

Extension of quantum symmetric simple exclusion process model for entanglement spreading in stochastic diffusive quantum systems

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We investigate out-of-equilibrium entanglement dynamics in a generalization of the so-called *QSSEP* model, which is a free-fermion chain with stochastic in space and time hopping amplitudes. In our setup, the noisy amplitudes are spatially modulated satisfying a ν -site translation invariance but retaining their randomness in time. For each noise realization, the dynamics preserves Gaussianity, which allows to obtain noise-averaged entanglement-related quantities. The statistics of the steady-state correlators satisfy nontrivial relationships that are of topological nature. They reflect the Haar invariance under multiplication with structured momentum-dependent random $SU(\nu)$ matrices. We discuss in detail the case with $\nu = 1$ and $\nu = 2$. For $\nu = 1$, i.e., spatially homogeneous noise, we show that the entanglement dynamics is describable by a stochastic generalization of the quasiparticle picture. Precisely, entanglement is propagated by pairs of quasiparticles. The entanglement content of the pairs is the same as for the deterministic chain. However, the trajectories of the quasiparticles are random walks, giving rise to diffusive entanglement growth.

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I. INTRODUCTION

Understanding universality in out-of-equilibrium quantum many-body systems coupled to an environment remains a formidable challenge. In the regime of weak system-environment interactions and Markovianity, the average-state evolution can be described effectively by the Lindblad master equation [1–3]. While traditional studies focused primarily on average-state dynamics, recent attention has shifted toward understanding environment-induced fluctuations. One of the most urgent questions is whether a general framework to describe the statistics of these fluctuations exists [4]. Indeed, unlike classical systems, for which a macroscopic fluctuation theory (*MFT*) has been formulated [5], its generalization to the quantum many-body realm remains elusive. Such a framework must account for intrinsically quantum features, for instance, entanglement and its statistics. In *isolated* one-dimensional integrable systems, entanglement and local conserved quantities spread ballistically after quantum quenches [6–11]. Numerical evidence suggests ballistic entanglement spreading also in nonintegrable systems [12], despite diffusive energy transport. Ballistic spreading of entanglement in chaotic systems is further supported in random unitary circuits, and in a so-called “membrane picture” [13,14]. Violations of this ballistic spreading paradigm are observed in some random unitary circuits [15] (see [16] for a recent result). For *open* quantum systems, the average state (e.g., obtained via Lindblad dynamics) typically saturates to a featureless,

i.e., classical-like, steady state. Yet, in quadratic fermionic and bosonic systems [17] the intermediate-time entanglement dynamics exhibits rich behavior, reminiscent of the quasiparticle picture in the unitary evolution [18–23]. Recently, there is mounting evidence that the statistics of entanglement fluctuations may harbour deeper universal signatures, such as entanglement-driven phase transitions [24,25].

Given this context, a paradigmatic model for studying fluctuating diffusive quantum dynamics is highly desirable. Such a model could serve as a playground for developing and validating effective descriptions, much like the classical symmetric exclusion process (*SEP*) played a pivotal role in establishing macroscopic fluctuation theory (*MFT*) [4]. Very recently, the so-called quantum symmetric simple exclusion process (*QSSEP*) emerged as a promising candidate [26–32]. The model describes a system of freely hopping fermions on a ring with noisy, in both space and time, hopping amplitudes. For each realization of the noise, the model is free-fermionic. The dynamics of the average state reduces to that of the classical symmetric simple exclusion process [33] (*SSEP*), whereas fluctuations are beyond the classical regime. Interestingly, the full statistics of the steady-state fluctuations can be determined [32], thanks to nontrivial topological constraints on the noise-averaged moments of the correlators. Extensions of the *QSSEP* include an asymmetric variant [34,35], and a boundary-driven bosonic version [36]. The probability distribution of the steady-state entanglement after a quantum quench from a product state was derived in Ref. [37] by using random matrix theory (see also Ref. [37]). Yet, the full-time entanglement dynamics remains an open problem.

Here, we investigate out-of-equilibrium entanglement dynamics in a generalized *QSSEP* model, in which we partially restore space-translation invariance, while retaining the randomness in time of the hopping amplitudes. Precisely, at each

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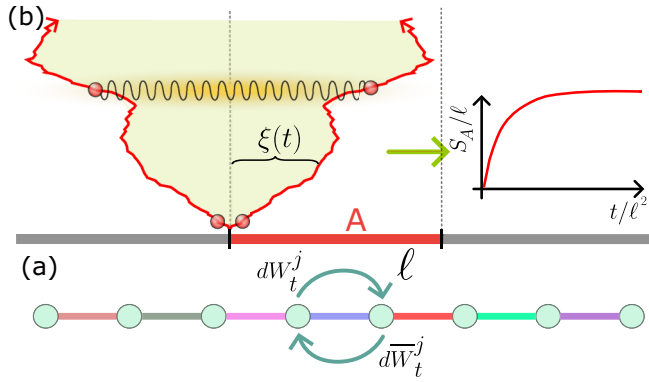


FIG. 1. Definition of the ν -QSSEP model (a). A system of noninteracting fermions evolves under the infinitesimal hopping Hamiltonian dH_t [cf. (2)] with space-time stochastic hopping amplitudes $dW_t^j, d\bar{W}_t^j$, the bar denoting complex conjugation. Here, dW_t^j are random variables *à la* Itô, which depend on time and on the position j . In the ν -QSSEP, we have $dW_t^j = dW_t^{j'}$ if $j = j' \bmod \nu$, with $\nu > 1$ an integer. (b) Quasiparticle picture for entanglement spreading after a quench for $\nu = 1$, i.e., spatially uniform noise. Pairs of entangled quasiparticles are produced after the quench. The entanglement content of the pair is the same as in the deterministic chain. The trajectories of the quasiparticles forming an entangled pair are mirror symmetric. All the pairs undergo the same dynamics. The entanglement between a subsystem A of size ℓ is proportional to $\min(2|\xi|, \ell)$, which is the number of pairs that are shared between A and its complement. The distance $\xi(t)$ traveled by the quasiparticles is stochastic and distributed with $P(\xi, t) = e^{-\xi^2/(4\gamma t)}/(2\sqrt{\pi\gamma t})$, giving rise to diffusive entanglement spreading.

time step the hopping amplitudes are defined in a unit cell of size ν , which is then repeated over the chain. The case with $\nu = 1$ corresponds to spatially uniform noise. For $\nu = 2$ one has different noise on even and odd sites. In the limit $\nu \rightarrow \infty$, one recovers the QSSEP. Our ν -QSSEP model is illustrated in Fig. 1(a). Similar to the QSSEP, the model is Gaussian for each realization of the noisy amplitudes. Hence, full information about the dynamics is encoded in the two-point fermionic correlation matrix. The stochastic differential equations governing the noise-averaged moments of the correlations close on themselves, similar to the QSSEP, and in contrast with other noisy fermionic models, such as fermions with density monitoring [38] (see also Ref. [39]). We derive these equations for arbitrary ν . We discuss in detail the case with $\nu = 1$ and $\nu = 2$. We show that the moments of the steady-state correlators satisfy nontrivial identities, and can be obtained as the Haar average with random momentum-dependent $SU(\nu)$ matrices.

Now, while introducing a spatially modulated noise might seem a bit artificial, there are several reasons why this is interesting. The main one is that unlike the original QSSEP, it is possible to obtain exact results for the intermediate time dynamics in the ν -QSSEP. Moreover, despite the structured noise the dynamics retains the main features of the QSSEP. Precisely, the dynamics of local observables and of the entanglement entropy is diffusive. As we are going to show, already for $\nu \sim 5$ the dynamics is numerically indistinguishable from that in the QSSEP. Furthermore, the study of quantum

dynamics in the ν -QSSEP can help unveil universal features. For instance, although the dynamics is diffusive for any ν , it exhibits quite nontrivial correlations patterns. Precisely, for $\nu = 1$ the dynamics exhibits only pairwise correlations, which underlies the straightforward applicability of the quasiparticle picture to describe entanglement dynamics. On the other hand, already for $\nu = 2$ correlations between more than two quasiparticles develop, and they can be described analytically [40]. It is likely that such multiparticles correlations are present also in the QSSEP, although it is difficult to characterize them analytically.

For $\nu = 1$, we show that it is possible to characterize analytically the full-time dynamics of the noise-averaged entanglement entropy, and in general of any sufficiently well-behaved function of the reduced density matrix of a subsystem A (see Fig. 1). The dynamics of entanglement-related quantities is understood in terms of a stochastic generalization of the quasiparticle picture [41]. Salient features of the quasiparticle picture are illustrated in Fig. 1(b). Similar to the deterministic model, entanglement is transported by pairs of correlated quasiparticles. Remarkably, the entanglement entropy shared by the members of the pair is the same as in the deterministic model. The trajectories of the quasiparticles are stochastic. At the leading order in the limit $t, \ell \rightarrow \infty$, with t/ℓ^2 fixed, and for each instance of the dynamics all the entangled pairs produced in the chain undergo the same random walk. Moreover, the trajectories of the quasiparticles forming an entangled pair are mirror symmetric with respect to the emission point [see Fig. 1(b)]. Thus, the entanglement dynamics depends only on the distance $\xi(t)$ from the emission point. The entanglement between a subregion A and the rest is proportional to the number of entangled pairs shared between A and its complement. Since for each noise realization all the entangled pairs perform the same random walk, the number of shared pairs is $\min(2|\xi|, \ell)$, which is reminiscent of the result in the deterministic model. After averaging over the noise realizations, one obtains the entanglement entropies as

$$\langle S_n \rangle = \int_{-\infty}^{\infty} d\xi \min(2|\xi|, \ell) s_n P(\xi, t). \quad (1)$$

As anticipated, s_n , with n an integer, are the *thermodynamic* Rényi entropies of the generalized Gibbs ensemble (GGE) that describes the steady state after the quench in the deterministic model [42,43]. In (1), $P(\xi, t)$ is the heat kernel $P(\xi, t) = 1/(2\sqrt{\pi\gamma t}) e^{-\xi^2/(4\gamma t)}$, with γ the square strength of the hopping amplitudes. Equation (1) implies that entanglement spreading is diffusive, in contrast with the ballistic dynamics in the deterministic chain. The stochastic quasiparticle picture (1) holds in the limit $t, \ell \rightarrow \infty$ with t/ℓ^2 fixed. Finally, we numerically verify that the diffusive entanglement scaling holds true for generic ν , even in the limit $\nu \rightarrow \infty$. Equation (1) captures qualitatively the entanglement spreading for generic ν , although it is exact only at $\nu = 1$. On the other hand, we numerically show that the entanglement dynamics for $\nu = 5$ is quite close to that at $\nu \rightarrow \infty$, at least for the quench from the Néel state. Equation (1) is, at least to the best of our knowledge, the first generalization of the quasiparticle picture to stochastic quantum dynamics.

The paper is structured as follows. In Sec. II, we introduce the ν -QSSEP model. In Sec. II A, we derive the equations

describing the dynamics of the moments of fermionic correlation functions in momentum space, which contain full information on the statistics of observables. In particular, in Sec. II B we focus on the ν -QSSEP with $\nu = 1$. In Sec. III, we discuss the moments of the correlators for $\nu = 2$. In Sec. III A, we show that the steady-state moments satisfy nontrivial relationships. In Sec. III B, we discuss the relationship between the steady-state values of the moments and the initial-state expectation values of conserved quantities. In Sec. III C, we discuss the Haar invariance of steady-state observables, focusing on the case with $\nu = 2$. In Sec. IV, we derive the stochastic quasiparticle picture for the entanglement dynamics. In Sec. V, we discuss numerical results supporting the results of Sec. IV, for the moments of the fermionic correlator (in Sec. V B) and for the von Neumann entropy (in Sec. V C). We conclude and discuss future directions in Sec. VI. In Appendix, we provide some details on the derivation of the quasiparticle picture presented in Sec. IV.

II. ν -QSSEP: DEFINITION OF THE MODEL

Here, we consider out-of-equilibrium dynamics in a generalization of the so-called QSSEP model, which is defined by the infinitesimal Hamiltonian dH_t defined as

$$dH_t = \sqrt{\gamma} \sum_{j=1}^L (c_{j+1}^\dagger c_j dW_t^j + c_j^\dagger c_{j+1} d\bar{W}_t^j), \quad (2)$$

where dW_t^j are random variables *à la* Itô, γ a coupling constant, and L the length of the chain. We employ periodic boundary conditions. The horizontal bar in $d\bar{W}_t^j$ denotes complex conjugation. Equation (2) describes a system of free fermions hopping in a chain with hopping amplitudes that are Brownian variables in both space and time. From (2), the infinitesimal Hamiltonian matrix dh_t is

$$[dh_t]_{nm} = \sqrt{\gamma} (dW_t^n \delta_{n-m,1} + d\bar{W}_t^m \delta_{m-n,1}). \quad (3)$$

We consider dynamics from an initial state $|\psi_0\rangle$ (quantum quench) that is not eigenstate of (2). The time-evolved wavefunction satisfies the equation as $|\psi_{t+dt}\rangle = e^{-idH_t} |\psi_t\rangle$. Here, we restrict ourselves to Gaussian initial states, i.e., satisfying Wick's theorem. It is straightforward to show that the dynamics under (2) preserves Gaussianity. From the evolution of the wavefunction, we obtain the density matrix ρ_t as

$$\rho_t = |\psi_t\rangle \langle \psi_t|. \quad (4)$$

In the original QSSEP model, the amplitudes dW_t^j are independent from site to site. Here, we define the ν -QSSEP by considering spatially modulated random amplitudes. Precisely, we require that $W_t^n = W_t^m$ for $n = m \bmod \nu$. For $\nu \geq L$, one recovers the standard QSSEP. For $\nu = 1$, the noise dW_t^j is the same on all sites, but random in time. For generic ν , the noise is defined in a ν -site unit cell. Our setup is depicted in Fig. 1(a). The hopping amplitudes satisfy the standard Itô relationships as

$$dW_t^n d\bar{W}_t^m = \delta_{n-m,0 \bmod \nu}, \quad \{dW_t^n\}_{\text{noise}} = 0. \quad (5)$$

The evolution of the noise-averaged density matrix [cf. (4)] is straightforwardly obtained by using (5) as

$$\begin{aligned} \langle \rho_{t+dt} \rangle &= \langle e^{idH_t} \rho_t e^{-idH_t} \rangle \\ &= \langle \rho_t \rangle + \gamma \sum_{\substack{j=j' \bmod \nu \\ \alpha=l,r}} \left(L_{\alpha,j} \langle \rho_t \rangle L_{\alpha,j'}^\dagger \right. \\ &\quad \left. - \frac{1}{2} \{L_{\alpha,j} L_{\alpha,j'}^\dagger, \langle \rho_t \rangle\} \right), \end{aligned} \quad (6)$$

where we defined the Lindblad operators

$$L_{l,j} = L_{r,j}^\dagger = c_{j+1}^\dagger c_j. \quad (7)$$

Equations (6) with (7) for $\nu \rightarrow \infty$ reduce to the standard Lindblad equation describing a system subjected to incoherent nearest-neighbour hopping. This is similar to the setup investigated in Ref. [44], although the standard Heisenberg term $[dH_t, \rho_t]$ is absent in (6) because of the noise average. For finite ν , the Lindblad operators (7) become nonlocal in space since j, j' can be different in (6).

In the following, we provide exact results for the steady-state value of observables after dynamics starting from arbitrary initial states. Moreover, for some low-entangled initial states we provide the full-time dynamics of entanglement-related quantities. Precisely, we consider dynamics starting from the fermionic Néel state $|N\rangle$ and the dimer state $|D\rangle$ defined as

$$|N\rangle = \prod_{j=1}^{L/2} c_{2j-1}^\dagger |0\rangle, \quad (8)$$

$$|D\rangle = \prod_{j=1}^{L/2} (c_{2j-1}^\dagger + c_{2j}^\dagger) / \sqrt{2} |0\rangle, \quad (9)$$

where $|0\rangle$ is the fermionic vacuum state.

A. Dynamics of correlation functions

Here, we are interested in the fermionic two-point correlation function G_{nm} defined as

$$G_{nm} = \langle \psi_t | c_n^\dagger c_m | \psi_t \rangle. \quad (10)$$

Since the dynamics under (2) preserves the Gaussianity of the initial state, G_{nm} contains full information about the time-dependent state ρ_t of the system. The time-dependent correlation function G is obtained from the evolved state $|\psi_t\rangle$ as

$$G(t+dt) = e^{-idh_t} G(t) e^{idh_t}, \quad (11)$$

where dh_t is given in (3). Equivalently, by expanding the exponentials in (11), up to second order one obtains

$$\begin{aligned} dG_{nm} &= \gamma [\Delta^{\text{dis}} G]_{nm} dt - i\sqrt{\gamma} (dW_t^n G_{n-1,m} \\ &\quad + d\bar{W}_t^{n+1} G_{n+1,m} - d\bar{W}_t^m G_{n,m-1} \\ &\quad - dW_t^{m+1} G_{n,m+1}). \end{aligned} \quad (12)$$

Formally, Eq. (12) is the same as in the QSSEP [27]. However, our definition of $\Delta^{\text{dis}} G$ is different, and it is given as

$$[\Delta^{\text{dis}} G]_{nm} := G_{n-1,m-1} - 2G_{nm} + G_{n+1,m+1} \quad (13)$$

for $n = m \bmod v$, and

$$[\Delta^{\text{dis}} G]_{nm} := -2G_{nm}, \quad (14)$$

for $n \neq m \bmod v$. Now, after averaging over the noise dW_t^n in Eq. (12), only the first row survives. For the *QSSEP*, i.e., in the limit $v \rightarrow \infty$, only the diagonal terms with $n = m$ in (13) are present, whereas for $n \neq m$ Eq. (14) remains the same. Instead, for the v -*QSSEP*, Eq. (12) together with (14) imply that $\langle G_{nm} \rangle_{\text{noise}}$ vanishes exponentially with time for $n - m \neq 0 \bmod v$, whereas it attains a nonzero steady-state value otherwise.

Here, we are interested in the dynamics of arbitrary functions of G_{nm} . To understand that, it is crucial to derive the dynamics of the noise-averaged moments of G_{nm} , such as the second moment $\langle G_{n_1 m_1} G_{n_2 m_2} \rangle$. By applying the standard Itô rule, this is obtained as $d\langle G_{n_1 m_1} G_{n_2 m_2} \rangle = \langle dG_{n_1 m_1} G_{n_2 m_2} + G_{n_1 m_1} dG_{n_2 m_2} + dG_{n_1 m_1} dG_{n_2 m_2} \rangle$, where one has to use (12) to compute each of the three terms. Notice that third term is nonzero solely because of the noise. One obtains

$$\begin{aligned} \gamma^{-1} \frac{d}{dt} \langle G_{n_1 m_1} G_{n_2 m_2} \rangle &= [\langle \Delta^{\text{dis}} G \rangle]_{n_1 m_1} G_{n_2 m_2} \\ &+ G_{n_1 m_1} [\langle \Delta^{\text{dis}} G \rangle]_{n_2 m_2} - \langle \delta_{n_1 \pm 1, n_2} G_{n_1 \pm 1, m_1} G_{n_2 \mp 1, m_2} \\ &- \delta_{n_1, m_2} G_{n_1 \pm 1, m_1} G_{n_2, m_2 \pm 1} - \delta_{m_1 - n_2, 0} G_{n_1, m_1 \pm 1} G_{n_2 \pm 1, m_2} \\ &+ \delta_{m_1, m_2 \mp 1} G_{n_1, m_1 \mp 1} G_{n_2, m_2 \pm 1} \rangle, \end{aligned} \quad (15)$$

where one has to sum over the $\pm$, and all the Kronecker delta functions are defined mod v . In (15), $\Delta^{\text{dis}} G$ is as defined in (13). By direct inspection of (15), it is straightforward to show that in the steady state at $t \rightarrow \infty$, only the correlators $\langle G_{nn} G_{mm} \rangle$ and $\langle G_{nm} G_{mn} \rangle$, where indices are equal mod v , are nonzero.

To proceed, it is convenient to exploit the invariance of (2) under translations by v sites, going to momentum space. Thus, we define the Fourier transformed correlator $G_{kq}^{\rho\sigma}$ as

$$G_{vn+\rho, vm+\sigma} = \frac{v}{L} \sum_{k,q} e^{-ikn+iqm} G_{k,q}^{\rho\sigma}, \quad (16)$$

with $k, q = 2\pi/(L/v)j$, with $j = 0, 1, \dots, L/v - 1$, and $\rho, \sigma = 0, 1, \dots, v - 1$ the unit-cell indices. At fixed k, q , $G_{kq}^{\rho\sigma}$ is a $v \times v$ matrix. In general, G_{kq} is not diagonal, except for dynamics starting from translational invariant initial states. We can perform the Fourier transform of Eqs. (13) and (14), obtaining for $\rho = \sigma$,

$$\begin{aligned} [\Delta^{\text{dis}} G]_{k,q}^{\rho\rho} &= (e^{i(k-q)} \delta_{\rho=0} + \delta_{\rho \neq 0}) G_{kq}^{\rho-1\rho-1} \\ &+ (e^{-i(k-q)} \delta_{\rho=v-1} + \delta_{\rho \neq v-1}) G_{kq}^{\rho+1\rho+1} - 2G_{kq}^{\rho\rho}, \end{aligned} \quad (17)$$

where $\delta_{\rho=0}$ and $\delta_{\rho \neq 0}$ enforce the conditions $\rho = 0$ and $\rho \neq 0$, respectively. For $\rho \neq \sigma$, we have

$$[\Delta^{\text{dis}} G]_{kq}^{\rho\sigma} = -2G_{kq}^{\rho\sigma}. \quad (18)$$

Thus, the evolution of the correlation function (12) in momentum space takes the simple form as

$$\begin{aligned} dG_{kq}^{\rho\sigma} &= \gamma dt [\Delta^{\text{dis}} G]_{kq}^{\rho\sigma} - i\sqrt{\gamma} [dW_t^\rho (\delta_{\rho=0} e^{ik} + \delta_{\rho \neq 0}) G_{kq}^{\rho-1\sigma} \\ &+ d\bar{W}_t^{\rho+1} (\delta_{\rho=v-1} e^{-ik} + \delta_{\rho \neq v-1}) G_{kq}^{\rho+1\sigma} \\ &- d\bar{W}_t^\sigma (\delta_{\sigma=0} e^{-iq} + \delta_{\sigma \neq 0}) G_{kq}^{\rho\sigma-1} \\ &- dW_t^{\sigma+1} (\delta_{\sigma=v-1} e^{iq} + \delta_{\sigma \neq v-1}) G_{kq}^{\rho\sigma+1}], \end{aligned} \quad (19)$$

where $[\Delta^{\text{dis}} G]_{kq}^{\rho\sigma}$ is defined in (17) and (18), and the indices $\rho \pm 1, \sigma \pm 1$ are computed mod v . Now, for $\rho \neq \sigma$, after taking the noise average, Eq. (19) gives the exponential decay as

$$\frac{d\langle G_{kq}^{\rho\sigma} \rangle}{dt} = -2\gamma \langle G_{kq}^{\rho\sigma} \rangle, \quad (20)$$

On the other hand, for $\rho = \sigma$, Eq. (19) gives

$$\begin{aligned} \frac{d\langle G_{kq}^{\rho\rho} \rangle}{dt} &= \gamma \langle (e^{i(k-q)} \delta_{\rho=0} + \delta_{\rho \neq 0}) G_{kq}^{\rho-1\rho-1} \\ &+ (e^{-i(k-q)} \delta_{\rho=v-1} + \delta_{\rho \neq v-1}) G_{kq}^{\rho+1\rho+1} - 2G_{kq}^{\rho\rho} \rangle. \end{aligned} \quad (21)$$

It is interesting to investigate the steady-state value of $G_{kq}^{\rho\rho}$. To have a nonzero steady-state value, the right-hand side of (21) has to vanish. From (21), one obtains a system of v equations. It is straightforward to check that a nonzero steady-state value is possible only for $k = q$. Moreover, one obtains that $G_{kk}^{\rho\rho}$ does not depend on ρ .

Similar to the *QSSEP* model [27], the steady-state value of the correlator and its moments can be obtained from the expectation value on the initial state of conserved quantities. For instance, by using the evolution equation (19), it is straightforward to check that N_k defined as

$$N_k = \sum_{\rho=0}^{v-1} G_{kk}^{\rho\rho} \quad (22)$$

is conserved during the dynamics for any k , even for a single realization of the noise. This implies that in the steady state, one has

$$\langle G_{kq}^{\rho\sigma} \rangle = \frac{N_k}{v} \delta_{kq} \delta_{\rho\sigma}. \quad (23)$$

The result (23) holds for any value of v . The fact that the nonzero steady-state values of the correlators can be determined from the conserved quantities remains true for higher moments of the correlation functions, and is reminiscent of a similar behavior in the *QSSEP* [27].

The building block to obtain the dynamics of the noise-averaged moments of $G_{kq}^{\rho\sigma}$ is the dynamics of the four-point function $G_{k_1, q_1}^{\rho_1 \sigma_1} G_{k_2, q_2}^{\rho_2 \sigma_2}$. By using (19), we obtain

$$d(G_{k_1, q_1}^{\rho_1 \sigma_1} G_{k_2, q_2}^{\rho_2 \sigma_2}) = (dG_{k_1, q_1}^{\rho_1 \sigma_1}) G_{k_2, q_2}^{\rho_2 \sigma_2} + G_{k_1, q_1}^{\rho_1 \sigma_1} (dG_{k_2, q_2}^{\rho_2 \sigma_2}) - \gamma K_{12} dt, \quad (24)$$

where we defined K_{ij} as

$$\begin{aligned}
K_{ij} = & -\delta_{\rho_i, \sigma_j} \tilde{G}_{k_i, q_i}^{\rho_i-1\sigma_i} \tilde{G}_{k_j, q_j}^{\rho_j\sigma_j-1} \\
& + \delta_{\rho_i, \rho_j+1} \tilde{G}_{k_i, q_i}^{\rho_i-1\sigma_i} \tilde{G}_{k_j, q_j}^{\rho_j+1\sigma_j} + \delta_{\rho_i+1, \rho_j} \tilde{G}_{k_i, q_i}^{\rho_i+1\sigma_i} \tilde{G}_{k_j, q_j}^{\rho_j-1\sigma_j} \\
& - \delta_{\rho_i, \sigma_j} \tilde{G}_{k_i, q_i}^{\rho_i+1\sigma_i} \tilde{G}_{k_j, q_j}^{\rho_j\sigma_j+1} - \delta_{\sigma_i, \rho_j} \tilde{G}_{k_i, q_i}^{\sigma_i-1} \tilde{G}_{k_j, q_j}^{\rho_j-1\sigma_j} \\
& + \delta_{\sigma_i, \sigma_j+1} \tilde{G}_{k_i, q_i}^{\sigma_i-1} \tilde{G}_{k_j, q_j}^{\rho_j\sigma_j+1} - \delta_{\sigma_i, \rho_j} \tilde{G}_{k_i, q_i}^{\sigma_i+1} \tilde{G}_{k_j, q_j}^{\rho_j+1\sigma_j} \\
& + \delta_{\sigma_i+1, \sigma_j} \tilde{G}_{k_i, q_i}^{\sigma_i+1} \tilde{G}_{k_j, q_j}^{\rho_j\sigma_j-1}, \tag{25}
\end{aligned}$$

with $\tilde{G}_{kq}^{\rho\sigma}$ defined as

$$\tilde{G}_{k,q}^{\rho-1,\sigma} = (\delta_{\rho=0}e^{ik} + \delta_{\rho \neq 0})G_{k,q}^{\rho-1\sigma}, \quad (26)$$

$$\tilde{G}_{k,q}^{\rho+1,\sigma} = (\delta_{\rho=v-1}e^{-ik} + \delta_{\rho \neq v-1})G_{k,q}^{\rho+1\sigma}, \quad (27)$$

$$\tilde{G}_{k,q}^{\rho,\sigma-1} = (\delta_{\sigma=0}e^{-iq} + \delta_{\sigma \neq 0})G_{k,q}^{\rho\sigma-1}, \quad (28)$$

$$\tilde{G}_{k,q}^{\rho,\sigma+1} = (\delta_{\sigma=v-1} e^{iq} + \delta_{\sigma \neq v-1}) G_{k,q}^{\rho\sigma+1}. \quad (29)$$

Again, the indices ρ_j, σ_j are defined mod ν . The term K_{12} in (24) originates from the application of the Itô rule to $dG_{k_1q_1}^{\rho_1\sigma_1}dG_{k_2q_2}^{\rho_2\sigma_2}$. The system of ν equations in (24) fully determines the dynamics of the product $G_{k_1q_1}G_{k_2q_2}$. Equation (24) is straightforwardly generalized to the higher moments of G_{kq} . Indeed, for the generic n th moment, we have that $d(G_{k_1q_1}^{\rho_1\sigma_1}G_{k_2q_2}^{\rho_2\sigma_2}\dots G_{k_nq_n}^{\rho_n\sigma_n})$ contains all the terms with a single $dG_{k_jq_j}^{\rho_j\sigma_j}$, as well as all the Itô terms as $dG_{k_jq_j}^{\rho_j\sigma_j}dG_{k_lq_l}^{\rho_l\sigma_l}$ with $j \neq l$. The former give rise to contributions of the form (12), whereas the latter give $\gamma K_{ij}dt$, with K_{ij} as in (25).

B. Full-time dynamics for $v = 1$

Before proceeding let us discuss the case with $\nu = 1$, which corresponds to the situation in which the hopping amplitudes dW_i^j do not depend on j , although they are random in time. Now, for any k, q , G_{ka} is a number, and Eq. (19) becomes

$$dG_{kq} = \{2\gamma[\cos(k-q) - 1]dt - i\sqrt{\gamma}(e^{ik} - e^{iq})dW_t - i\sqrt{\gamma}(e^{-ik} - e^{-iq})d\overline{W}_t\}G_{ka}, \quad (30)$$

from where it is clear that G_{kk} is conserved by the dynamics for any k , even before the noise average. In contrast with generic ν , for $\nu = 1$ it is straightforward to obtain the full-time dynamics of arbitrary noise-averaged moments of the correlators. For instance, for G_{kq} , by taking the average over $dW_t, d\bar{W}_t$ in (30) one obtains the equation for $\langle G_{kq} \rangle$, whose solution is $\langle G_{kq} \rangle = G_{kq}^{(0)} e^{\varepsilon t}$, with $G_{kq}^{(0)}$ the initial condition and the “energy” $\varepsilon = 2\gamma[\cos(k - q) - 1]$. $\langle G_{kq} \rangle$ decays exponentially with time except for $k = q$. Let us consider the dynamics of $\langle G_{k_1 q_1} G_{k_2 q_2} \rangle$. From (24), we obtain

$$\begin{aligned}
& d\langle G_{k_1 q_1} G_{k_2 q_2} \rangle \\
&= 2\gamma[-2 - \cos(k_1 - k_2) + \cos(k_1 - q_1) \\
&\quad + \cos(k_2 - q_1) + \cos(k_1 - q_2) + \cos(k_2 - q_2) \\
&\quad - \cos(q_1 - q_2)]\langle G_{k_1 q_1} G_{k_2 q_2} \rangle dt. \tag{31}
\end{aligned}$$

Equation (31) implies that $\langle G_{k_1 q_1} G_{k_2 q_2} \rangle$ decays exponentially with time, except for $G_{k_1 k_1} G_{k_2 k_2}$ and $G_{k_1 q_1} G_{q_1 k_1}$. It is useful to

introduce a pictorial notation for G_{kq} . Let us denote G_{kq} with a directed arrow as

$$G_{kq} = \text{diagram of a propagator with a loop}, \quad (32)$$

where the left and right endpoints denote the momenta k and q , respectively. It is easy to check that the quasimomenta configurations that correspond to correlators that have a nonzero steady-state value are given by the diagrams that at each vertex have the same number of ingoing and outgoing lines, which implies momentum conservation at each vertex. For instance, the only nonzero correlators $\langle G_{k_1 q_1} G_{k_2 q_2} \rangle$ correspond to the diagrams

$$\begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} \quad \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} \quad \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} \quad \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} \quad (33)$$

Let us now observe that, since Eq. (31) can be generalized to arbitrary moments of the correlators G_{kq} , one can obtain the full-time dynamics of arbitrary observables. However, the generalization of (31) become rapidly quite cumbersome. Still, as it will be clear in Sec. IV, for dynamics starting from translation-invariant initial states and in the long time limit only some diagrams contribute, which allows to determine the full-time entanglement dynamics.

III. DYNAMICS IN THE v -QSSEP WITH $v = 2$

Having discussed the case with $\nu = 1$, here we focus on the ν -QSSEP with $\nu = 2$. Precisely, we analyze the evolution equations (19), showing that the moments of the steady-state correlation functions satisfy a set of “topological” constraints, which allow to determine their value from initial-state data.

Let us first consider the two-point function $\langle G_{kq}^{\rho\sigma} \rangle$, where $\rho, \sigma = 0, 1$. Equation (19) gives

$$d\langle G_{ka}^{00} \rangle = -2\gamma \langle G_{ka}^{00} \rangle + \gamma(1 + e^{i(k-q)}) \langle G_{ka}^{11} \rangle dt, \quad (34)$$

$$d\langle G_{ka}^{11} \rangle = -2\gamma \langle G_{ka}^{11} \rangle + \gamma(1 + e^{-i(k-q)}) \langle G_{ka}^{00} \rangle dt. \quad (35)$$

Clearly, Eqs. (34) and (35) imply that $G_{kq}^{\rho\rho}$ is nonzero in the steady state only for $k = q$, as anticipated in Sec. II A for generic v . From (20), one has that $\langle G_{kq}^{\rho\sigma} \rangle$ vanishes exponentially for $\rho \neq \sigma$. By using (34), (35), and (20) one can obtain the exact full-time dynamics of $G_{kq}^{\rho\sigma}$. This is not straightforward for more complicated moments of the correlation functions. For instance, let us focus on the second moment $G_{k_1 q_1} G_{k_2 q_2}$.

A direct analysis of (24) shows that the only moments attaining nonzero steady-state values correspond to $\{\rho_1, \sigma_1, \rho_2, \sigma_2\} = \{0, 0, 0, 0\}, \{0, 0, 1, 1\}, \{0, 1, 1, 0\}$, together with those obtained by swapping $1 \leftrightarrow 0$. The dynamics of the noise-averaged moment $\langle G_{k_1 q_1}^{\rho_1 \sigma_1} G_{k_2 q_2}^{\rho_2 \sigma_2} \rangle$ is governed by a 6×6 matrix M as

$$\frac{d}{dt}\langle G_{k_1q_1}G_{k_2q_2}\rangle = \gamma \sum_{\rho_j, \sigma_j} M_{\rho_1\sigma_1\rho_2\sigma_2} \langle G_{k_1q_1}^{\rho_1\sigma_1} G_{k_2q_2}^{\rho_2\sigma_2} \rangle, \quad (36)$$

where the sum over ρ_j, σ_j is restricted to the six configurations above. The matrix M is constructed starting from any set of indices $\{\rho_1, \sigma_1, \rho_2, \sigma_2\}$ for which $G_{k_1 a_1}^{\rho_1 \sigma_1} G_{k_2 a_2}^{\rho_2 \sigma_2}$ is nonzero

in the steady state, and iteratively applying (24). One obtains

$$M = -5\mathbb{1}_6 + O_6 + \begin{pmatrix} 0 & e^{ik_2-iq_2} & e^{ik_2-iq_1} & e^{ik_1-iq_2} & e^{ik_1-iq_1} & -1 \\ e^{-ik_2+iq_2} & 0 & -2 - e^{-iq_1+iq_2} & -2 - e^{ik_1-ik_2} & -1 & e^{ik_1-iq_1} \\ e^{-ik_2+iq_1} & -2 - e^{iq_1-ik_2} & 0 & -1 & -2 - e^{ik_1-ik_2} & e^{ik_1-iq_2} \\ e^{-ik_1+iq_2} & -2 - e^{-ik_1+ik_2} & -1 & 0 & -2 - e^{-iq_1+iq_2} & e^{ik_2-iq_1} \\ e^{-ik_1+iq_1} & -1 & -2 - e^{-ik_1+ik_2} & -2 - e^{iq_1-iq_2} & 0 & e^{ik_2-iq_2} \\ -1 & e^{-ik_1+iq_1} & e^{-ik_1+iq_2} & e^{-ik_2+iq_1} & e^{-ik_2+iq_2} & 0 \end{pmatrix}, \quad (37)$$

where $\mathbb{1}_6$ is the identity matrix and O_6 is the matrix with all its entries equal to 1. One can check that the eigenvalues of (37) are nontrivial for generic k_j, q_j , implying that it is not straightforward to obtain the *exact* finite-time dynamics. As anticipated, for some momenta configurations k_j, q_j , Eq. (37) will have some zero eigenvalues, which correspond

to nonzero steady-state values of $\langle G_{k_1q_1}^{\rho_1\sigma_1} G_{k_2q_2}^{\rho_2\sigma_2} \rangle$. For completeness, let us also consider $G_{k_1q_1}^{\rho_1\sigma_1} G_{k_2q_2}^{\rho_2\sigma_2}$, with $\{\rho_1, \sigma_1, \rho_2, \sigma_2\} = \{0, 0, 1, 0\}, \{1, 0, 0, 0\}, \{1, 0, 1, 1\}, \{1, 1, 1, 0\}$, i.e., having an odd number of 1s. By employing (24) one can check that this set of correlators is closed under the dynamics. Now the matrix governing the dynamics is 4×4 , and it reads as

$$M = -5\mathbb{1}_4 + \mathbb{1}_4 \otimes \mathbb{1}_4 + \begin{pmatrix} 0 & -2 - e^{ik_1-ik_2} & e^{ik_1-iq_2} & e^{ik_1-iq_1} \\ -2 - e^{-ik_1+ik_2} & 0 & e^{ik_2-iq_2} & e^{ik_2-iq_1} \\ e^{-ik_1+iq_2} & e^{-ik_2+iq_2} & 0 & -2 - e^{-iq_1+iq_2} \\ e^{-ik_1+iq_1} & e^{-ik_2+iq_1} & -2 - e^{iq_1-iq_2} & 0 \end{pmatrix}. \quad (38)$$

Although the eigenvalues of (38) can be determined exactly, their expression is cumbersome, and we do not report it. However, one can check that all the eigenvalues of (38) are negative for any choice of k_j, q_j . This implies that all the correlators vanish exponentially at long times. Finally, we should observe that the dynamics of $G_{k_1q_1}^{01} G_{k_2q_2}^{00}$ is not accounted for in (38). However, it is easily obtained from $G_{k_1q_1}^{11} G_{k_2q_2}^{10}$ by swapping $1 \leftrightarrow 0$ and $(k_1, q_1) \leftrightarrow (k_2, q_2)$. In the next sections, we show that although for $\nu = 2$ it could be challenging to obtain the exact full-time dynamics of the generic moments of the correlators, their steady-state values satisfy some nontrivial constraints, which can be explained by random matrix theory.

A. Steady-state correlators

To identify the moments $\langle G_{k_1q_1}^{\rho_1\sigma_1} G_{k_2q_2}^{\rho_2\sigma_2} \rangle$ that attain nonzero steady-state values, one has to determine the kernel of M_6 in (37), i.e., the eigenvectors with zero eigenvalues, for all the configurations of the quasimomenta k_j, q_j . Now, one can straightforwardly check that Eq. (37) has zero eigenvalues only if all the momenta $k_j q_j$ appear an even number of times. To illustrate the correlators that are nonzero in the steady state we can employ the pictorial notation used for $\nu = 1$ in Sec. II B, ignoring the indices $\rho_j \sigma_j$ for now. The only quasimomenta configurations yielding nonzero steady-state results correspond to the diagrams

$$\text{Diagram 1}, \text{Diagram 2}, \text{Diagram 3}, \text{Diagram 4}, \quad (39)$$

The terms in (39) have different number of independent quasimomenta. Precisely, the two leftmost ones contain two independent momenta, and they represent $\langle G_{kk}^{\rho_1\sigma_1} G_{qq}^{\rho_2\sigma_2} \rangle$ and $\langle G_{kq}^{\rho_1\sigma_1} G_{qk}^{\rho_2\sigma_2} \rangle$, with $k \neq q$. The third term in (39) corresponds to $\langle G_{kk}^{\rho_1\sigma_1} G_{kk}^{\rho_2\sigma_2} \rangle$, where a single momentum k appears. The last term is $\langle G_{kq}^{\rho_1\sigma_1} G_{kq}^{\rho_2\sigma_2} \rangle$, with $k \neq q$. One can check that also for

higher moments of G_{kq} only the diagrams in which all the momenta appear an even number of times can yield nonzero steady-state values. Moreover, it is interesting to observe that all the diagrams in (39), except the last one, satisfy the rule that at each vertex the total number of incoming and outgoing lines is the same, reflecting conservation of momentum at each vertex. For $\nu = 1$, diagrams violating the momentum conservation rule were absent (see Sec. II B).

To fully determine which $\langle G_{k_1q_1}^{\rho_1\sigma_1} G_{k_2q_2}^{\rho_2\sigma_2} \rangle$ are nonzero in the steady state, one has to determine for each of the diagrams in (39) all the allowed values for the indices $\rho_j \sigma_j$. To proceed, let us first focus on the configurations satisfying momentum conservation at each vertex [the three leftmost diagrams in (39)]. By solving for the null space of (37) for each of the diagrams in (39), one can check that the eigenvectors components are nonzero only if at each vertex the number of incoming and outgoing 0s and 1s is the same. This means that vertices that satisfy momentum conservation also satisfy conservation of 0 and 1. We verified that this rule generalizes to higher-point functions, and it is also valid for generic ν . Moreover, further constraints on the indices ρ_j, σ_j exist.

To discuss them, it is convenient to introduce a pictorial notation, denoting with a directed red line the correlator $G_{kq}^{\rho\sigma}$ as , the endpoints being ρ and σ . Now, for each diagram of (39) there are several compatible red diagrams. The value of the red vertices is immaterial, the only constraint being that distinct vertices correspond to different values of ρ, σ . Moreover, for $\nu = 2$ the allowed diagrams representing ρ_j, σ_j can have at most two vertices, in contrast with the momentum diagrams [as, for instance, in (39)], which can have an arbitrary number of vertices. Interestingly, diagrams corresponding to nonzero steady-state moments of the correlator are not all independent but satisfy nontrivial relationships. These relationships are simpler if only vertices with only one ingoing and one outgoing momentum lines are present. For

instance, the diagram $\langle \text{diagram} \rangle$ gives

$$\langle \text{diagram} \rangle \rightarrow \langle \text{diagram} \rangle = \langle \text{diagram} \rangle, \quad (40)$$

where we restored the noise average $\langle \cdot \rangle$. Similarly, we obtain

$$\langle \text{diagram} \rangle \rightarrow \langle \text{diagram} \rangle = \langle \text{diagram} \rangle. \quad (41)$$

From (40) and (41), we conclude that all the moments that correspond to diagrams with vertices satisfying the conservation of momentum and of cell indices 0, 1 have the same value in the steady state. However, this does not remain true if there are vertices connecting more than two momentum lines. For instance, let us focus on the diagram with $k_1 = q_1 = k_2 = q_2$. Now, from the null space of (37) one obtains that

$$\langle \text{diagram} \rangle \rightarrow \langle \text{diagram} \rangle = \langle \text{diagram} \rangle + \langle \text{diagram} \rangle. \quad (42)$$

The identity (42) is the same as in the standard *QSSEP* (see Ref. [27]). However, we anticipate that for $\nu = 2$, more complicated moments of the correlator will satisfy only a subset of the relationships obtained in [27]. This is expected because for $\nu = 2$ only red diagrams with only two vertices are allowed, whereas for $\nu \rightarrow \infty$ diagrams with an arbitrary number of vertices can be present.

Finally, let us now consider the last diagram in (39), which violates the momentum conservation at the two vertices. From (37), one obtains the relations

$$\langle \text{diagram} \rangle \rightarrow \langle \text{diagram} \rangle = - \langle \text{diagram} \rangle. \quad (43)$$

Now, at each vertex the number of 0,1 is not preserved, suggesting that at a vertex violating the momentum conservation the conservation of ρ, σ is also violated. As we will discuss in the following, Eq. (43) is understood in the framework of random matrix theory.

For the following, it is convenient to change the notation denoting with a directed numbered line the correlator $G_{kq}^{\rho\sigma}$ as

$$G_{kq}^{\rho\sigma} := \text{diagram}, \quad a = \rho + 2\sigma, \quad a = \rho + 2\sigma. \quad (44)$$

This allows to rewrite the identities (40)–(42) as

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle \quad (45)$$

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle \quad (46)$$

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle + \langle \text{diagram} \rangle \quad (47)$$

where one can exchange $0 \leftrightarrow 3$ and $1 \leftrightarrow 2$. The identity (43) becomes

$$\langle \text{diagram} \rangle = - \langle \text{diagram} \rangle \quad (48)$$

To understand the behavior of higher moments of G_{kq} let us now consider $\prod_{j=1}^3 \langle G_{k_j q_j}^{\rho_j \sigma_j} \rangle$. As before, to have a nonzero steady-state value, each momentum has to appear an even number of times. This means that the maximum number of independent momenta is three. Let us denote as $\mathcal{C}_3$ the set of diagrams with three independent momenta. We have

$$\mathcal{C}_3 = \{ \text{diagram}, \text{diagram}, \text{diagram}, \text{diagram}, \text{diagram}, \text{diagram} \}. \quad (49)$$

The diagrams $\mathcal{C}_2$ with only two independent momenta are

$$\mathcal{C}_2 = \{ \text{diagram}, \text{diagram}, \text{diagram}, \text{diagram} \}. \quad (50)$$

Finally, if all the momenta are equal we obtain

$$\mathcal{C}_1 = \{ \text{diagram} \}. \quad (51)$$

Notice that in both (49) and (62) there are diagrams that do not satisfy the momentum conservation at all the vertices. To determine the allowed values for the indices ρ_j, σ_j , which give nonzero steady-state moments, one has to write the generalization of (24). One obtains a system of 20 equations. For the diagrams in class $\mathcal{C}_3$ we obtain the following relationships:

$$\langle \text{diagram} \rangle = - \langle \text{diagram} \rangle = \langle \text{diagram} \rangle = - \langle \text{diagram} \rangle \quad (52)$$

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle \quad (53)$$

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle = \langle \text{diagram} \rangle \quad (54)$$

$$\langle \text{diagram} \rangle = - \langle \text{diagram} \rangle \quad (55)$$

$$\langle \text{diagram} \rangle = \langle \text{diagram} \rangle. \quad (56)$$

Notice that here we do not show diagrams that are trivially related to the ones above. For instance, from the leftmost diagram in (52) one obtains an equivalent one by exchanging $0 \rightarrow 3$. In the last diagram (55) one can exchange $0 \leftrightarrow 3$. Again, the diagrams above have vertices with only two momentum lines. Except for the diagrams that violate momentum conservation, the identities above imply that for each diagram in (49), all the configurations of indices $\rho_j \sigma_j$ give the same

steady-state value. For the diagrams with two independent momenta, we obtain

$$\langle \text{diagram with } 0, 3, 0 \rangle = -\langle \text{diagram with } 2, 1, 0 \rangle \quad (57)$$

$$\langle \text{diagram with } 0, 0, 0 \rangle = \langle \text{diagram with } 0, 2, 1 \rangle \quad (58)$$

$$\langle \text{diagram with } 3, 0, 0 \rangle = \langle \text{diagram with } 0, 1, 2 \rangle \quad (59)$$

$$\langle \text{diagram with } 2, 1, 0 \rangle = \langle \text{diagram with } 0, 0, 0 \rangle - \langle \text{diagram with } 0, 1, 2 \rangle \quad (60)$$

The last identity (60) is obtained by applying the identity (42) to the left vertex, whereas the other constraints are similar to the ones for the diagrams in $\mathcal{C}_3$. The only diagram left with two independent momenta gives the relationships

$$\langle \text{diagram with } 0, 0, 0 \rangle = \langle \text{diagram with } 0, 3, 0 \rangle \quad (61)$$

$$\langle \text{diagram with } 0, 3, 0 \rangle = \langle \text{diagram with } 0, 0, 0 \rangle - \langle \text{diagram with } 1, 2, 0 \rangle. \quad (62)$$

From the first diagram in (61), i.e., the one with all the labels equal, one can obtain an equivalent one by exchanging $3 \leftrightarrow 0$. The same holds for the second diagram in (61). Again, the identities in (61) and (62) are obtained by applying (42). Finally, the diagram with all the momenta equal, i.e., in $\mathcal{C}_1$ [cf. (51)], satisfies the constraint

$$\langle \text{diagram with } 0, 0, 0 \rangle = \langle \text{diagram with } 3, 0, 0 \rangle + 2\langle \text{diagram with } 0, 1, 2 \rangle. \quad (63)$$

Now, Eq. (63) is the same as in the standard *QSSEP* [27]. However, in the original *QSSEP* there are three nontrivial relationships involving the third moment of the correlator, whereas for the ν -*QSSEP* with $\nu = 2$ only (63) survives. The reason is, again, that for $\nu = 2$ the indices ρ_j, σ_j can take only the values 0, 1.

Before concluding, to illustrate the type of constraints that one obtains for higher moments, we mention that for the fourth moment $\prod_{j=1}^4 G_{kk}^{\rho_j \sigma_j}$ with only one independent momentum k there are three identities as

$$3\langle \text{diagram with } 2, 1, 1, 2 \rangle = \langle \text{diagram with } 0, 0, 0, 0 \rangle - 4\langle \text{diagram with } 3, 0, 0, 0 \rangle + 3\langle \text{diagram with } 3, 3, 0, 0 \rangle \quad (64)$$

$$6\langle \text{diagram with } 2, 2, 1, 3 \rangle = \langle \text{diagram with } 0, 0, 0, 0 \rangle + 2\langle \text{diagram with } 3, 0, 0, 0 \rangle - 3\langle \text{diagram with } 3, 3, 0, 0 \rangle \quad (65)$$

$$3\langle \text{diagram with } 2, 1, 1, 2 \rangle = \langle \text{diagram with } 0, 0, 0, 0 \rangle - 2\langle \text{diagram with } 3, 0, 0, 0 \rangle. \quad (66)$$

Equations (64)–(66) are obtained by deriving the generalization of (36) and solving for the zero eigenvectors. An explicit calculation shows that one has to solve a system of 924 equations.

The identities derived above are a subset of the constraints that hold for the fourth moment of the real-space correlator in the *QSSEP* [27]. Precisely, by using the results of Ref. [27], one can obtain a set of 11 identities for the fourth moment $\langle \prod_{j=1}^4 G_{k_j}^{\rho_j \sigma_j} \rangle$. The corresponding diagrams contain more than two vertices. However, one can show that the subset of diagrams with only two vertices, satisfy (64)–(66).

B. Role of the conserved quantities

A crucial question is whether it is possible to determine all the nonzero steady-state moments of the correlators from initial-state data. For the two-point correlator this is the case because it can be written in terms of the initial occupations of the momentum modes N_k via (22). This holds true for any ν , as stressed in Sec. II A. Let us now discuss how to determine the steady-state expectation values of the second moments $\langle G_{k_1 q_1}^{\rho_1 \sigma_1} G_{k_2 q_2}^{\rho_2 \sigma_2} \rangle$. By using (19), one can build three conserved quantities $N_1^{kq}, N_2^{kq}, N_3^{kq}$ defined as

$$N_1^{kq} = \text{Tr}(G_{kk}^{\rho\sigma}) \text{Tr}(G_{qq}^{\rho\sigma}), \quad (67)$$

$$N_2^{kq} = \text{Tr} \left(\sum_{\sigma_1} G_{kq}^{\rho_1 \sigma_1} G_{qk}^{\sigma_1 \rho_2} \right), \quad (68)$$

$$N_3^{kq} = G_{kq}^{\rho\rho} G_{kq}^{\sigma\sigma} - G_{kq}^{\rho\sigma} G_{kq}^{\sigma\rho}, \quad (69)$$

where the traces are over the indices of the unit cell. These quantities are preserved even at the level of the single realization of the noise dW_t^ρ , as one can verify by using (19). Notice that the last term in (67) is the determinant of $G_{kq}^{\rho\sigma}$. Now, it is straightforward to rewrite N_j^{kq} as

$$N_1^{kq} = 2\langle \text{diagram with } 0, 3, 0 \rangle + 2\langle \text{diagram with } 0, 0, 0 \rangle \quad (70)$$

$$N_1^{kk} = 2\langle \text{diagram with } 0, 3, 0 \rangle + 2\langle \text{diagram with } 0, 0, 0 \rangle \quad (71)$$

$$N_2^{kq} = 2\langle \text{diagram with } 0, 0, 0 \rangle + 2\langle \text{diagram with } 1, 2, 0 \rangle \quad (72)$$

$$N_2^{kk} = 2\langle \text{diagram with } 0, 0, 0 \rangle + 2\langle \text{diagram with } 1, 2, 0 \rangle \quad (73)$$

$$N_3^{kq} = \langle \text{diagram with } 0, 0, 0 \rangle - \langle \text{diagram with } 1, 2, 0 \rangle, \quad (74)$$

Finally, by inverting (70)–(74) one can extract the steady-state values of the correlators in terms of N_j^{kq} . Similar relationships were derived for the standard *QSSEP* [27]. Now, it is natural to conjecture that, even for generic ν , the same strategy can be employed to obtain the steady-state expectation value of

arbitrary moments of the correlators in terms of initial-state expectation values.

C. Statistics of steady-state correlators

As for the *QSSEP* [27], the identities derived in the previous sections reflect the Haar invariance of the statistics of steady-state correlators. To highlight that, following Ref. [27], we consider the functions $Z_n(A)$ as

$$Z_n(A) = \langle (\text{Tr} GA)^n \rangle, \quad (75)$$

where the trace is over the both the unit-cell indices and momentum indices. Z_n are the coefficients of the generating function $Z(A) = \exp(\text{Tr} GA)$. By taking appropriate derivatives with respect to the matrix elements $A_{kq}^{\rho\sigma}$, one obtains all the moments of $G_{kq}^{\rho\sigma}$. For $n = 1$, one has

$$\left\langle \sum_{kq, \rho\sigma} G_{kq}^{\rho\sigma} A_{qk}^{\sigma\rho} \right\rangle = \frac{1}{2} \sum_{k, \rho} \text{Tr}(\langle G_{kk} \rangle) A_{kk}^{\rho\rho}. \quad (76)$$

For $n = 2$, we can expand $(\text{Tr} GA)^2$ in terms of the diagrams in (39). All the possible diagrams appear in the expansion. However, by using the identities (45)–(47), one obtains that

$$\begin{aligned} & \langle [\text{Tr}(GA)]^2 \rangle \\ &= \sum_{k \neq q} \left[\langle \text{Tr}(A_{qk} A_{kq}) \rangle + \langle \text{Tr}(A_{qk}) \text{Tr}(A_{kq}) \rangle \right. \\ & \quad + \langle \text{Tr}(A_{qk} A_{qk}) \rangle + \langle \text{Tr}(A_{qk}) \text{Tr}(A_{kk}) \rangle \\ & \quad \left. + \sum_k \left[\langle \text{Tr}(A_{kk} A_{kk}) \rangle + \langle \text{Tr}(A_{kk}) \text{Tr}(A_{kk}) \rangle \right] \right], \end{aligned} \quad (77)$$

where the trace is over the indices ρ, σ . All the prefactors of the traces in (77) can be rewritten in terms of the conserved quantities in (70)–(74).

Now, the crucial observation is that the terms that survive the noise average in Eq. (77) are the ones that are invariant under the transformation

$$A_{kq} \rightarrow V_k^\dagger A_{kq} V_q \quad (78)$$

with V_k, V_q random $SU(2)$ matrices. Notice that the terms $[\text{Tr}(A_{qk})]^2$ and $\sum_{\rho\sigma} (A_{qk}^{\rho\sigma} A_{qk}^{\sigma\rho})$ are not separately invariant under (78). However, the combination appearing in (77) is invariant owing to the identity (48), and it is the determinant of A_{kq} . One can check that a similar result holds for larger n and generic v . It is in general challenging to construct analytically $Z(A)$ for generic $SU(v)$. Moreover, for small v a direct computation of the noise average is convenient, as we show in the following.

The invariance under (78) implies that the steady-state noise average of a generic moment $\prod_{j=1}^n G_{k_j q_j}^{\rho_j \sigma_j}$ with fixed $k_j, q_j, \rho_j, \sigma_j$ can be computed as

$$\left\langle \prod_{j=1}^n G_{k_j q_j}^{\rho_j \sigma_j} \right\rangle = \int d\eta(V) \prod_l [V_{k_l}^\dagger G_{k_l q_l} V_{q_l}]^{\rho_l \sigma_l}, \quad (79)$$

where V_k are random $SU(2)$ matrices, and we denote by $d\eta(V)$ the corresponding Haar measure. Precisely, a random $SU(2)$

matrix V_k can be parametrized as

$$V_k = \begin{pmatrix} \rho_k e^{i\phi_k} & e^{-i\psi_k} \sqrt{1 - \rho_k^2} \\ -e^{i\psi_k} \sqrt{1 - \rho_k^2} & e^{-i\phi_k} \rho_k \end{pmatrix} \quad (80)$$

with $\phi_k, \psi_k \in [0, 2\pi]$, and $0 \leq \rho_k \leq 1$. The average with the Haar measure over V_k is written as

$$\int d\eta(V_k) = \int_0^1 2\rho_k d\rho_k \int_0^{2\pi} \frac{d\phi_k}{2\pi} \int_0^{2\pi} \frac{d\psi_k}{2\pi}. \quad (81)$$

The integral in (79) can be written in terms of quantities that are invariant under the Haar average. Similar decomposition rules exist for generic $SU(N)$ [45,46]. Precisely, we employ the key formula

$$\int d\eta(V) V_{i_1 j_1} \cdots V_{i_k j_k} = \frac{1}{(k+1)!} \sum_{\text{pairings } P} \prod_{(a,b) \in P} \varepsilon_{i_a, i_b} \varepsilon_{j_a, j_b}, \quad (82)$$

where ε_{ij} is the completely antisymmetric two-dimensional tensor with $\varepsilon_{12} = 1$, and the sum is over all the possible ways of pairing the indices. Clearly, the Haar average of the product of an odd number of terms V_{ij} vanishes. The average of the product of an even number of terms V_{ij}^* is obtained from (82) by using that $V_{ij}^* = \varepsilon_{ip} V_{ps} \varepsilon_{sj}^{-1}$, where repeated indices are summed over, and $\varepsilon_{sj}^{-1} = \varepsilon_{js}$. Let us consider the general case of a product of $n + m$ $SU(2)$ random matrices containing both V and V^* terms. One has

$$\int d\eta(V) V_{i_1 j_1} \cdots V_{i_n j_n} V_{i_{n+1} j_{n+1}}^* \cdots V_{i_{n+m} j_{n+m}}^* = \frac{1}{[(n+m)/2 + 1]!} \sum_{\text{pairings } P} \prod_{(a,b) \in P} K_{i_a, j_a, i_b, j_b}, \quad (83)$$

where $K_{i_a, j_a, i_b, j_b} = \varepsilon_{i_a i_b} \varepsilon_{j_a j_b}$ for VV and V^*V^* pairings, and $K_{i_a, j_a, i_b, j_b} = \delta_{i_a i_b} \delta_{j_a j_b}$ for VV^* pairings. Notice that for the Haar average (83) to be nonzero $n + m$ has to be even. One can now apply (82) and (83) to the generic steady-state value of $\langle G_{k_1 q_1} G_{k_2 q_2} \cdots G_{k_n q_n} \rangle$. Indeed, the result is obtained by performing the Haar average of $[V_{k_1}^\dagger G_{k_1 q_1} V_{q_1}]^{\rho_1 \sigma_1} [V_{k_2}^\dagger G_{k_2 q_2} V_{q_2}]^{\rho_2 \sigma_2} \cdots [V_{k_n}^\dagger G_{k_n q_n} V_{q_n}]^{\rho_n \sigma_n}$ over the $SU(2)$ random matrices V_{k_j}, V_{q_j} . This can be easily done by first grouping the matrices V_k with the same momentum label and then using (82) and (83), exploiting the fact that the Haar averages over random matrices with different momentum labels factorize. Notice that for the pairings involving $V_{k_j}^\dagger$ one has to take a matrix transposition in (82) and (83). It is natural to expect that the results above generalize to arbitrary v , although the identities (82) and (83) become more cumbersome.

Let us now discuss the consequences of Haar invariance on steady-state expectation values of arbitrary functions of the real-space correlators. Precisely, let us consider a generic function $\mathcal{F}(z)$ of G_{ij} . This includes functions of the correlator G_A restricted to a subregion A , such as the entanglement entropy. It is clear that the steady-state value of $\text{Tr} \mathcal{F}(G)$ is obtained as

$$\langle \text{Tr} \mathcal{F}(G) \rangle = \int_{\mathcal{E}} d\eta(V) \text{Tr} \mathcal{F}(V^\dagger G^{(0)} V), \quad (84)$$

where $G^{(0)}$ is the initial correlator, and $\mathcal{E}$ is the ensemble of $L \times L$ random matrices obtained as the inverse Fourier transform of random block diagonal $SU(\nu)$ matrices V_k as

$$V_k = \mathbb{1}_{L/\nu} \otimes M_k, \quad (85)$$

where M_k is a random $SU(\nu)$ matrix. For $\nu = 1$, V_k is a diagonal matrix with a random momentum-dependent phase on the diagonal. A similar result holds for the *QSSEP* [27], and it was used in Ref. [37] to determine the steady-state statistics of the entanglement.

IV. FULL-TIME ENTANGLEMENT DYNAMICS FOR $\nu = 1$

In the following, we show that for $\nu = 1$ it is possible to obtain the full-time dynamics or arbitrary entanglement-related quantities in the space-time scaling limit $t, \ell \rightarrow \infty$ with t/ℓ^2 fixed. Although here we restrict ourselves to $\nu = 1$, it should be possible to generalize our results to $\nu > 1$, at least for moderate ν . The starting point are the results of Sec. II B. Let us start by computing the moments M_n of the reduced correlation matrix defined as

$$M_n = \text{Tr}((G_A^n)). \quad (86)$$

To illustrate the strategy of the derivation, let us consider the case with $n = 2$. One has to determine the large ℓ, t behavior of the expression

$$M_2 = \frac{1}{L^2} \sum_{n,m} \sum_{\substack{k_1, q_1 \\ k_2, q_2}} e^{-i(k_1 - q_2)n + i(q_1 - k_2)m} \langle G_{k_1 q_1} G_{k_2 q_2} \rangle, \quad (87)$$

where $\langle G_{k_1 q_1} G_{k_2 q_2} \rangle$ evolves with time, except for $k_1 = q_1$ and $k_2 = q_2$, or $k_1 = q_2, k_2 = q_1$, which correspond to conserved quantities. In the following, we will consider the thermodynamic limit $L \rightarrow \infty$, replacing $1/L \sum_k \rightarrow \int_{-\pi}^{\pi} dk/(2\pi)$.

To perform the sum over $n, m \in [1, \ell]$ we use the trivial identity as [47]

$$\sum_{m=1}^{\ell} e^{imk} = \frac{\ell}{4} \int_{-1}^1 d\xi w(k) e^{i(\ell\xi + \ell+1)k/2}, \quad (88)$$

with

$$w(k) = \frac{k}{\sin(k/2)}. \quad (89)$$

Here, we focus on the quench dynamics from initial states that exhibit two-site translation invariance, such as the Néel and the dimer state [see (8) and (9)]. This means that G_{kq} is almost diagonal. Precisely, here we focus on G_{kq} of the form

$$G_{kq} = \frac{1}{2} f(k) \delta_{kq} + \frac{1}{2} g(k) \delta_{k, q+\pi}, \quad (90)$$

with $g(k - \pi) = g^*(k)$, to ensure Hermiticity. The case of the Néel state corresponds to $f(k) = g(k) = 1$, whereas the dimer state corresponds to $f(k) = 1 + \cos(k)$ and $g(k) = -i \sin(k)$. The evolution of the correlator $\langle G_{k_1 q_1} G_{k_2 q_2} \rangle$ is given as [cf. (31)]

$$\frac{d}{dt} \langle G_{k_1 q_1} G_{k_2 q_2} \rangle = \varepsilon_{k_1 q_1 k_2 q_2} \langle G_{k_1 q_1} G_{k_2 q_2} \rangle. \quad (91)$$

The “energies” $\varepsilon_{k_1 q_1 k_2 q_2}$ can take only few values because G_{kq} is almost diagonal. Precisely, one has

$$\varepsilon_{k_1 q_1 k_2 q_2} = -8\gamma(1 + \cos(k_1 - k_2)), \quad (92)$$

for $q_1 = k_1 \pm \pi$ and $q_2 = k_2 \pm \pi$, and

$$\varepsilon_{k_1 q_1 k_2 q_2} = -4\gamma \quad (93)$$

for $q_1 = k_1$ or $q_2 = k_2$. The solution of (92) is straightforward and it gives an exponential decay, which can be used in (87). We report the detailed calculation for $n = 2$ in Appendix. Let us discuss the general strategy. For generic n , M_n is determined as

$$M_n = \left(\frac{\ell}{4}\right)^n \prod_{j=1}^n \int \frac{dk_j}{2\pi} \frac{dq_j}{2\pi} \prod_{j'=1}^n \int d\xi_{j'} e^{i\ell \sum_{p=1}^n \xi_p (q_{p-1} - k_p)/2} \times \prod_{p=1}^n w(q_{p-1} - k_p) G_{k_p, q_p} e^{t\varepsilon(\{k_j, q_j\})}, \quad (94)$$

where $w(x)$ is defined in (88), and the energies are given by (92) and (93) for $n = 2$. For $n > 2$, $\varepsilon_{k_j q_j}$ can be determined by considering $d\langle G_{k_1 q_1} G_{k_2 q_2} \cdots G_{k_n q_n} \rangle$ and applying (30) to each of the $G_{k_j q_j}$ and (31) to all the pairs $G_{k_r q_r} G_{k_s q_s}$. In writing the phase in (94), we used the fact that for generic one-site and two-site translation invariant initial states one has $G_{k_p, q_p} \propto \delta_{k_p - q_p, 0 \bmod \pi}$, which allows us to neglect the term $\ell + 2$ in the complex exponentials (88) because it would give a trivial phase. Let us now consider the large ℓ limit. One can check that the integration over ξ_j gives a nonzero contribution only if there is an even number of momenta q_j such that $q_j = k_j \pm \pi$ and the total π shift of the q_j vanishes. From now on we define two momenta $k_j, k_{j'}$ paired if $q_j = k_j \pm \pi$ and $q_{j'} = k_{j'} \mp \pi$. The reason why one can restrict to the terms with an even number of paired momenta is that the integration over ξ_j gives in the large ℓ limit a factor $\delta(q_{j-1} - k_j)$. One can check that the “chain” of delta functions arising from the integrals over all the ξ_j gives a nonzero result only in the presence of an even number of paired momenta. It is important to stress that terms that do not have an even number of paired momenta give a subleading contribution in the limit $\ell, t \rightarrow \infty$, although they are nonzero, except for the quench from the Néel state, for which they vanish. We anticipate (see Fig. 3) that this is the reason why for the Néel state finite-size corrections are smaller.

Let us now discuss the structure of the energies ε . It is easy to check that they depend in a simple way on the paired momenta. For the case with $2p$ paired momenta one has

$$\varepsilon = -8\gamma \left(p + \sum_{\text{paired } j, j'} \cos(k_j - k_{j'}) \right), \quad (95)$$

where the sum is over all the paired $k_j, k_{j'}$ momenta. In the following, we consider the case in which at least two momenta are paired, i.e., we neglect the momenta configurations with $k_j = q_j$ for any j , which do not evolve with time. Now, the fact that momenta have to be paired implies that the phase factor $e^{i\ell \sum_p \xi_p (q_{p-1} - k_p)}$ contains π shifts. It is convenient to remove them by changing variables as follows. For the case with k_j and $k_{j'}$ paired one can remove the π shifts by redefining $k'_l = k_l \mp \pi$ for $l \in [j + 1, j']$. The same change of variables can be iterated to remove the π shifts associated with all the

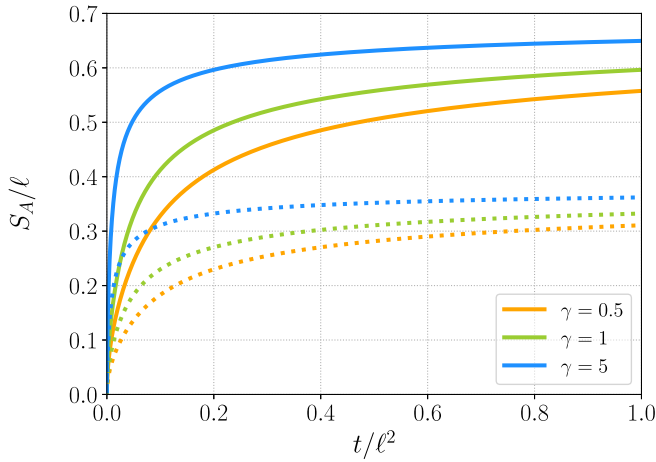


FIG. 2. Entanglement dynamics in the ν -QSSEP with $\nu = 1$ after the quench from the fermionic Néel state (continuous line) and the dimer state (dotted line). We show results for several values of the noise rate γ , plotting S_A/ℓ vs t/ℓ^2 , with ℓ the size of the subsystem.

paired momenta. This change of variables will lead to some minus signs in the energies ε and in the G_{kq} .

It is also convenient to change variables in (94) defining ζ_j as

$$\zeta_0 = \xi_1, \quad \zeta_j = \xi_{i+1} - \xi_i, \quad i \in [1, n-1]. \quad (96)$$

Now, we can rewrite (94) as

$$\begin{aligned} M_n &\simeq \left(\frac{\ell}{4}\right)^n \prod_{j=1}^n \int \frac{dk_j}{2\pi} \prod_{j'=1}^{n-1} \int d\zeta_{j'} \mu(\{\zeta_j\}) \\ &\times e^{i\ell \sum_{p=1}^{n-1} \zeta_p (k_p - k_n)/2} \prod_{p=1}^n w(k_{p-1} - k_p) \\ &\times \sum_{\text{pairings}} [G'_{k_1, q_1} G'_{k_2, q_2} \cdots G'_{k_n, q_n} e^{t\varepsilon'(\{k_j, q_j\})}], \end{aligned} \quad (97)$$

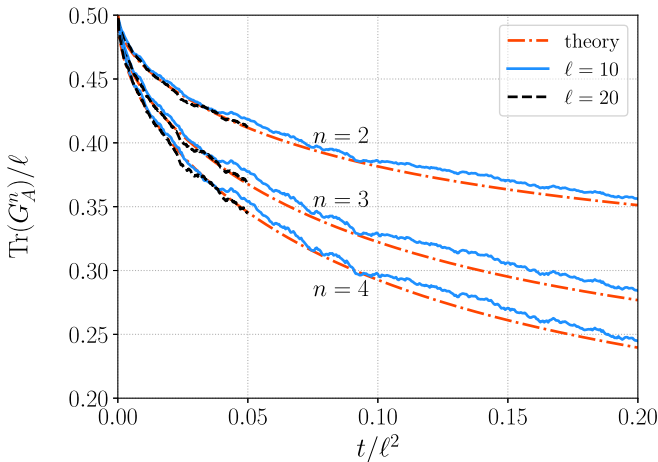


FIG. 3. Moments $M_n = \text{Tr}(G_A^n)$ after the quench from the Néel state in the ν -QSSEP with $\nu = 1$ and $\gamma = 0.5$. We plot M_n/ℓ vs t/ℓ^2 , for $\ell = 20, 40, 80$. The continuous lines are exact numerical data averaged over ~ 500 noise realizations. The dashed-dotted lines are the theoretical predictions.

where we used the fact that the integrand does not depend on ζ_0 , and in the second row we sum over all the possible ways of pairing the momenta k_j . In (97), the $\simeq$ is to stress that we are ignoring the terms that do not contain paired momenta, and hence do not evolve with time. We will restore these contributions at the end of the calculation. The prime in $G'_{k_j q_j}$ and in ε' is to take into account the effect of redefining k_j to remove the π shifts. From (90), one has that each G_{kq} contributes with a factor $f(k)$ for each momentum that is not paired and a factor $g(k)$ for each paired one. Moreover, at least for the Néel state and the dimer state the change of variable to remove the π shifts does not have any effect on $f(k)$, whereas it leads to a change of sign in $g(k)$. In (97), $\mu(\{\zeta_j\})$ is the result of the integration over ζ_0 , and is defined as

$$\mu(\{\zeta_j\}) = \max \left[0, \min_{j \in [0, n]} \left(1 - \sum_{l=1}^j \zeta_l \right) + \min_{j \in [0, n]} \left(1 + \sum_{l=1}^j \zeta_l \right) \right], \quad (98)$$

Now, in the large- ℓ limit, one can use the stationary phase approximation [48]. The stationarity condition with respect to ζ_j gives that $k_j \rightarrow k_n$ for any j . In this limit, we can replace $w(k_{p-1} - k_p) \rightarrow 2$. Moreover, we can also replace $k_j \rightarrow k_n$ in the product $G'_{k_1, q_1} G'_{k_2, q_2} \cdots G'_{k_n, q_n}$. However, we have to keep distinct the momenta in the exponents in (97), although we can expand $\cos(k_j - k_l) \approx 1 - (k_j - k_l)^2/2$.

Let us discuss how to proceed focusing on the contributions with two paired momenta. For instance, let us consider the case with k_j and $k_{j'}$ ($j < j'$) paired as

$$q_j = k_j \pm \pi, \quad q_{j'} = k_{j'} \mp \pi, \quad q_r = k_r \quad r \neq j, j'. \quad (99)$$

Now, Eq. (97) becomes

$$\begin{aligned} M_n^{(2p)} &= \left(\frac{\ell}{2}\right)^n \prod_{j=1}^n \left(\int \frac{dk_j}{2\pi} \int d\zeta_j \right) \mu(\{\zeta_j\}) \\ &\times e^{i\ell \sum_{p=1}^{n-1} \zeta_p (k_p - k_n)/2} \frac{|g(k_n)|^2}{2^n} \sum_{r < r'} f^{n-2-r'+r}(k_n) \\ &\times f^{r'-r}(k_n + \pi) e^{-4\gamma t(k_r - k_{r'})^2}, \end{aligned} \quad (100)$$

where the functions $f(x), g(x)$ are defined in (90), and the superscript $2p$ is to stress that we restrict to the situation with only two paired momenta. Here one obtains the term $|g(k_n)|^2$ because of the condition $q_j = k_j \pm \pi$ and $q_{j'} = k_{j'} \mp \pi$ and after the change of variable on k_j to remove the π shifts. Now, each integration over a momentum k_j that is not paired gives $2/\ell \delta(\zeta_j)$, and we obtain

$$\begin{aligned} M_n^{(2p)} &= \frac{1}{2^n} \frac{\ell^3}{8} \sum_{j < j'} \int \frac{dk_j}{2\pi} \int \frac{dk_{j'}}{2\pi} \left(\prod_{s=1}^{n-1} \int d\zeta_s \right) \mu(\{\zeta_s\}) \\ &\times \prod_{r \neq j, j'} \delta(\zeta_r) f^{n-2-j'+j}(k_n) f^{j'-j}(k_n + \pi) |g(k_n)|^2 \\ &\times e^{-i\ell \sum_{p=1}^{n-1} \zeta_p k_n/2 + i\ell \zeta_j k_j/2 + i\ell \zeta_{j'} k_{j'}/2} e^{-4\gamma t(k_j - k_{j'})^2}, \end{aligned} \quad (101)$$

In the last row in (101), we expanded the exponent since $k_j \rightarrow k_n$ for any j in the limit $\ell, t \rightarrow \infty$. The integration over

the Dirac deltas is trivial. Furthermore, one can integrate over k_j and $k_{j'}$, to obtain

$$M_n^{(2p)} = 2^{-n} \frac{\ell^2}{4} \sum_{j < j'} \int \frac{k_n}{2\pi} \int d\zeta_j \int d\zeta_{j'} \delta(\zeta_j + \zeta_{j'}) \times f^{n-2-j'+j}(k_n) f^{j'-j}(k_n + \pi) |g(k_n)|^2 \mu(\zeta_j, \zeta_{j'}) \times e^{-i\ell(\zeta_j + \zeta_{j'})k_n/2} \frac{e^{-\ell^2 \zeta_j^2 / (64\gamma t)}}{4\sqrt{\pi\gamma t}}. \quad (102)$$

After integrating over $\zeta_{j'}$, we obtain

$$M_n^{(2p)} = \frac{1}{2^n} \frac{\ell^2}{4} \int \frac{dk_n}{2\pi} \sum_{j < j'} f^{n-2-j'+j}(k_n) f^{j'-j}(k_n + \pi) \times |g(k_n)|^2 \int d\zeta_j \mu(\zeta_j, -\zeta_j) \frac{e^{-\ell^2 \zeta_j^2 / (64\gamma t)}}{4\sqrt{\pi\gamma t}}, \quad (103)$$

Finally, since $\mu(\zeta_j, \zeta_{j'}) = (2 + \zeta_j)\Theta(-\zeta_j) + (2 - \zeta_j)\Theta(\zeta_j)$ and $-2 \leq \zeta_j \leq 2$, the integration over ζ_j in (103) gives

$$M_n^{(2p)} = \frac{1}{2^n} \sum_{j < j'} \int \frac{dk}{2\pi} f^{n-2-j'+j}(k) f^{j'-j}(k + \pi) |g(k)|^2 \times \left[4\sqrt{\frac{\gamma t}{\pi}} (e^{-\ell^2 / (16\gamma t)} - 1) + \ell \operatorname{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right], \quad (104)$$

where we replaced $k_n \rightarrow k$, and $\operatorname{Erf}(x)$ is the error function. Notice that for $n = 2$, Eq. (104) is the full result because there are only terms with two pairings, i.e., $j = 1, j' = 2$. The term in the last row in (104), which encodes the dynamics does not depend on k . This means that the quasiparticles exhibit the same diffusive dynamics irrespective of k . In contrast, in the standard quasiparticle picture entanglement spreads ballistically, and quasiparticles with different k have different group velocities [41].

Let us now discuss the case in which there are more than two paired momenta in (97). First, let us focus on the contribution of the product $\prod_{j=1}^n G'_{k_j q_j}$. As we stressed for the case of two paired momenta, one can replace in the limit $\ell, t \rightarrow \infty$ $k_j \rightarrow k_n$. The contribution of the product is a polynomial in powers of $f(k_n), f(k_n + \pi), g(k_n), g(k_n + \pi)$. Actually, one can check that the contribution of the term with $2r$ pairings is the prefactor of x^{2r} in the expansion around $x = 0$ of $\operatorname{Tr} Q^{2r}(x)$, with the matrix $Q(x)$ defined as

$$Q(x) = \frac{1}{2} \begin{pmatrix} f(k + \pi) & xg(k + \pi) \\ xg(k) & f(k) \end{pmatrix}. \quad (105)$$

It is convenient to diagonalize Q , obtaining the eigenvalues as

$$\lambda_{\pm} = \frac{1}{4} [f(k) + f(k + \pi) \pm (f^2(k) + f^2(k + \pi) - 2f(k)f(k + \pi) + 4x^2 g(k)g(k + \pi))^{\frac{1}{2}}]. \quad (106)$$

To perform the integration over the momenta k_j in (97) we use the trivial formula

$$\int \frac{dk_j}{2\pi} e^{i\ell \sum_{p=1}^{n-1} \zeta_p / 2 (k_p - k_n) - 4\gamma t (\sum_p \sigma_p k_p)^2} = \frac{1}{4\sqrt{\pi\gamma t}} e^{-\ell^2 \zeta_j^2 / (4\gamma t) - i\ell k_n (\sigma_j \zeta_j + \sum_p \zeta_p) + i\ell / 2 \sum_{p \neq j} k_p (\zeta_p + \sigma_p \zeta_j)}. \quad (107)$$

In the exponential in the left-hand side in (107), $\sigma_p = \pm 1$ takes into account the signs of the momenta k_j obtained when expanding ε' [cf. (97)] in the limit $k_j \rightarrow k_n$. As it is clear from Eq. (107), a single integration over any of the paired momenta is sufficient to decouple them, leaving in the exponent only linear terms in k_j . One can easily integrate over all the remaining $k_p \neq k_n$ in (107), obtaining

$$\left(\frac{2}{\ell}\right)^{n-2} \prod_{p \neq j} \delta(\zeta_p - \sigma_p \zeta_j). \quad (108)$$

The remaining integration over ζ_j can be easily performed. One obtains that all the terms with any nonzero number of paired momenta give rise to the same dynamics, i.e., the second row in (104). Notice that although this might look surprising, it also happens for the quench in the deterministic chain, where all the terms with paired momenta give $\min(|v_k|t, \ell)$, which describes the ballistic propagation of entangled quasiparticles [21]. Unlike the case of the deterministic chain, the propagation of quasiparticles is now diffusive. In conclusion, based on the derivation for $n \leq 4$, we conjecture the general formula for M_n as

$$M_n = \frac{\ell}{2} \int \frac{dk}{2\pi} [\lambda_-^n + \lambda_+^n - 2^{-n} f^n(k) - 2^{-n} f^n(k + \pi)] \times \left[4\sqrt{\frac{\gamma t}{\pi}} \frac{\ell}{2} (e^{-\ell^2 / (16\gamma t)} - 1) + \operatorname{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right] + \frac{\ell}{2} \int \frac{dk}{2\pi} [2^{-n} f^n(k) + 2^{-n} f^n(k + \pi)]. \quad (109)$$

The first term in (109) is $\operatorname{Tr} Q^n(1) = \lambda_+^n + \lambda_-^n$, where we subtracted the terms with no paired momenta because they do not evolve, and we used that the terms with paired momenta exhibit the same dynamics.

The last term in (109) is the steady-state value of M_n . This corresponds to the momenta configurations for which ε vanishes. For instance, for $n = 2$ one has $G_{kk} G_{qq}$ and $G_{kq} G_{qk}$, where k, q can be equal. To extract the steady-state value of M_n one has to compute a multidimensional integral similar to (94), although without time-dependent terms. The large- ℓ analysis is performed in a similar way. One obtains that at the leading order in the limit $\ell \rightarrow \infty$ only the terms with $k_j = q_j$ contribute. For instance, this means that for $n = 2$ the terms with $G_{kq} G_{qk}$ do not contribute. This also means that the steady-state value of M_n , and hence the entanglement entropy, is obtained from G_{kk} , i.e., the density of fermionic modes, which defines the GGE describing the quench in the deterministic chain. Crucially, this holds for the infinite chain, i.e., if one takes the thermodynamic limit $L \rightarrow \infty$ first, and then the limit $\ell \rightarrow \infty$. For instance, if one takes the limit $L \rightarrow \infty$ with ℓ/L finite, the terms originating from $G_{kq} G_{qk}$ contribute to the steady-state correlators. For the Néel quench, we have $f(k) = f(k + \pi) = 1$ and $g(k) = g(k + \pi) = 1$,

implying that $\lambda_+ = 1$ and $\lambda_- = 0$. For the quench from the dimer state, we have $f(k) = 1 + \cos(k)$ and $g(k) = -i \sin(k)$. Again, one has $\lambda_+ = 1$ and $\lambda_- = 0$.

Equation (109) allows one to determine the dynamics of $\text{Tr}\mathcal{F}(G_A)$ for a generic function $\mathcal{F}(x)$ that admits a Taylor expansion around $x = 0$. One obtains

$$\begin{aligned} \text{Tr}\mathcal{F}(G_A) &= \frac{\ell}{2} \left[4\sqrt{\frac{\gamma t}{\pi \ell^2}} (e^{-\ell^2/(16\gamma t)} - 1) + \text{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right] \\ &\times \int \frac{dk}{2\pi} [\mathcal{F}(\lambda_+) + \mathcal{F}(\lambda_-) - \mathcal{F}(f(k)/2) \\ &- \mathcal{F}(f(k + \pi)/2)] + \frac{\ell}{2} \int \frac{dk}{2\pi} \\ &\times [\mathcal{F}(f(k)/2) + \mathcal{F}(f(k + \pi)/2)]. \end{aligned} \quad (110)$$

It is convenient to rewrite Eq. (110) as

$$\begin{aligned} \text{Tr}\mathcal{F}(G_A) &= \frac{1}{2} \int_{-\infty}^{\infty} d\xi P(\xi) \int \frac{dk}{2\pi} \\ &\times \{ \max[\ell - 2|\xi|, 0] [\mathcal{F}(\lambda_+) + \mathcal{F}(\lambda_-)] \\ &+ \min(2|\xi|, \ell) [\mathcal{F}(f(k)/2) + \mathcal{F}(f(k + \pi)/2)] \}. \end{aligned} \quad (111)$$

Here, we defined $P(\xi, t)$ as

$$P(\xi, t) = \frac{1}{2\sqrt{\pi\gamma t}} e^{-\xi^2/(4\gamma t)}. \quad (112)$$

Equation (111) admits a simple hydrodynamic interpretation in terms of quasiparticles. As in the standard quasiparticle picture for entanglement spreading, the starting point is that entangled pairs of quasiparticles are produced after the quench. Unlike in the deterministic chain, quasiparticles spread diffusively, and their trajectories do not depend on k . In (111), ξ is interpreted as the distance of the quasiparticles from the point at which they were initially emitted. Equation (111) suggests that the two members of the entangled pairs perform symmetric (“mirrored”) random walks with respect to the emission point (see Fig. 1). The fact that at any given time the hopping amplitudes do not depend on the position suggests that all the pairs undergo the same dynamics, for each realization of the noise. Thus, the term $\max(\ell - 2|\xi|, 0)$ in (111) counts the number of entangled quasiparticles that at time t traveled a distance $|\xi|$ and are within A , whereas $\min(2|\xi|, \ell)$ is the number of shared entangled pairs, i.e., such that at time t only one member of the pair is in A . Although the kinematic structure of (111) is the same as in the deterministic chain [20,21,47], the dynamics is stochastic, and ξ is a random variable distributed with $P(x, t)$ [cf. (112)]. Finally, the contributions $\mathcal{F}(\lambda_+) + \mathcal{F}(\lambda_-)$ and $\mathcal{F}(f(k)/2) + \mathcal{F}(f(k + \pi)/2)$ of the pairs to $\text{Tr}\mathcal{F}(G_A)$ are the same as in the deterministic chain. One can check that by replacing $P(\xi) \rightarrow \delta(\xi - v_k t)$, with v_k the group velocity of the quasiparticles, one recovers the result for the quench in the deterministic chain.

From (111) one obtains the dynamics of the Rényi entropies and the von Neumann entropy by choosing $\mathcal{F}(x)$ as

$$\mathcal{F}_n(x) = \frac{1}{1-n} \ln [x^n + (1-x)^n], \quad (113)$$

$$\mathcal{F}_1(x) = -x \ln(x) - (1-x) \ln(1-x), \quad (114)$$

respectively. For (113) and (114), $\mathcal{F}(1) = \mathcal{F}(0) = 0$, which reflects that the full-system entropies are zero because the state is pure for each realization of the hopping amplitudes. Thus, for the entropies Eq. (109) can be rewritten as

$$S_n = \int_{-\infty}^{\infty} d\xi \min(2|\xi|, \ell) s_n P(\xi), \quad (115)$$

where

$$s_n = \int \frac{dk}{2\pi} \mathcal{F}_n\left(\frac{f(k)}{2}\right). \quad (116)$$

Equation (115) provides a stochastic generalization of the quasiparticle picture for entanglement spreading.

In Fig. 2, we plot Eq. (115) for the quench from the Néel state and the dimer state [cfs. (8) and (9)]. The results are reported with the continuous lines and the dotted lines, respectively. For the Néel quench, the Rényi entropy shared by the members of the entangled pairs is $s_n = \ln(2)$ for any n , whereas for the quench from the dimer state one has [21]

$$s_n = \frac{1}{1-n} \ln \left[\left(\frac{1 + \cos(k)}{2} \right)^n + \left(\frac{1 - \cos(k)}{2} \right)^n \right]. \quad (117)$$

The von Neumann entropy s_1 is obtained by taking the limit $n \rightarrow 1$. In Fig. 2, we plot the von Neumann entropy S_A/ℓ versus t/ℓ^2 , with ℓ the size of interval A (see Fig. 1). The growth of S_A is diffusive for both quenches and for any γ . The saturation value for the quench from the dimer state is $S_A/\ell \rightarrow \ln(2)/2$, whereas for the Néel quench one has $\ln(2)$. Although the asymptotic value of the von Neumann entropy at long times does not depend on γ , Fig. 2 shows that it is approached faster for larger γ .

V. NUMERICAL RESULTS

Here, we discuss numerical results for the dynamics in the ν -QSSEP, focusing on the case with $\nu = 1$. In Sec. V A, we detail the numerical strategy to simulate the full-time dynamics. We discuss the dynamics of the moments $M_n = \text{Tr}(G_A^n)$ in Sec. V B. Then, we provide numerical evidence supporting the validity of the stochastic quasiparticle picture (115) for the von Neumann entropy after the quench from the Néel state and the dimer state in Sec. V C.

A. Numerical method

Here, we employ the strategy of Ref. [38]. We generate the hopping amplitudes dW_t as $dW_t^r + idW_t^i$ with dW_t^r and dW_t^i real random numbers distributed with a Gaussian distribution with zero mean and variance $\gamma dt/2$. The generic initial state is represented as

$$|\psi_0\rangle = \prod_{k=1}^N \sum_{j=1}^L U_{jk}^{(0)} c_j^\dagger |0\rangle, \quad (118)$$

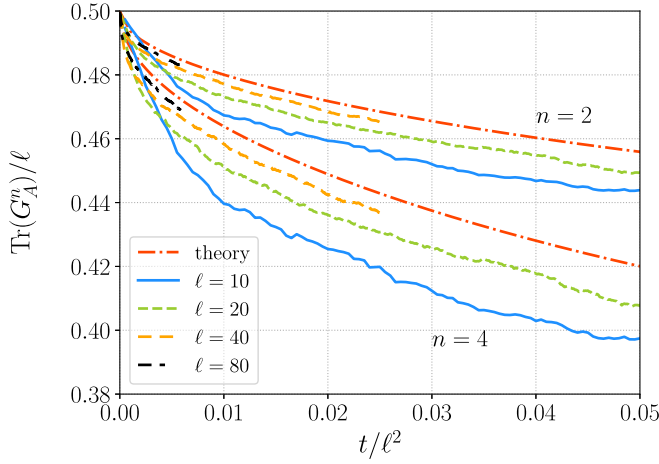


FIG. 4. Same as in Fig. 3 for the quench from the dimer state. Finite-size effects are stronger than in the Néel quench.

with $U_{jk}^{(0)}$ a $L \times N$ matrix, and N the number of fermions in the chain. We require that $[U^{(0)}]^\dagger U = \mathbb{1}_N$, with $\mathbb{1}_N$ the $N \times N$ identity matrix. One can repeatedly apply the infinitesimal evolution operator e^{-idH_t} , with dH_t given in (2). The state (118) keeps the same form as in (118) during the dynamics. The evolved U^t is obtained by first calculating $W^{t+dt} = e^{-idh_t} U^t$, with dh_t given in (3). To conclude the iteration step we perform a QR decomposition as $W^{t+dt} = QR$, and set $U^{t+dt} = Q$. Notice that the last step is not needed here because the dynamics, even for finite dt , preserves the condition $U^\dagger U = \mathbb{1}_N$. The correlation matrix C_{ij} is obtained as [38]

$$C_{ij} = \langle \psi_t | c_i^\dagger c_j | \psi_t \rangle = U^t [U^t]^\dagger. \quad (119)$$

In our simulations, we employ $dt = 0.05$. We checked that the numerical results do not change significantly if we consider smaller dt . Finally, from the correlation function (119) we obtain entanglement-related quantities via the Peschel trick [49].

B. Moments M_n

In Fig. 3, we plot M_n with $n = 2, 3, 4$ for the quench from the Néel state. The continuous and the dotted lines are exact numerical data for $\ell = 10, 20$, respectively. The data are for a chain with $L = 200$ and M_n are averaged over 500 realizations of the hopping amplitudes dW_t^j . We checked that boundary effects caused by the finite L are negligible, at least for the times $t/\ell^2 \approx 0.2$ that we consider. All the data are for fixed $\gamma = 1/2$. In the figure we plot M_n/ℓ versus t/ℓ^2 . At $t = 0$ one has $M_n = 1/2$ for any n . At long times M_n decay to zero. The dash-dotted line is the prediction (109), which holds in the limit $\ell, t \rightarrow \infty$ with t/ℓ^2 fixed. Even for $\ell = 10, 20$, finite-size effects are mild, and the agreement between the numerical data and the asymptotic result is perfect.

Let us now discuss the quench from the dimer state (9). We show numerical data for M_2 and M_4 in Fig. 4. The data are for fixed $\gamma = 1/2$ and for a chain with $L = 400$. In contrast with the quench from the Néel state (see Fig. 3) we observe sizable finite-size corrections. The qualitative behavior is similar as for the Néel quench. At $t = 0$ one has $M_n = 1/2$, and M_n

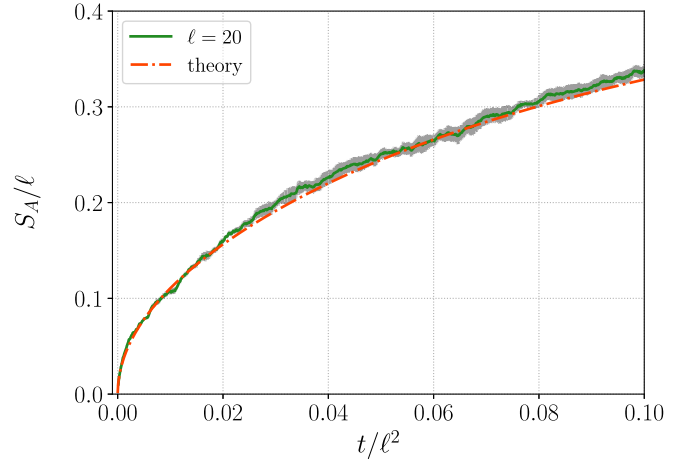


FIG. 5. Dynamics of the von Neumann entropy of a subsystem of length ℓ plotted as a function of t/ℓ^2 in the ν -QSSEP with $\nu = 1$ and noise strength $\gamma = 0.5$. The dynamics is from the Néel state. The continuous and the dotted lines are numerical results averaged over 1000 realizations of the dynamics. The dashed-dotted line is the analytic result in the limit $\ell, t \rightarrow \infty$, with the ratio t/ℓ^2 .

decrease upon increasing t . To understand the corrections we plot data for several subsystem sizes up to $\ell = 80$. Upon increasing ℓ the data approach the asymptotic result (dashed-dotted line in the figure). Already the result for $\ell = 80$ is quite close to the asymptotic result. A fit of the data at fixed t/ℓ^2 suggests that corrections are compatible with a $1/\ell$ decay, which is the same behavior as in the deterministic chain.

C. von Neumann entropy

Let us now discuss the von Neumann entropy. In Fig. 5, we plot S_A/ℓ versus t/ℓ^2 for the quench from the Néel state. As in Fig. 3, the data are for fixed $\gamma = 1/2$. The continuous line is the exact numerical result for S_A for $\ell = 20$. The gray shading is the standard deviation of the mean computed by averaging the data over the noise realizations. The dashed-dotted line is (115) with $s_n = \ln(2)$. As we observed for M_n (see Fig. 3), the agreement with the theory is impressive, despite the “small” $\ell = 20$.

Figure 6 shows results for S_A for the quench from the dimer state (9). Deviations between the exact numerical data and the theoretical prediction in the large ℓ, t limit are larger compared with the Néel quench. The finite-size corrections are investigated in the inset by plotting S_A/ℓ versus $1/\ell$ for fixed $t/\ell^2 = 0.00625$. The star symbol at $\ell \rightarrow \infty$ is the theoretical prediction. The dotted line is a fit to $a + b/\ell$ with a fixed and b a fitting parameter. Finally, it is interesting to investigate the dependence on ν of the entanglement dynamics. In Fig. 7, we consider the quench from the Néel state in the ν -QSSEP for fixed $\gamma = 1/2$ and $\nu = 5$ and $\nu \rightarrow \infty$. We show results for $\ell = 10$ (continuous lines) and $\ell = 20$ (dashed lines). The dashed-dotted line is the theoretical result (115) for $\nu = 1$. Interestingly, the data for $\nu = 5$ and $\nu \rightarrow \infty$ for both $\ell = 10$ and $\ell = 20$ exhibit a satisfactory collapse, which means that $\nu = 5$ is a good approximation for the dynamics in the QSSEP, i.e., for $\nu \rightarrow \infty$, at least for the quench from the

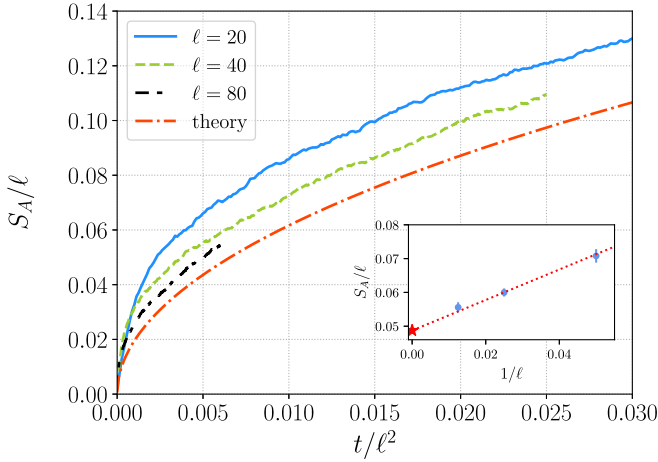


FIG. 6. The same as in Fig. 5 for the quench from the dimer state. Finite-size corrections are larger compared to the dynamics from the Néel state (see Fig. 5). (Inset) data for S_A/ℓ at fixed $t/\ell^2 = 0.625$ plotted as a function of $1/\ell$. The star symbol is the analytic result in the limit $\ell \rightarrow \infty$. The dotted line is a fit to $a + b/\ell$, with a fixed and b a fitting parameter.

Néel state. On the other hand, Eq. (115) captures correctly the qualitative behavior, although it is not quantitatively accurate.

VI. CONCLUSIONS

We investigated the out-of-equilibrium dynamics in a minimal modification of the so-called quantum symmetric simple exclusion process (*QSSEP*), which is a tight-binding fermionic chain with space-time random complex hopping amplitudes. Here, we considered the ν -*QSSEP* in which the hopping amplitudes retain their randomness in time but are defined in a unit cell of ν sites. We derived the dynamics of the generic moment of the fermionic correlation function in momentum space. We showed that the statistics of the

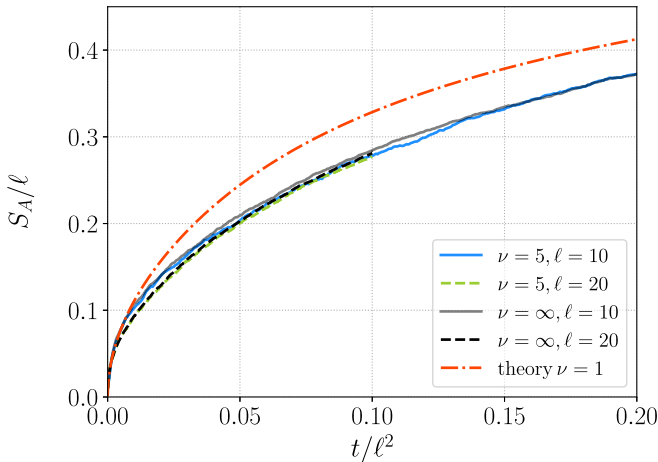


FIG. 7. Entanglement dynamics after the quench from the Néel state in the ν -*QSSEP* model at noise rate $\gamma = 0.5$. The continuous and the dashed lines are exact numerical data for $\nu = 5$ and $\ell = 10$ and $\ell = 20$, respectively. The dashed-dotted line is the analytical result for $\nu = 1$ in the large ℓ , t limit.

moments in the steady state is invariant under the Haar average with structured random matrices. The latter are obtained in terms of momentum-dependent $SU(\nu)$ random matrices. As a consequence, the moments of the fermionic correlators satisfy an intriguing set of relationships that we characterized in detail for $\nu = 2$. For $\nu = 1$, we characterized analytically the full-time dynamics of entanglement-related quantities, in the space-time scaling limit of long times and large subsystem sizes. In contrast with quantum quenches in the deterministic tight-binding chain, the entanglement spreading is diffusive for any ν . For $\nu = 1$, the entanglement dynamics is quantitatively described by a stochastic generalization of the quasiparticle picture. In this modified quasiparticle picture pairs of entangled quasiparticles are produced at each point in space. The two members of the pair undergo two symmetric random walks in opposite directions. All the entangled pairs produced in the system undergo the same dynamics. The distance from the emission point of each pair is random, and it is distributed with the heat kernel with diffusion constant γ , i.e., the strength of the hopping rate.

Let us now discuss some possible directions for future work. First, here we analyzed in detail only the dynamics for $\nu = 1$ and the steady state for $\nu = 2$. Clearly, it would be interesting to investigate the full-time dynamics of the entanglement entropy for $\nu > 1$. For moderate values of ν , for instance for $\nu = 2$, this should be feasible, and it would allow to understand how to generalize the stochastic quasiparticle picture to generic ν . The key ingredients are the eigenvalues $\varepsilon_{\{k_j, q_j\}}$ governing the dynamics of the moments of the correlation functions. Although already for the second moment determining the exact eigenvalues requires diagonalizing (37), which is 6×6 , this should be possible in the limit in which all the momenta become equal, which is expected to govern the long-time dynamics. Furthermore, here we considered dynamics from low-entanglement initial states, with almost diagonal correlation functions in momentum space. It would be interesting to explore dynamics from initial states with more complex correlations. Moreover, it should be possible to generalize the stochastic quasiparticle picture to the dynamics of other entanglement-related quantities, such as the entanglement negativity [50], or even the entanglement Hamiltonian [51]. Finally, Eq. (2) can be generalized to the interacting case. Although the interacting model is unlikely to be solvable, the dynamics could be studied by employing the time-dependent density matrix renormalization group [52] (*tDRMG*). It would be interesting to investigate whether interactions modify the diffusive nature of entanglement spreading, and whether they give rise to entanglement phase transitions.

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DATA AVAILABILITY

Data that are not openly available are available upon reasonable request from the authors.

APPENDIX: DETAILED DERIVATION OF M_n FOR SMALL n

Here, we provide details on the derivation of the large ℓ, t behavior of the moments M_n of G_A [cf. (86)] for $n = 2, 3, 4$. Let us consider the case of the quench from the Néel [cf. (8)] state [see Eq. (109) for the result for M_n]. For M_2 , from (94), we obtain that

$$\begin{aligned} \langle \text{Tr}(G_A^2) \rangle &\simeq \frac{1}{4} \left(\frac{\ell}{4} \right)^2 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \int_{-1}^1 d\xi_1 \int_{-1}^1 d\xi_2 \\ &\times e^{-8\gamma t(1-\cos(k_1-k_2)) - i\ell(\xi_1-\xi_2)(k_1-k_2)/2} w^2(k_1-k_2). \end{aligned} \quad (\text{A1})$$

Here, we used that the only nonzero entries of $G_{k_1 q_1} G_{k_2 q_2}$ are for $q_j = k_j$ and $q_j = k_j \pm \pi$. Moreover, we used that in the large ℓ, t limit stationarity with respect ξ_j implies that k_1 and k_2 are ‘‘paired’’, meaning that both q_1 and q_2 are shifted as $q_1 = k_1 \pm \pi$ and $q_2 = k_2 \mp \pi$. Other configurations, for instance with $q_1 = k_1$ and $q_2 = q_2$ are subleading in the limit $\ell \rightarrow \infty$. Actually, for the quench from the Néel state, since G_{kq} does not depend on the quasimomenta, these terms vanish exactly. This reflects that finite- ℓ corrections are smaller for the quench from the Néel state. Finally, in (A1) we used that $\langle G_{k_1 k_1 + \pi} G_{k_2 k_2 - \pi} \rangle(t) = 1/4 e^{-8\gamma t(1+\cos(k_1-k_2))}$. The phase factor $e^{i\ell\xi_1(q_2-k_1)/2 + i\ell\xi_2(q_1-k_2)/2}$ would give $e^{i\ell\xi_1(k_2 \pm \pi - k_1)/2 + i\ell\xi_2(k_1 \mp \pi - k_2)/2}$ after using that k_1 and k_2 have to be paired. By changing variable $k_1 \rightarrow k_1 \pm \pi$ one obtains (A1). Notice that in the exponent in (A1) one has $-8\gamma t(1 - \cos(k_1 - k_2))$ instead of $-8\gamma t(1 + \cos(k_1 - k_2))$ [cf. (92)], because of the change of variables.

Moreover, the $\simeq$ in (A1) is to stress that we are considering only the terms that evolve with time, ignoring the non-evolving ones, which correspond to $k_1 = q_1, k_2 = q_2, q_1 = k_2, q_2 = k_1$, and $k_1 = k_2, q_1 = q_2$. By changing variables as $\zeta_0 = \xi_1$ and $\zeta_1 = \xi_1 - \xi_2$, we obtain

$$\begin{aligned} \langle \text{Tr}(G_A^2) \rangle &\simeq \frac{1}{4} \left(\frac{\ell}{4} \right)^2 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \int_{-2}^2 d\zeta_1 \\ &\times e^{-8\gamma t(1-\cos(k_1-k_2)) - i\ell\zeta_1(k_1-k_2)/2} w^2(k_1-k_2) \mu(\zeta_1), \end{aligned} \quad (\text{A2})$$

where we used that the integrand does not depend on ζ_0 . One should observe that a crucial difference with the dynamics under the deterministic tight-binding chain [21] is that the momenta k_1, k_2 are coupled in a nonlinear way by the $\cos(k_1 - k_2)$ term. This gives rise to diffusive entanglement spreading. The integration over ζ_0 gives $\mu(\zeta_1)$, which is defined as

$$\mu(\zeta_1) = \max(0, \min(1, 1 - \zeta_1) + \min(1, 1 + \zeta_1)). \quad (\text{A3})$$

Crucially, in the large ℓ, t limit one can imagine of treating the integral in (A2) via the stationary phase approximation [48]. The stationarity condition with respect to ζ_1 implies that only the region $k_1 \approx k_2$ contributes in the integral. This allows us to expand the $\cos(k_1 - k_2)$, and replace $w(k_1 - k_2) \rightarrow 2$. We obtain

$$\begin{aligned} \langle \text{Tr}(G_A^2) \rangle &\simeq \left(\frac{\ell}{4} \right)^2 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \int_{-2}^2 d\zeta_1 \\ &\times \mu(\zeta_1) e^{-4\gamma t(k_1-k_2)^2 - i\ell\zeta_1(k_1-k_2)/2}. \end{aligned} \quad (\text{A4})$$

We can perform the integral over k_1 , which gives

$$\begin{aligned} \langle \text{Tr}(G_A^2) \rangle &\simeq \left(\frac{\ell}{4} \right)^2 \int \frac{dk_2}{2\pi} \int_{-2}^2 d\zeta_1 \mu(\zeta_1) \frac{i}{8\sqrt{\pi\gamma t}} \\ &\times e^{-\ell^2 \zeta_1^2 / (64\gamma t)} \left[\text{Erfi} \left(\frac{\ell\zeta_1 + 16i\gamma t(k_2 - \pi)}{8\sqrt{\gamma t}} \right) \right. \\ &\left. - \text{Erfi} \left(\frac{\ell\zeta_1 + 16i\gamma t(k_2 + \pi)}{8\sqrt{\gamma t}} \right) \right], \end{aligned} \quad (\text{A5})$$

where $\text{Erfi}(x) = -i\text{Erf}(ix)$, with $\text{Erf}(x)$ the error function. It is now clear that the dynamics is diffusive since the combination ℓ^2/t appears in (A5). We can simplify Eq. (A5) by taking the limit $\ell, t \rightarrow \infty$ with $\ell/\sqrt{t}$ fixed to extract the leading behavior of (A5). In this limit the dependence on k_2 disappears, and we obtain

$$\langle \text{Tr}(G_A^2) \rangle \simeq \left(\frac{\ell}{4} \right)^2 \int \frac{dk_2}{2\pi} \int d\zeta_1 \mu(\zeta_1) \frac{e^{-\ell^2 \zeta_1^2 / (64\gamma t)}}{4\sqrt{\pi\gamma t}}. \quad (\text{A6})$$

Finally, the integration over k_2 is trivial, and the integral over ζ_1 gives

$$\langle \text{Tr}(G_A^2) \rangle \simeq \ell \left[\sqrt{\frac{\gamma t}{\pi \ell^2}} (e^{-\ell^2 / (16\gamma t)} - 1) + \frac{1}{4} \text{Erf} \left(\frac{\ell}{4\sqrt{\gamma t}} \right) \right]. \quad (\text{A7})$$

Let us now discuss the steady-state value of $\langle \text{Tr}(G_A^n) \rangle$, which we have to add to (A7) to recover the full result. We have

$$\langle \text{Tr}(G_A^2) \rangle = \frac{1}{L^2} \sum_{k_j, q_j} \sum_{n, m=1}^{\ell} e^{-i(k_1-q_2)n + i(q_1-k_2)m} G_{k_1 q_1}(t) G_{k_2 q_2}(t). \quad (\text{A8})$$

Now, the only terms attaining a nonzero value for $t \rightarrow \infty$ correspond to $k_1 = q_1 = k_2 = q_2$, $k_1 = q_1 \neq k_2 = q_2$, and $k_1 = q_2 \neq k_2 = q_1$. Crucially, because of the fact that G_{kq} is almost diagonal, reflecting the translation invariance of the initial states, one can check that only the terms with $k_1 = q_1 \neq k_2 = q_2$ contribute at the leading order in the limit $L \rightarrow \infty$ and $\ell \rightarrow \infty$. For instance, the term with $k_1 = q_2$ and $k_2 = q_1$ would give

$$(\ell/L)^2 \sum_{k, q} G_{kq} G_{qk} \sim \frac{\ell^2}{L}. \quad (\text{A9})$$

Now, Eq. (A9) vanishes if one takes the thermodynamic limit $L \rightarrow \infty$ first and then the limit $\ell \rightarrow \infty$. One can check that for translation invariant states, such as the Néel state and the dimer state, this generalizes to arbitrary n . Precisely, the

steady-state value of M_n is determined by the “diagonal” terms $G_{k_1 k_1} G_{k_2 k_2} \cdots G_{k_n k_n}$. This means that

$$M_n \xrightarrow{L, t, \ell \rightarrow \infty} \ell \int_{-\pi}^{\pi} \frac{dk}{2\pi} G_{kk}^n, \quad (\text{A10})$$

where the limit $L, t, \ell \rightarrow \infty$ is taken in the specified order. Equation (A10) implies that the steady-state value of M_n is determined by the same generalized Gibbs ensemble (GGE) that describes the steady state after the quench with the deterministic tight-binding Hamiltonian.

For the quench from the Néel state one has $G_{kk} = 1/2$. From (A7), we obtain

$$M_2 = \ell \left[\sqrt{\frac{\gamma t}{\pi \ell^2}} (e^{-\ell^2/(16\gamma t)} - 1) + \frac{1}{4} \text{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right] + \frac{\ell}{4}, \quad (\text{A11})$$

where $\ell/4$ is the steady-state contribution.

Let us now consider the case of M_3 . We now have to consider the average $\langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} \rangle$. Again, let us consider only the time-dependent terms in the dynamics, i.e., neglecting the steady-state value of M_3 . Now, from (19) and (24) we have

$$\frac{d\langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} \rangle}{dt} = \varepsilon_{\{k_j q_j\}} \langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} \rangle. \quad (\text{A12})$$

Now if there is only one π shift $q_j = k_j \pm \pi$ for some $j \in [1, 3]$ one has $\varepsilon = -4$. The configuration with two shifts as $q_j = k_j \pm \pi$ and $q_{j'} = k_{j'} \pm \pi$ corresponds to energy $\varepsilon = -8(1 + \cos(k_j - k_{j'}))$. If all the q_j are shifted, one has $\varepsilon = -2(6 + 4\cos(k_1 - k_2) + 4\cos(k_1 - k_3) + 4\cos(k_2 - k_3))$. Again, we observe that in the large ℓ limit, within the stationary phase approximation, only the terms with $q_j = k_j \pm \pi$ and $q_{j'} = k_{j'} \mp \pi$, i.e., with $k_j, k_{j'}$ paired, contribute. We have

$$\begin{aligned} M_3 &\simeq \left(\frac{\ell}{4}\right)^3 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \int \frac{dk_3}{2\pi} \int d\xi_1 \int d\xi_2 \int d\xi_3 \\ &\times 2^{-3} e^{i\ell(\xi_1 - \xi_2)k_2/2 + i\ell(\xi_2 - \xi_3)k_3/2 + i\ell(\xi_3 - \xi_1)k_1/2} w(k_2 - k_1) \\ &\times w(k_3 - k_2)w(k_1 - k_3) \sum_{i < j} e^{-8\gamma t(1 - \cos(k_i - k_j))}, \end{aligned} \quad (\text{A13})$$

where the factor 2^{-3} originates from $G_{kq} = 1/2\delta_{kq} \bmod \pi$ for the Néel state. The terms in the sum in (A13) correspond to the three different pairings of the momenta $q_j = k_j \pm \pi$, $q_{j'} = k_{j'} \mp \pi$. Similar to the case of M_2 , we redefined the quasimomenta k_j to remove the π shifts from the exponent in the second row in (A13). This also leads to the term $1 - \cos(k_i - k_j)$ in the exponent in the last row. Again, in the large t, ℓ limit only the region $k_i \approx k_j$ contributes in the integral, and we can expand the cosine functions, to obtain

$$\begin{aligned} M_3 &\simeq \left(\frac{\ell}{4}\right)^3 \int \frac{dk_1}{2\pi} \int \frac{dk_2}{2\pi} \int \frac{dk_3}{2\pi} \int d\zeta_1 \int d\zeta_2 \mu(\{\zeta_j\}) \\ &\times e^{i\ell\zeta_1(k_1 - k_3)/2 + i\ell\zeta_2(k_2 - k_3)/2} \sum_{i < j=1}^3 e^{-4\gamma t(k_i - k_j)^2}, \end{aligned} \quad (\text{A14})$$

where we also changed variables from the ξ_j to the ζ_j [cf. (96)], we performed the integral with respect to ζ_0 , and

replaced $w(k_i - k_j) \rightarrow 2$. Finally, one can check that the three terms in (A14) give the same result, and we obtain that

$$M_3 = 3\ell \left[\frac{1}{2} \sqrt{\frac{\gamma t}{\pi \ell^2}} (e^{-\ell^2/(16\gamma t)} - 1) + \frac{1}{8} \text{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right] + \frac{\ell}{8}, \quad (\text{A15})$$

where the last term is the steady-state contribution at $t \rightarrow \infty$.

Let us now consider M_n with $n = 4$. The dynamics of $\langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} G_{k_4 q_4} \rangle$ is governed by

$$d\langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} G_{k_4 q_4} \rangle = \varepsilon_{\{k_j q_j\}} \langle G_{k_1 q_1} G_{k_2 q_2} G_{k_3 q_3} G_{k_4 q_4} \rangle dt. \quad (\text{A16})$$

The energies $\varepsilon_{\{k_j, q_j\}}$ vanish if $k_j = q_j$ for all values of j . If only one of the q_j is shifted as $q_j = k_j \pm \pi$, then one has $\varepsilon_{\{k_j q_j\}} = -4$. If one has $q_j = k_j \pm \pi$ and $q_{j'} = k_{j'} \pm \pi$ for some $j \neq j'$, then $\varepsilon_{\{k_j q_j\}} = -8(1 + \cos(k_j - k_{j'}))$. Finally, if $q_j = k_j \pm \pi$ for all the values of j one has $\varepsilon = -8(2 + \sum_{j < j'} \cos(k_j - k_{j'}))$.

Again, by employing the stationary phase approximation, one can verify that only the terms with an even number of paired momenta give a nonzero contribution in the limit $t, \ell \rightarrow \infty$. For $n = 4$, one can have two or four paired momenta. The case with two pairings corresponds to $q_j = k_j + \pi$ and $q_{j'} = k_{j'} - \pi$ for some $j \neq j'$. The terms with two pairings are treated as for M_2 , and give all the same contribution. Let us consider the case with four pairings focusing on the case with $q_{1,2} = k_{1,2} + \pi$, and $q_{3,4} = k_{3,4} - \pi$. It is straightforward to check that the result depends only on the number of pairings. Let us denote the contribution to M_4 originating from the case with four pairings as $M_4^{(4p)}$. $M_4^{(4p)}$ is given as

$$\begin{aligned} M_4^{(4p)} &= \left(\frac{\ell}{4}\right)^4 \prod_{j=1}^4 \int \frac{dk_j}{2\pi} \int d\xi_j \{ e^{i\ell(\xi_1 - \xi_4)k_4/2} \\ &\times e^{+i\ell(\xi_4 - \xi_3)(k_3 - \pi)/2 + i\ell(\xi_3 - \xi_2)k_2/2 + i\ell(\xi_2 - \xi_1)(k_1 + \pi)/2} \\ &\times G_{k_1, k_1 + \pi} G_{k_2, k_2 + \pi} G_{k_3, k_3 - \pi} G_{k_4, k_4 - \pi} \\ &\times w(k_4 - \pi - k_1)w(k_1 + \pi - k_2) \\ &\times w(k_2 + \pi - k_3)w(k_3 - \pi - k_4) \\ &\times e^{-8\gamma t[2 + \cos(k_1 - k_2) + \cos(k_1 - k_3) + \cos(k_1 - k_4) + \cos(k_2 - k_3)]} \\ &\times e^{-8\gamma t[\cos(k_2 - k_4) + \cos(k_3 - k_4)]} \}. \end{aligned} \quad (\text{A17})$$

Now, we can change variables by redefining k_1 and k_3 , to remove all the π shifts. Let us also change variables to the ζ_j [cf. (96)]. After integrating over ζ_0 , we obtain

$$\begin{aligned} M_4^{(4p)} &= \left(\frac{\ell}{4}\right)^4 \prod_{j=1}^4 \int \frac{dk_j}{2\pi} \int d\zeta_1 \int d\zeta_2 \int d\zeta_3 \mu(\zeta_1, \zeta_2, \zeta_3) \\ &\times \{ 2^{-4} e^{i\ell(k_3 - k_4)\zeta_3/2} e^{+i\ell(k_2 - k_4)\zeta_2/2 + i\ell(k_1 - k_4)\zeta_1/2} \\ &\times w(k_4 - k_1)w(k_1 - k_2)w(k_2 - k_3)w(k_3 - k_4) \\ &\times e^{-8\gamma t(2 - \cos(k_1 - k_2) + \cos(k_1 - k_3) - \cos(k_1 - k_4) - \cos(k_2 - k_3))} \\ &\times e^{-8\gamma t[\cos(k_2 - k_4) - \cos(k_3 - k_4)]} \}. \end{aligned} \quad (\text{A18})$$

Again, in the large t, ℓ limit we argue that only the region with all the k_j equal contributes to the integral, which

allows us to expand the cosine functions in (A18), and replace $w(k_i - k_j) \rightarrow 2$. The integration over k_j is trivial, and it gives

$$M_4^{(4p)} = \left(\frac{\ell}{4}\right)^4 \int_{-2}^2 d\zeta_1 d\zeta_2 d\zeta_3 \mu(\zeta_1, \zeta_2, \zeta_3) \frac{2^3}{\ell^3} \delta(\zeta_1 + \zeta_2) \times \delta(\zeta_2 + \zeta_3) \delta(\zeta_1 - \zeta_3) \frac{e^{-\ell^2 \zeta_3^2 / (64\gamma t)}}{8\sqrt{\pi\gamma t}}. \quad (\text{A19})$$

In conclusion, after performing the integrals in ζ_j , we obtain

$$M_4^{(4p)} = \frac{\ell}{2} \left[\frac{1}{2} \sqrt{\frac{\gamma t}{\pi \ell^2}} (e^{-\ell^2 / (16\gamma t)} - 1) + \frac{1}{8} \text{Erf}\left(\frac{\ell}{4\sqrt{\gamma t}}\right) \right]. \quad (\text{A20})$$

Now, it is clear that the time-dependent term is of the same form as for M_2 and M_3 , apart from the prefactors.

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