

Manipulating non-Abelian anyons of Kitaev quantum spin liquids with local magnetic fields

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Non-Abelian anyonic excitations of quantum spin liquids have potential for application to topological quantum computation, but designing logical operations requires developing protocols to faithfully create, move, and read out such quasiparticles. In this paper, we test two such protocols for manipulating Z_2 fluxes (“visons”) of ferromagnetic and antiferromagnetic Kitaev models perturbed by a small uniform magnetic field. We show that a protocol that employs a driving magnetic field locally applied to the sites of the lattice suffers from significant unavoidable errors. These errors, accumulating over time, prevent reaching high probabilities of generating and displacing flux pairs. However, individual spin couplings can also be turned on and off, thus it is possible to create and displace fluxes with nearly 100% probability. We show that braiding non-Abelian anyons can be achieved with good precision with our second protocol. We comment on the feasibility of our protocols in solid-state devices and quantum emulators.

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I. INTRODUCTION

A central challenge in realizing large-scale quantum computation is the problem of decoherence—the loss of quantum coherence due to environmental interactions [1–5]. Quantum error correction codes have been developed to combat this issue [6–12], but they typically rely on encoding logical qubits across multiple physical qubits, which significantly hampers scalability. This motivates the search for qubits that are intrinsically robust against noise.

Topological quantum computation [13–17] provides a promising route to overcome decoherence. In certain topologically ordered systems, such as the Kitaev toric code [14], the ground-state manifold decomposes into distinct sectors [18]. Transitions between these sectors require the application of operator strings the lengths of which scale with system size. As a result, quantum superpositions spanning different sectors are naturally shielded from decoherence in macroscopic systems. Environmental noise is unlikely to generate the long, nonlocal operator sequences needed to disrupt such encoded quantum information. Furthermore, local errors can be detected and corrected via measurements of local observables known as stabilizers [14].

In this computational framework, quantum operations are implemented by braiding anyonic excitations [19]. When one anyon is adiabatically moved around another, the quantum state undergoes a unitary transformation within the ground-

state manifold, effectively realizing a quantum gate. Thus, quantum algorithms can be constructed by designing sequences of braiding operations.

The Kitaev honeycomb model [20–24] supports both Abelian and non-Abelian anyons [20,25] and offers a platform for their generation and manipulation. These anyons emerge as fluxlike excitations (also referred to as vortices or visons) of an effective gauge field resulting from the fractionalization of spin degrees of freedom. In the absence of an external magnetic field, these fluxes are conserved.

Recent neutron scattering experiments in RuCl_3 suggest the presence of a Kitaev quantum spin-liquid phase [26–28], and the observation of a robust thermal Hall effect [29–31] provides additional support for this scenario. This opens up the exciting possibility of performing topological quantum computation in solid-state platforms. In the first part of this work, we explore protocols that utilize minimal perturbations—specifically, local time-dependent magnetic fields applied to sequences of atomic sites—to manipulate visons in two distinct phases of the Kitaev model: the A_z phase and the B phase. These fields can be implemented using local magnetic scanning probes. We demonstrate that visons can be generated and displaced in both phases, although the probability for success is relatively low in the B phase. However, as the length scale of the manipulation increases, cumulative errors grow large enough to impede accurate braiding, leading to exponentially low success probabilities for flux manipulation. In addition, we point out another crucial constraint which makes flux manipulation in the Kitaev candidate material RuCl_3 very challenging: the lifetime of the fluxes is not long enough compared to the requirements of adiabatic transport.

In the second part of this work, we propose a protocol that displaces visons—and the attached Majorana zero mode—with low error by adiabatically decoupling a single spin from the rest of the system and applying a local magnetic field to it. The primary target platforms are quantum emulators [32–34],

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such as Rydberg atom arrays trapped in optical lattices, which have recently emerged as promising systems for simulating strongly correlated models, including the Kitaev model [35]. These platforms offer fine control over individual qubits and their interactions, enabling both the realization of the model and the operations necessary for anyon braiding.

This paper is organized as follows. In Sec. II, we review the Kitaev model and its solution. In Sec. III, we calculate the magnetic field induced phase transition points of the Kitaev model within the mean-field approximation. In Sec. IV, we present our protocol for flux manipulation in solid-state devices, and explain why it is unlikely to succeed. In Sec. V, we present our protocol for flux manipulation in quantum emulators. We draw conclusions in Sec. VI.

II. THE MODEL

In this section, we discuss the definition of a Majorana zero mode, their relation to the fluxes in the Kitaev model, and the sources of error in their transport. However, before doing so we present the model in which the protocols are implemented. The Hamiltonian of the system is $\hat{H}_0 = \hat{H}_K + \hat{H}_h + \hat{H}_\kappa$, where

$$\hat{H}_K = \sum_{\langle j,k \rangle} J_{\alpha_{jk}} \sigma_j^{\alpha_{jk}} \sigma_k^{\alpha_{jk}} \quad (1)$$

is the Kitaev Hamiltonian [20] which couples nearest-neighbor spins $\langle j, k \rangle$ on the honeycomb lattice. Here, σ_i^α denotes the α th Pauli matrix acting on the spin-1/2 Hilbert space of site i , while α_{jk} takes the values x, y, z depending on the direction of the bond $\langle j, k \rangle$. The local Zeeman coupling to the external field is

$$\hat{H}_h = \sum_j \mathbf{h}_j \cdot \boldsymbol{\sigma}_j, \quad (2)$$

while

$$\hat{H}_\kappa = \kappa \sum_{ijk} \sigma_i^{\alpha_{ij}} \sigma_j^{\beta} \sigma_k^{\alpha_{jk}} \quad (3)$$

is a next-nearest-neighbor interaction between spins on three consecutive sites i, j , and k . Here, β stands for the direction of the third bond which connects neither i and j nor j and k . The quasiparticles of the Kitaev model acquire non-Abelian anyonic statistics when the three-spin interaction $\hat{H}_\kappa$ is present. In solid-state systems, the three-spin interaction can arise from a magnetic field perturbation with nonzero components in all three x, y, z directions [20].

We note, however, that a magnetic field breaks the conservation of fluxes (see the definition below), while H_κ does not. Therefore, a uniform magnetic field and the three-spin term H_κ have very different impacts on the dynamics of fluxes. Thus, in this paper we study numerically the model defined above in the presence of both Zeeman and three-spin perturbation. To keep the model as general as possible, the magnitude of the Ising-like coupling $J_{\alpha_{jk}}$ is also allowed to depend on the bond direction. Hereafter, we scale energies with the smallest of the three J_α s (termed “ J ”) and we set $\hbar = 1$. Therefore, times are in units of J^{-1} .

The model (1) can be exactly solved by mapping Pauli matrices into Majorana operators. As in Ref. [20], we introduce four Majorana operators b_j^x, b_j^y, b_j^z , and c_j at each lattice site

j , such that $\sigma_j^\alpha = ib_j^\alpha c_j$. In the Majorana representation, the Kitaev Hamiltonian becomes

$$\hat{H}_K = -i \sum_{jk} J_{\alpha_{jk}} u_{jk}^{\alpha_{jk}} c_j c_k \quad (4)$$

where all $u_{jk}^{\alpha_{jk}} = ib_j^{\alpha_{jk}} b_k^{\alpha_{jk}}$ commute with $\hat{H}_K$ and are therefore constants of motions of the unperturbed Kitaev model. Their eigenvalues are ± 1 . Therefore, the Hamiltonian (4) describes the dynamics of Majorana fermions c , hereafter called matter fermions, in the presence of a $\mathbb{Z}_2$ gauge potential.

The Majorana representation doubles the Hilbert space of the original Kitaev model [20]. The component of any state belonging to the unphysical Hilbert space is removed by acting on the wave function with an opportune projector operator [20]. To define this, we introduce the gauge transformation operator $D_j = b_j^x b_j^y b_j^z c_j$, which satisfies $D_j = 1$ for physical states (for which it coincides with $-i\sigma_j^x \sigma_j^y \sigma_j^z$) and $D_j = -1$ for unphysical ones [20]. The sought projector is then constructed as

$$P = \prod_j (1 + D_j)/2 \quad (5)$$

where the product is over all lattice sites j . We will comment more on gauge invariance in Sec. III, where we discuss our mean-field approximation.

The Kitaev Hamiltonian in Eq. (1) possesses a set of conserved quantities W_p , each defined on a hexagonal plaquette p . In the Majorana representation they are $W_p = \prod_{\langle j,k \rangle} u_{jk}^{\alpha_{jk}}$, where $\langle j, k \rangle$ cycles through the six sides of the hexagonal plaquette p in the counterclockwise direction. Owing to Lieb’s theorem [36], the ground state of the Kitaev model must satisfy $W_p = 1$ for any plaquette p . This state is said to be “flux free.” In this paper, such a state is defined by setting $u_{ij}^{\alpha_{ij}} = -1$ for i (j) belonging to the A (B) sublattice. When $W_p = -1$, the plaquette p is said to be threaded by a flux. After inserting D_j at the middle spin j we get

$$\hat{H}_\kappa = \kappa \sum_{ijk} i u_{ij} D_j u_{jk} c_i c_k, \quad (6)$$

therefore we effectively have a next-nearest-neighbor interaction. In the flux-free sector, this interaction opens up a gap between the matter-fermion bands, which become topologically nontrivial. In the presence of this interaction, each flux carries a Majorana zero mode [20]. This means that, when two fluxes are located in plaquettes p and p' ($W_p = W_{p'} = -1$), the diagonalization of the Hamiltonian $\hat{H} = \hat{H}_K + \hat{H}_\kappa$ yields

$$\hat{H} = \sum_j E_j (a_j^\dagger a_j - a_j a_j^\dagger), \quad (7)$$

where one of the eigenvalues is exponentially small:

$$\epsilon = E_0 \propto e^{-L/\xi}. \quad (8)$$

Here, L is the distance between p and p' , while ξ is the localization length which is inversely proportional to the gap of the system, i.e., $\xi = |\sqrt{3}J/\Delta|$, where $\Delta = 6\sqrt{3}\kappa$ is the Majorana gap [20]. All other modes have energies above the gap. A pictorial illustration of Majorana zero modes for the Kitaev model is provided in Fig. 1. We can define the Majorana

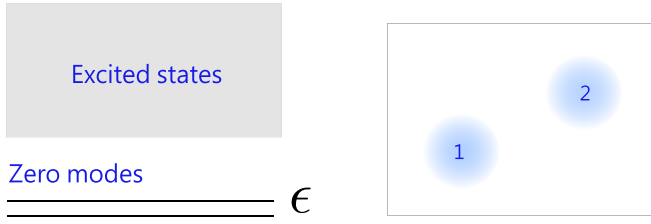


FIG. 1. A pictorial illustration of the spectrum and zero-mode wave function of the Kitaev model in the presence of two fluxes. The Majorana zero modes are exponentially localized around the fluxes.

operators in terms of the fermion creation and annihilation operators $a_0^\dagger$ and a_0 as

$$\gamma_1 = \frac{1}{2}(a_0 + a_0^\dagger), \quad (9)$$

$$\gamma_2 = \frac{i}{2}(a_0 - a_0^\dagger). \quad (10)$$

These are called Majorana zero modes. The wave functions of the zero modes decay exponentially to zero far enough away from both p and p' .

The logical qubits used for quantum computation are stored in the zero modes, i.e., the qubit states are $|a_0^\dagger a_0 = 1\rangle$ and $|a_0^\dagger a_0 = 0\rangle$. Their energy difference is on the order of ϵ . Therefore, as the system evolves, the two states in the zero-energy subspace acquire a dynamical phase difference proportional to ϵt , which implies that the state of the topological qubit is only secure on a timescale of order $1/\epsilon$. We call this the “dynamical error.” It follows that any braiding protocol must be executed faster than this timescale.

Another source of error is due to transport of zero modes not being perfectly adiabatic. In this case, the zero mode is mixed with excited states.

Suppose that at time $t = 0$ there are two zero modes γ_1 and γ_2 separated by a distance L . After time T , the flux at 2 is brought to an adjacent plaquette, which we call 3. If the zero mode had been created directly in plaquette 3, it would be represented by γ_3 . However, because any realistic protocol can only be adiabatic to a certain degree, one has

$$U(T)^\dagger \gamma_1 U(T) = p_1 \gamma_1 + q_1 \gamma_3 + \delta_1, \quad (11)$$

$$U(T)^\dagger \gamma_2 U(T) = p_2 \gamma_3 + q_2 \gamma_1 + \delta_2. \quad (12)$$

Here, $U(T)$ is the time evolution operator that displaces the flux from plaquette 2 to plaquette 3, q_1 and q_2 result from dynamical error, and $\delta_{1,2}$ are the errors due to nonadiabaticity, i.e., the mixing with states outside the zero-energy manifold. We call p_2 the *Majorana transport probability*, and for protocols that are close to adiabatic, they are expected to be close to 1. The parameters q_1 and q_2 can be made small by increasing the separation L , while δ_1 and δ_2 can be improved only by approaching the adiabatic limit.

III. MEAN-FIELD APPROXIMATION

The magnetic field $\mathbf{h}_j$ breaks the exact solvability of the model, i.e., the ground state of $\hat{H}_0$ does not satisfy $W_p = 1$ for all p . Thus, to determine the ground state, we resort to a mean-field decoupling of $\hat{H}_K$. We decouple all four-operator

terms in Eq. (4) and six-operator terms in Eq. (6) using Wick’s theorem, and neglect all normal-ordered terms with more than two operators [37]. In this way, the Hamiltonian $\hat{H}_0$ is reduced to a quadratic form in the Majorana fermions, $\hat{H}_{\text{MF}} = \sum H_{\text{MF}}^{ij} m_i m_j$, where the matrix H_{MF} depends on the values of mean fields as well as the couplings J and κ . Hereafter, to make our notation more compact, we denote with $\{m_j\}$ the set of all b_i^γ and c_i Majorana fermions (note that j here does not denote lattice sites as before). Note that the mean-field decoupling is performed in a single chosen gauge. However, different gauges produce the same physical results upon projection into the physical Hilbert space, just as in the normal Kitaev model.

To find the ground state of the Hamiltonian H_0 treated within the mean-field approximation, we derive and solve static mean-field self-consistent equations as follows. First, we assume an initial value for $\langle m_i m_j \rangle$ and diagonalize the Hamiltonian $\hat{H}_{\text{MF}}$. We obtain $\hat{H}_{\text{MF}} = \sum_k \epsilon_k a_k^\dagger a_k$, where k labels the eigenmodes of the system and the m_i ’s are related to the a_k ’s according to $m_i = S_{ik} a_k$. Here, S_{ik} is a unitary matrix. Since we are looking for the lowest-energy state, the average $\langle a_k^\dagger a_k \rangle$ is equal to 1 whenever $\epsilon_k < 0$, and zero otherwise. It follows that $\langle m_i m_j \rangle = \sum_{\epsilon_k < 0} S_{ik}^* S_{jk}$. This gives us a new ansatz for the mean fields, which we plug back into $\hat{H}_{\text{MF}}$. We repeat the process until convergence. We stop the iterations when the maximum of the absolute values of the differences between the averages for two consecutive steps is below 10^{-6} .

The uniform magnetic field is taken to be $\mathbf{h} = h\hat{\mathbf{n}}$, where $\mathbf{n} = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}$. At large values of h we expect spins to polarize in the $\hat{\mathbf{n}}$ direction, and the systems to be no longer in a spin-liquid phase. In the next section, we determine when this happens by solving an equilibrium mean-field self-consistency equation and we calculate the spin polarization that arises from $\mathbf{h}$. To avoid polarizing the system, in the following sections we will use small values of h .

The procedure described above yields a mean-field state $|\psi_{\text{MF}}\rangle$ that does not belong to the physical subspace. To obtain the gauge-invariant expectation value of a physical operator O , in principle one should evaluate

$$\langle O \rangle_g = \frac{\langle POP \rangle}{\langle P \rangle}, \quad (13)$$

where $\langle \cdot \rangle_g$ denotes the gauge-invariant average, $\langle \cdot \rangle$ is the average taken with respect to $|\psi_{\text{MF}}\rangle$, and P is the projector onto the physical subspace [see Eq. (5)]. Since any physical operator O commutes with P , this expression can be written as

$$\langle O \rangle_g = \frac{\sum_{i,j,\dots} \langle O(1 + D_i + D_i D_j + \dots) \rangle}{\sum_{i,j,\dots} \langle 1 + D_i + D_i D_j + \dots \rangle}. \quad (14)$$

We write G_k for any product of gauge transformations appearing in the sums above. A given G_k contributes nontrivially only if it does *not* flip any bond operator $\hat{u}_{ij}$ for which $|\psi_{\text{MF}}\rangle$ is an eigenstate. Consequently, when $|\psi_{\text{MF}}\rangle$ lies entirely within a single gauge sector, we have $\langle O \rangle_g = \langle O \rangle$. One might note that the global gauge transformation $G = \prod_i D_i$ also leaves all bond variables unchanged; however, G is equivalent (up to Majorana reordering) to the total fermion-parity operator [38]. Since our mean-field states always have even fermion

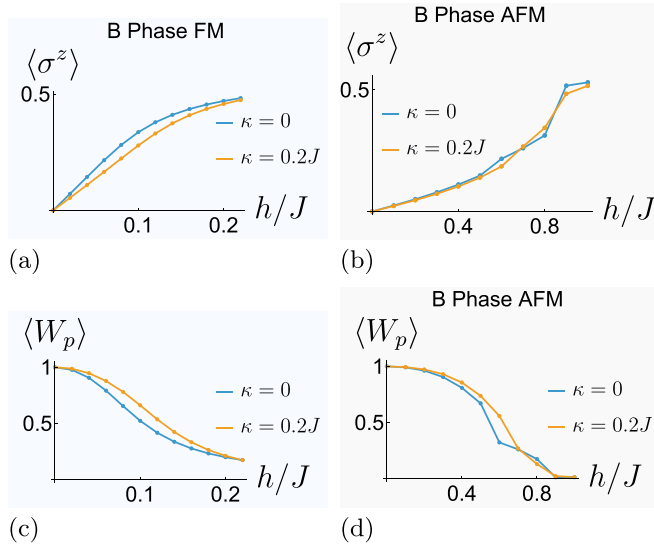


FIG. 2. (a) The spin polarization of the B phase along the direction of the external magnetic fields in the FM model for two values of κ . (b) Same as panel (a) but for the AFM model. (c) The average of the flux operator $\langle W_p \rangle$ as a function of the magnetic field h for the B phase of the FM model. (d) Same as panel (c) but for the AFM model. A clear phase transition occurs in the AFM model at $h \approx 0.8J$, as evidenced by the behavior of both average magnetization and flux operator. No phase transition occurs in the FM case. (c), (d) $\langle W_p \rangle$ is independent of p because of the periodicity of the system.

parity, it follows that $G|\psi_{\text{MF}}\rangle = |\psi_{\text{MF}}\rangle$, and G does not affect the expectation values.

Furthermore, when the system is only weakly perturbed by a magnetic field h , the state remains approximately an eigenstate of the $\hat{u}_{ij}$ operators. In this regime, gauge mixing is negligible, which justifies the approximation

$$\langle O \rangle_g \approx \langle O \rangle. \quad (15)$$

Phase transitions

Solid-state implementations of the Kitaev model require a uniform magnetic field h to open a gap in the excitation spectrum. Non-Kitaev interactions cause α -RuCl₃ to acquire zig-zag magnetic order at low temperatures, which can be suppressed by a uniform magnetic field, before becoming polarized at high fields [39,40]. It is therefore important to identify suitable values of h that preserve the anyonic statistics of the Kitaev spin liquid. To this end, we first estimate the critical field strengths separating the spin-liquid and polarized phases. Although evaluating these critical fields formally extends our mean-field approximation beyond the regime where neglecting gauge projection is quantitatively reliable, this does not affect our conclusions: our aim is simply to choose a sufficiently small h that ensures the system remains in the spin-liquid phase, rather than to accurately determine the full phase diagram. We also compare our estimates with results from more sophisticated numerical studies that locate the phase boundary more accurately, further supporting the validity of our chosen field values.

We consider a lattice of 64 unit cells. We employ periodic boundary conditions and we study the so-called A_z and B

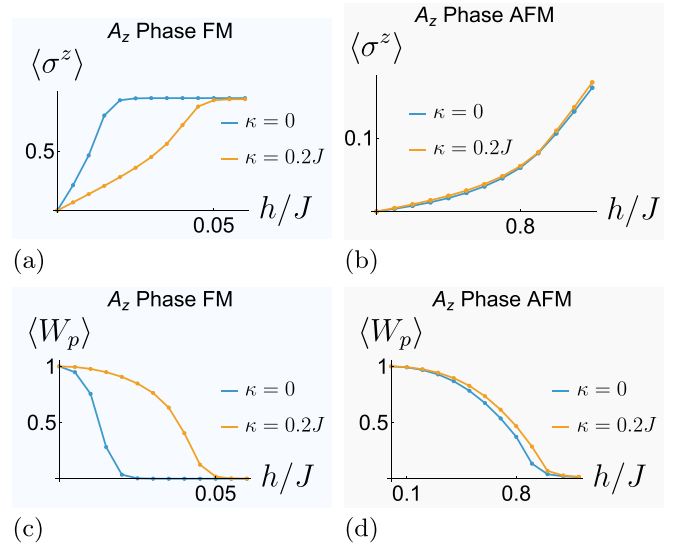


FIG. 3. (a) The spin polarization of the A_z phase along the z direction in the FM model. (b) Same as panel (a) but for the AFM model. (c) The average of the flux operator $\langle W_p \rangle$ as a function of the magnetic field h for the A_z phase of the FM model. The vanishing of $\langle W_p \rangle$ signals a phase transition at $h \approx 0.02J$. (d) Same as panel (c) but for the AFM model. A phase transition occurs in the AFM model at $h \approx J$, as evidenced by the vanishing of the average flux. In panels (c) and (d) $\langle W_p \rangle$ is independent of p because of the periodicity of the system.

phases of the Kitaev model, for both ferromagnetic (FM) and antiferromagnetic (AFM) couplings. The A_z phase is gapped and it is obtained for $|J_z| > |J_x| + |J_y|$. In the absence of a magnetic field, the excitations of the A_z phase are Abelian anyons [20]. As the gap size is of the order of J_z a small magnetic field $h \ll J_z$ will not close it. Therefore, the excitations in the A_z phase in the presence of a small magnetic field are still Abelian anyons. The B phase is gapless and exists when $|J_x|$, $|J_y|$, and $|J_z|$ all satisfy the triangle inequality $|J_\alpha| < |J_\beta| + |J_\gamma|$, where $\{\alpha, \beta, \gamma\}$ is any permutation of the three indices $\{x, y, z\}$. The excitations of this phase are non-Abelian anyons [20]. We set $J_x = J_y = J_z = J$ for the B phase and $J_x = J_y = J$, $J_z = 2.1J$ for the A_z phase, where J can be either FM, i.e., $J < 0$, or AFM, i.e., $J > 0$.

The spin polarization along z , $\langle \sigma_z \rangle$, for the B phase of the FM [AFM] model is shown in Fig. 2(a) [Fig. 2(b)] for two values of κ . In Figs. 2(c) and 2(d) we show instead the average value of the flux operator W_p of the FM and AFM models, respectively. In both cases, W_p is expected to approach zero, since it becomes equally likely to have one or no vortex in any plaquette at high magnetic fields. Similarly, Fig. 3(a) [Fig. 3(b)] shows $\langle \sigma_z \rangle$ for the A_z phase of the FM [AFM] model, while Fig. 3(c) [Fig. 3(d)] shows the average value of W_p of the FM [AFM] model.

In the B phase of the FM model, we find that the polarization rises smoothly. The presence of κ slightly decreases spin polarization while also increasing flux excitations at a given magnetic field, suggesting that κ stabilizes the spin-liquid phase against a magnetic field perturbation, despite our calculations not showing a clear transition. The absence of a clear phase transition is at odds with what is reported in

literature. Reference [25] finds a phase transition at $h \approx 0.056J$ (scaled to our units) using infinite density-matrix renormalization group. Calculations on finite clusters tend to find higher values for the phase transition point; for instance, Ref. [41] finds a singularity in the first derivative of the magnetization at around $h \approx 0.085J$ using density-matrix renormalization group methods. References [42,43] find $h \approx 0.08J$ using exact diagonalization, while Ref. [44] finds a change of Chern number at $h \approx 0.072J$ using mean field. The major difference with our study is that the mean-field approach of Ref. [44] enforces gauge invariance. We believe lack of gauge invariance in our calculation to be the reason why we do not find a clear phase transition: at high magnetic fields, the gauge sectors are strongly mixed, and our approximation of ignoring gauge projection no longer holds. However, in agreement with Ref. [41], we find that in both cases W_p vanishes smoothly. Both curves, in Fig. 2(c) and in Ref. [41], show an initially flat region where W_p does not change much, followed by a steep descent, to then smoothly converge to zero in the limit of infinite h . These contrasting results show that, though our mean-field approximation has limitation in describing certain features of thermodynamic quantities, it is capable of capturing the general behavior of fluxes. This lends credibility to the studies of the rest of this paper. The phase transition points in Refs. [25,41–44] suggest that a choice of $h = 0.02J$ for the B phase safely puts us in the spin-liquid regime.

In the A_z phase of the FM model, we find that the phase transition occurs at a much lower field, with $h_c \approx 0.02J$ at $\kappa = 0$ and $h_c \approx 0.05J$ at $\kappa = 0.2J$. These suggest that a value of $h = 0.005J$ will ensure that we stay in the spin-liquid regime. Interestingly, we also find that the κ term stabilizes the spin-liquid phase to a much larger extent than in the B phase. A simple perturbative argument can explain this behavior: the Kitaev model is perturbatively connected to the toric code in the A_z phase [20], with a Hamiltonian of the form $-J_{\text{eff},1} \sum_i Q_i$, where $J_{\text{eff},1} = J_x^2 J_y^2 / (16|J_z|^3)$, and the summation is done over all hexagonal plaquettes of the model. The Q_i operators become the familiar A_s and B_p operators of the toric code [14] under a unitary transformation and rearrangement of the lattice [20]. As we explain in the Appendix, the κ term also produces a toric-code Hamiltonian perturbatively: $H = -J_{\text{eff},2} \sum_i Q_i$, where $J_{\text{eff},2} = \kappa^2 / (2|J_z|)$. The magnetic field, on the other hand, maps to an effective “magnetic field” on the toric code in second order $h_{\text{eff}} \propto h^2$. Now, the phase transition point of the toric code is scaled by the energy: $h_{\text{eff},c} \propto J_{\text{eff}}$. It follows that $h_{\text{eff},c,\kappa=0} / h_{\text{eff},c,\kappa=0.2J} \sim J_{\text{eff},1} / (J_{\text{eff},1} + J_{\text{eff},2}) \approx 0.4$, which implies that the phase transition points for the real magnetic fields are $h_{c,\kappa=0} / h_{c,\kappa=0.2J} \approx 0.6$. The phase transition points we found in Figs. 3(a) and 3(c) roughly satisfy $h_{c,\kappa=0} / h_{c,\kappa=0.2J} \approx 0.5$, which is reasonably consistent with the perturbative argument given above.

Contrary to the previous cases, in the AFM model we find clear signs of phase transitions in both B and A_z phases. Furthermore, the curves for $\kappa = 0$ and $0.2J$ almost overlap with each other in the spin-liquid phases of both A_z and B phases of the AFM model, with the only difference being that in the B phase, κ makes the phase transition occur at a slightly lower magnetic field. In the AFM B phase, at $h \approx 0.85J$, the spin polarization jumps discontinuously, as shown in Fig. 2(b).

At the same critical field, the average flux W_p vanishes exactly, and remains equal to zero for higher fields, as shown in Fig. 2(d). In the AFM A_z phase, the jump in polarization which occurs at $h \approx J$ is less clear, as shown in Fig. 3(b). It is easier to identify the same critical point in Fig. 3(d), which shows that the average flux W_p clearly vanishes at $h \approx J$ and remains zero for higher fields. However, we do not observe the intermediate phase in the AFM Kitaev model that has been reported in several previous studies employing more sophisticated methods [45–49]. We attribute this discrepancy to the fact that our approach is not gauge invariant. Indeed, other mean-field studies that explicitly enforce gauge invariance in the field-perturbed Kitaev model [44,50] do find a gapped intermediate phase. As discussed in Sec. III, our approximation of neglecting gauge projection is valid only when the system remains weakly mixed between different gauge sectors. This condition is violated at strong magnetic fields, where the vison density approaches zero, indicating significant gauge mixing and, consequently, the breakdown of our approximation. We stress again that these limitations do not invalidate the calculations performed in the following sections. In fact, we work at magnetic fields small enough for gauge invariance to be approximately enforced, and for the system to remain in a quantum spin-liquid phase.

IV. A PROTOCOL FOR SOLID-STATE ENVIRONMENTS

In this section, we study the efficacy of a protocol for creating and moving fluxes in which only local magnetic fields are applied. This is roughly what would happen in a solid-state system manipulated by local magnetic probes. We begin by considering the probe to be infinitely sharp, and to be able to address a single bond. A discussion of the role of the size of the probe is given below.

In the Kitaev model the spin operator σ_j^α anticommutes with the bond operator u_{jk}^α . Therefore, applying the operator σ_j^α to the ground state creates two vortices in the two plaquettes that share the bond $\langle j, k \rangle$. This suggests that one can apply a local (Zeeman) magnetic field to drive the system out of equilibrium and manipulate vortices. We therefore perturb the system with the time-dependent Hamiltonian

$$\hat{V}(t) = \sum_j \delta \mathbf{h}_j(t) \cdot \boldsymbol{\sigma}_j, \quad (16)$$

where $\delta \mathbf{h}_j(t)$ is in the $\hat{\mathbf{z}}$ direction (note that its magnitude is not bound to be small with respect to J).

The Hamiltonian (16) generates the time evolution of mean fields, which is found by solving the equation of motion for $M_{jk}(t) = \langle \psi(t) | m_j m_k | \psi(t) \rangle$, i.e.,

$$\frac{dM}{dt} = 4i[H_{\text{MF}} + V(t), M], \quad (17)$$

where $V_{ij}(t)$ is the matrix representation of the driving Hamiltonian: $\hat{V}(t) = \sum_{ij} V_{ij}(t) m_i m_j$. Note that in this calculation the mean fields are not evolved adiabatically, i.e., we do not solve the mean-field self-consistent equations at each time step. Therefore the mean fields do not remain in equilibrium during the time evolution.

In the next subsections we discuss flux generation and displacement in both the A_z and B phases of the FM and AFM

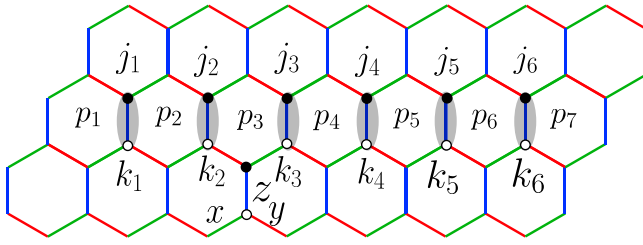


FIG. 4. Illustration of the Kitaev model and of the first flux manipulation protocol. The bonds of the honeycomb lattice are classified into three types, called x , y , and z , colored in green, red, and blue, respectively. Empty (filled) circles are sites of the A (B) sublattice, denoted as j_ℓ (k_ℓ). Sites are connected by bonds which are highlighted in dark gray. Fluxes are localized at plaquettes, which are denoted as p_ℓ . At $t = 0$, a magnetic field in the z direction is applied simultaneously at sites j_1 and k_1 and turned off after time T . We repeat this procedure on the sites j_ℓ and k_ℓ , for $\ell = 2, \dots, 6$. At the end of the protocol, a pair of fluxes is created in p_1 and p_7 .

model. With reference to Fig. 4, we study a protocol to create a pair of vortices in plaquettes p_1 and p_7 . We recall that the averages of bond operators on the six bonds highlighted in Fig. 4, $u_l = u_{j_k l}$, where $l = 1, \dots, 6$, are equal to -1 in the flux free sector, so that W_p is positive everywhere. Thus, we need to flip all six bonds $u_1, u_2, \dots, u_6$ to create the pair of vortices in the end plaquettes of the string (the only two for which W_p will be negative). The protocol we use to achieve the creation of vortices can be summarized as follows. First, we turn on a driving field δh_ℓ with the same sign on two spin sites $\ell = j_1$ and k_1 , which causes the bond-operator average u_1 to increase. We turn off the driving magnetic field after a time T for which the average of the bond operator u_1 is maximal. Repeating the process for all other bonds, we eventually create a flux pair at p_1 and p_7 .

A note on gauge invariance. The bond variables u_l are not gauge-invariant quantities; strictly speaking, one should evaluate the gauge-invariant expectation value of the physical flux operator $\langle W_{p_l} \rangle_g$ instead. Recall that the gauge-invariant average is defined as

$$\langle W_{p_l} \rangle_g = \frac{\sum_k \langle W_{p_l} G_k \rangle}{\sum_k \langle G_k \rangle}, \quad (18)$$

where G_k denotes any product of local gauge transformations. With reference to Fig. 4, the only gauge transformations G_k that do not vanish in the above sums are those which flip exclusively a subset of the six bonds $u_1, u_2, \dots, u_6$, since the mean-field state $\psi(t)$ remains approximately an eigenstate of all other bonds. However, no such local gauge transformation exists. Therefore, it is safe to neglect the gauge projection and to use the approximation

$$\langle W_{p_l} \rangle_g \approx \langle W_{p_l} \rangle. \quad (19)$$

A. Flux pair generation

We first examine the probability of creating a pair of fluxes in neighboring plaquettes. We first assume that the magnetic field δh can be turned on and off instantaneously; that is, we perform a quantum quench. We study the dependence on the magnetic field δh applied to the two spins on the

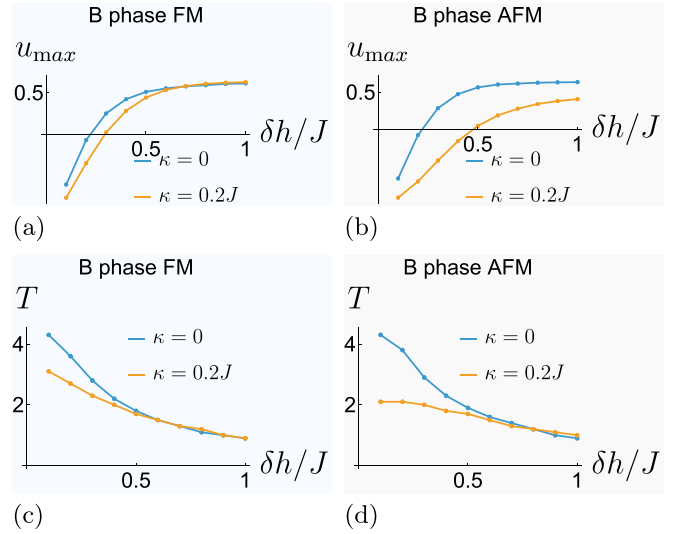


FIG. 5. (a) The maximum value reached by the bond mean field in the B phase of the FM model for two values of κ . (b) Same as panel (a) but for the AFM model with a staggered driving field. (c) The time T taken to reach maximum u as a function of the magnetic field δh for the B phase of the FM model. (d) Same as panel (c) but for the AFM model and a staggered field.

first bond in Fig. 4, and record the maximum value $u_{\max}$ reached by u_1 and the time T it takes to do so. For all our calculations u_1 starts off negative [i.e., $u_1(t=0) = -1$]; therefore $W_{p_1} \approx W_{p_2} \approx -u_1$. The probability of flux creation, given by $(1 - W_p)/2$, is large for positive values of $u_{\max}$ approaching unity.

In the B phase of the FM model we choose a uniform field $h = 0.02J$, well below any phase transition point, which increases the gap and stabilizes the spin-liquid phase without destroying the quantum order. As shown in Fig. 5(a), for $\kappa = 0$ the maximum bond mean field $u_{\max}$ reached becomes higher with larger values of the driving magnetic field δh , and it saturates for $\delta h > 0.5J$ at a value $u_{\max} \approx 0.5$. This corresponds to approximately 75% probability of flux pair creation. When $\kappa \neq 0$, we find that $u_{\max}$ decreases, but in the limit of large $\delta h > J$, $u_{\max}$ also saturates at the same value $u_{\max} = 0.5$. For both values of κ used in the figure, the time T it takes for u_1 to reach the maximum decreases with δh [Fig. 5(c)]. For small values of the driving field δh , the maximum is reached significantly faster when κ is finite. However, this difference disappears for sufficiently large δh .

In the A_z phase of the FM model, we find that we can achieve a much higher probability of flux creation [Fig. 6(a)]. We choose a uniform field $h = 0.005J$ to be well below the phase transition occurring at $h \simeq 0.02J$. For $\kappa = 0$, $u_{\max}$ remains approximately equal to 0.98 for most values of δh . The probability of flux creation in this case is approximately independent of δh and equal to 99%. Conversely, for $\kappa = 0.2J$, we find the probability of creating a flux pair to be significantly lower at low values of δh . As δh increases, the difference between the values achieved by $u_{\max}$ for $\kappa \neq 0$ and $\kappa = 0$ decreases considerably. As in the B phase, the time T taken for u_1 to reach the maximum value decreases when δh is made larger [Fig. 6(c)]. The time needed for finite κ is slightly

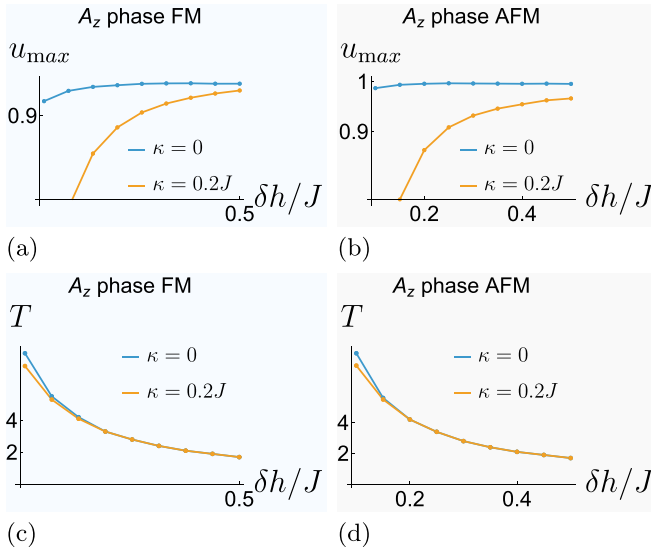


FIG. 6. (a) The maximum value reached for u in the A_z phase of the FM model for two values of κ . (b) Same as panel (a) but for the AFM model in a staggered driving field. (c) The time T taken to reach maximum u as a function of the magnetic field δh for the A_z phase of the FM model. (d) Same as panel (c) but for the AFM model and a staggered driving field.

shorter than what we found for $\kappa = 0$. As δh is increased, the difference between these two times disappears.

In Fig. 7 we show that if the driving field δh is turned on and off smoothly over a time t_0 in the B phase of the FM model the maximum bond mean field $u_{\max}$ is unaffected. However, the total time T taken for the protocol to complete the task becomes longer. The A_z phase behaves similarly.

In the AFM model, we find that the above protocol needs modification. We focus on the B phase, as modifications are similar for the A_z phase. If the field δh_ℓ takes the same value on both sites $\ell = j_1$ and k_1 (which belong to different sublattices), the value of the bond operator u_1 remains very close to -1 . Thus, almost no flux will be created. However, if the driving magnetic field takes opposite signs on the two sublattices, then we find that the maximum $u_{\max}$ is almost identical to that of the ferromagnetic model. This is not by

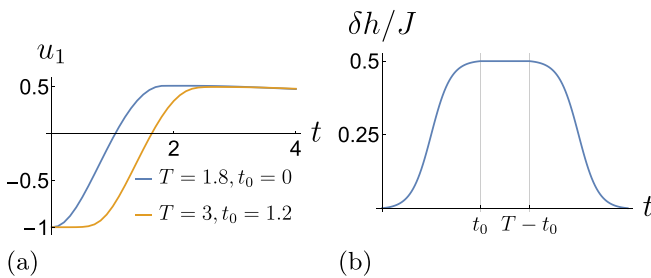


FIG. 7. (a) The value reached by the bond mean field is unaffected by the switching-on time. The blue curve is obtained by instantaneously switching on and off the driving field. The yellow curve is obtained with δh following the temporal profile in panel (b). The switching-on part of the function between $0 < t < t_0$ is proportional to $[1 + e^{-10(t/t_0 - 0.5)}]^{-1}$.

chance, and it is due to a property of the model that allows us to translate almost perfectly the results found for the FM model with a uniform field to the AFM one with a *staggered* magnetic field. This feature of the Kitaev model has been noted in previous studies [51–53].

To understand this behavior, we define the operator $O_A = \prod_{j \in A} \theta_j$, where $\theta_j \equiv b_j^x b_j^y b_j^z$, and the product is taken over the A sublattice sites only. Note that, because the θ_j 's anticommute with each other, the order of the terms in the product must be kept fixed throughout the calculation. The operator O_A anticommutes with the spins in the A sublattice, but commutes with the B sublattice spins. Since O_A is unitary, we can use it to transform every observable $\hat{X}$ as $\hat{X} \rightarrow O_A^\dagger \hat{X} O_A$, and every state $|\psi\rangle$ as $|\psi\rangle \rightarrow O_A^\dagger |\psi\rangle$. In particular, the Hamiltonian transforms as

$$O_A^\dagger \hat{H}_K O_A = -\hat{H}_K, \quad (20)$$

therefore O_A turns a ferromagnetic Kitaev model into an antiferromagnetic one, and vice versa. The driving Hamiltonian (16) gets transformed into

$$O_A^\dagger V O_A = \sum_i (-\delta h_{i,A} \sigma_{i,A} + \delta h_{i,B} \sigma_{i,B}), \quad (21)$$

where i here denotes the unit cell to which the spins of the two sublattices belong. Therefore, the operator O_A reverses the sign of the magnetic field $\delta h_{i,A}$ applied on A sublattice sites, while keeping the field on B sublattice sites unchanged. It must be emphasized that the operator O_A is not gauge invariant, and is therefore not a physical operator. It follows that the FM and AFM Kitaev models are not related by a unitary transformation.

Thus, the FM model with a small uniform magnetic field h and a driving field $\delta h = \delta h_A = \delta h_B$ that has the same sign on the two sublattices is mapped into the AFM one with staggered h and δh . This fact indicates that our results for the FM model can be carried over to the AFM one with little modifications, as shown by the similarity between FM and AFM results in Figs. 5 and 6. The small discrepancy between the FM and the AFM model with $\delta h_A = -\delta h_B$ is due to the fact that the sign of the uniform field h has been kept the same on both sublattices.

B. Flux creation probability

In Fig. 8, we show the time evolution of the bond variable u_1 following the sudden application of a constant driving field δh at $t = 0$ on the two spins k_1 and j_1 . When the system is in a gapped regime and the magnetic field is not too large, $\langle u_1(t) \rangle$ displays nearly sinusoidal oscillations, indicating that the dynamics can be qualitatively captured by a two-state model.

Let $|00\rangle$ denote the gauge-sector state in the flux-free configuration and $|\pi\pi\rangle = b_{k_1}^z |00\rangle$ the state with two fluxes. For simplicity we assume that the magnetic field is only applied on the spin k_1 .

Then

$$ib_{k_1} c_{k_1} |00\rangle |\phi_{00}\rangle = i|\pi\pi\rangle c_{k_1} |\phi_{00}\rangle, \quad (22)$$

$$ib_{k_1} c_{k_1} |\pi\pi\rangle |\phi_{\pi\pi}\rangle = -i|00\rangle c_{k_1} |\phi_{\pi\pi}\rangle. \quad (23)$$

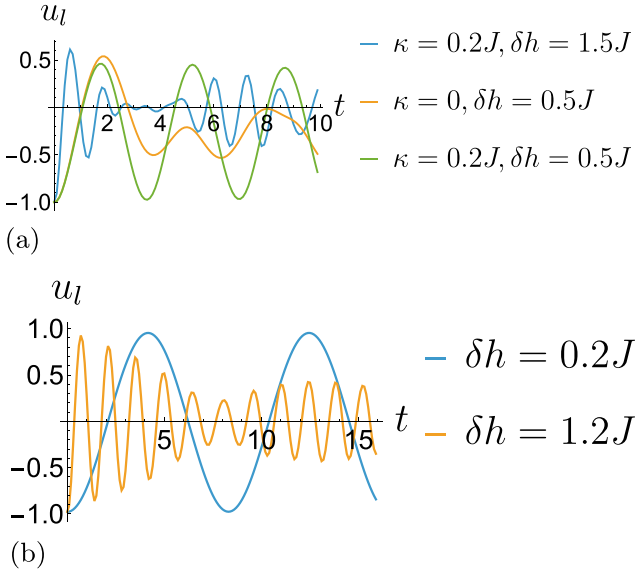


FIG. 8. Time evolution of u_l under a constant driving field δh applied at $t = 0$ on two spins k_1 and j_1 . (a) In the B phase, when $\kappa = 0.2J$ and $\delta h = 0.5J$, the system is gapped and the time evolution of u_l is sinusoidal. For larger magnetic fields or when the gap closes, the dynamics becomes irregular, signaling the breakdown of the two-state picture. (b) In the A_z phase, the system exhibits sinusoidal oscillations at small $\delta h = 0.2J$, while stronger fields destroy the two-level behavior.

The full state of the system can be expressed as

$$|\psi(t)\rangle = \alpha |00\rangle |\phi_{00}\rangle + i\beta |\pi\pi\rangle c_{k_1} |\phi_{\pi\pi}\rangle, \quad (24)$$

where $|\phi_{00}\rangle$ and $|\phi_{\pi\pi}\rangle$ are matter-sector states (which have the same fermion parity due to the c_{k_1} factor), and α and β are time-dependent amplitudes. Grouping the matter states into a two-component vector $\phi = (\phi_{00}, \phi_{\pi\pi})$, their evolution is governed by

$$H = \begin{pmatrix} H_{00} & \delta h \\ \delta h & H_{\pi\pi} \end{pmatrix}, \quad (25)$$

where H_{00} and $H_{\pi\pi}$ are the matter-sector Hamiltonians in the flux-free and two-flux configurations, respectively.

To obtain a purely two-state Hamiltonian, we assume that the time evolution of the c fermions in the 00 and $\pi\pi$ sectors is slow and can be neglected. We then replace H_{00} and $H_{\pi\pi}$ by constants, with $H_{\pi\pi}$ corresponding to the two-flux energy cost, i.e., the flux gap Δ_f . This yields the effective model

$$H_{\text{eff}} \approx \begin{pmatrix} 0 & \delta h \\ \delta h & \Delta_f \end{pmatrix}, \quad (26)$$

which describes Rabi-like oscillations between the $|00\rangle$ and $|\pi\pi\rangle$ states. The maximum transition probability is

$$P = \frac{4\delta h^2}{\Delta_f^2 + 4\delta h^2}. \quad (27)$$

Equation (27) shows that a smaller flux gap leads to a higher flux-pair creation probability. In the B phase, the flux gap is $\Delta_f \approx 0.26J$ [20,54], and its dependence on κ is quadratic and weak. In contrast, in the A_z phase, as derived

perturbatively in the Appendix,

$$\Delta_f = \frac{\kappa^2}{|J_z|} + \frac{J_x^2 J_y^2}{8|J_z|^3}, \quad (28)$$

which for $\kappa = 0.2J$ and $J_z = 2.1J$ gives $\Delta_f \approx 0.016J$, significantly smaller than in the B phase. This naturally explains why our protocol achieves a much higher flux-creation success probability in the A_z phase. Furthermore, since Δ_f increases with κ , Eq. (27) is consistent with the behavior observed in Fig. 6, where the flux-creation probability is lower for $\kappa = 0.2J$ than for $\kappa = 0$.

Finally, although Eq. (27) seems to suggest that $P \rightarrow 1$ for $\delta h \rightarrow \infty$, in this limit the strong magnetic field induces rapid oscillations within the matter sector, invalidating the two-state approximation—as indeed seen in Fig. 8.

C. Flux displacement

We now characterize the capability of the “solid-state protocol” defined above to move fluxes. This is essential for achieving anyon braiding. We focus on the FM model for simplicity, as results for the AFM model are similar, albeit staggered magnetic fields are needed as shown above. In the B phase, we use $\delta h = 0.5J$, $T = 1.8J^{-1}$, $t_0 = 0$, and $h = 0.02J$. We keep $\kappa = 0$ for our calculations in this section, as models with finite values of κ behave similarly.

To move a flux, we apply the procedure described at the beginning of this section on the next two sites, $\ell = j_2$ and k_2 . This results in the (partial) flipping of the value of u_2 . Repeating the process for all other bonds, we eventually create a flux pair at p_1 and p_7 . As in the case of flux creation, the displacement is only achieved with a certain probability.

The results of this protocol are shown in Figs. 9(a) and 9(b). In Fig. 9(a) [as well as in Fig. 5(a)] we show that applying the driving fields δh_{j_1} and δh_{k_1} for a time T flips the value of u_1 , although not completely. For the parameters given above, we reach the maximum value $u_1 \approx 0.5$, which corresponds to $W_{p_1} = W_{p_2} = -0.5$. This corresponds to generating a state which, in each plaquette p , is in a superposition between a vortex and no-vortex state. As discussed in Sec. IV A, the probability of creating the vortex pair is $(1 - W_p)/2 \approx 0.75$, while the probability to have no vortex is $(1 + W_p)/2 \approx 0.25$.

Repeating this procedure for all other bonds in the string, we see that we can only achieve $u_l \approx 0.2$ for $l = 2, \dots, 6$. Thus, for intermediate plaquettes the probability of having a vortex is very close to one-half. The value of the flux operator is relatively large and negative in the first and last plaquettes. This corresponds to a probability of having a flux equal to ≈ 0.7 and ≈ 0.58 in the first and last plaquettes, respectively. We note that the vortices we create are quite stable, as shown by the flat time evolution of the u'_l s shown in Fig. 9(c).

We then apply the same protocol to create a pair of fluxes in the A_z phase of the FM model. In this case, we use the parameters $T = 4.1J^{-1}$, $h = 0.005J$, and $\delta h = 0.2J$. The results are shown in Figs. 10(a) and 10(b). In Fig. 10(a) we show that, starting from the uniform flux-free configuration and applying the driving fields δh_{j_1} and δh_{k_1} for a time T , the value of u_1 is almost completely flipped. For the parameters used, as expected from Fig. 6(a), we can reach $u_1 \approx 0.97$, which corresponds to generating fluxes in p_1 and p_2 with

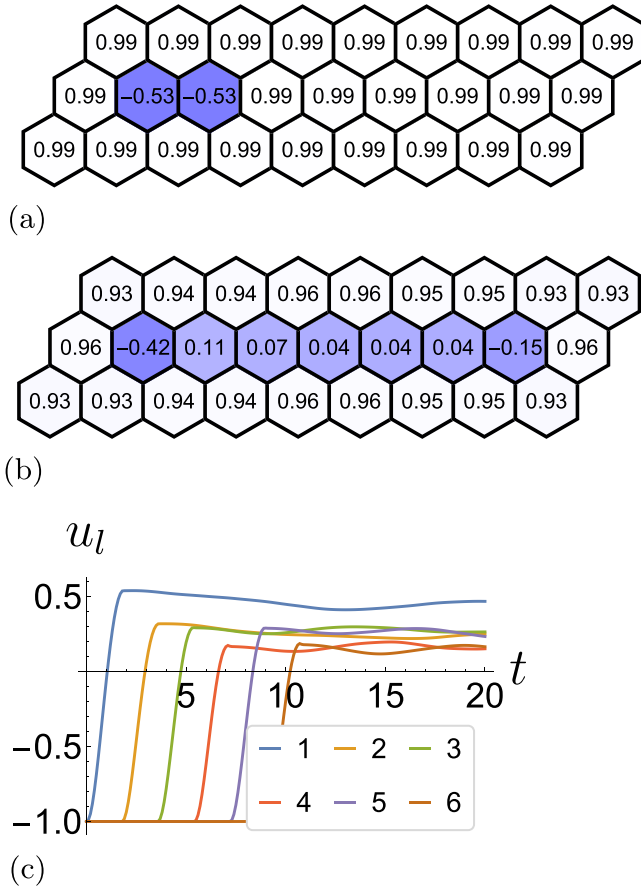


FIG. 9. Flux manipulation protocol applied to the B phase of the FM model starting from a configuration with no fluxes and $\kappa = 0$. (a) The flux configuration at time $t = T$: the magnetic field applied on the first bond (see Fig. 4) has a superposition between flux and no-flux states in plaquettes p_1 and p_2 . (b) The flux configuration at the end of the protocol ($t = 20J^{-1}$). All plaquettes along the chain are in a superposition between states with and without fluxes. (c) The time evolution of the bond mean fields u_l ($\ell = 1, \dots, 6$) between plaquettes p_ℓ and $p_{\ell+1}$. Mean fields become quite stable after the removal of the driving field.

probability ≈ 1 . Repeating this procedure for all other bonds in the string, we see that we can achieve $u_l \approx 1$ for $l = 2, \dots, 6$. Thus, the intermediate plaquettes remain almost flux free in this case, and fluxes are almost completely localized to the end plaquettes of the string. The probability of having a flux is equal to ≈ 0.93 and ≈ 0.97 for the first and last plaquettes, respectively. The vortices we create are very stable, as shown by the flat time evolution of the u_l 's shown in Fig. 10(c). Clearly, our protocol works much better in the A_z phase than the B phase.

In the flux manipulation results presented above, we assumed zero value of the Haldane term κ . A finite value of κ in general worsens the flux pair creation probability. However, we find that, if we start the system in a state that already contains two separated fluxes and apply our protocol, then a finite κ leads to a higher chance of success to move the preexisting fluxes (see Fig. 11). This result is consistent with what is reported in Ref. [55]. In this case, we initialize the FM system in the B phase in a state that contains two separate

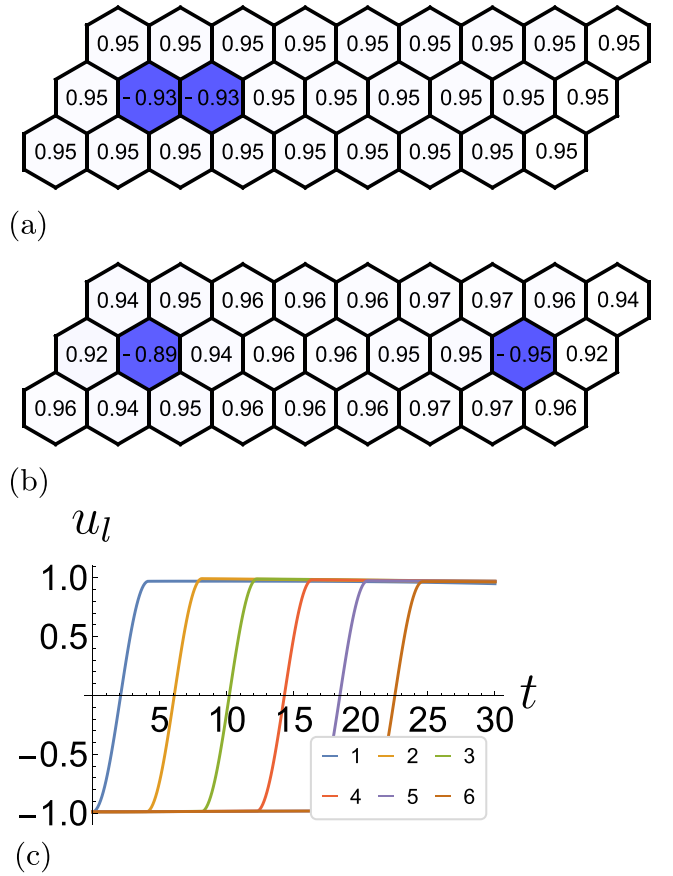


FIG. 10. Flux manipulation protocol applied to the A_z phase of the FM model starting from a configuration with no fluxes and $\kappa = 0$. (a) The flux configuration at time $t = T$. The magnetic field applied on the first bond (see Fig. 4) has created two nearly quantized fluxes in the neighboring plaquettes p_1 and p_2 . (b) The flux configuration at the end of the protocol ($t = 30J^{-1}$). The plaquettes at the end of the chain contain an almost quantized flux, while the others are close to a no-flux state. (c) The time evolution of the bond mean fields u_l ($\ell = 1, \dots, 6$) between plaquettes p_ℓ and $p_{\ell+1}$. Mean fields become very stable after the removal of the driving field.

fluxes. One flux is shown in Fig. 11(a), while its partner is not shown and is located five unit cells down the vertical axis. We choose a finite $\kappa = 0.1J$, and magnetic fields $\delta h = 0.2J$ and $h = 0.01J$ for this calculation. The time it takes for a single bond to flip is $T = 6.4J^{-1}$. As before, we apply the magnetic field on two spins k_1 and j_1 for a time T , after which u_1 reaches 0.87, which corresponds to a flux flipping probability $\approx 93\%$. We then repeat the procedure on the other bonds, and in the final state [Fig. 11(b)] we can achieve $W_p = -0.92$ on the last plaquette, which corresponds to a probability $\approx 96\%$ of having a flux.

D. Feasibility of the protocol in solid-state devices

In this section, we address the problem of feasibility of the above protocol in a realistic setup. We assume that the local magnetic field is applied with a scanning superconducting quantum interference device (SQUID) probe. The SQUID excites a number of bonds in the region where the magnetic field is applied. The typical SQUID size [56] is around 50–100 nm,

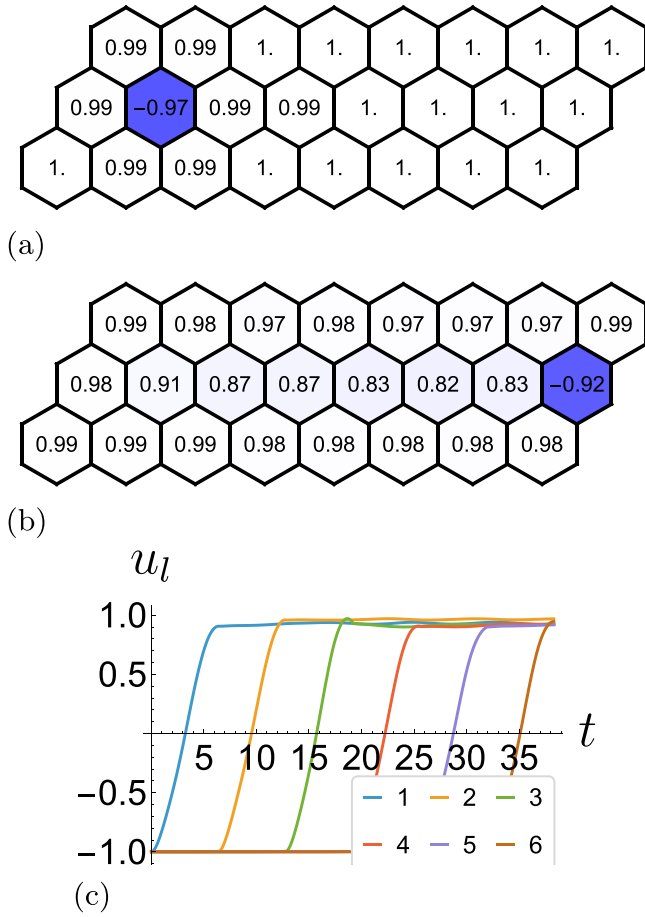


FIG. 11. Flux manipulation protocol applied to the B phase of the FM model starting from a configuration with two fluxes and for a finite value of κ . (a) The flux configuration at time $t = 0$ (the second flux is displaced vertically and is not shown in this figure). (b) The flux configuration at the end of the protocol ($t = 38J^{-1}$). (c) The time evolution of the bond variables u_l ($\ell = 1, \dots, 6$) between plaquettes p_ℓ and $p_{\ell+1}$. When κ is finite and the protocol is started with well-quantized fluxes the probability of success becomes much higher. Moreover, mean fields become very stable after the removal of the driving field. For comparison, see Figs. 9(b) and 9(c).

so the field is applied to a region covering approximately 100×100 spins (assuming the SQUID to be roughly a square for simplicity). To capture the effect of the tip, we apply our protocol to a sizable region covering several sites, as shown in Fig. 14, and extrapolate our findings to larger probe areas. We expect the qualitative features of this result to remain valid even if the tip was chosen to be of realistic size.

We focus on the A_z phase: we do not present results for the B phase because the protocol has little chance of success, as shown above. In the A_z phase we find that the response of single bonds is largely independent of the tip size. That is, the peak value of u_j is about the same for all bonds within the probe region and is reached in similar amounts of time. The states of the system can then be described by “strings.” A string ($p_1 p_2$) is defined as the state with exactly two fluxes in plaquettes p_1 and p_2 (which can be far away from each other), and nowhere else in the middle. Such a state is created by flipping the mean fields u_j along a string of bonds connecting

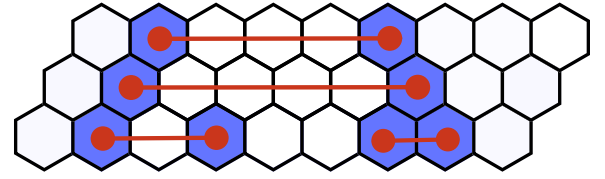


FIG. 12. Illustration for a configuration of “strings.” The true state is a superposition of such configurations.

p_1 and p_2 . For example, Fig. 12 shows four strings in the horizontal direction, the topmost of the length 4, the middle one of length 5, and the two lowermost of length 2 and 1. This is but one way to define the strings. In fact, the string representation is not unique, because a string can be chosen in arbitrary ways, as long as it connects p_1 and p_2 . Furthermore, in general, a state can have more than one pair of fluxes, and can therefore be represented by several strings.

The goal of our protocol is to create a pair of fluxes in spatially separate regions. With reference to Fig. 13, suppose the tip sweeps from point p_1 to point p_2 . Let R_1 and R_2 be disk-shaped regions around p_1 and p_2 of radius r , where r is also the radius of the tip. Let P be the middle region swept by the tip, which excludes R_1 and R_2 . We say that the protocol is successful if there is a single net flux in R_1 and a single net flux in R_2 , and nowhere else. This final state corresponds to string configurations that start in R_1 and end in R_2 , hence there is an odd number of string end points in R_1 and R_2 ,

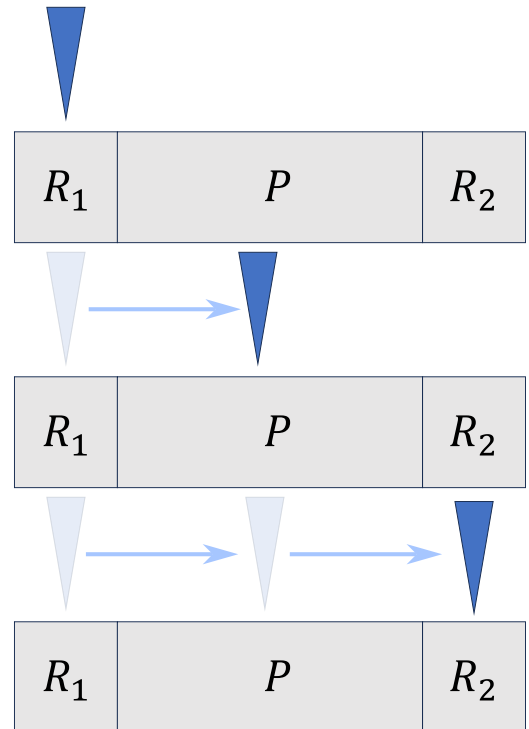


FIG. 13. A pictorial illustration of a practical realization of the protocol. The blue triangles represent the tip, which sweeps from region R_1 through P to R_2 . The protocol is successful when a single net flux is generated in both R_1 and R_2 , and no flux is left behind in P . The magnetic field is only applied to the region below the tip.

respectively. If the protocol is successful, no string starts or ends in the region P .

Let us assume that the number of rows in the region P is r_P , and the number of bonds in each row is ℓ_P . We define a as the probability that all the bonds in one row are flipped, and b as the probability that none of them are flipped. They are given by

$$a = p_{\text{flip}}^{\ell_P}, \quad (29)$$

$$b = (1 - p_{\text{flip}})^{\ell_P} \quad (30)$$

where p_{flip} is the probability to flip a single bond. For the protocol to succeed, all the bonds in an odd number of rows must be flipped, and every other bond must remain the same. These conditions ensure that there is an odd net number of fluxes in R_1 and R_2 , and that there are no broken strings that start or terminate in the middle region P . The final probability is given by the sum over such configurations:

$$p_{\text{succ}} = \sum_{1 \leq k \leq r_P, k \text{ odd}} \binom{r_P}{k} a^k b^{r_P-k} \quad (31)$$

$$= \frac{(b+a)^{r_P} - (b-a)^{r_P}}{2}. \quad (32)$$

For even r_P , this probability is maximum at $a = b = 0.5$, where $p_{\text{succ}} = 0.5$. This can only occur if $\ell_P = 1$, which is when there is only one bond in each row of P . For realistic situations, the probabilities a and b will be exponentially suppressed: $a, b \approx 1/2^{\ell_P}$. For the large systems we are considering, they become practically zero. For odd r_P , the maximum probability occurs at $a = 1, b = 0$, where $p_{\text{succ}} = 1$. This is the situation where bonds are all flipped, and the number of fluxes in R_1 and R_2 is odd because the number of rows is odd.

For small b , the success probability can be approximated by

$$p_{\text{succ}} = \begin{cases} a^{r_P}, & r_P \text{ odd} \\ r_P a^{r_P-1} b, & r_P \text{ even} \end{cases} \quad (33)$$

For a realistic estimate of p_{succ} , we put $r_P = 200$, $\ell_P = 2000$, and we use $p_{\text{flip}} \approx 0.98$. These numbers give $a \approx 2.8 \times 10^{-18}$ and $b \approx 0$. Therefore, even with a good flip probability, the success probability is still almost zero due to the large system size. In practice, there will always be strings that break in the region P , making the protocol unsuccessful according to our definition.

However, the situation may not be as bad as it looks. In the A_z phase the flip probability is high. Therefore, although the protocol leaves many broken strings in the region P , their end points are very close to each other. If two string end points are close enough to each other, over time it is possible for them to merge. To numerically estimate the effect of string self-healing, we assume that end points that are within a distance d will annihilate each other. For simplicity, we assume that all three regions R_1, R_2 , and P are one dimensional. In our numerical calculations, we set their lengths to be $\ell_1 = \ell_2 = 20$ and $\ell_P = 1960$. We take a bond flip probability of $p = 90\%$, as found in the A_z phase. We then generate random lists of flux configurations and count how many pairs of end points can be found within a distance d . We find that, for $d = 2$, the success

rate is only 17%; for $d = 3$, however, it is 86%. For $d = 4$, the success rate is 98%. However, in the B phase the bond-flip probability is too low ($\approx 60\%$) and therefore the success rate is below 1% for all $d = 2, 3, 4$.

E. Instability of fluxes

Apart from the probability argument presented above, we point out another crucial drawback of solid-state implementation of topological quantum architectures. Due to the presence of terms that do not conserve fluxes, such as the small magnetic field h , the fluxes gain a finite lifetime τ . If the time for adiabatic transport of fluxes is T_{total} , the following must hold:

$$\hbar/\Delta \ll T_{\text{total}} \ll \tau \quad (34)$$

where Δ is the gap. The first part of this inequality comes from requirements of adiabatic transport, while the second part ensures that the fluxes survive until the end of the protocol. Let us now estimate the constraints in Eq. (34) for a realistic system. We take RuCl_3 as an example. In this case [57,58], the gap that can be generated is $\Delta \approx 0.1J \approx 1$ meV for an out-of-plane field of 10 T, which gives $\hbar/\Delta \approx 4.15$ ps. The flux lifetime, on the other hand, is inversely proportional to the flux hopping parameter t_h , which is proportional to the magnetic field to the first power, $t_h \propto h$, or second power, $t_h \propto h^2$, for the FM and AFM models, respectively [51]. For a magnetic field of 10 T, and an effective magnetic moment of $\mu \approx 0.13$ meV/T, one can estimate $\tau \approx 3.19$ ps. Therefore the upper and lower bound for T_{total} are of the same order of magnitude, which implies that vision manipulation is likely to be extremely challenging in this system.

V. A PROTOCOL FOR QUANTUM EMULATORS

In this section, we present a protocol that can displace Majorana zero modes with very high probability. The protocol requires turning on and off interaction terms in $\hat{H}_K$ and $\hat{H}_\kappa$, which is impossible in a solid-state environment. However, this is possible in quantum emulators [32–34]. Emulators also lessen the requirements of extreme (atomic size) locality of applied fields that plague solid-state implementation. This in turn leads to much higher success probabilities.

The protocol works as follows. Suppose, for instance, that we are trying to move a flux from plaquette p_1 to p_2 , as shown in Fig. 15(a). In the first step, the spin k_1 is slowly removed from the Hamiltonian, i.e., all the couplings involving it [orange-shaded in Fig. 15(a)] are slowly turned off between $0 < t < t_0$. During this switching-off process, the couplings are assumed to depend on time according to

$$f_{i_0}(t) = \frac{a}{1 + e^{5-10t/t_0}} + b \quad (35)$$

where a and b are chosen such that $f_{i_0}(0) = 1$ and $f_{i_0}(t_0) = 0$. This curve is shown in yellow in Fig. 15(b).

Next, once the bonds are completely turned off, a magnetic field $\delta h(t)\sigma_{k_1}^z$ is applied to the spin k_1 . This generates a time evolution

$$\begin{aligned} \partial_t \langle b_{k_1}^z b_{j_1}^z \rangle &= -2\delta h(t) \langle b_{j_1}^z c_1 \rangle, \\ \partial_t \langle b_{j_1}^z c_{k_1} \rangle &= 2\delta h(t) \langle b_{k_1}^z b_{j_1}^z \rangle. \end{aligned} \quad (36)$$

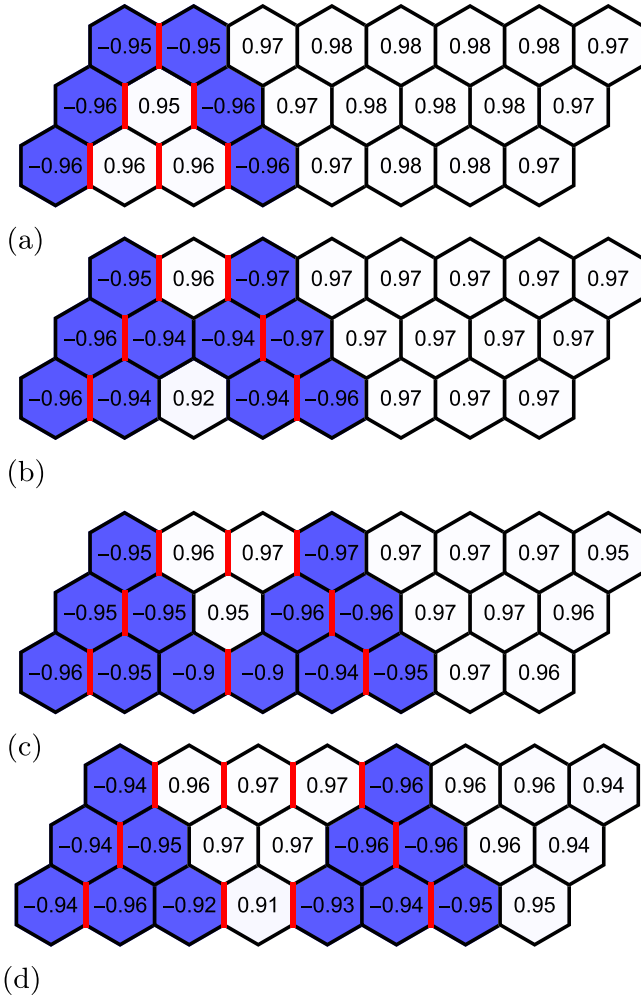


FIG. 14. Results when the same A_z phase protocol is used on a larger region. Red bonds are flipped bonds ($u_l < 0$). The “tip” covers the 12 spins in the bonds shown in red in panel (a). It applies the magnetic field for T , and then it moves by one unit cell to the right, applies the field for another T , and so on. (a)–(d) The result after T , $2T$, $3T$, and $4T$.

These equations take a closed form because the interactions between the spin k_1 and the rest of the system have been turned off. The solution of the system of equations (36) is

$$\langle b_{k_1}^z b_{j_1}^z \rangle_t = \langle b_{k_1}^z b_{j_1}^z \rangle_{t_0} \cos \int_{t_0}^t 2\delta h(\tau) d\tau \quad (37)$$

where we used the condition $\langle b_{j_1}^z c_{k_1} \rangle_{t_0} = 0$. This equation implies that the value of $u_1 = \langle ib_{k_1}^z b_{j_1}^z \rangle$ is completely flipped when the integral $\int_{t_0}^t 2\delta h(\tau) d\tau = \pi$. In the remaining part of this paper we will assume that $\delta h(t) = \delta h_0$ whenever $t_0 < t < T - t_0$, i.e., the magnetic field perturbation is turned on and off infinitely fast, as shown by the blue curve in Fig. 15(b). We use

$$\delta h_0 = \frac{\pi}{2(T - 2t_0)} \quad (38)$$

for the strength of the magnetic field perturbation, which ensures that at time $t = T - t_0$, the value of $\langle b_1^z b_2^z \rangle$ is completely flipped. We also fix $T - 2t_0 = 5J^{-1}$. Note that $\delta h(t)$ does

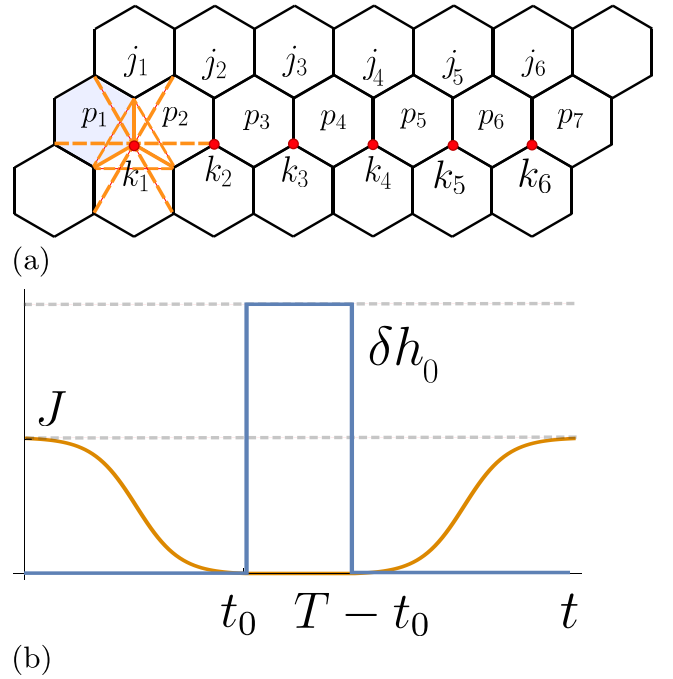


FIG. 15. (a) Illustration of the second flux manipulation protocol. Red circles are sites of the A sublattice, denoted as k_ℓ . Nearest-neighbor couplings are shown in black. The blue shaded plaquette p_1 stands for a flux. At $t = 0$, a time-dependent magnetic field in the z direction is applied to spin k_1 . Its time dependence follows the blue curve in panel (b). The orange shaded bonds (i.e., all couplings to spin k_1) are turned off between $0 < t < t_0$ and back on between $T - t_0 < t < T$ according to the orange curve in panel (b). We repeat this procedure on the sites j_ℓ and k_ℓ , for $\ell = 2, \dots, 6$. At the end of the protocol, the flux at p_1 is moved to p_7 . (b) Time evolution of the magnetic field δh (blue) and of the bond strengths (orange).

not need to be constant. Any $\delta h(t)$ such that the integral in Eq. (37) equals π at time $t = T - t_0$ will produce the same result. In other words, the purpose of the magnetic field is to implement the unitary operator $e^{i\frac{\pi}{2}\sigma_{k_1}^z} = i\sigma_{k_1}^z$ on the spin k_1 , and the particular shape of the magnetic field is irrelevant here.

Lastly, for the final step, we slowly reconnect the spin to the rest of the system. We do so by bringing the orange-shaded couplings in Fig. 15, over the time $T - t_0 < t < T$, back to their original value. After all three steps are performed, the flux is transported one unit cell to the right, and the Majorana zero mode moves with it. We can repeat this procedure as many times as necessary to move the Majorana zero mode to any desired location.

Note that the time evolution in Eq. (36) involves the mean field $\langle b_{k_1}^z b_{j_1}^z \rangle$ defined on the bond between spins k_1 and j_1 . Such mean field is not bound to necessarily vanish because, although the bond has been cut off, the two spins can still be entangled. However, requiring them to be entangled puts a constraint on the time T taken by the protocol to run. In fact, T must be shorter than the typical times it takes for entanglement to decay (e.g., shorter than the decoherence times of the emulator). Typical optical tweezer pick up times are on the order of 1 ms [59]. The decoherence times, on the other hand, depends on the type of system. State-of-the-art hyperfine qubits [60]

can achieve decoherence times on the order of 10 s, while Rydberg atom arrays have decoherence time on the order of 1 ms [61]. Therefore, realistic implementation of our protocol is feasible, albeit challenging in certain types of systems.

Our protocol, which operates by effectively removing a spin from the system, may recall earlier studies showing that a single vacancy in the Kitaev model can bind a Majorana zero mode [62–65]. In our case, once a spin is removed, the Majorana zero mode associated with it becomes localized at the vacancy site. From the viewpoint of the remaining lattice, there is simultaneously both a flux and a vacancy at this location. The zero mode originating from the flux and that associated with the vacancy effectively “cancel” each other, leaving behind a single Majorana zero mode bound to the removed spin.

A. Error rates

To determine the error rate of our protocol, we apply it to a periodic system with 8×20 unit cells, and we create a pair of fluxes at unit cells (1,1) and (5,10). We call the Majorana zero mode they carry $\hat{\gamma}_1$ and $\hat{\gamma}_2$, and their wave functions γ_1 and γ_2 , respectively. We move γ_1 by one unit cell to (2,1) while keeping γ_2 stationary. We call β_1 the wave function of the Majorana zero mode located at unit cell (2,1). Its wave function is obtained by diagonalizing the Hamiltonian with fluxes in unit cells (2,1) and (5,10). For perfect transport, we would get $\gamma_1(T) = \beta_1$.

However, this is not the case, as the transport of the zero mode from (1,1) to (2,1) introduces some error. At the end of the protocol, the wave function of the zero mode γ_1 becomes

$$\gamma_1(T) = p_1 \beta_1 + \gamma_{1,\text{rest}} + \delta_1 \quad (39)$$

where $p_1 \beta_1$ is the projection of $\gamma_1(T)$ onto β_1 , $\gamma_{1,\text{rest}}$ is the projection of γ_1 onto γ_2 , and δ_1 is the component of $\gamma_1(T)$ outside the zero-energy subspace. To test the protocol, we run it for a range of values of t_0 . The resulting errors are shown in Fig. 16(a). We find that $|\gamma_{1,\text{rest}}|^2$ slowly increases. (Hereafter, to shorten our notation, we use $|\psi|^2$ to denote the sum of the moduli squared of all the components of the wave function ψ .) This is expected since the total time taken by the protocol ($T = 2t_0 + 5J^{-1}$) increases linearly and the dynamical phase due to the interaction of Majorana fermions scales as $T\epsilon$. Here, ϵ is the interaction energy between γ_1 and γ_2 as defined in Eq. (8). The nonadiabatic error $|\delta_1|^2$, on the other hand, shows an oscillating exponential decrease with t_0 . This shows that the nonadiabatic error rate of our protocol can be exponentially suppressed by increasing the bond turn-off time t_0 .

To assess the dependence of the dynamical error $|\gamma_{1,\text{rest}}|^2$ on the distance between fluxes, we calculate the error rates for a system with 8×30 unit cells. We place the mobile zero mode γ_1 at unit cell (1,1) and the stationary mode γ_2 at unit cell $(d, 1)$. We then move γ_1 by one unit cell to $(2,1)$, and calculate the error rates. We keep $t_0 = 12J^{-1}$ and $T = 2t_0 + 5$ for this calculation. The result is shown in Fig. 16(b). The figure shows that the nonadiabatic error barely changes with d , while the dynamical error decays exponentially.

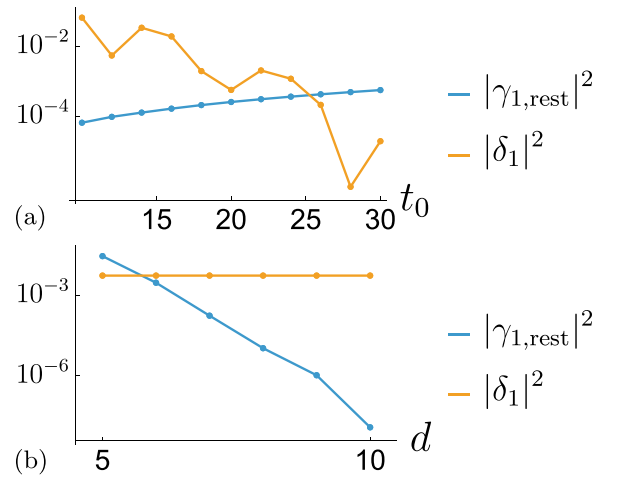


FIG. 16. (a) Dynamical and nonadiabatic errors as a function of the switching-on time t_0 . The nonadiabatic error $|\delta_1|^2$ decays exponentially, while the dynamical error rises slowly. (b) Dynamical and nonadiabatic errors as a function of the distance between fluxes γ_1 and γ_2 . The dynamical error decays exponentially with distance.

B. Braiding Majorana zero modes

To demonstrate the feasibility of a simple quantum gate with our protocol, we consider a system with 15×24 unit cells and place four fluxes in the locations shown in Fig. 17. We set $\kappa = 0.2J$, and put the system in the isotropic point of the ferromagnetic phase ($J_x = J_y = J_z = -J$). We try two braiding times $t_0 = 20J^{-1}$ and $30J^{-1}$. We then perform braiding with our protocol along the green and yellow paths in Fig. 17, and compare the modes $\gamma_{1,2,3,4}(t)$ with the modes

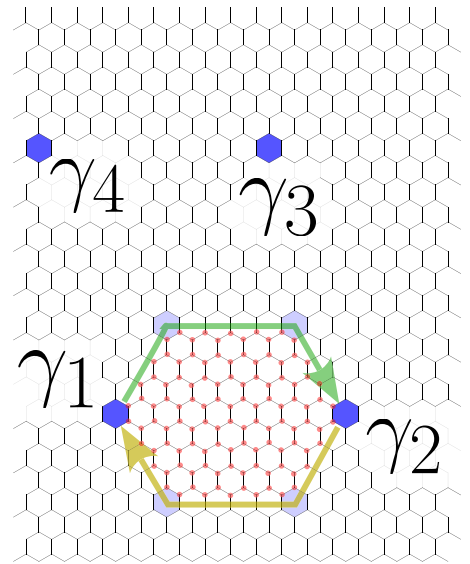


FIG. 17. A periodic system with 15×24 unit cells with four Majorana zero modes $\gamma_{1,2,3,4}$ placed at time $t = 0$ in the four locations shown here. We exchange γ_1 with γ_2 by moving γ_1 along the green path and γ_2 along the yellow path, and then compare the modes with the ones we started with. The sites shaded in red are the ones where a gauge transformation must be applied to the final modes in order to compare with the initial modes.

$\beta_{1,2,3,4}(t)$. The $\beta_i(t)$ modes are obtained by diagonalizing the Hamiltonian under the same gauge configuration as what the braiding process requires at time t .

The protocol ends at $T_f = 13T$, when the two Majorana zero modes $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are exchanged. In the ideal case, this operation would lead to [66]

$$\hat{\gamma}_1(T_f) = -\hat{\gamma}_2, \quad (40)$$

$$\hat{\gamma}_2(T_f) = \hat{\gamma}_1 \quad (41)$$

with $\hat{\gamma}_3$ and $\hat{\gamma}_4$ unchanged. In other words, a quantum gate

$$\hat{\tau} = e^{-\pi \hat{\gamma}_1 \hat{\gamma}_2 / 4} \quad (42)$$

is applied. Here, we write γ_1 for $\gamma_1(0)$ for simplicity.

For the intermediate states, we write

$$\gamma_i(t) = p_i(t)\beta_i(t) + \gamma_{i,\text{rest}}(t) + \delta_i(t) \quad (43)$$

where $\beta_i(t)$ denotes the Majorana mode that γ_i would evolve into if the braiding were perfect, while $\delta_i(t)$ is its components outside the zero-energy subspace. Finally, $\gamma_{i,\text{rest}}(t)$ is the “undesired” components of $\gamma_i(t)$ inside the zero-energy subspace, i.e., the portion of the zero-energy Hilbert space spanned by β_j with $j \neq i$.

In Figs. 18(a) and 18(b) we plot the squared norms of the nonadiabatic error operators, $|\delta_i(t)|^2$, for Majorana modes $i = 1, \dots, 4$ for both $t_0 = 20J^{-1}$ and $30J^{-1}$. For both values of t_0 , we find that the nonadiabatic errors are near zero for the stationary fluxes γ_3 and γ_4 . This is expected, since the perturbation does not directly act on them. We also find that the nonadiabatic error for the mobile fluxes γ_1 and γ_2 for $t_0 = 30J^{-1}$ is lower by two orders of magnitude than that for $t_0 = 20J^{-1}$. This is consistent with our previous test, which showed an exponentially decaying nonadiabatic error rate for larger t_0 .

In Figs. 18(c) and 18(d) we plot the dynamical errors $|\gamma_{i,\text{rest}}|^2$ as a function of time. We find that the dynamical errors are larger for $t_0 = 30J^{-1}$. This behavior is also expected, as the interaction between fluxes introduces a phase on the order $\sim e^{i\epsilon T_f}$ between the states in the zero-energy subspace, and $T = 2t_0 + 5J^{-1}$ is larger for $t_0 = 30J^{-1}$ than for $20J^{-1}$. For both times, we find that dynamical errors increase faster around $t \approx 7T$ than any other time. We believe that this behavior is due to γ_1 and γ_3 being insufficiently separated at time $t \approx 7T$, leading to a larger ϵ .

In Figs. 18(e) and 18(f) we plot the probabilities $p_i(t)$ for both values of t_0 , and we find that $t_0 = 20J^{-1}$ behaves slightly better than $t_0 = 30J^{-1}$. This is due to the larger dynamical errors for longer braiding times in our finite-size system, as the nonadiabatic errors are negligibly small for $t_0 = 30$. For $t_0 = 20J^{-1}$, the total errors are larger for the mobile fluxes γ_1 and γ_2 , and smaller for the stationary fluxes γ_3 and γ_4 . This is because the nonadiabatic error is near zero for γ_3 and γ_4 . On the other hand, the errors for $t_0 = 30$ are completely dominated by dynamical errors. Such errors can be exponentially suppressed by separating the Majorana zero modes by a large enough distance. Therefore, it is reasonable to speculate that, for $t_0 = 30J^{-1}$ and for a larger system with longer distance between the fluxes, the dynamical errors could be reduced. In turn, this would lead to improved results for p_i . We note that, while making the system larger is theoretically possible,

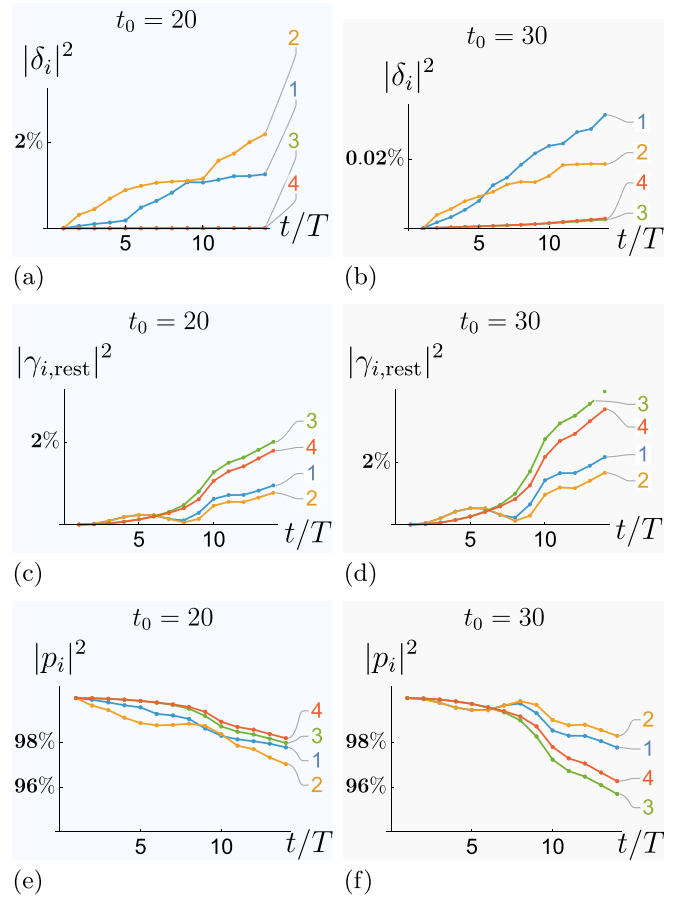


FIG. 18. (a), (b) Error $|\delta_i(t)|^2$ for Majorana fermion $i = 1, \dots, 4$ due to nonadiabaticity, plotted as a function of the braiding time t for (a) $t_0 = 20J^{-1}$ and (b) $t_0 = 30J^{-1}$. Time is measured in units of T , the time taken for the flux to move one unit cell. A larger T is shown to greatly reduce nonadiabaticity errors. (c), (d) Errors $|\gamma_{i,\text{rest}}|^2$ due to interactions between the fluxes. The case for $t_0 = 30J^{-1}$ has larger errors because the interaction phase is proportional to $\sim \exp(i\epsilon T_f)$. (e), (f) Success probability $p_i(t)$ as a function of time.

it may not be experimentally feasible. Hence, achieving a high probability of success of a quantum gate requires fine-tuning operation times.

VI. CONCLUSIONS

We have numerically studied the Kitaev model in the presence of uniform and local driving magnetic fields with a combination of mean-field and exact-diagonalization techniques.

In the first part of this paper, we have studied the effect of a time-dependent and local driving magnetic field with a mean-field approach. We have demonstrated that such magnetic field can be used to create and move vortices in both the FM and AFM Kitaev model. We found the optimal parameters for such processes, and discussed the quality of the fluxes produced. We also have shown that there is a one-to-one correspondence between the FM and AFM Kitaev models. We assessed the feasibility of our protocol in solid-state devices by modeling the effect of a local scanning tip. We found that

the success probability is too low, even for the A phase. Even though the success probability for single bond flips can be large, the fact that the tip scans over a large region almost guarantees that lone fluxes are created. Thus, the protocol is doomed to fail somewhere along the way. Additionally, we point out that in typical Kitaev candidate materials such as RuCl_3 , the lifetime of fluxes may be too short for any protocol to succeed.

In the second part of this paper, we presented an improved protocol to manipulate Majorana zero modes trapped by fluxes of the Kitaev model. This protocol is obtained by applying both time-dependent magnetic fields and temporarily cutting off bonds around a given site, while keeping it entangled to its neighbors. We discussed error rates and found that nonadiabatic transition amplitudes can be exponentially suppressed by slowly cutting off bonds. We then simulated a simple quantum gate by performing braiding on a system with four Majoranas and analyzed the errors produced in the process. We found that nonadiabatic transport and interactions between fluxes are the major sources of error. Nonadiabatic and interactions errors can be exponentially suppressed by increasing the algorithm execution time and the size of the system, respectively. However, realistic implementations must consider the finite size of the system, while algorithms must run in a finite amount of time. Therefore, it is necessary to carefully consider the tradeoff between different sources of error.

It is of course hard to envision how such protocol could be implemented in a solid-state environment. The operation of turning the bonds on and off is very difficult, if not impossible, in such devices. On the other hand, in emulators based on superconducting qubits or cold atoms, it may be possible to apply the equivalent of a magnetic field to a single bond and turn off bond strengths [67,68]. Provided that such operations are possible, and that one can construct a realization of the Kitaev Hamiltonian [69], our second protocol could make the braiding of non-Abelian anyons possible.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: PERTURBATION IN THE A_z PHASE

In this section, we show that the chiral three-spin term κ generates an effective toric-code Hamiltonian in the A_z phase. We focus on the ferromagnetic case; the antiferromagnetic case is analogous. In the A_z regime, where $|J_z| > |J_x| + |J_y|$,

following Ref. [20], the two spins connected by each z bond form a ferromagnetic dimer. We define the low-energy pseudospin basis as

$$|s_i^z = +1\rangle = |\uparrow_{iA}\uparrow_{iB}\rangle, \quad (\text{A1})$$

$$|s_i^z = -1\rangle = |\downarrow_{iA}\downarrow_{iB}\rangle, \quad (\text{A2})$$

where s_i^z denotes the effective spin operator on the i th z bond, and iA and iB label the two physical spins forming that bond. The unperturbed Hamiltonian is

$$H_0 = -J_z \sum_{\langle ij \rangle_z} \sigma_i^z \sigma_j^z, \quad (\text{A3})$$

the ground-state manifold of which consists of the ferromagnetic dimer states defined above. We denote the corresponding unperturbed Green's function as $G'_0(E) = (E - H_0)^{-1}$.

1. Magnetic field perturbation

The Zeeman term is

$$V_h = h \sum_{i,\alpha} \sigma_i^\alpha. \quad (\text{A4})$$

First-order perturbation by h_x or h_y excites the system out of the ground-state manifold, while h_z produces only a constant shift. At second order, one obtains four contributions per dimer:

$$PV_h G'_0 V_h P = -\frac{h^2}{2|J_z|} \sum_i (\sigma_{iA}^x \sigma_{iB}^x + \sigma_{iA}^y \sigma_{iB}^y + \sigma_{iA}^x \sigma_{iB}^y + \sigma_{iA}^y \sigma_{iB}^x), \quad (\text{A5})$$

where $\sigma_{iA}^x \sigma_{iB}^x \sim s_i^x$ and $\sigma_{iA}^y \sigma_{iB}^y \sim -s_i^x$ cancel each other, leaving $\sigma_{iA}^x \sigma_{iB}^y$ and $\sigma_{iA}^y \sigma_{iB}^x$, both mapping to s_i^y . Hence,

$$H_{\text{eff},h}^{(2)} = -\frac{h^2}{|J_z|} \sum_i s_i^y. \quad (\text{A6})$$

2. Kitaev perturbation

The J_x and J_y terms of the Kitaev Hamiltonian are

$$V_J = \sum_{\langle ij \rangle_\alpha} J_\alpha \sigma_i^\alpha \sigma_j^\alpha, \quad (\text{A7})$$

where the sum runs over x - and y -type bonds. The lowest nontrivial contribution arises at fourth order in perturbation theory:

$$H_{\text{eff},J}^{(4)} = PV_J G'_0 V_J G'_0 V_J G'_0 V_J P = -\frac{J_x^2 J_y^2}{16|J_z|^3} \sum_p Q_p, \quad (\text{A8})$$

where $Q_p = s_a^y s_b^z s_c^y s_d^z$, with a, b, c , and d labeling the z -bond dimers on the plaquette shown in Fig. 19. Here P denotes projection onto the low-energy dimer manifold.

3. Three-spin perturbation

The chiral three-spin term is

$$V_\kappa = \kappa \sum_{\langle i,j,k \rangle_\Delta} \sigma_i^{\alpha_{ij}} \sigma_j^\beta \sigma_k^{\alpha_{jk}}, \quad (\text{A9})$$

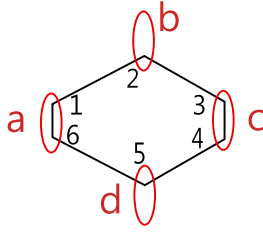


FIG. 19. A distorted plaquette of the Kitaev model. The physical spins are labeled by 1–6, while the z -bond dimers are labeled a, b, c, d in red.

as defined in Eq. (3). The leading nontrivial contribution arises at second order:

$$P(\kappa \sigma_1^x \sigma_2^z \sigma_3^y) G'_0(\kappa \sigma_4^x \sigma_5^z \sigma_6^y) P. \quad (\text{A10})$$

This process maps to

$$H_{\text{eff},\kappa}^{(2)} = -\frac{2\kappa^2}{4|J_z|} \sum_{a,b,c,d \in p} s_a^y s_b^z s_c^y s_d^z = -\frac{\kappa^2}{2|J_z|} \sum_p Q_p, \quad (\text{A11})$$

where the first sum runs over the four dimers on plaquette p , and the second sum is over all plaquettes. The factor of 2 in the numerator accounts for the two possible operator orderings

(123)(456) and (456)(123), while the denominator $4|J_z|$ arises because each three-spin term excites two dimers (a and c), costing $4|J_z|$ in energy.

Acting once with V_κ flips a single σ^z spin on a z dimer, creating two broken dimers and exciting the system out of the low-energy manifold at energy $\approx 2|J_z|$. Acting twice on adjacent triangles around a plaquette returns the system to the ground-state manifold, producing the plaquette operator Q_p .

4. Resulting low-energy Hamiltonian

Collecting all terms, the effective Hamiltonian in the A_z phase is

$$H_{\text{eff}} = J_{\text{eff}} \sum_p Q_p + h_{\text{eff}} \sum_i s_i^y, \quad (\text{A12})$$

with

$$J_{\text{eff}} = -\frac{\kappa^2}{2|J_z|} - \frac{J_x^2 J_y^2}{16|J_z|^3}, \quad h_{\text{eff}} = -\frac{h^2}{|J_z|}. \quad (\text{A13})$$

The operator s_i^y anticommutes with the two plaquette operators Q_p located directly above and below dimer i . Consequently, as h_{eff} increases, the system undergoes a transition from the toric-code phase to a polarized phase once the two terms become comparable.

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- $$\begin{aligned} \Upsilon(m_{q_1}, m_{q_2}, m_{q_3}, m_{q_4}) = & \langle m_{q_1} m_{q_2} \rangle \langle m_{q_3} m_{q_4} \rangle \\ & + \langle m_{q_1} m_{q_4} \rangle \langle m_{q_2} m_{q_3} \rangle \\ & - \langle m_{q_1} m_{q_3} \rangle \langle m_{q_2} m_{q_4} \rangle \end{aligned}$$
- where $p_1 < p_2$ are now picked from $\{1, 2, 3, 4, 5, 6\}$, and $q_1 < q_2 < q_3 < q_4$ are the rest of the numbers.
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