r} + (1 + e^{i(\delta k' - \delta k)}) \widetilde{\langle S^z \rangle} (\delta k' - \delta k) c_{\delta k, l}^\dagger c_{\delta k', l} \\ &\quad + (1 - e^{i(\delta k' - \delta k)}) \widetilde{\langle S^z \rangle} (\pi + (\delta k' - \delta k)) c_{\delta k, r}^\dagger c_{\delta k', l} + (1 - e^{i(\delta k' - \delta k)}) \widetilde{\langle S^z \rangle} (\pi + (\delta k' - \delta k)) c_{\delta k, l}^\dagger c_{\delta k', r}) \\ &\quad - \sum_i \langle S_i^z \rangle. \end{aligned} \quad (8)$$

By using the earlier intuition, now the helicity cross-connecting terms due to the perturbation actually cancel out due to the $(1 - e^{i(\delta k' - \delta k)})$ term in the antiferromagnetic off-chain spin configuration sector. Thus the ferromagnetic off-chain spin configuration is selected as the ground state sector perturbatively in the XY limit in the presence of triangular motifs.

E. Bipartite X-Ising

This last case is the most unusual and interesting. It is the X-Ising variant on lattices without triangular motifs. The ground state here turns out to be an Ising ordered/disordered state for the on-chain spins coexisting with a very short-ranged liquid-like (classical) paramagnetic state for the off-chain spins! In other words, all the conserved off-chain spin sectors have the same spectrum as indicated in Fig. 2(b). This arises due to an additional structure present in this case that was absent in the previous cases. The structure is the presence of additional extensively large conserved quantities apart from the off-chain spins that are mutually anticommuting when there is a shared site. General consequences of this structure form the subject of Ref. [28]. The additional conserved quantities are products of three X-component operators from sites on each of the Z-Ising segments [orange solid lines in Fig. 2(b)], i.e., $S_i^x S_j^x S_k^x$ where $\{i, j, k\}$ forms a Z-Ising segment $J_{Z1}(S_i^z S_j^z + S_j^z S_k^z)$ that is *disconnected* from other Z-Ising segments. Since there are an extensive number of these operators coming from all the Ising segments, this leads to the extensive degeneracy associated with the off-chain spin configurations. The extensive scaling with system size of these additional conserved quantities can depend on

the model, i.e., here they scale as L^2 same as the conserved off-chain spins. There can be variants where they scale as L for example [29]. This should not be thought of as violating the third law of thermodynamics [30], but rather is well-understood as capturing the relevant physics of real systems operating at the relevant temperature scales in a physical setting or experiment [31] similar to classical spin ices [32]. In the presence of other (even smaller) couplings, a natural guess is to presume the existence of a (possibly gapless) “slow” mode primarily composed of the off-chain spins as excitations on top of the ground state, apart from the gapped Majorana excitations from the on-chain spins which will be higher in energy compared to the presumed slow mode. The solvable limit can be said to have an extensive number of zero modes from the perspective of this expectation. The above structure also implies that the absence of triangular motifs is not an absolute requirement for this physics; rather one only requires Z-Ising (closed or open) segments with at least one conserved spin in the spirit of the above constructions connected through solely X-Ising (or solely YY-Ising) segments as shown in Fig. 7. Also it must be mentioned before proceeding further, that the results discussed above pertaining to the particular chain-based construction of Fig. 2(b) is essentially a rediscovery and restatement of the result found in Ref. [33]. The authors of Ref. [33] arrived at them with different motivations coming from orbital physics in transition metal systems and did not discuss explicitly the classical spin liquid aspect of the off-chain spins as well as the phenomenological consequence analogous to the Kitaev honeycomb model that we describe next.

Given the above massively degenerate ground state manifold, one now expects a two-step entropy release similar

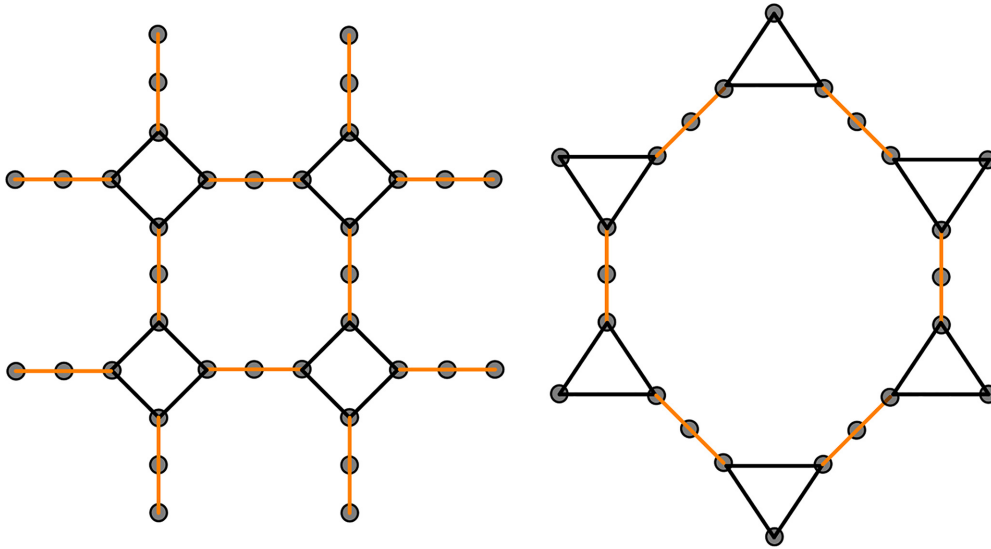


FIG. 7. Some additional constructions not based on the one-dimensional chain geometry.

to that in the Kitaev spin liquids in presence of thermal fluctuations [35,36] with residual entropy effects as temperature goes to zero. The first step would correspond to the entropy release due to the off-chain spins and the second step to the on-chain spins. The relative magnitude of the two steps will depend on the ratio of the number of off-chain spins to on-chain spins with it being 1:2 for the lattice geometry in this case [Fig. 2(b)]. Thus these states can be considered as simpler variants of the Kitaev spin liquid. As differences, the conserved off-chain spins are geometrically independent and site-local in our constructions, whereas in the Kitaev model, the conserved plaquette operators are “interlocked” with common sites. The excitations also have a stripy nature in all of the above models, whereas the Majorana excitations in the Kitaev spin liquid are fully two-dimensional in nature. One can have nonstripy constructions as well with localized excitations throughout as shown Fig. 7. The conserved quantities of the Kitaev model manifest as Z_2 flux background for these Majorana fermions. Being fluxes in two dimensions, they are relevant for the energetics of the excitations. In the model considered here, the conserved off-chain spins only lead to gauge artifacts for the stripy Majorana excitations. Thus the gauge artifact argument described in the beginning gives another perspective on the extensive degeneracy of this model. If there are additional conserved sites on the on-chain X-Ising spin chains, then the resultant models will be rendered effectively classical. The models considered here can therefore be said to be “in between” effectively classical states and fully quantum spin liquids as shown in Fig. 8.

IV. SUMMARY AND DISCUSSION

In this paper, we constructed a host of frustrated bond-dependent $S = \frac{1}{2}$ models in two (and higher) dimensions that are solvable by the Jordan-Wigner transformation Sec. II. We focused on the ground state properties and spectral computations for the constructed models in Sec. III. The various cases were categorized as: (1) Bipartite XY constructions as

in Sec. III B which hosts a gapped Ising antiferromagnetic ground state with gapped on-chain spinon excitations. (2) Nonbipartite XY and X-Ising constructions as in Sec. III C which host ferromagnetic states along with, respectively, on-chain Luttinger liquid states with gapless spinon excitations, and on-chain Ising ordered or disordered states with gapped Majorana excitations away from the quantum critical point ($\frac{J_{Z1}}{J_{X1}} = 0.5$). (3) Bipartite X-Ising constructions as in Sec. III E which hosts an unusual coexistence state of a classical spin liquid (with extensive number of “zero modes”) on the off-chain sites and quantum Ising order or disorder on the on-chain sites. Details on the numerical methods and perturbative arguments for ground state sector selection within the solvable parameter regime were given in Secs. III A and III D.

One can also calculate on-chain transverse spin-spin correlations $\langle \sigma_i^+ \sigma_j^- + \text{H.c.} \rangle$ using known techniques [37]. Computation of the full partition function seems nontrivial, especially summing over the conserved off-chain spin sectors and is an open question. The solvability is also preserved in presence of magnetic field along appropriately chosen quantization axes. It is desirable to study various other aspects of these models is left to the future such as (1) thermodynamical quantities at finite temperatures, (2) dynamical phenomena in presence of quenches and drives, and (3) investigation of the stability of the reported magnetic states outside the solvable parameter regime. We comment on this last issue some more in what follows.

A. Stability considerations

It is also important to consider the stability of the grounds states discussed here with respect to natural deformations of the Hamiltonian, i.e., off-chain XY couplings and on-chain Z-Ising couplings. For the XY variants with on-chain Z-Ising couplings as deformations, one can invoke Luttinger liquid arguments to argue for stability [2,38]. The bipartite XY case would be stabler to on-chain Z-Ising deformations due to the presence of the gap in the on-chain spinless fermionic excitations. The same would be expected for the nonbipartite and bipartite X-Ising case due to the gap in the on-chain

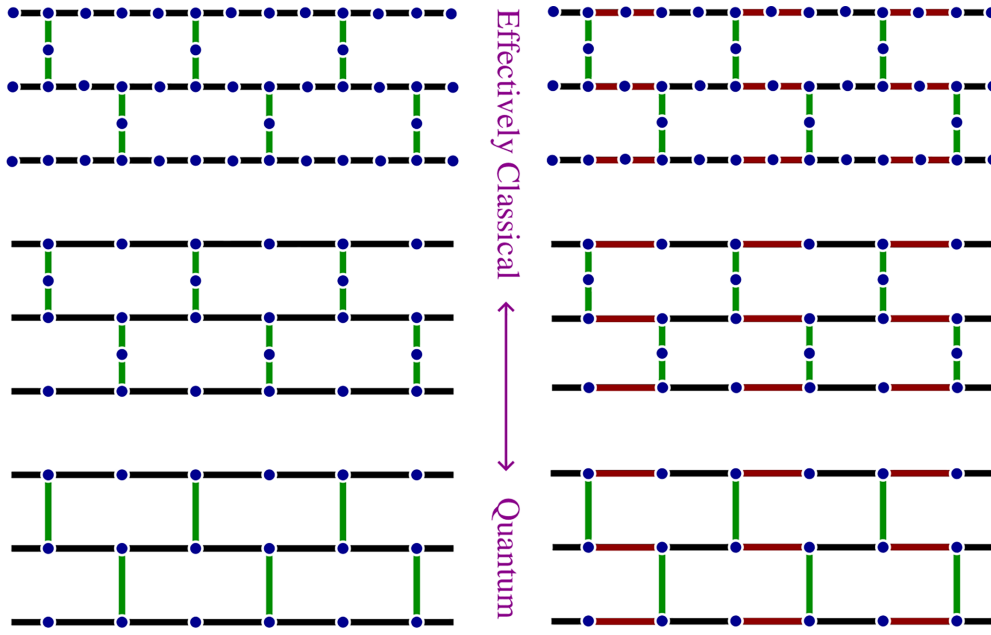


FIG. 8. A top-to-bottom sequence of models with increasing levels of quantum behavior. Black lines represent X-Ising couplings. Red lines represent YY-Ising couplings. Green lines represent Z-Ising couplings. Blue dots represent $S = \frac{1}{2}$ degree of freedom. Left sequence has the model of Fig. 2(b) at the middle. Right sequence has the Kitaev model at the bottom. The top effectively classical model in this sequence was studied by Mizoguchi [34]. Lieb lattice variants of the top row of brickwork lattices can also be envisaged.

spectrum in presence of Ising order or disorder. Stability to off-chain deformations in the first three cases of bipartite XY, nonbipartite XY, and X-Ising models is evidenced by the presence of the sector gap as shown in Figs. 3 and 5. This is because it is precisely these deformations such as off-chain XY or X-Ising couplings that would lead to quantum fluctuations in the off-chain spin Z-values that were otherwise conserved by construction. Presence of the sector gap suggests that if the energy scale associated with these fluctuations is lesser than the gap, then the ground states should be stable till they become large enough. For the particular case of nonbipartite XY construction with gapless on-chain excitations, one may worry that the interchain interactions induced through the gapped off-chain excitations—by integrating out the off-chain spin degrees of freedom—may destabilize the system and lead to a gapped state for the on-chain spectrum as well [39,40]. However, given the perturbative stability arguments in both $\frac{J_{XY}}{J_{Z1}} \ll 1$ and $\frac{J_{XY}}{J_{Z1}} \gg 1$ limits for the gapless on-chain state within the solvable bond-dependent parameter regime as discussed previously in Sec. III D, we conjecture that additional off-chain perturbations will lead to a locking of the spin superfluid order parameter across the chains in the ground state. This suggests that in presence of off-chain perturbations a long-range ordered spin superfluid ground state—without a gap in the thermodynamic limit—obtains in the nonbipartite XY construction up to a finite range of such local perturbations, outside of which a gap may open. Even in the solvable regime, one may wonder if finite (low) temperatures can lead to a locking of the spin superfluid order across the chains via an order-by-disorder mechanism.

Note that the excitations on top of a long-ranged ordered spin superfluid ground state in two dimensions are bosonic magnons. Even so, the fermionic spinon excitation physics

discussed in this work can play a role in a “dimensional reduction” sense, i.e., in an appropriate regime of temperature, the coupling ratio $\frac{J_{XY}}{J_{Z1}}$ and other couplings (including an external magnetic field in Z direction) that take us out of the solvable regime, the effective description may still be in terms of on-chain spinon excitations and localized spin flip excitations with majority weight on the off-chain spins. A recent experimental example of a quasi-two-dimensional copper based quantum magnet showing such dimensional reduction phenomenology can be found in Ref. [41]. See also Ref. [42] in the context of two-dimensional $SU(2)$ -symmetric Heisenberg magnets with magnonic excitations “where a broad spectral-weight continuum at wave vector $q = (\pi, 0)$ was interpreted as deconfined spinons” in relation to neutron-scattering experiments on another copper based compound [43].

Explicit verification of the above arguments may be done using various Quantum Monte Carlo or Matrix product based methods. In the case of the bipartite X-Ising model with an extensive degeneracy from the paramagnetic off-chain spins, the off-chain deformations would be a possible mechanism for the development of the conjectured slow mode and the associated restoration of the third law of thermodynamics.

B. Physical realization

We end with some remarks with a view towards physical realization. Firstly, the models discussed here in the JW-mapped fermionic form are reminiscent of Falicov-Kimball models where also itinerant electrons or Bloch waves interact with static or localized electrons. They have been invoked to study a variety of phenomena such as mixed-valence and metal-insulator transitions. More details on these electronic

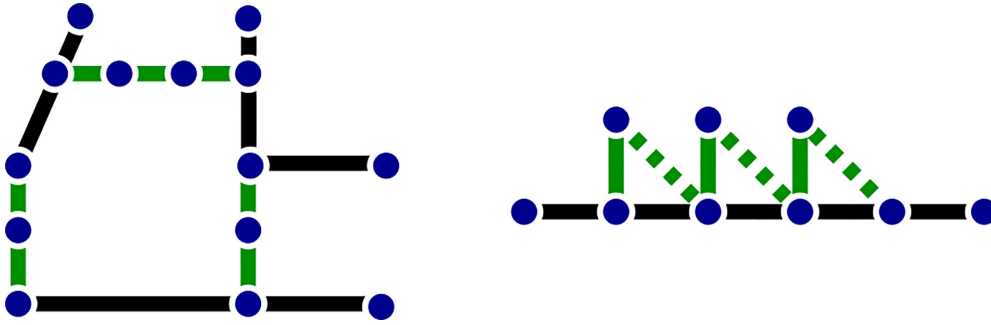


FIG. 9. Some imagined quantum architectures following the color scheme of Fig. 8. The dashed green Z-Ising couplings can be switched on or off.

phenomenon can be found in the following reviews [44–47]. See this Ref. [48] for an example of Falicov-Kimball models with Majorana excitations. The models discussed in this paper realize aspects of Falicov-Kimball physics, however and it must be emphasized, in a nonelectronic context and only after a *nonlocal* transformation. In other words, the spin-spin correlations encode Falicov-Kimball type effects involving the itinerant electrons in a highly nonlinear way. Furthermore, the possible geometries allowed by the embedding mechanism laid down here admits on-site potentials that act as a “shared” background for sites on different chains that are somewhat unnatural in the electronic context that motivated the Falicov-Kimball model in the first place, i.e., “as a set of localized states centered at the sites of the metallic ions in the crystal” [49].

In a fully localized context without any itinerant electrons, one may imagine physical realizations such as an insulator material with localized $S = \frac{1}{2}$ moments with a crystal structure and quantum chemistry that accommodates a grid structure of easy-axis and easy-plane spin chains as the case may be. If the nonbipartite XY case could be so realized, it would allow for highly anisotropic spin transport along the spin chains which can have practical applications such as in spin-based classical technologies. This could be an interesting avenue of exploration for the materials design and search community as alternative to the Kitaev material class with bond-dependent magnetic couplings.

But perhaps the more natural application of the ideas laid out in this work may be in the realm of quantum technologies involving artificial platforms with controllable qubits. The

grid structure proposed here can provide a blueprint for a bus architecture for quantum information traffic. The conserved off-chain spins can sequester the spin information to the on-chain spin chains. Some imagined architectures are shown in Fig. 9. For example the right panel of Fig. 9 could be a use case where turning on and off the dashed Z-Ising couplings would act as a valve for spin information flow. Interestingly, such an architecture has been recently reported in Ref. [50] with a different application as its focus.

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