Inequivalence of Landau-Lifshitz and Landau-Lifshitz-Gilbert dynamics for a single quantum spin

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We examine the relation between the quantum Landau-Lifshitz equation (q-LL) [Wieser, Phys. Rev. Lett. **110**, 147201 (2013)] and quantum Landau-Lifshitz-Gilbert equation (q-LLG) [Liu *et al.*, Phys. Rev. Lett. **133**, 266704 (2024)], two nonlinear purity preserving master equations that extend classical atomistic spin dynamics into the quantum regime. While the classical LL and LLG counterparts for any number of spins are known to be equivalent, i.e., give identical spin trajectories up to a rescaling of the time parameter, the quantum formulations are equivalent only in certain cases, such as for pure states or for arbitrary single spin-$\frac{1}{2}$ states. Here, we demonstrate that this equivalence breaks down even at the level of a single spin, provided $s \geqslant 1$. Focusing on a spin-1 particle in an anisotropic crystal field, we show that the q-LL and q-LLG equations generate inequivalent time evolution. We introduce temporal rescaling misfits that quantify the inequivalence of the two types of dynamics. Although our results highlight fundamental differences in dissipation mechanisms encoded in these equations, the resulting trajectories remain qualitatively similar for this system.

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I. INTRODUCTION

Atomistic spin dynamics is a computational approach used to simulate and understand the magnetic behavior of materials at the atomic scale [1,2]. In atomistic spin dynamics, each atom is assigned a magnetic moment (or site centered "atomic spin"), and the time evolution of these spins is calculated using equations derived from classical mechanics, such as the Landau-Lifshitz (LL) [3] or Landau-Lifshitz-Gilbert (LLG) [4] equation. Notably, the LL and LLG equations are equivalent, i.e., give identical spin trajectories up to a rescaling of the time parameter, for any number of spins [5].

More recently, quantum analogs of these equations have been explored. The quantum Landau-Lifshitz (q-LL) equation, introduced in Ref. [6], is formulated as a nonlinear master equation, with the dissipative "double-bracket" term explicitly involving the spin Hamiltonian. In contrast, the dissipative term in the quantum Landau-Lifshitz-Gilbert (q-LLG) equation, proposed in Ref. [7], takes the form of a "single-bracket" expression that involves the density operator and its time derivative.

Although these equations look fundamentally different, as outlined explicitly in the next section, they can produce equivalent dynamics in certain special cases. These include single spin states that can be expressed in terms of spin operator components, as well as arbitrary spin systems in pure state evolution. As discussed in Ref. [7], the equivalence typically breaks down in the case of multispin systems in nonpure states.

In this work, we demonstrate that the dynamics described by the q-LL and q-LLG equations can exhibit fundamental differences, even at the level of a single spin, provided $s \geqslant 1$. To illustrate this inequivalence, we analyze the simplest nontrivial scenario: a single spin-1 particle. While the classical LL and LLG equations yield equivalent dynamics, the divergent behavior of their quantum counterparts is due to the difference in the underlying commutator structure. We identify two primary sources of this inequivalence in the $s = 1$ system: (i) the presence of an anisotropic crystal field, which breaks the rotational symmetry of the spin Hamiltonian, and (ii) the choice of initial quantum state, which can strongly influence the spin trajectory under each dynamical framework. These findings highlight that the quantum treatment of atomistic spin dynamics introduces qualitatively new behavior, even in minimal models, with implications for quantum spin dynamics in magnetic systems. Examples of materials relevant for this investigation involve lanthanide impurities (with an open $4f$ shell) in a nonmagnetic rare-earth host [8], as well as NV centers discussed for quantum technologies [9].

II. QUANTUM ANALOG SPIN DYNAMICS

In the q-LL and q-LLG formulations, the classical spin vectors are replaced by the density operator ϱ_t, which encodes

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the quantum state of the system at time t. The evolution is described by master equations that combine unitary dynamics (via the spin Hamiltonian) with dissipative terms that mimic Gilbert damping. These dissipative terms are constructed using nonlinear operator bracket structures designed to preserve physical properties such as positivity and purity conservation.

Explicitly, the q-LL and q-LLG equations take the form (we put $\hbar = 1$ from now on) [6]

$$\dot{\varrho}_t = i[\varrho_t, H] - \kappa[\varrho_t, [\varrho_t, H]] \tag{1}$$

and [7]

$$\dot{\varrho}_t = i[\varrho_t, H] + i\kappa[\varrho_t, \dot{\varrho}_t], \tag{2}$$

respectively. Here, κ is the Gilbert damping describing the rate of dissipation in the system.

The key issue we address in this work is the relation between these two equations. Specifically, we focus on a single spin in nonpure states. In this case, the two equations are equivalent up to a rescaling of time, provided the density operator is of *spin type*, i.e., can be expressed in terms of the spin operators $\mathbf{S} = (S_x, S_y, S_z)$, throughout the time evolution. In the case of $s = 1$, such a state takes the form

$$\varrho_t = \tfrac{1}{3}(\hat{1} + \mathbf{m}_t \cdot \mathbf{S}), \tag{3}$$

with $\mathbf{m}_t \propto \langle \mathbf{S} \rangle_t$ corresponding to the magnetization associated with the spin. This vector must satisfy $|\mathbf{m}_t| = |\mathbf{m}_0| \equiv m_0 \leqslant 1$ in order for the physical requirement $\varrho_t \geqslant 0$ to hold [10]. Note that the purity $\mathrm{Tr}\,\varrho_t^2$ increases monotonically with m_0, but its maximal value $\mathrm{Tr}\,\varrho_t^2 = \frac{5}{9}$ obtained for $m_0 = 1$ corresponds to a nonpure state [11]. Equation (2) can be turned into the q-LL form given by Eq. (1) under the rescaling [7]

$$t \mapsto \left[1 + \tfrac{1}{9}\kappa^2 m_0^2\right]t \tag{4}$$

for dynamics described by Eq. (3).

While a time rescaling is always true for the dynamics of a single spin-$\frac{1}{2}$, the most general ϱ_t for $s \geqslant 1$ would require the use of a $(2s+1)^2 - 1$ dimensional *coherence vector* $\vec{x}_t$ [12]. This is vastly different from the restricted class of states given by Eq. (3), being fully parametrized by the magnetization $\mathbf{m}_t \in \mathbb{R}^3$, and this difference would no longer guarantee equivalence between the q-LL and q-LLG dynamics. For a single spin-1, the most general state, the *qutrit-type* state, takes the form [13]

$$\varrho_t = \tfrac{1}{3}(\hat{1} + \sqrt{3}\vec{x}_t \cdot \vec{\lambda}), \tag{5}$$

with $\vec{\lambda} = (\lambda_1, \ldots, \lambda_8)$ a vector of the eight traceless Gell-Mann operators (see Appendix A for details), and $\vec{x}_t \in \mathbb{R}^8$. The coherence vector $\vec{x}_t$ satisfies $0 \leqslant |\vec{x}_t|^2 \leqslant 1$, with $|\vec{x}_t|^2 = 1$ for pure states and $|\vec{x}_t|^2 = 0$ for the maximally mixed state. Note that the purity conservation implies $|\vec{x}_t| = |\vec{x}_0|$.

III. DYNAMICS OF A SINGLE SPIN-1

A. Spin model

Although the Zeeman term $-\gamma_g \mathbf{B} \cdot \mathbf{S}$, γ_g the gyromagnetic ratio, fully describes the Hamiltonian for isolated spin-$\frac{1}{2}$ particles, material systems with $s \geqslant 1$ can experience nontrivial anisotropy effects that significantly influence the dynamics, even for a single spin prepared in a spin-type state [cf. Eq. (3)]. Here, we examine q-LL and q-LLG dynamics of a spin-1 system in an anisotropic environment that is exposed to a time-independent external magnetic field $\mathbf{B}$.

To model this system, we use the Hamiltonian

$$H = -\gamma_g \mathbf{B} \cdot \mathbf{S} + K_\perp S_x^2 + K_\parallel S_z^2, \tag{6}$$

with $\mathbf{S} = (S_x, S_y, S_z)$ spin operators and $K_\perp, K_\parallel$ components of the anisotropy tensor along the principal axes. In the following, we take $K_\parallel < 0 < K_\perp$, making the z axis the easy axis along which the anisotropy tends to align the spin. Note that the direction of the external magnetic field $\mathbf{B}$ does not need to coincide with any of the principal axes.

B. The role of anisotropy

To elucidate the role of anisotropy for the inequivalence of the q-LL and q-LLG dynamics, we express H in Eq. (6) in terms of the eight Gell-Mann operators. This is achieved by making use of the relations

$$\begin{aligned} S_x &= \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_6), \\ S_y &= \frac{1}{\sqrt{2}}(\lambda_2 + \lambda_7), \\ S_z &= \frac{1}{2}(\lambda_3 + \sqrt{3}\lambda_8), \end{aligned} \tag{7}$$

from which we find

$$\begin{aligned} S_x^2 &= \frac{2}{3}\hat{1} - \frac{1}{4}\lambda_3 + \frac{1}{2}\lambda_4 + \frac{1}{4\sqrt{3}}\lambda_8, \\ S_z^2 &= \frac{4}{3}\hat{1} + \frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8. \end{aligned} \tag{8}$$

By inserting Eqs. (7) and (8) into Eq. (6), we obtain

$$\begin{aligned} H = &-\frac{\gamma_g B_x}{\sqrt{2}}(\lambda_1 + \lambda_6) - \frac{\gamma_g B_y}{\sqrt{2}}(\lambda_2 + \lambda_7) - \frac{\gamma_g B_z}{2}(\lambda_3 + \sqrt{3}\lambda_8) \\ &+ K_\perp\left(-\frac{1}{4}\lambda_3 + \frac{1}{2}\lambda_4 + \frac{1}{4\sqrt{3}}\lambda_8\right) \\ &+ K_\parallel\left(\frac{1}{2}\lambda_3 - \frac{1}{2\sqrt{3}}\lambda_8\right), \end{aligned} \tag{9}$$

where we have omitted unimportant terms that are $\propto \hat{1}$.

The anisotropy contributions to the Hamiltonian imply that an initial state of the spin type [cf. Eq. (3)] would evolve into a qutrit-type state [cf. Eq. (5)], thereby opening up for inequivalent q-LL and q-LLG dynamics. To illustrate this point, we consider the special case where $K_\perp = 2K_\parallel$, for which the Hamiltonian takes the form

$$H = -\gamma_g \mathbf{B} \cdot \mathbf{S} + K_\parallel \lambda_4. \tag{10}$$

We calculate the infinitesimal change of the density operator in the q-LL case as

$$\begin{aligned} \varrho_{\delta t}^{q\text{-LL}} - \varrho_0 = (&-i\gamma_g[\varrho_0, \mathbf{B} \cdot \mathbf{S}] + iK_\parallel[\varrho_0, \lambda_4] \\ &+ \kappa\gamma_g[\varrho_0, [\varrho_0, \mathbf{B} \cdot \mathbf{S}]] - \kappa K_\parallel[\varrho_0, [\varrho_0, \lambda_4]])\delta t \\ &+ O(\delta t^2), \end{aligned} \tag{11}$$

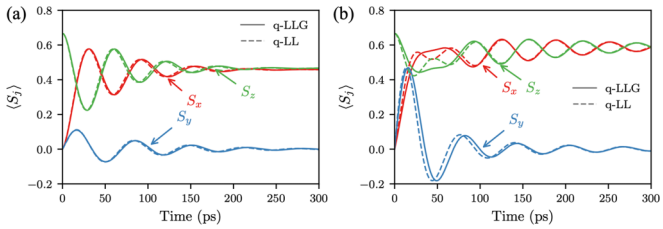


FIG. 1. The q-LLG (solid lines) and q-LL (dashed lines) dynamics of the spin components. (a) shows the time evolution for the spin-type initial state in Eq. (16) with $m_0 = 1$ and in the presence of anisotropy [case (i)]. (b) shows the time evolution for the qutrit-type initial state in Eq. (17) with $p = \frac{5}{6}$ but without anisotropy [case (ii)]. These parameter values are chosen so that the two initial states correspond to $\langle \mathbf{S} \rangle = (0, 0, \frac{2}{3})$. An oblique magnetic field $\mathbf{B} = \frac{B_0}{\sqrt{2}}(1, 0, 1)$ with $|\mathbf{B}| = B_0$ defining the timescale $(\gamma_g B_0)^{-1} \sim 10$ ps and Gilbert damping $\kappa = 0.5$ are chosen. The anisotropy parameters in case (i) are $K_\perp/(\gamma_g B_0) = 0.3$ and $K_\parallel/(\gamma_g B_0) = -0.1$.

and in the q-LLG case as

$$\varrho_{\delta t}^{q\text{-LLG}} - \varrho_0 = \varrho_{\delta t}^{q\text{-LL}} - \varrho_0 + (i\kappa^2\gamma_g[\varrho_0, [\varrho_0, [\varrho_0, \mathbf{B}\cdot\mathbf{S}]]] - i\kappa^2 K_\parallel[\varrho_0, [\varrho_0, [\varrho_0, \lambda_4]]])\delta t + O(\delta t^2), \tag{12}$$

where we have used Eq. (11) to single out the extra contributions in the q-LLG expression. Suppose now that the system is prepared in the spin-type state $\varrho_0 = \frac{1}{3}(\hat{1} + m_0 S_z)$. By using the algebra of the spin and Gell-Mann operators, one obtains the Zeeman contributions,

$$[\varrho_0, [\varrho_0, [\varrho_0, \mathbf{B}\cdot\mathbf{S}]]] \propto [\varrho_0, \mathbf{B}\cdot\mathbf{S}], \tag{13}$$

and anisotropy contributions,

$$[\varrho_0, [\varrho_0, [\varrho_0, \lambda_4]]] \propto \lambda_5, \tag{14}$$

respectively. Thus, the Zeeman triple commutator in the q-LLG expression can be absorbed into the unitary term $i[H, \varrho_0]$ and would therefore just contribute to the rescaling of time. In contrast, the anisotropy term generates a new direction $\propto\lambda_5$ in SU(3) that makes the change $\varrho_0 \mapsto \varrho_{\delta t}^{q\text{-LLG}}$ fundamentally different from that in the q-LL case. This shows that the two formulations are inequivalent as a consequence of the spin anisotropy in this setting.

C. Numerical analysis

To demonstrate the inequivalence of the q-LL and q-LLG dynamics for the spin-1 system, we numerically compare them for specific choices of parameters and initial states. To fully characterize the spin state in the simulations, we use the spin components $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, as well as the covariance matrix $\mathbb{C}$ defined as [14]

$$\mathbb{C}_{jk} = \tfrac{1}{2}\langle\{S_j - \langle S_j\rangle, S_k - \langle S_k\rangle\}\rangle = \tfrac{1}{2}\langle\{S_j, S_k\}\rangle - \langle S_j\rangle\langle S_k\rangle, \quad j,k = x,y,z. \tag{15}$$

While the former defines the magnetization associated with the spin, the latter tracks the quantum fluctuations along the evolution path.

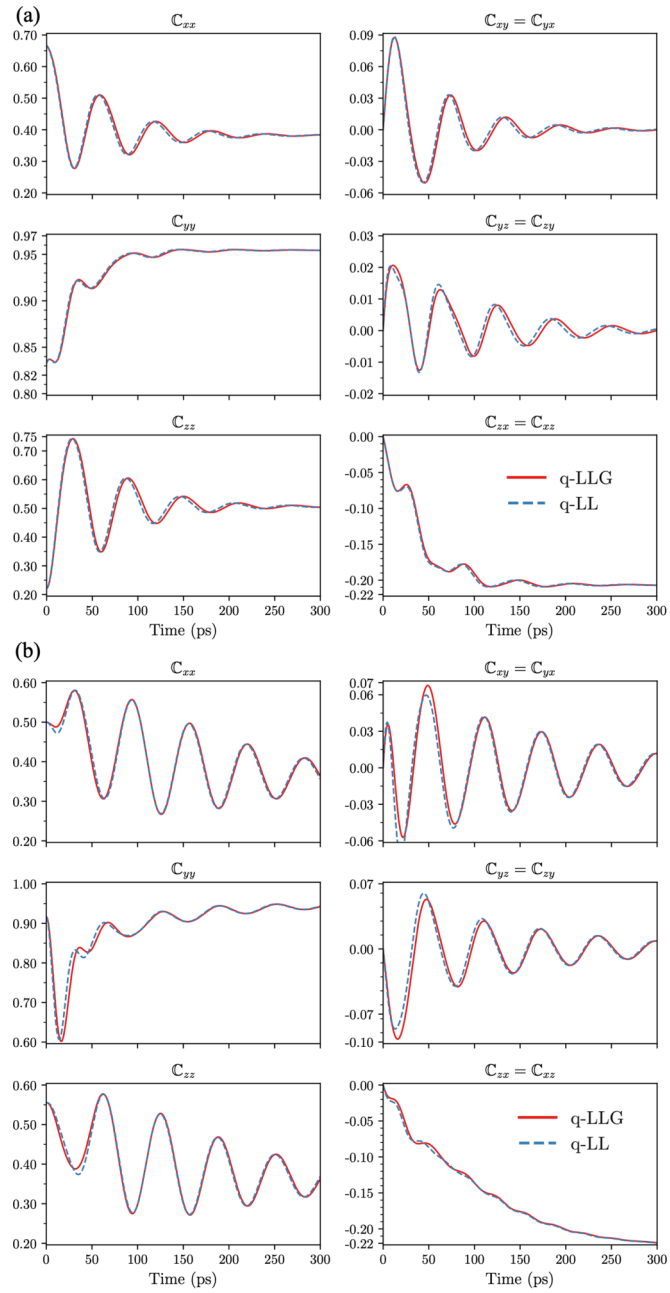


FIG. 2. The q-LLG (solid red lines) and q-LL (dashed blue lines) time evolution of the covariance matrix elements $\mathbb{C}_{jk} = \frac{1}{2}\langle\{S_j, S_k\}\rangle - \langle S_j\rangle\langle S_k\rangle$, $j,k = x,y,z$. (a) shows the spin-type initial state in Eq. (16) with $m_0 = 1$ and in the presence of anisotropy [case (i)]. (b) shows the qutrit-type initial state in Eq. (17) with $p = \frac{5}{6}$ but without anisotropy [case (ii)]. All numerical values of the dynamical parameters used are identical to those in Fig. 1.

The covariance matrix can be pictured as an ellipsoid surrounding the tip of the evolving spin vector $\langle \mathbf{S} \rangle(t) = \mathrm{Tr}(\mathbf{S}\varrho_t)$. The direction and length of the ellipsoid's half axes are the normalized eigenvectors and squared eigenvalues, respectively, of the covariance matrix. Thus, the volume of the ellipsoid is given by $V_e(t) = \frac{4\pi}{3}\sqrt{\det \mathbb{C}(t)}$. An important feature of $V_e(t)$ is that it vanishes when the spin state has no uncertainty (zero variance) in one direction. This makes $V_e(t)$ a useful scalar quantifier of total uncertainty

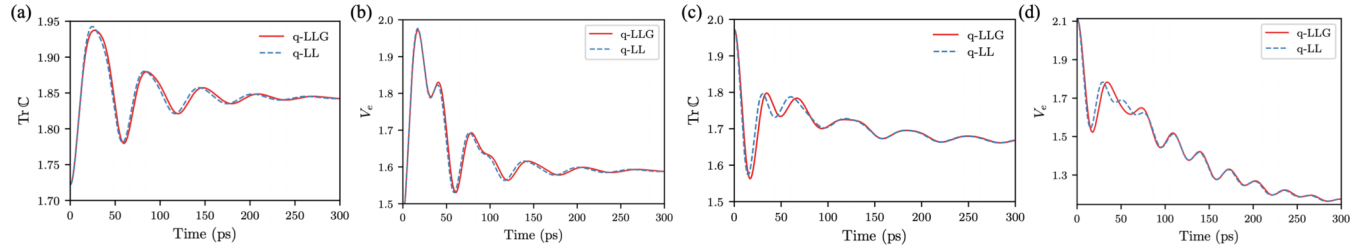


FIG. 3. The total variance $(\Delta S)^2_{\rm tot}(t) = (\Delta S_x)^2(t) + (\Delta S_y)^2(t) + (\Delta S_z)^2(t) = {\rm Tr}\,\mathbb{C}(t)$ and total quantum fluctuations $V_e(t) = \frac{4\pi}{3}\sqrt{\det\,\mathbb{C}(t)}$ under q-LLG and q-LL dynamics of two spin-1 systems. (a) and (b) show these quantities for the spin-type initial state with anisotropy [case (i)]. (c) and (d) show these quantities for the qutrit type without anisotropy [case (ii)]. All numerical values of the dynamical parameters used are identical to those in Fig. 1.

and correlations among the spin components: A larger determinant means greater total quantum fluctuations in all directions. A second useful quantity is ${\rm Tr}\,\mathbb{C}(t)$, which is the total variance $(\Delta S)^2_{\rm tot}(t) = (\Delta S_x)^2(t) + (\Delta S_y)^2(t) + (\Delta S_z)^2(t)$ with $\Delta S_j^2(t) = {\rm Tr}(S_j^2\varrho_t) - [{\rm Tr}(S_j\varrho_t)]^2$, $j = x, y, z$. This quantity satisfies $1 \leqslant (\Delta S)^2_{\rm tot} \leqslant 2$, with the lower bound corresponding to a "quasiclassical" spin coherent state [15] and the upper bound any state for which $\langle \mathbf{S}\rangle$ vanishes (for further details, see Appendix B).

We will now numerically examine two distinct cases:

(i) The spin is initially prepared in the spin-type state,

$$\varrho_0 = \tfrac{1}{3}\big(\hat{1} + m_0 S_z\big), \tag{16}$$

and evolves under the influence of anisotropy and an applied field.

(ii) The spin is prepared in a qutrit-type state,

$$\begin{aligned}\varrho_0 &= p|1;1\rangle\langle 1;1| + (1-p)|1;-1\rangle\langle 1;-1| \\ &= \tfrac{1}{3}[\hat{1} + (3p-2)S_z + \lambda_3],\end{aligned} \tag{17}$$

and evolves in the absence of anisotropy but an applied field.

For case (i), we choose $m_0 = 1$, corresponding to a nonpure spin-type state for which the purity takes the maximal value ${\rm Tr}\,\varrho_0^2 = \frac{5}{9}$ [11]. We further use the anisotropies $K_\perp/(\gamma_g B_0) = 0.3$ and $K_\parallel/(\gamma_g B_0) = -0.1$. For case (ii), we choose $p = \frac{5}{6}$, and put $K_\perp = K_\parallel = 0$. In both simulations, we choose the precession timescale $(\gamma_g B_0)^{-1} \sim 10$ ps, and Gilbert damping $\kappa = 0.5$. The external magnetic field is taken to be $\mathbf{B} = \frac{B_0}{\sqrt{2}}(1, 0, 1)$, i.e., an oblique field at 45° to the z axis. The two cases (i) and (ii) are complementary in the sense that the former shows the effect of anisotropy and the latter shows the effect of the initial state on the inequivalence between the q-LL and q-LLG dynamics. While the parameters m_0 and p are chosen so that the two initial states correspond to $\langle \mathbf{S}\rangle = (0, 0, \frac{2}{3})$, the corresponding covariance matrices differ at $t = 0$.

Figure 1 shows the time evolution of the spin vector components $\langle S_j\rangle$, $j = x, y, z$, and Fig. 2 shows the corresponding covariance matrix elements, in the case of q-LLG and q-LL dynamics. In addition, the total uncertainty and total variance

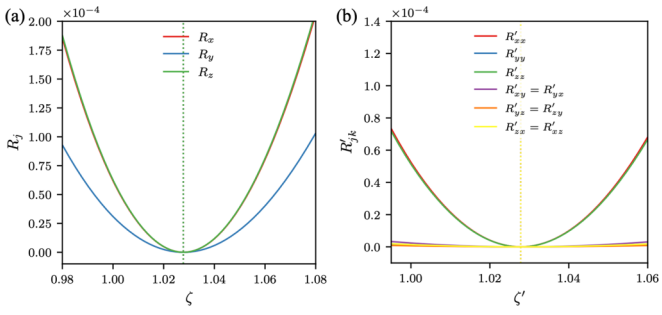


FIG. 4. The temporal rescaling misfits of (a) spin components and (b) covariance matrix elements. Here, we consider a spin-type initial state $m_0 = 1$ in Eq. (16), vanishing anisotropy, and an oblique magnetic field, $\mathbf{B} = \frac{B_0}{\sqrt{2}}(1, 0, 1)$. The plots confirm the known equivalence [7] between the q-LL and q-LLG dynamics with rescaling factor $1 + \frac{1}{9}0.5^2 \approx 1.028$ in this system.

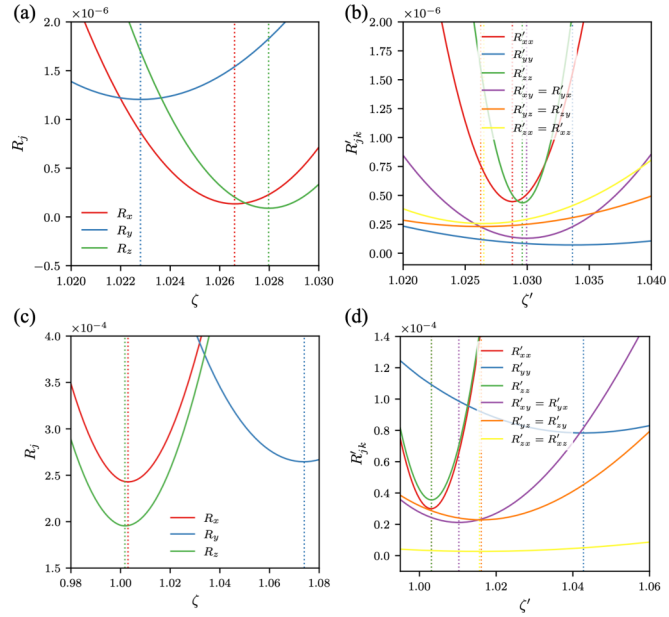


FIG. 5. The temporal rescaling misfits of spin components and covariance matrix elements. The dotted lines mark the minimal values of each time rescaling factors. (a) and (b) are temporal rescaling misfits of spin components and covariance matrix elements of the spin-type initial state with anisotropy [case (i)]. (c) and (d) are the temporal rescaling misfits of spin components and covariance matrix elements of the qutrit type without anisotropy [case (ii)]. The lack of common minima for the various vector components and matrix elements demonstrates the inequivalence between the q-LL and q-LLG dynamics. All numerical values of the dynamical parameters used are identical to those in Fig. 1.

of the spin-1 states obtained by solving the two equations are shown in Fig. 3. Note that two different starting states are considered in these figures. The plots in Figs. 2 and 3 show that while these two types of dynamics match closely, despite the relatively large Gilbert damping, one can discern their inequivalence in the details.

To quantitatively confirm that the dynamics of q-LLG and q-LL are indeed inequivalent under time rescaling, we introduce and numerically compute the *temporal rescaling misfits* (from q-LLG to q-LL) of the spin vector components, defined as

$$R_j(\zeta) = \frac{1}{N_{\text{grid}}} \sum_t \left[S_j^{(q\text{-LLG})}(\zeta t) - S_j^{(q\text{-LL})}(t) \right]^2, \quad (18)$$

as well as of the covariance matrix elements,

$$R'_{ij}(\zeta') = \frac{1}{N_{\text{grid}}} \sum_t \left[\mathbb{C}_{ij}^{(q\text{-LLG})}(\zeta' t) - \mathbb{C}_{ij}^{(q\text{-LL})}(t) \right]^2, \quad (19)$$

both summing over the time grid ($N_{\text{grid}} = 5 \times 10^4$ equidistant grid points). Here, ζ and ζ' are temporal scale factors. For the two types of evolution to be equivalent, it must hold that all minimal temporal rescaling factors ζ, ζ' are the same for all vector components $\langle S_j \rangle$ and covariance matrix elements $\mathbb{C}_{jk}$. Figure 4 shows the rescaling misfit plots for vanishing anisotropy and spin-type initial states, confirming the known equivalence [7] between q-LL and q-LLG dynamics with a rescaling factor $1 + \frac{1}{9}0.25^2 \approx 1.028$ [cf. Eq. (4) with $m_0 = 1$]. On the other hand, Fig. 5 shows that the minimal temporal rescaling factors are different for the various components of $\langle \mathbf{S} \rangle$ and elements of $\mathbb{C}$. In other words, there is no uniform time rescaling, neither for the spin vector, nor for the covariance matrix, despite the initial state being of spin type or in the absence of spin anisotropy. These two examples for a single spin-1 are direct proof of q-LLG and q-LL equations being indeed two inequivalent dynamical processes, even though the overall behavior of the spin is very similar in the two descriptions.

To further test the above noted similarity, we simulate a less symmetric setting, where the qutrit-type state in Eq. (17) with $p = \frac{5}{6}$ is combined with nonvanishing anisotropy. Figure 6 shows the spin components and the corresponding temporal rescaling misfits. Although the difference between the two types of dynamics is more clear than in Figs. 1 and 5, the overall behavior remains similar.

IV. CONCLUSIONS

We have examined the q-LL and q-LLG dynamics of a single spin-1 particle in an anisotropic crystal environment and found that the two frameworks are fundamentally inequivalent. Specifically, the time evolution generated by q-LL and q-LLG cannot be related by any simple rescaling of time. This is in contrast to the classical atomistic spin dynamics framework, where the Landau-Lifshitz and Landau-Lifshitz-Gilbert equations are equivalent up to time rescaling for any number of spins.

The inequivalence highlights a distinct nonclassical feature of quantum spin dynamics that emerges even in the minimal setting of a single spin with $s = 1$. Despite their inequivalence, both the q-LL and q-LLG dynamics appear to produce similar trajectories in the system considered here. This similarity has been seen also in the multispin-$\frac{1}{2}$ cases reported in Ref. [16]. This suggests that, although the underlying commutator structures differ, the physical predictions closely match, an observation that warrants further investigation in other spin models.

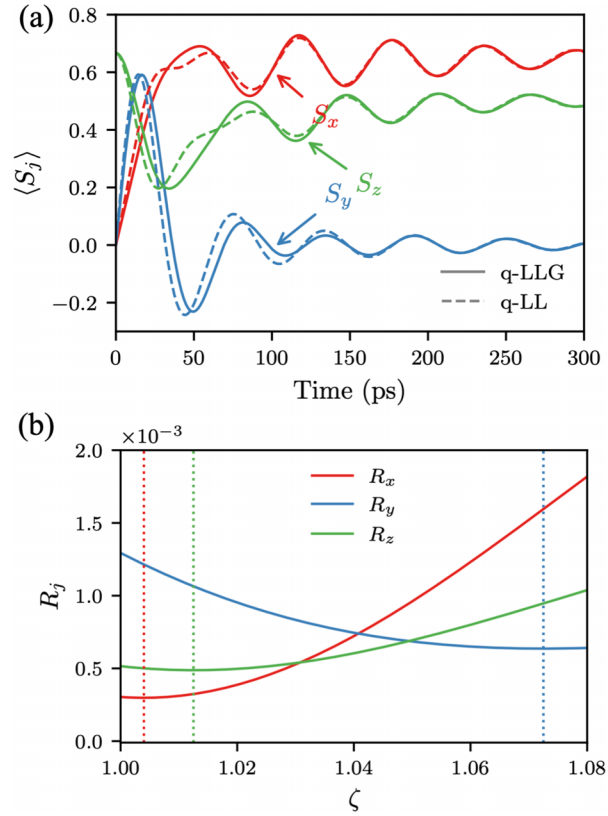


FIG. 6. (a) The q-LLG (solid lines) and q-LL (dashed lines) dynamics of the spin components for the qutrit-type initial state in Eq. (17) with $p = \frac{5}{6}$ and nonvanishing anisotropy. (b) The combined qutrit-type initial state and nonvanishing anisotropy results in temporal rescaling misfits R_j which are larger than those shown in Fig. 5(c) by almost one order of magnitude. All numerical values of the dynamical parameters used are identical to those in Fig. 1.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: SPIN-1 AND GELL-MANN OPERATORS

In terms of the standard spin-1 basis $\{|1;1\rangle, |1;0\rangle, |1;-1\rangle\}$, the spin operators read

$$\begin{aligned} S_x &= \frac{1}{\sqrt{2}}(|1;1\rangle\langle 1;0| + |1;0\rangle\langle 1;-1| + \text{H.c.}), \\ S_y &= \frac{1}{\sqrt{2}}(-i|1;1\rangle\langle 1;0| - i|1;0\rangle\langle 1;-1| + \text{H.c.}), \\ S_z &= |1;1\rangle\langle 1;1| - |1;-1\rangle\langle 1;-1|. \end{aligned} \tag{A1}$$

Similarly, the Gell-Mann operators are taken to be

$$\begin{aligned} \lambda_1 &= |1;1\rangle\langle 1;0| + |1;0\rangle\langle 1;1|, \\ \lambda_2 &= -i|1;1\rangle\langle 1;0| + i|1;0\rangle\langle 1;1|, \\ \lambda_3 &= |1;1\rangle\langle 1;1| - |1;-1\rangle\langle 1;-1|, \\ \lambda_4 &= |1;1\rangle\langle 1;-1| + |1;-1\rangle\langle 1;1|, \\ \lambda_5 &= -i|1;1\rangle\langle 1;-1| + i|1;-1\rangle\langle 1;1|, \\ \lambda_6 &= |1;0\rangle\langle 1;-1| + |1;-1\rangle\langle 1;0|, \\ \lambda_7 &= -i|1;0\rangle\langle 1;-1| + i|1;-1\rangle\langle 1;0|, \\ \lambda_8 &= \frac{1}{\sqrt{3}}(|1;1\rangle\langle 1;1| + |1;0\rangle\langle 1;0| \\ &\quad -2|1;-1\rangle\langle 1;-1|). \end{aligned} \tag{A2}$$

APPENDIX B: UPPER AND LOWER BOUND OF TOTAL VARIANCE

First note that

$$(\Delta S)^2_{\text{tot}} = \langle \mathbf{S}^2 \rangle - |\langle \mathbf{S} \rangle|^2. \tag{B1}$$

We recall that for a spin-1 system,

$$\begin{aligned} \mathbf{S}^2 &= S_x^2 + S_y^2 + S_z^2 = 2\hat{1} \\ &\Rightarrow \langle \mathbf{S}^2 \rangle = 2. \end{aligned} \tag{B2}$$

In other words, the upper and lower bounds of the total variance is completely determined by $|\langle \mathbf{S} \rangle|$. This quantity satisfies the inequality

$$0 \leqslant |\langle \mathbf{S} \rangle| \leqslant 1. \tag{B3}$$

The lower bound is obtained, e.g., by a random mixture $\frac{1}{3}\hat{1}$ or a pure "anticoherent" state [17] that can take the form $a|1;1\rangle + b|1;0\rangle - a|1;-1\rangle$, a, b real numbers such that $2a^2 + b^2 = 1$. The upper bound characterizes a spin-coherent state [15], which is a pure state with maximal projection along some arbitrary direction. We thus conclude that

$$1 \leqslant (\Delta S)^2_{\text{tot}} \leqslant 2. \tag{B4}$$

[1] V. P. Antropov, M. I. Katsnelson, B. N. Harmon, M. van Schilfgaarde, and D. Kusnezov, Spin dynamics in magnets: Equation of motion and finite temperature effects, Phys. Rev. B **54**, 1019 (1996).

[2] O. Eriksson, A. Bergman, L. Bergqvist, and J. Hellsvik, *Atomistic Spin Dynamics: Foundations and Applications* (Oxford University Press, Oxford, UK, 2017).

[3] L. D. Landau and E. M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowjetunion **8**, 153 (1935).

[4] T. L. Gilbert, A phenomenological theory of damping in ferromagnetic materials, IEEE Trans. Magn. **40**, 3443 (2004).

[5] M. Lakshmanan and K. Nakamura, Landau-lifshitz equation of ferromagnetism: Exact treatment of the Gilbert damping, Phys. Rev. Lett. **53**, 2497 (1984).

[6] R. Wieser, Comparison of quantum and classical relaxation in spin dynamics, Phys. Rev. Lett. **110**, 147201 (2013).

[7] Y. Liu, I. P. Miranda, L. Johnson, A. Bergman, A. Delin, D. Thonig, M. Pereiro, O. Eriksson, V. Azimi-Mousolou, and E. Sjöqvist, Quantum analog of Landau-Lifshitz-Gilbert dynamics, Phys. Rev. Lett. **133**, 266704 (2024).

[8] M. Pivetta, F. Patthey, I. Di Marco, A. Subramonian, O. Eriksson, S. Rusponi, and H. Brune, Measuring the intra-atomic exchange energy in rare-earth adatoms, Phys. Rev. X **10**, 031054 (2020).

[9] F. Jelezko, T. Gaebel, I. Popa, A. Gruber, and J. Wrachtrup, Observation of coherent oscillations in a single electron spin, Phys. Rev. Lett. **92**, 076401 (2004).

[10] To see this, we combine

$$\varrho_t \geqslant 0 \Rightarrow \langle \varrho_t \rangle = \frac{1}{3}(1 + \langle \mathbf{m}_t \cdot \mathbf{S} \rangle) \geqslant 0$$

and $\langle \mathbf{m}_t \cdot \mathbf{S} \rangle \geqslant -|\mathbf{m}_t|$.

[11] By using the identity

$$\text{Tr}(S_j S_k) = \left. \frac{s(s+1)(2s+1)}{3} \right|_{s=1} \delta_{jk} = 2\delta_{jk}$$

and the inequality $|\mathbf{m}_t| \leqslant 1$, we obtain

$$\text{Tr}\varrho_t^2 = \tfrac{1}{3} + \tfrac{2}{9}|\mathbf{m}_t|^2 \leqslant \tfrac{5}{9}.$$

[12] R. A. Bertlmann and P. Krammer, Bloch vectors for qudits, J. Phys. A: Math. Theor. **41**, 235303 (2008).

[13] Arvind, K. S. Mallesh, and N. Mukunda, A generalized Pancharatnam geometric phase formula for three-level quantum systems, J. Phys. A: Math. Gen. **30**, 2417 (1997).

[14] Bharath H. M., M. Boguslawski, M. Barrios, L. Xin, and M. S. Chapman, Exploring non-abelian geometric phases in spin-1 ultracold atoms, Phys. Rev. Lett. **123**, 173202 (2019).

[15] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1995), pp. 328–329.

[16] D. Mirzaei, B. Hashemi, and V. Azimi-Mousolou, LREI: A fast numerical solver for quantum Landau-Lifshitz equations, arXiv:2508.21200.

[17] J. Zimba, "Anticoherent" spin states via the Majorana representation, Electron. J. Theor. Phys. **3**, 143 (2006).