

Charge pumps, boundary modes, and the necessity of unnecessary criticality

Abhishodh Prakash^{1,2,*} and S. A. Parameswaran^{1,†}

¹*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford OX1 3PU, United Kingdom*

²*Harish-Chandra Research Institute, a CI of Homi Bhabha National Institute, Prayagraj (Allahabad) 211019, India*



(Received 20 March 2025; revised 25 September 2025; accepted 4 December 2025; published 29 December 2025)

We link the presence of “unnecessary” quantum critical surfaces within a single gapped phase of matter to the nontrivial topology of *families* of gapped Hamiltonians that encircle the critical surface. We study a specific set of one-dimensional spin models where each such family forms a one-parameter loop in a two-dimensional phase diagram. Foliating the noncritical region by such loops identifies “radial” and “angular” coordinates in the phase diagram that respectively parametrize different families and different members of a single family. We show that each one-parameter family is a generalized Thouless charge pump, all with the same topological index, and hence the gapped phase undergoes one or more nontrivial boundary phase transitions as we vary the angular coordinate in a loop through members of one family. Tuning the radial coordinate generates loci of boundary critical points that terminate at the end points of the bulk unnecessary critical line within the gapped phase. We discuss broader implications of our results and possible extensions to higher dimensions.

DOI: [10.1103/kztj-jcyc](https://doi.org/10.1103/kztj-jcyc)

Understanding changes in the dynamics and thermodynamics as the parameters of a system are varied is a central problem in many-body physics. For quantum systems near absolute zero, our interest here, these properties are encoded in the ground state and low-lying excitations, so a natural focus is on surfaces in parameter space on which the excitation gap closes. A much-studied example involves quantum criticality [1,2], where the gap closes as a *single* parameter is tuned—a situation intermediate between stable phases whose gaplessness is robust to all changes in parameters, and multicriticality, accessed by tuning multiple parameters. The most interesting scenario is when the gap vanishes continuously as the critical surface is approached. This signals a vanishing scale for fluctuations, validating a long-wavelength field theory description. This in turn captures the scaling behavior of correlation functions, that can be studied within the renormalization group (RG) framework [3].

Conventional wisdom holds that quantum criticality is tied to a transition between different phases of matter. Such distinctions could be based on symmetry breaking in ground-state observables *à la* Landau, or could stem from topological features [4] of the ground-state wave function that can be diagnosed using nonlocal properties such as entanglement [5,6], as in several contemporary examples. However, recent work has identified instances where this linkage fails: Quantum criticality emerges *within a single phase of matter* [7–14]. The resulting phase diagram resembles that of a liquid-gas transition—it involves a critical surface that terminates within

a phase and can hence be avoided by a suitably chosen path in parameter space—but with the difference that it is continuous rather than first order.

Such “unnecessary critical points” cannot be diagnosed by the usual tactic of studying gapped Hamiltonians straddling the critical surface: One can always find an adiabatic path between any two such points along which the bulk energy gap remains open, and so no sharp distinction can be drawn between Hamiltonians on the two “sides.” There is as yet no general understanding of the criteria for the emergence and stability of unnecessary criticality.

Here, we argue that “unnecessary” criticality within a region of parameter space can be necessitated by the topological properties of the *families* of gapped Hamiltonians that enclose the region [15]. In the example we study here, this family can be viewed as a closed loop in a two-dimensional (2D) parameter space: While each of its members taken in isolation belongs to the same trivial phase, the one-parameter family of Hamiltonians over the closed loop represents a nontrivial pump [16–21] that transports symmetry charge across the system. It has been previously argued [22] that this guarantees that the bulk gap closes somewhere within the region enclosed by the pump. However, this picture does not explain how a critical line rather than a multicritical point might arise, or discuss other aspects of the phase diagram—questions we address in this Letter, using a strategy we now outline. We first show that a class of spin chains previously introduced by us [10] exemplifies the link between charge pumps and unnecessary criticality. We explain how distinct variants of the pumping cycle (extracted through exact calculations on a one-parameter family of zero-correlation-length Hamiltonians) can lead to distinct unnecessary critical scenarios in its interior (which is amenable to a field theoretic analysis). In doing so, we prove a conjectured link [10] between stable boundary modes and bulk unnecessary criticality in our specific setting, and use this along with the bulk-boundary correspondence for Hamiltonian

*Contact author: abhishodhprakash@hri.res.in, he/him/his

†Contact author: sid.parameswaran@physics.ox.ac.uk

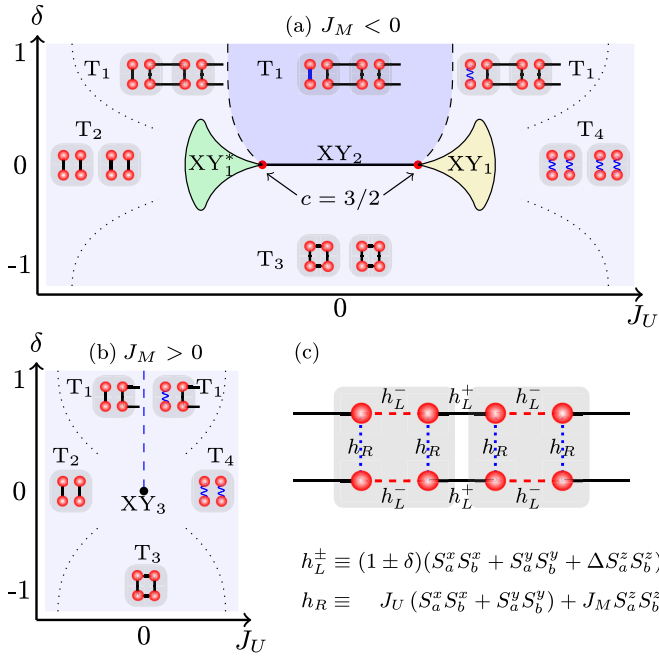


FIG. 1. Unnecessary critical (a) and multicritical (b) phase diagrams in a spin ladder (c), for $J_M \approx \mp 5.2$, $\Delta \approx -0.05$. (a), (b) show the evolution of ground states near one end of an open ladder using graphical notation described in the text. Over the pumping cycle the ladder remains in a single bulk gapped phase, but its boundary transforms under distinct irreducible representations of layer exchange ($\mathbb{Z}_2^L$) and spin-rotation [$O(2)$] symmetry for $\delta > 0$ and $J_U < -|J_M|$ or $J_U > |J_M|$. These are separated by a single boundary transition [dashed line in (b)] or a nontrivial boundary phase involving a 2D boundary irrep of $O(2)$ [blue shaded region in (a)]. Dotted lines in (a) and (b) represent crossovers between “RG basins” of symmetric field theory vacua $T_{1,\dots,4}$ of Table I.

families [22–25] to sharpen the conjecture in the general case. We close with a discussion of the broader implications of our work, and possible extensions of these ideas to higher dimensions.

Model, symmetries, and phase diagrams. We consider the Hamiltonian [10] $H = H_{L,1} + H_{L,2} + H_R$, where

$$H_{L,\alpha} = \sum_{j=1}^L J_j(\delta) (S_{\alpha j}^x S_{\alpha j+1}^x + S_{\alpha j}^y S_{\alpha j+1}^y + \Delta S_{\alpha j}^z S_{\alpha j+1}^z),$$

$$H_R = \sum_j [J_M S_{1j}^z S_{2j}^z + J_U (S_{1j}^x S_{2j}^x + S_{1j}^y S_{2j}^y)], \quad (1)$$

with $\vec{S} = \frac{1}{2} \vec{\sigma}$ in terms of the Pauli matrices σ^μ . Equation (1) describes a spin ladder [Fig. 1(c)] each of whose legs is an XXZ spin chain with a staggered coupling $J_j(\delta) = 1 + (-1)^j \delta$, and rung couplings J_U, J_M , and is invariant under the following symmetries: (i) $U(1)$ spin rotations around the S^z axis, under which $S_{\alpha j}^x \pm i S_{\alpha j}^y \equiv S_{\alpha j}^\pm \mapsto e^{\pm i\chi} S_{\alpha j}^\pm$; (ii) $\mathbb{Z}_2^R$ spin reflections, $S_{\alpha j}^\pm \mapsto S_{\alpha j}^\mp$, $S_{\alpha j}^z \mapsto -S_{\alpha j}^z$; and (iii) $\mathbb{Z}_2^L$ “leg exchange,” $S_{\alpha j}^\mu \mapsto S_{\bar{\alpha} j}^\mu$, with $\bar{1} = 2, \bar{2} = 1$. Together, these symmetries form the non-Abelian group $[U(1) \rtimes \mathbb{Z}_2^R] \times \mathbb{Z}_2^L \cong O(2) \times \mathbb{Z}_2^L$.

Varying the bond dimerization δ and rung coupling J_U yields various possible phase diagrams shown in Fig. 1, depending on the (fixed) values of Δ, J_M . Figure 1(a) shows the unnecessary critical behavior that was a focus of Ref. [10], where a second-order critical surface accessible by tuning a single parameter is stable to arbitrary symmetry-preserving perturbations, and terminates in extended regions of gapless phase. Criticality can be avoided by a suitable path in parameter space that, crucially, remains within a single gapped bulk phase. However, the phase diagram has a striking feature [10]: the presence of robust edge modes [shaded region in Fig. 1(a)] that do not enjoy any symmetry protection. These edge modes disappear via the bulk unnecessary critical transition or a boundary transition. Adopting the unconventional view of treating bulk and boundary phase structures on the same footing would give a compelling rationalization for the presence of the critical line, leading to the conjecture [10] that the presence of edge modes and the merger of bulk and boundary transitions is a generic feature of unnecessary critical phase diagrams, consistent with several other examples [7–9, 11, 26].

We demonstrate the validity of this conjecture for the models of Eq. (1), both for the unnecessary critical line in Fig. 1(a) and its deformation to a single “unnecessary multicritical” point [Fig. 1(b)] (Ref. [27] discusses other cases involving spontaneous symmetry breaking). Along the way, we make contact with progress in the general program of classifying topological phases parametrized over any manifold and in arbitrary dimension [15, 19–25, 28–38]. This classification depends on the dimensions of both space d and parameter space d_p ; our case corresponds to $d = 1, d_p = 2$, but we will set the stage more generally. Let us take the region of bulk gap closing to lie within some compact domain around the origin of parameter space. Each $d_p - 1$ surface that encloses the critical region without intersecting it constitutes a $(d_p - 1)$ -dimensional family of d -dimensional gapped Hamiltonians; for obvious reasons, we refer to the $d_p - 1$ parameters that label Hamiltonians within a family as “angular,” and the single parameter labeling different families as “radial.” The topological classification of gapped families generalizes Thouless’s notion of a $d = 1, d_p = 2$ pump of $U(1)$ charge [16] to higher dimensions, more parameters, and other symmetries [39]. One result of the classification is that a parameter-space region enclosed by a nontrivial pump must host some loci of “diabolical points” [22] where the bulk gap closes, necessitating the superficially unnecessary criticality. Another is a generalized bulk-boundary correspondence, wherein a gap on the $(d - 1)$ -dimensional boundary closes for one or more members of each family, forming loci of boundary transitions within the same bulk phase as we move “radially inward” between families. These terminate at end points of the unnecessary critical surface, consistent with the conjecture of Ref. [10].

Despite its intuitive appeal, it is unclear if this picture holds for Eq. (1) and (here and more generally) whether it offers any insight on different possible criticality patterns in the interior of the phase diagram. To that end, we explore these questions in an exactly solvable limit of Eq. (1), and then show how aspects of the pump and the boundary physics are imprinted on the long-wavelength Luttinger liquid (LL) field theory in the near-critical regime.

Charge pump and boundary criticality. We begin by focusing on the periphery of the phase diagrams in Fig. 1, where $|J_U| \rightarrow \infty$ or $|\delta| = 1$; the angular parameter is chosen to evolve through this loop. For these values, H decomposes into a sum of decoupled four-spin problems: Thus, the phase diagram is encircled by an exactly solvable “fixed-point family” of Hamiltonians. It is convenient to introduce a graphical notation for four distinct rung states: the two Bell pairs (in the S^z basis)

$$|\uparrow\uparrow\rangle = \frac{1}{\sqrt{2}} \left(\left| \begin{smallmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{smallmatrix} \right\rangle \right), \quad |\downarrow\downarrow\rangle = \frac{1}{\sqrt{2}} \left(\left| \begin{smallmatrix} \uparrow & \uparrow \\ \downarrow & \downarrow \end{smallmatrix} \right\rangle - \left| \begin{smallmatrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{smallmatrix} \right\rangle \right) \text{ and}$$

the “doublet” of states $|\uparrow\downarrow\rangle = \left| \begin{smallmatrix} \uparrow & \downarrow \\ \uparrow & \downarrow \end{smallmatrix} \right\rangle, \left| \begin{smallmatrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{smallmatrix} \right\rangle$. These transform

in different irreducible representations (irreps) of $O(2) \cong [U(1) \times \mathbb{Z}_2^R]$, distinguished, e.g., by the action of $\mathbb{Z}_2^R$: The Bell pairs form one-dimensional irreps with $\mathbb{Z}_2^R$ eigenvalue ± 1 , respectively, while the states in the doublet are exchanged by $\mathbb{Z}_2^R$, forming a two-dimensional irrep. Meanwhile, the antisymmetric Bell pair changes sign under $\mathbb{Z}_2^L$ leg exchange, which leaves the other rung states invariant. Finally, we also introduce the “plaquette state”

$$|\square\rangle = \alpha |\uparrow\uparrow\rangle + \beta |\downarrow\downarrow\rangle + \gamma \left(\left| \begin{smallmatrix} \uparrow & \downarrow \\ \uparrow & \downarrow \end{smallmatrix} \right\rangle + \left| \begin{smallmatrix} \uparrow & \downarrow \\ \downarrow & \uparrow \end{smallmatrix} \right\rangle \right), \text{ where}$$

α, β, γ depend on Δ, J_M, J_U in a complicated way. This is invariant under all symmetries of H and reduces to $|\uparrow\downarrow\rangle$ and

$|\downarrow\uparrow\rangle$ for $J_U \rightarrow \pm\infty$, respectively.

For $4\Delta > |J_M| - \sqrt{J_M^2 + 16}$ and periodic boundary conditions, the ground state $|\text{GS}(J_U, \delta)\rangle$ is unique [27], and smoothly evolves between four representative points,

$$|\text{GS}(\infty, \delta)\rangle = \cdots \quad \text{[Diagram: chain of vertical bonds]} \quad \cdots, \quad (2a)$$

$$|\text{GS}(0, 1)\rangle = \cdots \quad \text{[Diagram: chain of horizontal bonds]} \quad \cdots, \quad (2b)$$

$$|\text{GS}(-\infty, \delta)\rangle = \cdots \quad \text{[Diagram: chain of vertical bonds with alternating colors]} \quad \cdots, \quad (2c)$$

$$|\text{GS}(0, -1)\rangle = \cdots \quad \text{[Diagram: chain of horizontal bonds with alternating colors]} \quad \cdots, \quad (2d)$$

while the bulk gap remains open: As advertised, we can avoid criticality while remaining in a single bulk phase.

However, as we now show, in an *open* ladder, the boundary undergoes a phase transition (properly, a level crossing in $d = 0$) that can be understood in terms of symmetry. A more formal argument that Eqs. (2a)–(2d) describe a nontrivial family may be possible, e.g., by using “suspension isomorphism” to construct generalized charge pumps [25], but the argument here is more intuitive and relates directly to one of our objectives.

To proceed, we consider a semi-infinite chain and fix a fiducial four-site unit cell, indicated by the gray boxes in Fig. 1. Examining the four limiting cases (2a)–(2d), we see that introducing a boundary does not affect the fixed-point family ground states except along the $\delta = 1$ line. Here, the two spins at the boundary of the ladder decouple from the bulk

and are governed by the effective Hamiltonian

$$H_\partial(J_M, J_U) = J_U(S_1^x S_2^x + S_1^y S_2^y) + J_M S_1^z S_2^z. \quad (3)$$

The ground state of Eq. (3) is the $\mathbb{Z}_2^{L,R}$ -even state $|\uparrow\uparrow\rangle$ for

$J_U < -|J_M|$, and the $\mathbb{Z}_2^{L,R}$ -odd state $|\uparrow\downarrow\rangle$ for $J_U > |J_M|$. Since the bulk is a product of symmetric plaquettes, as we cycle through the fixed-point family, the ground state of the semi-infinite ladder must undergo a level crossing somewhere along the $\delta = 1, J_U \in (-|J_M|, |J_M|)$ line between distinct sectors of either $\mathbb{Z}_2^L$ or $O(2)$ (the latter follows from the change in $\mathbb{Z}_2^R$ eigenvalue). This is a purely boundary phenomenon, absent in the periodic case, and captures the fact that the cycle transfers symmetry charge across the system: The family is a nontrivial pump of both $\mathbb{Z}_2^L$ and $\mathbb{Z}_2^R$ [40]. Although the nontrivial pump only requires preserving either $\mathbb{Z}_2^L$ or $\mathbb{Z}_2^R$, the larger symmetry group is needed to stabilize the unnecessary critical line [10,27].

There are two possible ways in which the boundary charge can change along $\delta = 1$. For $J_M > 0$, we get a single direct level crossing (a boundary phase transition) at J_U between the two boundary charge sectors described above, at which the ground state is twofold degenerate. This can be rationalized in terms of either symmetry. However, the non-Abelian nature of $O(2)$ admits a second possibility for $J_M < 0$. In this case, there is a region of finite extent $J_U < |J_M|$ (a “boundary phase”) where the ground state of Eq. (3) transforms in the $O(2)$ doublet irrep $|\uparrow\downarrow\rangle$, i.e., there is a twofold degenerate

boundary mode in the open system. The semi-infinite ladder thus undergoes two distinct level crossings at $J_U = \pm|J_M|$, at which its ground state is threefold degenerate. If we now tune parameters “radially inwards” towards the critical region, the symmetry argument above continues to hold even though the model is no longer exactly solvable [27].

Field theories near criticality. We now explore how the charge pump emerges near the unnecessary critical region, which admits a bosonized description [27,41]. The resulting theory is a two-component Luttinger liquid, $H = H[\theta_1, \phi_1] + H[\theta_2, \phi_2] + H_g$, with

$$H[\theta, \phi] = \frac{v}{2\pi} \int dx \left(\frac{1}{4K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right), \quad (4)$$

and

$$H_g = \int dx \left(\frac{J_M}{4\pi^2} \partial_x \phi_1 \partial_x \phi_2 + \sum_{\mathcal{O}} g_{\mathcal{O}} \mathcal{O}(x) \right). \quad (5)$$

H_g includes an exactly marginal operator with coupling $\propto J_M$ and various operators of the form $\mathcal{O} \propto \cos(\Theta)$ where Θ is some combination of fields $\phi_\alpha, \theta_\alpha$. To eliminate spurious symmetries relative to Eq. (1) requires several such $\mathcal{O}$ ’s [10,27], but the bulk phases in Fig. 1 can be understood in terms of the RG flows of just four: $\mathcal{U} \equiv \sum_\alpha \cos(\phi_\alpha)$, $\mathcal{V}_\pm \equiv \cos(\phi_1 \pm \phi_2)$, and $\mathcal{W}_- \equiv \cos(\theta_1 - \theta_2)$, with bare couplings $g_{\mathcal{U}} \propto \delta$, $g_{\mathcal{V}_\pm} \propto \mp J_M$, and $g_{\mathcal{W}_-} \propto J_U$. The scaling dimensions of these operators (denoted $[\cdots]$) are related, $[\mathcal{W}_-] = 1/[\mathcal{V}_-]$, and $[\mathcal{U}] = ([\mathcal{V}_+] + [\mathcal{V}_-])/4$, and depend on microscopic couplings [27]. As J_M is tuned, modifying $[\mathcal{V}_\pm]$, different sets of operators

TABLE I. Distinct Luttinger liquid vacua corresponding to the same bulk gapped phase. Depending on the operator scaling dimensions and parameter choices, the flow to strong coupling pins different combinations of fields.

Vacuum	Scaling dimensions	Parameters	Pinned fields
T_1	$[\mathcal{U}] < [\mathcal{W}_-]$	$\delta > 0$	$\langle \phi_1 \rangle = \langle \phi_2 \rangle = \pi$
T_2	$[\mathcal{U}] > [\mathcal{W}_-], [\mathcal{V}_+] < 2$	$J_U < 0$	$\langle \phi_1 + \phi_2 \rangle = 0, \langle \theta_1 - \theta_2 \rangle = 0$
T_3	$[\mathcal{U}] < [\mathcal{W}_-]$	$\delta < 0$	$\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$
T_4	$[\mathcal{U}] > [\mathcal{W}_-], [\mathcal{V}_+] < 2$	$J_U > 0$	$\langle \phi_1 + \phi_2 \rangle = 0, \langle \theta_1 - \theta_2 \rangle = \pi$

become relevant (i.e., $[\mathcal{O}] < 2$) and flow to strong coupling, pinning different field combinations with or without opening a gap. Each gapless case is described by a single-component Luttinger liquid $H[\theta, \phi]$, but the long-wavelength degrees of freedom, and hence the action of symmetries on (θ, ϕ) , differs depending on the combination of operators “left behind” as others flow to strong coupling [10,27]. This ensures that they cannot be connected without changing universality class or encountering an intermediate phase [42,43]. Of these, XY_2 describes the unnecessary critical line in Fig. 1(a) and XY_3 the unnecessary multicritical point in Fig. 1(b), respectively, reached by tuning one or two parameters, whereas the phases XY_1 and XY_1^* require no fine tuning.

The trivial gapped phase emerges when both sectors of the Luttinger liquid are gapped while preserving symmetry. This can happen in four distinct ways ($T_{1,\dots,4}$; Table I); while these “vacua” represent distinct basins of attraction under RG, they can all be adiabatically connected without any bulk phase transitions [27,44].

However, the boundary physics changes nontrivially as the gapped family evolves through the T_1 vacuum where the pinned fields take values $\langle \phi_\alpha \rangle = \pi$. To see this, we first observe that we can model a semi-infinite chain in the T_1 vacuum for $x > 0$ with open boundary conditions at $x = 0$ as an infinite system where $g_U \propto \delta$ is smoothly modulated in space with $g_U(x \rightarrow \pm\infty) \leq 0$, such that the system lies in the T_3 vacuum as $x \rightarrow -\infty$ and in the T_1 vacuum as $x \rightarrow \infty$ [45,46]. This pins $\langle \phi_\alpha \rangle$ to their asymptotic $T_{1,3}$ vacuum values of 0, π as $x \rightarrow \mp\infty$, but they will deviate from these values near the interface where g_U vanishes. The interpolation between the vacua is controlled by the competition of the remaining cosines $\mathcal{V}_+, \mathcal{V}_-$ and $\mathcal{W}_-$ near the interface.

For $J_M < 0$, there are two possibilities. When $\mathcal{V}_- = \cos(\phi_1 - \phi_2)$ dominates, since $g_{\mathcal{V}_-} \propto J_M < 0$, configurations where $\phi_1 = \phi_2$ [Fig. 2(a)] are favored, so that $\phi_1 + \phi_2$ evolves smoothly from 0 to $\pm 2\pi$. Although the two vacua are invariant under all symmetries, the interpolations are exchanged by $\mathbb{Z}_2^R$ and involve a $U(1)$ charge $Q = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \partial_x (\phi_1 + \phi_2) = \pm 1$ localized near the interface. We conclude that the interface transforms as a two-dimensional irrep of $O(2)$ [47]. Since irreps are stable, we expect a finite window (a “phase”) of boundary degeneracy, consistent with the physics for $\delta \lesssim 1$ and small J_U in Fig. 1(a). In contrast, when $\mathcal{W}_- = \cos(\theta_1 - \theta_2)$ dominates, $\theta_1 - \theta_2$ is pinned to 0 or π depending on the sign of $g_{\mathcal{W}_-}$. This forces its conjugate $\phi_1 - \phi_2$ to fluctuate at the interface, but allows $\phi_1 + \phi_2$ to have a definite value; although this could apparently be pinned to either 0 or π depending on the sign of $g_{\mathcal{V}_+}$, the latter is frustrated by the asymptotic vacua [27]. Since for $\phi_1 + \phi_2 = 0$, $\phi_{1,2}$ wind in opposite senses, we conclude that the system tunnels back and

forth between $U(1)$ -neutral configurations where $(\phi_1, \phi_2) = (\pm\pi/2, \mp\pi/2)$ near the interface [Fig. 2(b)]. For $\theta_1 - \theta_2 = 0$ ($g_{\mathcal{W}_-} \propto J_U < 0$), this leads to a unique $\mathbb{Z}_2^L$ -even ground state, whereas for $\theta_1 - \theta_2 = \pi$ ($g_{\mathcal{W}_-} \propto J_U > 0$), it leads to a unique $\mathbb{Z}_2^L$ -odd ground state, consistent with Fig. 1(a) for $\delta \lesssim 1$ and large $|J_U|$.

The two possibilities above meet in a threefold degenerate crossing, since the two states in the $O(2)$ doublet and the $\mathbb{Z}_2^L$ -even/odd singlet are distinct on symmetry grounds and hence do not experience level repulsion. The crossings involving the even and odd singlets respectively correspond to the line of boundary transitions in Fig. 1(a) for $J_U < 0$ and $J_U > 0$, which terminate at the bulk multicritical points (between the XY_2 line and the XY_1/XY_1^* phases resulting from the competition between the same operators $\mathcal{W}_-, \mathcal{V}_-$ in the bulk). These are described by the central charge $c = 3/2$ conformal field theories (CFTs) that the two-component Luttinger liquid theory flows to when $[\mathcal{W}_-] = [\mathcal{V}_-]$, $[\mathcal{V}_+] > 2$ and $g_U = 0$ [27,44,48,49].

For $J_M > 0$ and parameters consistent with Fig. 1(b), it is possible to show that $\mathcal{W}_-$ dominates ($[\mathcal{W}_-] < [\mathcal{V}_-]$ near the multicritical point) always, so generically we only have the first possibility above: The boundary is $\mathbb{Z}_2^L$ even for $g_{\mathcal{W}_-} \propto J_U < 0$, and $\mathbb{Z}_2^L$ odd for $g_{\mathcal{W}_-} \propto J_U > 0$. Evidently, tuning J_U to zero leads to a twofold degenerate boundary level crossing where the boundary $\mathbb{Z}_2^L$ charge changes; this will terminate at the XY_3 multicritical point as in Fig. 1(b).

In the crossover region for large $J_U < 0$ ($J_U > 0$), $\theta_1 - \theta_2$ is pinned at the edge of the T_1 vacuum to the same value as in the *bulk* of the adjacent T_2 (T_4) vacuum. Thus, by changing δ from +1 to -1 within $T_{2,4}$, $\mathbb{Z}_2^L$ charge can be transferred continuously from the boundary into the bulk.

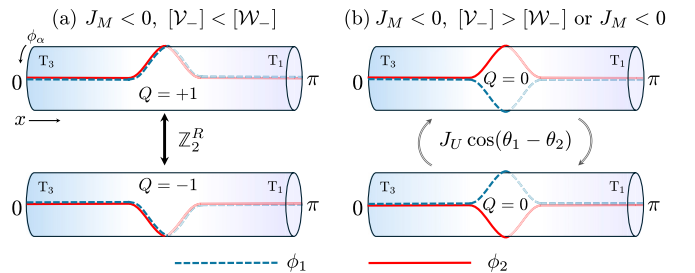


FIG. 2. Interpolations at the $T_3 - T_1$ interface. For $J_M < 0$ and $[\mathcal{V}_-] < [\mathcal{W}_-]$, $\phi_{1,2}$ wind in the same sense, so $\phi_1 + \phi_2$ evolves smoothly from 0 to $\pm 2\pi$; this binds $U(1)$ charge $Q = \pm 1$ to the interface, which is hence an $O(2)$ doublet. For $J_M < 0$ and $[\mathcal{V}_-] > [\mathcal{W}_-]$ or $J_M > 0$, $\phi_{1,2}$ wind oppositely; tunneling between the resulting $Q = 0$ configurations via $J_U \cos(\theta_1 - \theta_2)$ leads to a unique ground state whose $\mathbb{Z}_2^L$ charge is $-\text{sgn}(J_U)$.

This allows us to smoothly go between T_1 and T_3 , completing the pump cycle, and underscoring the essential role of the $T_{2,4}$ vacua. We have thus reproduced the essential features of the pump entirely within the field theory, as promised.

Discussion. In this Letter, we have explicitly demonstrated a link between unnecessary criticality in the interior of a phase diagram and a nontrivial “pump” property of the family of gapped Hamiltonians that encircles the critical region, in the setting of a concrete one-dimensional model. Our exact microscopic arguments far from criticality were mirrored by field theory calculations proximate to it. In both cases, we demonstrated the pumping of a $\mathbb{Z}_2$ symmetry charge by examining the changes at the boundaries of an open system. In the latter case, our analysis related these properties to the presence of multiple RG fixed points corresponding to the same bulk gapped phase—a principle with possible utility beyond the specific questions studied here. These arguments naturally demonstrate a bulk-boundary correspondence that argues that terminuses of unnecessary critical surfaces source lines of boundary transitions in any gapped family of Hamiltonians that encircles them.

An outstanding puzzle is to identify a topological invariant that characterizes the gapped family in the bulk, without resorting to a boundary analysis, similar to the classification of Thouless pumps of $U(1)$ charge. While this is a difficult problem in general, the exactly solvable limit may be especially amenable to matrix-product state methods [34–38] or the techniques of Ref. [25].

What broader lessons might our $d = 1$ example teach us? Most obviously, we see that the presence of a nontrivial pump (or its higher-dimensional generalizations) at the periphery of the parameter space is at least a *sufficient* condition for a gapless point in the interior—indeed, this is implicit in the notion that such a nontrivial gapped family defines a noncontractible loop in parameter space. However, the presence of unnecessary criticality—as opposed to *multicriticality*—cannot be deduced quite so readily. Here, we show the bulk-boundary correspondence plays a key role in one dimension: While in its basic incarnation it is a corollary of the pump property, our work sharpens the link between the boundary and bulk phase structures. To wit, the presence of a boundary *critical* point is matched to bulk unnecessary *multicriticality*, and a boundary *phase* to bulk *criticality* [27]. We conjecture that this is a feature also in $d > 1$, but defer its exploration to future work.

Acknowledgments. We thank Ryan Thorngren, Nick Jones, Senthil Todadri, Paul Fendley, Fabian Essler, Yuchi He, Gurkirat Singh, Chris Hooley, and Michele Fava for helpful discussions and correspondence, and Michele Fava for collaboration on a related work [10]. We acknowledge support from the European Research Council under the European Union Horizon 2020 Research and Innovation Programme, Grant Agreement No. 804213-TMCS, and from a Gutzwiller Fellowship at the Max Planck Institute for Complex Systems, where much of this work was completed (S.A.P.).

Data availability. No data were created or analyzed in this study.

-
- [1] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, UK, 2009).
 - [2] S. L. Sondhi, S. M. Girvin, J. P. Carini, and D. Shahar, Continuous quantum phase transitions, *Rev. Mod. Phys.* **69**, 315 (1997).
 - [3] K. G. Wilson, The renormalization group and critical phenomena, *Rev. Mod. Phys.* **55**, 583 (1983).
 - [4] F. D. M. Haldane, Nobel lecture: Topological quantum matter, *Rev. Mod. Phys.* **89**, 040502 (2017).
 - [5] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Entanglement spectrum of a topological phase in one dimension, *Phys. Rev. B* **81**, 064439 (2010).
 - [6] F. Pollmann and A. M. Turner, Detection of symmetry-protected topological phases in one dimension, *Phys. Rev. B* **86**, 125441 (2012).
 - [7] F. Anfuso and A. Rosch, String order and adiabatic continuity of Haldane chains and band insulators, *Phys. Rev. B* **75**, 144420 (2007).
 - [8] Z. Bi and T. Senthil, Adventure in topological phase transitions in $3+1$ -D: Non-Abelian deconfined quantum criticalities and a possible duality, *Phys. Rev. X* **9**, 021034 (2019).
 - [9] C.-M. Jian and C. Xu, Generic “unnecessary” quantum critical points with minimal degrees of freedom, *Phys. Rev. B* **101**, 035118 (2020).
 - [10] A. Prakash, M. Fava, and S. A. Parameswaran, Multiversality and unnecessary criticality in one dimension, *Phys. Rev. Lett.* **130**, 256401 (2023).
 - [11] R. Verresen, J. Bibo, and F. Pollmann, Quotient symmetry protected topological phenomena, [arXiv:2102.08967](https://arxiv.org/abs/2102.08967).
 - [12] Y. He, D. M. Kennes, C. Karrasch, and R. Rausch, Terminable transitions in a topological fermionic ladder, *Phys. Rev. Lett.* **132**, 136501 (2024).
 - [13] Y. Zhang, X.-Y. Song, and T. Senthil, Dirac spin liquid as an “unnecessary” quantum critical point on square lattice antiferromagnets, *SciPost Phys. Core* **8**, 024 (2025).
 - [14] A. Prakash and N. G. Jones, Classical origins of Landau-incompatible transitions, *Phys. Rev. Lett.* **134**, 097103 (2025).
 - [15] A. Kitaev, Differential forms on the space of statistical mechanics models (2019), talk at the conference in celebration of Dan Freed’s 60th birthday, <https://web.ma.utexas.edu/topqft/talkslides/kitaev.pdf>.
 - [16] D. J. Thouless, Quantization of particle transport, *Phys. Rev. B* **27**, 6083 (1983).
 - [17] J. C. Y. Teo and C. L. Kane, Topological defects and gapless modes in insulators and superconductors, *Phys. Rev. B* **82**, 115120 (2010).
 - [18] Y. Kuno and Y. Hatsugai, Interaction-induced topological charge pump, *Phys. Rev. Res.* **2**, 042024(R) (2020).

- [19] S. Ohyama, K. Shiozaki, and M. Sato, Generalized Thouless pumps in $(1+1)$ -dimensional interacting fermionic systems, *Phys. Rev. B* **106**, 165115 (2022).
- [20] S. Bachmann, W. D. Roeck, M. Fraas, and T. Jappens, A classification of G -charge Thouless pumps in 1D invertible states, *Commun. Math. Phys.* **405**, 157 (2024).
- [21] A. Kapustin and L. Spodyneiko, Higher-dimensional generalizations of the Thouless charge pump, [arXiv:2003.09519](https://arxiv.org/abs/2003.09519).
- [22] P.-S. Hsin, A. Kapustin, and R. Thorngren, Berry phase in quantum field theory: Diabolical points and boundary phenomena, *Phys. Rev. B* **102**, 245113 (2020).
- [23] C. Cordova, D. S. Freed, H. T. Lam, and N. Seiberg, Anomalies in the space of coupling constants and their dynamical applications I, *SciPost Phys.* **8**, 001 (2020).
- [24] C. Cordova, D. S. Freed, H. T. Lam, and N. Seiberg, Anomalies in the space of coupling constants and their dynamical applications II, *SciPost Phys.* **8**, 002 (2020).
- [25] X. Wen, M. Qi, A. Beaudry, J. Moreno, M. J. Pflaum, D. Spiegel, A. Vishwanath, and M. Hermele, Flow of higher Berry curvature and bulk-boundary correspondence in parametrized quantum systems, *Phys. Rev. B* **108**, 125147 (2023).
- [26] S. Moudgalya and F. Pollmann, Fragility of symmetry-protected topological order on a Hubbard ladder, *Phys. Rev. B* **91**, 155128 (2015).
- [27] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/kztj-jcyc> for more details, which includes Refs. [50–61].
- [28] K. Shiozaki, Adiabatic cycles of quantum spin systems, *Phys. Rev. B* **106**, 125108 (2022).
- [29] M. Qi, D. T. Stephen, X. Wen, D. Spiegel, M. J. Pflaum, A. Beaudry, and M. Hermele, Charting the space of ground states with tensor networks, *SciPost Phys.* **18**, 168 (2025).
- [30] A. Beaudry, M. Hermele, J. Moreno, M. J. Pflaum, M. Qi, and D. D. Spiegel, Homotopical foundations of parametrized quantum spin systems, *Rev. Math. Phys.* **36**, 2460003 (2024).
- [31] P.-S. Hsin and Z. Wang, On topology of the moduli space of gapped Hamiltonians for topological phases, *J. Math. Phys.* **64**, 041901 (2023).
- [32] D. Aasen, Z. Wang, and M. B. Hastings, Adiabatic paths of Hamiltonians, symmetries of topological order, and automorphism codes, *Phys. Rev. B* **106**, 085122 (2022).
- [33] A. Kapustin and L. Spodyneiko, Higher-dimensional generalizations of Berry curvature, *Phys. Rev. B* **101**, 235130 (2020).
- [34] S. Ohyama and S. Ryu, Higher structures in matrix product states, *Phys. Rev. B* **109**, 115152 (2024).
- [35] S. Ohyama and S. Ryu, Higher Berry connection for matrix product states, *Phys. Rev. B* **111**, 035121 (2025).
- [36] S. Ohyama and S. Ryu, Higher Berry phase from projected entangled pair states in $(2+1)$ dimensions, *Phys. Rev. B* **111**, 045112 (2025).
- [37] O. E. Sommer, X. Wen, and A. Vishwanath, Higher Berry curvature from the wave function. I. Schmidt decomposition and matrix product states, *Phys. Rev. Lett.* **134**, 146601 (2025).
- [38] O. E. Sommer, A. Vishwanath, and X. Wen, Higher Berry curvature from the wave function II: Locally parameterized states beyond one dimension, *Phys. Rev. B* **111**, 155110 (2025).
- [39] Formally, such pumps are deformation classes of Hamiltonians parametrized over the $d_F - 1$ “angular” variables, just as gapped topological phases are deformation classes of gapped Hamiltonians over a point.
- [40] The boundary charge returns to its original value at the end of a cycle without a second boundary level crossing, as charge is pumped through the bulk. Since this is impossible in a purely $d = 0$ system, the boundary family is “anomalous” [22–24].
- [41] T. Giamarchi, *Quantum Physics in One Dimension*, International Series of Monographs on Physics (Clarendon Press, Oxford, UK, 2004).
- [42] R. Verresen, R. Thorngren, N. G. Jones, and F. Pollmann, Gapless topological phases and symmetry-enriched quantum criticality, *Phys. Rev. X* **11**, 041059 (2021).
- [43] S. Mondal, A. Agarwala, T. Mishra, and A. Prakash, Symmetry-enriched criticality in a coupled spin ladder, *Phys. Rev. B* **108**, 245135 (2023).
- [44] P. Lecheminant, A. O. Gogolin, and A. A. Nersisyan, Criticality in self-dual sine-Gordon models, *Nucl. Phys. B* **639**, 502 (2002).
- [45] A. Keselman and E. Berg, Gapless symmetry-protected topological phase of fermions in one dimension, *Phys. Rev. B* **91**, 235309 (2015).
- [46] This follows from the fact that open boundary conditions on each XXZ leg corresponds to $\phi_\alpha(x=0, t) = 0$ in a semi-infinite chain [62,63].
- [47] Recall that 1D $O(2)$ irreps are $U(1)$ neutral, whereas 2D irreps consist of charge $\pm q$ states which are exchanged by $\mathbb{Z}_2^R$ [27,64].
- [48] H. J. Schulz, Phase diagrams and correlation exponents for quantum spin chains of arbitrary spin quantum number, *Phys. Rev. B* **34**, 6372 (1986).
- [49] H.-L. Zhang, H.-Z. Li, S. Yang, and X.-J. Yu, Quantum phase transition and critical behavior between the gapless topological phases, *Phys. Rev. A* **109**, 062226 (2024).
- [50] R. Dijkgraaf, E. Verlinde, and H. Verlinde, $C = 1$ conformal field theories on Riemann surfaces, *Commun. Math. Phys.* **115**, 649 (1988).
- [51] R. Bishop, P. Li, R. Zinke, R. Darradi, J. Richter, D. Farnell, and J. Schulenburg, The spin-half XXZ antiferromagnet on the square lattice revisited: A high-order coupled cluster treatment, *J. Magn. Magn. Mater.* **428**, 178 (2017).
- [52] X.-Y. Song, C. Wang, A. Vishwanath, and Y.-C. He, Unifying description of competing orders in two-dimensional quantum magnets, *Nat. Commun.* **10**, 4254 (2019).
- [53] G. Delfino and G. Mussardo, Non-integrable aspects of the multi-frequency sine-Gordon model, *Nucl. Phys. B* **516**, 675 (1998).
- [54] M. Fabrizio, A. Gogolin, and A. Nersisyan, Critical properties of the double-frequency sine-Gordon model with applications, *Nucl. Phys. B* **580**, 647 (2000).
- [55] M. Nakamura, Identification of topologically different valence bond states in spin ladders, *Phys. B: Condens. Matter* **329-333**, 1000 (2003).
- [56] F. Haldane, Demonstration of the “Luttinger liquid” character of Bethe-ansatz-soluble models of 1-D quantum fluids, *Phys. Lett. A* **81**, 153 (1981).
- [57] R. Thorngren, A. Vishwanath, and R. Verresen, Intrinsically gapless topological phases, *Phys. Rev. B* **104**, 075132 (2021).
- [58] L. Fidkowski and A. Kitaev, Effects of interactions on the topological classification of free fermion systems, *Phys. Rev. B* **81**, 134509 (2010).

- [59] X. Chen, Z.-C. Gu, and X.-G. Wen, Complete classification of one-dimensional gapped quantum phases in interacting spin systems, [Phys. Rev. B **84**, 235128 \(2011\)](#).
- [60] H. C. Po, L. Fidkowski, T. Morimoto, A. C. Potter, and A. Vishwanath, Chiral Floquet phases of many-body localized bosons, [Phys. Rev. X **6**, 041070 \(2016\)](#).
- [61] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, [Rev. Mod. Phys. **83**, 1057 \(2011\)](#).
- [62] S. Eggert and I. Affleck, Magnetic impurities in half-integer-spin Heisenberg antiferromagnetic chains, [Phys. Rev. B **46**, 10866 \(1992\)](#).
- [63] I. Affleck, Edge magnetic field in the XXZ spin-chain, [J. Phys. A: Math. Gen. **31**, 2761 \(1998\)](#).
- [64] W.-K. Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985), Vol. 1.