

Nonstabilizerness in kinetically constrained Rydberg atom arrays

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Nonstabilizer states are a fundamental resource for universal quantum computation. However, despite broad significance in quantum computing, the emergence of many-body nonstabilizerness in interacting quantum systems remains poorly understood due to its analytical intractability. Here we show that Rydberg atom arrays provide a natural reservoir of nonstabilizerness that extends beyond single qubits and arises from quantum correlations generated by the Rydberg blockade. We demonstrate that this nonstabilizerness can be experimentally accessed using two complementary methods, either performing quench dynamics or via adiabatic ground state preparation. Using the analytical framework based on matrix product states, we explain the origin of Rydberg nonstabilizerness via a quantum circuit decomposition of the wave function.

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I. INTRODUCTION

Concepts from quantum information theory have become indispensable tools for understanding many-body quantum systems. For example, quantum entanglement is now central to the understanding of topological order [1–4] and the nonequilibrium dynamics of interacting quantum systems [5–8] (see Ref. [9] for a review). Moreover, low-energy eigenstates of quantum Hamiltonians typically have a limited amount of entanglement, making them amenable to variational Ansätze such as tensor networks and the density-matrix renormalization group [10–12].

Entanglement, however, is not the only resource required for large-scale, fault tolerant, quantum computation. The implementation of a universal set of gates is a major challenge—typically only the Clifford group of multiqubit Pauli gates is feasible, a set of gates that can be efficiently simulated classically [13,14]. One approach to universal quantum computation is to inject nonstabilizer or magic states into the circuit [15–17]. From a practical standpoint, this raises the question how such special states can be conveniently generated. While there has been much progress in understanding the nonstabilizerness of *few-qubit* systems [16,18], analogous property in *many-qubit* Hamiltonian and circuit systems is a subject of active investigation [19–29]. Consequently, many basic questions remain open, e.g., whether nonstabilizerness can play a similar role to entanglement in characterizing the properties of many-body systems.

Unfortunately, quantifying the nonstabilizerness of generic wave functions is very costly. In this work, we focus on the stabilizer Rényi entropy (SRE) [22,30], which has recently been proposed as a measure of nonstabilizerness for many-qubit wave functions (we note there are related local measures such as robustness of magic (RoM) [18,31,32] and mana

entropies for qudit systems [33,34]). The SRE, while still exponentially hard to evaluate in general, can be approximated via Monte Carlo methods [24,35,36]. Moreover, for a class of matrix product states (MPSs) with a low bond dimension, the SRE can be expressed in closed form [23,37,38]. Nevertheless, it remains unclear what information about the wave function is contained in its SRE. Studies of quantum spin chains and related models have empirically found that many-body ground states can exhibit varying levels of nonstabilizerness but generally do not approach the upper bound of the SRE, even at quantum critical points [24]. Moreover, a microscopic explanation of the origin of the SRE for a given state is lacking. Thus, it is important to identify analytically tractable models where enhancement of SRE compared to a single qubit can be analytically understood.

In this work, we study nonstabilizerness in the PXP model—an effective model of one-dimensional (1D) Rydberg atom quantum simulators [39–41]. This model has the advantage that both its ground state and certain kinds of far-from-equilibrium quench dynamics can be accurately described using a manifold of low-bond dimension MPSs [42,43]. While the ground states of gapped systems in 1D are well-known to be captured by finite bond dimension MPS, the ability to describe highly excited quench dynamics in the same way is due to the PXP model hosting a set of nonthermalizing eigenstates known as quantum many-body scars (QMBSs) [44–47]. We analytically calculate the SRE across the MPS manifold associated with QMBS dynamics of the PXP model, finding that the manifold hosts regions of large nonstabilizerness. The nonstabilizerness can be understood microscopically from the corresponding unitary circuit that generates the states in the manifold. Finally, we demonstrate that the MPS manifold is relevant for the physics of the full 1D Rydberg model realized in experiments, and we propose two protocols that can be used to access the nonstabilizer states, either by performing the global quench or by adiabatic ground state preparation.

The remainder of this paper is organized as follows. In Sec. II, we review the concept of stabilizer Rényi entropy

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for MPS states—the main quantity of interest in this work. In Sec. III, we introduce the PXP model of Rydberg atom arrays and the MPS Ansatz that describes its nonequilibrium behavior under quench dynamics from certain initial states. Focusing on this MPS variational manifold, we then analyze the nonstabilizerness that can be generated under time evolution and we identify its microscopic origin. Two experimental protocols for observing the nonstabilizerness in Rydberg atom arrays are discussed in Sec. IV. Our conclusions are presented in Sec. V, while the Appendixes contain further technical details of the MPS calculations, a discussion of other measures of nonstabilizerness, and a study of nonstabilizerness of the eigenstates of the PXP model.

II. STABILIZER RÉNYI ENTROPY OF MPS STATES

For a pure state $|\psi\rangle$ of N spin-1/2 particles, a useful measure of nonstabilizerness is the SRE of order n [22]:

$$M^{(n)}(|\psi\rangle) = (1-n)^{-1} \ln \sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^{2n}}{2^N}, \quad (1)$$

where $\mathcal{P}_N$ denotes the set of all N strings of Pauli matrices $\{\sigma^\alpha\} = \{\mathbb{I}, \sigma^x, \sigma^y, \sigma^z\}$. The SRE is zero iff $|\psi\rangle$ is a stabilizer, it is invariant under Clifford unitaries, and additive under tensor product [22].

The cost of directly evaluating Eq. (1) scales as 4^N , which rapidly becomes intractable. Instead, the SRE can be approximated by Monte Carlo sampling over Pauli strings [24,35], which scales more favorably but may require many samples to obtain accurate statistics. Finally, the SRE can be calculated directly using MPS techniques, naively scaling like χ^{6n} in the bond-dimension χ of the MPS [23]. We utilize the latter approach in this work as we focus on low- χ MPS states for which we can obtain analytical insight into the SRE.

A. Replica MPS method for evaluating SRE

We consider a translation-invariant MPS state $|\psi(A)\rangle$, defined on an infinite lattice with $(d=2)$ -dimensional local Hilbert space,

$$|\psi(A)\rangle = \sum_{\{\sigma_j\}} \text{tr}(\cdots A^{\sigma_{j-1}} A^{\sigma_j} A^{\sigma_{j+1}} \cdots) |\{\sigma_j\}\rangle, \quad (2)$$

where A^{σ_j} is a set of d matrices of dimensions $\chi \times \chi$, with χ being the MPS bond dimension. To calculate the SRE of $|\psi(A)\rangle$, we employ the replica trick method from Ref. [23] that we briefly review in this section.

To calculate the n th order SRE, we begin by creating a $2n$ -fold replica of the state $|\phi^{(n)}\rangle = |\psi\rangle^{\otimes 2n}$ with physical dimension $d' = d^{2n}$ and bond dimension $\chi' = \chi^{2n}$. Below we will mainly be interested in the simplest $n=2$ case. Let us define tensors

$$B_{i,j}^{\sigma^s} = (A_{ij}^\sigma)^{\otimes 2n}, \quad \Lambda_k^{(n)} = \frac{1}{2} \sum_{\alpha=0}^3 (\sigma_k^\alpha)^{\otimes 2n}, \quad (3)$$

where $B_{i,j}^{\sigma^s}$ is a $2n$ -fold copy of A and $\Lambda_k^{(n)}$ acts over every $2n$ replica of $|\psi\rangle$ on a physical site k . With these tensors, the expectation value over all the N -qubit Pauli strings can be seen

as a single expectation value:

$$\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^{2n}}{2^N} = \langle \phi^{(n)} | \Lambda_1^{(n)} \otimes \cdots \otimes \Lambda_N^{(n)} | \phi^{(n)} \rangle, \quad (4)$$

which we will denote as $\langle \phi^{(n)} | \mathbf{\Lambda}^{(n)} | \phi^{(n)} \rangle$. Thus, the many-body SRE is

$$M^{(n)}(|\psi\rangle) = \frac{1}{1-n} \ln \langle \phi^{(n)} | \mathbf{\Lambda}^{(n)} | \phi^{(n)} \rangle, \quad (5)$$

and the SRE density can be defined as

$$m^{(n)} = M^{(n)}/N. \quad (6)$$

Below we will always use the SRE density $m^{(n)}$ as it is an intensive quantity that can also be defined for a state on an infinite lattice, discussed next.

The SRE in the thermodynamic limit is obtained from the modified $\chi^{4n} \times \chi^{4n}$ transfer matrix:

$$\tau_{(ik),(jl)} = \sum_{s,s'} B_{i,j}^{\sigma^s} (\Lambda_1^{(n)} \otimes \cdots \otimes \Lambda_N^{(n)})^{\sigma^s, \sigma^{s'}} \bar{B}_{k,l}^{\sigma^{s'}}. \quad (7)$$

Denoting the dominant eigenvalue of τ as $\lambda_0^{(n)}$, the SRE density in the thermodynamic limit is given by

$$m^{(n)}(|\psi(A)\rangle) = \frac{1}{1-n} \ln \lambda_0^{(n)}. \quad (8)$$

As defined here, the SRE is upper bounded by

$$m^{(n)} \leq (1/N) \ln \mathcal{D}, \quad (9)$$

where $\mathcal{D}$ is the Hilbert space dimension [22].

Performing the SRE calculation directly is computationally demanding for $\chi \gtrsim 8$. Thus, for larger bond dimensions we make use of a reformulated version of the replica trick via conversion to the Pauli basis [38]. This allows for truncation of the bond dimension during the calculation and the key steps are outlined in Appendix A.

B. Global versus local nonstabilizerness

The SRE, Eq. (1), is intrinsically a global property of the wave function as it depends upon N -site products of Pauli matrices, and the latter can be extended throughout the entire system. In contrast, physical observables are typically spread over a few physical sites and therefore can be measured on a finite subsystem described by the reduced density matrix. We will now address the extent to which the nonstabilizerness of the MPS Ansatz is stored in local or global degrees of freedom.

The SRE is a basis-dependent quantity, hence it can vary even for uncorrelated product states. For this reason, it is common to consider long-range nonstabilizerness, defined as the minimal nonstabilizerness after performing arbitrary local rotations on every site of the system. As an example, for the MPS Ansatz that will be used in the PXP model in Sec. III below, the only relevant rotation that can reduce the SRE is the y axis rotation for spins on the even and odd sublattice:

$$|\psi\rangle \rightarrow |\tilde{\psi}(\gamma_o, \gamma_e)\rangle = \bigotimes_j (e^{i\gamma_o \sigma_{2j-1}^y} \otimes e^{i\gamma_e \sigma_{2j}^y}) |\psi\rangle. \quad (10)$$

The long-range SRE is then defined by minimizing over the local basis rotations:

$$m_L^{(2)} = \min_{\{\gamma_o, \gamma_e\}} m^{(2)}(|\tilde{\psi}(\gamma_o, \gamma_e)\rangle). \quad (11)$$

Another way to distinguish local from global nonstabilizer-ness is via the nonstabilizer-ness of a two-site reduced density matrix ρ_2 . Since ρ_2 is generally a mixed state, the calculation of the SRE density changes from the definition given in Eq. (1) to [22]

$$m^{(2)}(\rho_2) = -\frac{1}{2} \ln \left(\frac{\sum_{P \in \mathcal{P}_2} |\text{tr}(\rho_2 P)|^4}{\sum_{P \in \mathcal{P}_2} |\text{tr}(\rho_2 P)|^2} \right). \quad (12)$$

Therefore, the mixed state magic is computed from the expectation values of the 16 two-qubit Pauli operators in the state ρ_2 . For simplicity, we will restrict to two-site reduced density matrices and denote ρ_2 simply as ρ .

We note that SREs are not strong magic monotones, i.e., they do increase under nondeterministic stabilizer protocols [48]. However, they are widely used as they are relatively easy to compute for large systems. In Appendix C, we consider the RoM of a two-qubit reduced density matrix ρ_2 as a method of quantifying local magic. Unlike SREs, the RoM is a strong magic monotone but is typically reserved for few qubit systems.

III. NONSTABILIZERNESS IN THE PXP MODEL

In the remainder of this paper, we focus on the physical manifestations of nonstabilizer-ness in Rydberg atom arrays [49]. The latter are described by the kinetically constrained 1D spin-1/2 PXP model [40,50,51]:

$$H_{\text{PXP}} = \frac{\Omega}{2} \sum_{j=1}^N P_{j-1} \sigma_j^x P_{j+1}, \quad (13)$$

where $\Omega = 1$ is the Rabi frequency describing the flipping of each atom between its ground state $|0\rangle$ and the excited Rydberg state $|1\rangle$, N is the total number of atoms, and $P_j = (1 - \sigma_j^z)/2$ is the projector on the $|0\rangle$ state. With open boundary conditions (OBCs), we set $P_0 = P_{N+1} = \mathbb{1}$. The PXP model describes the low-energy physics in the Rydberg blockade regime [49], a phenomenon where neighboring excitations of the atoms, such as $|\dots 11 \dots\rangle$, are energetically forbidden. The blockade is imposed globally with the projector $\mathcal{P} = \bigotimes_j (\mathbb{1} - |11\rangle\langle 11|_{j,j+1})$ or, equivalently, in the local form using the P_j operators as in Eq. (13).

A. PXP dynamics and MPS Ansatz

When the PXP model is quenched from typical initial states such as $|0\rangle = |0000\dots\rangle$, rapid thermalization is observed [39], consistent with the system being chaotic [40]. Nevertheless, the PXP model also hosts a small number of nonthermal eigenstates which are evenly distributed in energy and possess anomalously low entanglement entropy [40,41,52–57]. These QMBS eigenstates have a high overlap with the $|\mathbb{Z}_2\rangle = |0101\dots\rangle$ state, leading to a suppressed growth of entanglement entropy and periodic revivals in local observables when the system is quenched from that initial state [39].

The short-time dynamics of the PXP model can be understood semiclassically using an MPS Ansatz with $\chi = 2$ with the help of the time-dependent variational principle (TDVP) [42,43]. The Ansatz is physically motivated by applying the Rydberg blockade projector $\mathcal{P}$ to a product of spin coherent states,

$$|\psi(\boldsymbol{\theta}, \boldsymbol{\phi})\rangle = \mathcal{P} \bigotimes_j (\cos(\theta_j/2)|0\rangle_j - ie^{i\phi_j} \sin(\theta_j/2)|1\rangle_j), \quad (14)$$

where θ_j, ϕ_j are the Bloch sphere angles of the j th spin. The resulting state $|\psi(\boldsymbol{\theta}, \boldsymbol{\phi})\rangle$ can be equivalently expressed as a $\chi = 2$ MPS [42]:

$$A^0(\theta_j, \phi_j) = \begin{pmatrix} \cos(\theta_j/2) & 0 \\ \sin(\theta_j/2) & 0 \end{pmatrix}, \quad A^1(\theta_j, \phi_j) = \begin{pmatrix} 0 & -ie^{i\phi_j} \\ 0 & 0 \end{pmatrix}, \quad (15)$$

which defines a continuous manifold $\mathcal{M} = \text{span}\{|\psi(\boldsymbol{\theta}, \boldsymbol{\phi})\rangle \mid \forall \theta_j, \phi_j\}$. The manifold $\mathcal{M}$, by definition, includes the states $|0\rangle$ and $|\mathbb{Z}_2\rangle$, which are representatives of generic (thermalizing) and scarred (nonthermalizing) initial conditions.

We note that the Rydberg blockade projector increases the periodicity of $\mathcal{M}$ to $T = 4\pi$ [42]. As we are primarily interested in $|0\rangle$ and $|\mathbb{Z}_2\rangle$ initial states, we assume two-site periodicity of the angles, leaving the angles on even and odd sites, (θ_e, ϕ_e) and (θ_o, ϕ_o) , as our only parameters. Furthermore, we can set $\phi_e = \phi_o = 0$ due to energy conservation [42]. Quantum dynamics can then be projected to $\mathcal{M}$ using TDVP. This projection distorts the Schrödinger equation, resulting in a system of two nonlinear differential equations for θ_e and θ_o :

$$\begin{aligned} \dot{\theta}_o &= \cos\left(\frac{\theta_e}{2}\right) + \sin\left(\frac{\theta_o}{2}\right) \tan\left(\frac{\theta_e}{2}\right) \cos^2\left(\frac{\theta_o}{2}\right), \\ \dot{\theta}_e &= \cos\left(\frac{\theta_o}{2}\right) + \sin\left(\frac{\theta_e}{2}\right) \tan\left(\frac{\theta_o}{2}\right) \cos^2\left(\frac{\theta_e}{2}\right), \end{aligned} \quad (16)$$

see Ref. [42] for a derivation. This elegant description of the dynamics in terms of low- χ MPS states will allow us to gain analytical insight into the out-of-equilibrium behavior of nonstabilizer-ness in the PXP model, Eq. (13).

B. Nonstabilizer-ness of MPS Ansatz

The dynamics of SRE density $m^{(2)}$, Eq. (8), when the PXP model is quenched from $|0\rangle$ and $|\mathbb{Z}_2\rangle$ initial states is presented in Fig. 1(a). We obtain the time-evolved state in MPS representation for various bond dimensions using a numerical implementation of TDVP [58] and then evaluate its SRE according to Eq. (4). For the scarred $|\mathbb{Z}_2\rangle$ state, we see a complex pattern in the dynamics of SRE, with a single large peak in between two smaller peaks. This data is well-converged already with small χ . The complex dynamics of SRE should be compared with the independent spin precession generated by $H = \sum_j \sigma_j^x/2$, which is also shown in Fig. 1(a). By contrast, the thermalizing $|0\rangle$ state displays a rapid increase in SRE, exceeding the values reached by the $|\mathbb{Z}_2\rangle$ state. Following this initial increase, the SRE of the $|0\rangle$ initial state remains relatively stable, despite the continuously increasing entanglement entropy found in Ref. [40].

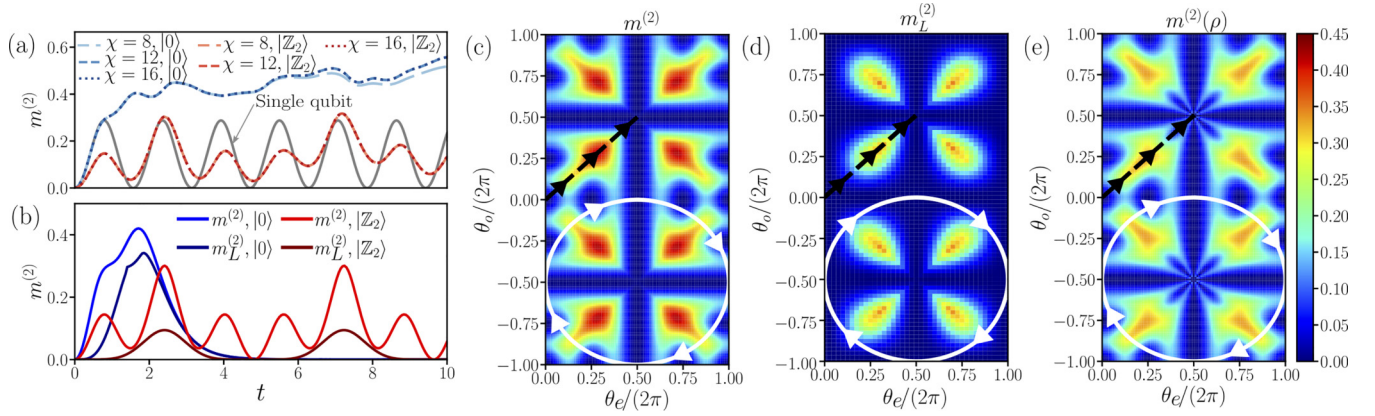


FIG. 1. (a), (b) The dynamics of $m^{(2)}$ following the quench from the $|0\rangle$ and $|\mathbb{Z}_2\rangle$ initial states in the PXP model. (a) Full quantum dynamics obtained numerically using TDVP for $N = 51$ spins and different bond dimensions χ indicated in the legend. Single-qubit precession is shown in gray for comparison. (b) The dynamics projected to $\mathcal{M}$ with $\chi = 2$, obtained analytically by integrating the equations of motions (16). For comparison, we also include the long-range nonstabilizerness, $m_L^{(2)}$, in Eq. (11). (c)–(e) Phase diagram of nonstabilizerness across the MPS manifold $\mathcal{M}$, plotted as a function of θ_e and θ_o angles. The three panels correspond, respectively, to $m^{(2)}$, its long-range version $m_L^{(2)}$ in Eq. (11), and the two-site mixed state nonstabilizerness $m^{(2)}(\rho)$ in Eq. (12). The trajectories traversed by the $|\mathbb{Z}_2\rangle$ and $|0\rangle$ states are shown by solid white and black dashed lines, respectively.

The dynamics projected into $\mathcal{M}$ lead to perfectly periodic evolution of the SRE when starting in the $|\mathbb{Z}_2\rangle$ state, see Fig. 1(b). Although the full dynamics in Fig. 1(a) is not exactly periodic, the TDVP representation of $m^{(2)}$ within $\mathcal{M}$ still shows excellent agreement. In particular, the distinctive pattern of a large peak surrounded by two smaller peaks in $m^{(2)}$ is fully reproduced within $\mathcal{M}$, and we will provide its explanation in Sec. III C below. For the $|0\rangle$ initial state, however, the agreement between full dynamics and $\mathcal{M}$ is only good up to times $t \approx 2$, after which the TDVP clearly no longer captures the full SRE dynamics. This is expected due to the large leakage of the dynamics outside the manifold [42]. In fact, at late times the evolution of the $|0\rangle$ state becomes perpendicular to the Ansatz and $m^{(2)}$ is stuck near zero. Nevertheless, comparing Figs. 1(a) and 1(b), we see that projection into $\mathcal{M}$ captures well the early-time enhancement of the SRE, which will be our main focus below.

Since nonstabilizerness is basis dependent, *a priori* it is not clear if the behavior seen in Fig. 1(a) is truly a many-body phenomenon or if it can be removed through a local unitary rotation as discussed in Sec. II B. In Fig. 1(b) the long-range SRE, $m_L^{(2)}$ in Eq. (11), for the MPS Ansatz trajectories of the $|0\rangle$ and $|\mathbb{Z}_2\rangle$ states are shown. For the $|0\rangle$ state, we see the long-range SRE is slightly smaller than the SRE but not significantly different, indicating that most of the SRE is long-range for this trajectory. For the $|\mathbb{Z}_2\rangle$ state, on the other hand, we see a more pronounced difference between $m^{(2)}$ and $m_L^{(2)}$. The distinctive three-peak behavior seen in the SRE disappears for the long-range SRE, with only a single residual peak, much reduced in magnitude. For this reason, we argue that the two smaller peaks of the SRE in the $|\mathbb{Z}_2\rangle$ case can be considered primarily local, i.e., similar in nature to the nonstabilizerness of a product state. The larger central peak, however, is a genuine many-body phenomenon, arising from the correlations due to the Rydberg blockade.

Figure 1(c) shows the SRE across the manifold $\mathcal{M}$ plotted as a function of θ_e and θ_o angles. When either angle is zero, the MPS Ansatz reduces to a simple product state, with one

of the spins fixed to $|0\rangle$ and the other rotating in the yz plane. Hence, the SRE reaches a maximum whenever it is furthest from the other spin being an σ^y or σ^z eigenstate, i.e., at $\theta_{o/e} = \pi/4, 3\pi/4, \dots$. Cases $\theta_{o/e} = \pi$ are also easy to understand, since one of the sites in the unit cell is guaranteed to be occupied, hence the other must be unoccupied due to the Rydberg blockade. Since this corresponds to the $|\mathbb{Z}_2\rangle$ product state, the SRE is zero along these lines. The nontrivial feature of the diagram in Fig. 1(c) is the four arrowlike structures pointing toward the center of each quadrant. The SRE maxima lie at the centers of these arrows along the $\theta_e = \pm\theta_o$ line with a large value of $m^{(2)} \approx 0.42$. Note that the latter is considerably larger than the quantum Ising model at its critical point [24,36].

The TDVP trajectory followed by the $|\mathbb{Z}_2\rangle$ state in $\mathcal{M}$ is shown by a white solid line in Fig. 1(c), moving periodically between high and low SRE regions. By contrast, the $|0\rangle$ state follows the black dashed line in Fig. 1(c), moving diagonally from $(\theta_o, \theta_e) = (0, 0)$ to $(\theta_o, \theta_e) = (\pi, \pi)$, passing through the state with maximal SRE before exiting the manifold. Recalling Fig. 1(b), it is clear that the repeating pattern of three SRE peaks is due to the $|\mathbb{Z}_2\rangle$ trajectory cutting through the arrow-shaped regions in Fig. 1(c). The two smaller peaks in the SRE are from the sides of the arrow shape, suggesting they arise due to local nonstabilizerness, whereas the larger peak is a many-body effect. By contrast, the dynamics initialized in the $|0\rangle$ state directly flows toward the global nonstabilizerness maximum $m^{(2)} \approx 0.42$ within $\mathcal{M}$, close to the value in the full model in Fig. 1(a). Thus, high nonstabilizerness is created under PXP dynamics within $\mathcal{M}$, given that SRE is upper bounded by $m^{(2)} \lesssim 0.48$ due to the PXP Hilbert space dimension growing as the Fibonacci number [41].

Figure 1(d) shows the long-range SRE, Eq. (11), across the entire MPS manifold. We see that the arrow-head structure vanishes for the long-range SRE, demonstrating that only the central areas of the arrowheads are nonlocal. On the other hand, the SRE of a two-site reduced density matrix, Eq. (12), plotted in Fig. 1(e), accurately recreates the

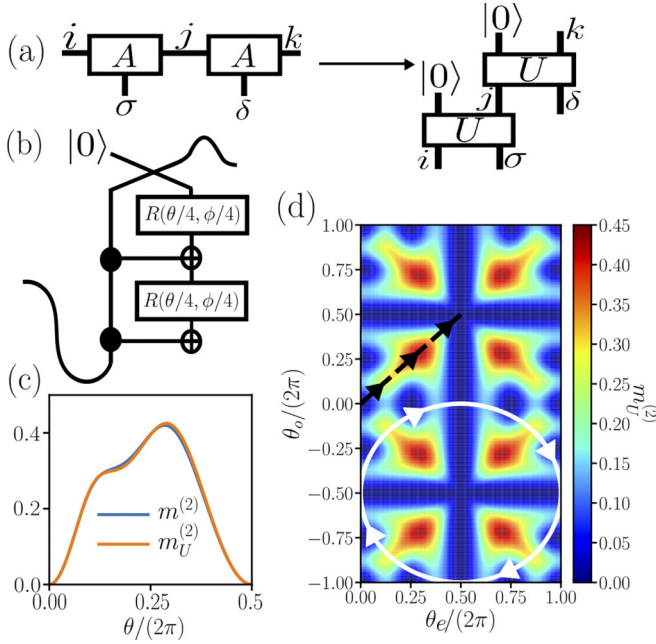


FIG. 2. (a), (b) Decomposing the MPS tensors into unitary rotations (a), with the resulting PXP unitary circuit shown in (b). (c) The SRE $m^{(2)}$ of the MPS Ansatz along the $|0\rangle$ trajectory is well-described by $m_U^{(2)}$, Eq. (19), the nonstabilizer of the input state ρ_{in} after being acted on by the unitary generated by the Ansatz. (d) The SRE from the PXP unitary Ansatz, $m_U^{(2)}$, across the entire manifold $\mathcal{M}$.

structures obtained in the full SRE calculation. The two-site reduced density matrix is generally in good agreement with $m^{(2)}$ in Fig. 1(c), with some quantitative differences, e.g., the peak SRE for the former is at around $m^{(2)}(\rho) \approx 0.32$, while for the latter it is considerably larger, $m^{(2)} \approx 0.42$.

In summary, comparing Figs. 1(c)–1(e), we conclude that a two-site reduced density matrix contains sufficient information to qualitatively reproduce the many-body nonstabilizer that emerges after the quench from the $|0\rangle$ initial state in the PXP model. While the two-site reduced density matrix, in principle, also captures the behavior of nonstabilizer from the $|\mathbb{Z}_2\rangle$ initial state, in this case the nonstabilizer is largely due to local basis rotations.

C. The origin of nonstabilizer

To provide an intuitive understanding of large SRE in Fig. 1(c), we utilize a quantum circuit description. Any MPS can be rewritten to satisfy the canonical condition [11,59]

$$\sum_{\sigma} (A^{\sigma})^{\dagger} A^{\sigma} = \mathbb{1}_{\chi}, \quad (17)$$

which can be thought of as χ columns of a $d\chi \times d\chi$ unitary matrix. The product structure of the MPS wave function, $\dots A^{\sigma_j} A^{\sigma_{j+1}} \dots$, is reproduced by composing these unitary matrices in a staircase pattern as shown in Fig. 2(a), with one leg contracted with a dummy state $|0\rangle$ to select the first χ columns of the unitary. The Ansatz described in Eq. (15) is no exception—it can be constructed from the two-qubit unitary

given in Fig. 2(b):

$$U_{\text{MPS}} = \text{CNOT}_{\theta/4, \phi/4}^{(2)} \text{CNOT}_{\theta/4, \phi/4}^{(2)} \text{SWAP}, \quad (18)$$

where $R_{\theta, \phi}^{(2)} = \exp(i(\pi/2)\mathbf{n} \cdot \boldsymbol{\sigma})$ is a rotation of the second qubit around the axis $\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Note that the θ and ϕ parameters in Eq. (18) are the same as in Eq. (15).

U_{MPS} in Eq. (18) is simply an arbitrary controlled rotation on the second qubit, conditional on the first being $|0\rangle$, and combined with a SWAP gate. The unitary is made up of two CNOT gates and one SWAP gate which are Clifford gates, and two local rotations which are non-Clifford and responsible for the nonstabilizer of the MPS wave function. The subtlety, however, is that the non-Clifford unitaries are only relevant if they act nontrivially on the input state. The first qubit always acts upon $|0\rangle$ by construction. In the thermodynamic limit, the second qubit acts on ρ_R , the right eigenvalue of the MPS transfer matrix $E = \sum_{\sigma} A^{\sigma} \otimes \bar{A}^{\sigma}$. Thus, the nonstabilizer of the MPS can be estimated as the nonstabilizer of $\rho_{\text{in}} = |0\rangle\langle 0| \otimes \rho_R$ with U_{MPS} applied to it, minus the nonstabilizer of ρ_{in} ,

$$m_U^{(2)} = m^{(2)}(U_{\text{MPS}} \rho_{\text{in}} U_{\text{MPS}}^{\dagger}) - m^{(2)}(\rho_R), \quad (19)$$

where we have taken advantage of the SRE additivity under the tensor product to write $m^{(2)}(\rho_{\text{in}}) = m^{(2)}(\rho_R)$. Due to the inherent gauge freedom in our MPS, U_{MPS} is only unique up to a unitary transformation, $U_{\text{MPS}} \rightarrow (u \otimes \mathbb{I}_d) U_{\text{MPS}} (u \otimes \mathbb{I}_d)$, where u is an arbitrary $\chi \times \chi$ unitary matrix. Since u can be non-Clifford, it can change: $m_U^{(2)}$. The gauge chosen throughout this paper is the only natural choice for the PXP model: the form of ρ_{in} and U_{MPS} guarantees that the $|11\rangle$ component of the MPS remains zero, ensuring that the Rydberg blockade constraint is also obeyed by the unitary generating the MPS.

In Fig. 2(c), we compare the nonstabilizer obtained from the MPS Ansatz in Fig. 1(b) against $m_U^{(2)}$ in Eq. (19). We observe excellent agreement between the two over the range in which we traverse diagonally through one of the arrow-shaped regions, especially at the two extremes of the θ range. In the region of high nonstabilizer, our Ansatz in Eq. (19) very slightly deviates from the exact $m^{(2)}$, implying that U_{MPS} is not the exact unitary and further corrections may be needed to exactly capture the SRE across the entire manifold $\mathcal{M}$. In Fig. 2(d), $m_U^{(2)}$ is shown for the full two-site unit-cell Ansatz. We see that, despite small deviations, it is significantly closer to the full SRE than two-site SRE and accurately recreates the SRE of the PXP manifold.

IV. EXPERIMENTAL PROTOCOLS

Finally, in this section we outline two protocols for experimentally accessing the nonstabilizer states in Fig. 1. The PXP model, Eq. (13), arises as a limiting case of the more general Hamiltonian describing a 1D array of Rydberg atoms [39],

$$H_{\text{Ryd}} = \frac{\Omega}{2} \sum_j \sigma_j^y + \Delta \sum_j n_j + V \sum_{i < j} \frac{n_i n_j}{|i - j|^{\alpha}}, \quad (20)$$

where $n_j = |1\rangle\langle 1|_j$ counts local excitations, Δ is the chemical potential, and V is the overall strength of the van der Waals interactions. Due to the fast decay of the interactions ($\alpha = 6$),

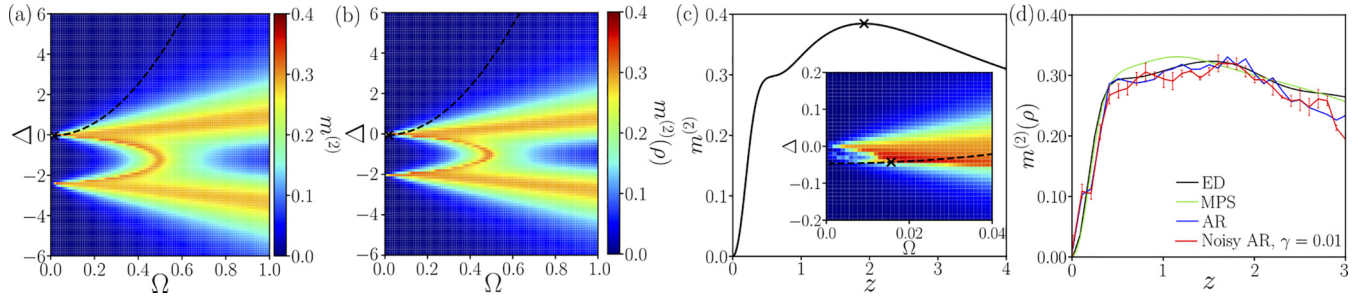


FIG. 3. (a), (b) SRE of the ground state of the Rydberg model in Eq. (20) with $V=1$ and varying Δ and Ω , with interactions truncated at next-nearest neighbors. The ground state of $N = 51$ atoms was obtained using density matrix renormalization group (DMRG) based on ITensor Library [60] at a bond dimension of $\chi = 8$. (a) The SRE $m^{(2)}$, while (b) is the two-site reduced density matrix SRE $m^{(2)}(\rho)$. Dashed line represents the trajectory of the effective model H_0 in Eq. (22), with the cross denoting the point of maximum nonstabilizerness. (c) Nonstabilizerness in the effective model H_0 , Eq. (21), as a function of z . The maximum value $m_2 \approx 0.38$ is comparable to Fig. 1 and attained at $z \approx 2$ (cross). Inset is a magnified section of the phase diagram in (a). (d) Two-site reduced density matrix SRE $m^{(2)}(\rho)$ for the ground state of the Rydberg model computed by four different methods: MPS (green line), ED (black line), clean adiabatic ramping (AR) protocol with sampling in Bloqade (blue line), and Bloqade's stochastic noise simulation with noise amplitude $\gamma = 0.01$ (red symbols). All simulations were performed for $N = 11$ atoms while the Bloqade simulations were performed with a lattice spacing of $3.12 \mu\text{m}$ and $n_{\text{shots}} = 1000$. The error bars represent the statistical error in averaging the noisy results over ten runs.

we will neglect interaction terms beyond next-nearest neighbors. The Hamiltonian H_{Ryd} reduces to the PXP model in the regime $V \gg \Omega, \Delta$ [40,50,51].

In the strong blockade regime $V \gg \Omega, \Delta$, the nonstabilizerness in Fig. 1 can be accessed by repeating the same type of quench experiments performed in Ref. [39]. While measuring the global SRE may not be feasible, the SRE can be approximated locally using a two-site reduced density matrix, Eq. (12), as we have demonstrated in Fig. 1. Note that, for the ground state of Eq. (20) to be consistent with states obtained through dynamics in Fig. 1(c), where $\langle \sum_j \sigma_j^x \rangle = 0$, we have oriented the Rabi flip term in Eq. (20) along the y axis. This, however, does not affect the SRE value.

Alternatively, it is possible to access the nonstabilizerness in Fig. 1 by adiabatic ground state preparation. With the reparametrizations

$$\Omega = 2V/(2^\alpha z), \quad \Delta = -(2V/2^\alpha)(3 - 1/z^2), \quad (21)$$

the ground state of H_{Ryd} can be approximated with the ground state of the following Hamiltonian [61,62]:

$$H_0 = \sum_{j=1}^N P_{j-1}(\sigma_j^y + zP_j + z^{-1}n_j)P_{j+1}. \quad (22)$$

The advantage of H_0 is that it is frustration-free and its ground state is known exactly for any value of z [61]. In Appendix B, we prove that the ground state of H_0 is, in fact, equivalent to the MPS state in Eq. (15) with

$$\theta = \theta_0 = \theta_e, \quad z = \sin(\theta/2)/\cos^2(\theta/2). \quad (23)$$

Thus, the nonstabilizerness in Fig. 1 can be generated by simply preparing the ground state of H_0 with $z \approx 2$.

In Fig. 3(a), we show the SRE of the ground state of H_{Ryd} for $N = 51$ atoms as a function of Ω and Δ , with fixed $V = 1$. We overlay the trajectory of H_0 by the dashed black line and mark the point of maximal nonstabilizerness with a cross. We observe a large W-shaped band of nonstabilizerness which becomes more pronounced at larger Ω with the trajectory touching the tip of this band. A similar phase diagram is

observed in Fig. 3(b), where we plot the two-site reduced density matrix SRE for the ground state of the Rydberg model. We find this local measure of magic produces an excellent qualitative agreement with the full calculation of the SRE in Fig. 3(a), capturing the W-shaped band emanating from $\Delta \approx 0$.

The black dashed line in Figs. 3(a) and 3(b) represents the effective Hamiltonian for the MPS Ansatz, H_0 , whose trajectory is analyzed in more detail in Fig. 3(c). The SRE of the trajectory of H_0 attains a maximum value of $m^{(2)} \approx 0.38$ at $z \approx 2$, corresponding to $\Omega \approx 0.0156$ and $\Delta \approx -0.0429$. Similarly, the maximal value of the two-site density matrix SRE, $m^{(2)}(\rho) \approx 0.32$, in Fig. 3(b) is comparable to the actual SRE maximum of $m^{(2)} \approx 0.38$, suggesting that quantum state tomography would allow us to accurately approximate the SRE.

Finally, in Fig. 3(d) we demonstrate that it is possible to detect nonstabilizerness by preparing the states along the trajectory described by the Hamiltonian H_0 and by measuring the mixed state magic of two sites at the center of the chain. To assess the impact of experimental imperfections on the measurements of nonstabilizerness, we perform realistic small-scale simulations of QuEra's Rydberg atom platform using the Bloqade interface [63]. We determine the ground state $|\psi\rangle$ of the Rydberg Hamiltonian along the desired trajectory using exact diagonalization (ED) and by adiabatically ramping (AR) from the polarized state $|0\rangle$. We have assumed a maximum ramp time of $4 \mu\text{s}$ [63]. To adiabatically prepare the ground state along the H_0 trajectory, we consider two separate ramp sequences depending on the sign of Δ . For Ω , we consider a quick linear ramp to the desired Ω_i . When we ramp to negative Δ , we also consider a linear function where we ramp from Δ_{prep} to the desired Δ_i . When ramping to a positive Δ_i , we consider a sigmoid ramp with curvature k ,

$$\Delta(t) = \frac{t - kt}{k - 2k|t| + 1}, \quad (24)$$

with curvature $k = 0.98$. Second, we shift the inflection point such that it lies at $(t, \Delta) = (2, 0)$ and we rescale $t \in (0, 2)$

and $t \in (2, 4)$ to match Δ_{prep} and Δ_i , respectively. To mimic the experimental protocol, we perform two-site tomography by sampling the probability distribution $|\psi|^2$ and calculating the 16 possible Pauli expectation values for two sites located at the center of the chain. Using these measurements, the definition given in Eq. (12) and standard statistical error analysis, we obtain estimates of the $n = 2$ mixed state SRE. To simulate experimental noise, we consider a depolarizing noise channel with error rate γ during the adiabatic ramping procedure and optimize the ramp parameters over the trajectory to obtain the mixed state SRE.

Figure 3(d) compares the mixed-state nonstabilizerlessness $m^{(2)}(\rho)$ obtained using ED and adiabatic ramping through noiseless and noisy channels with sampling (averaged over ten runs). We also show the MPS results for $\chi = 100$ for comparison and the error bars represent the average error over the ten runs. We note that the difference between the peak of nonstabilizerlessness in Fig. 3(d) and $z \approx 2$ in Fig. 3(c) is due to finite-size effects and divergences in regions of large global nonstabilizerlessness. We find good agreement between the expected ED results and the experimental simulations in the regions of high nonstabilizerlessness. This suggests that our protocol can be implemented in practice, even after accounting for experimental noise and readout errors.

V. CONCLUSIONS

We showed that kinetic constraints stemming from the Rydberg blockade can lead to many-body nonstabilizerlessness. While we focused on the strong blockade regime described by the PXP model, our approach can be directly generalized to other blockade regimes explored in recent experiments [64]. While in the main text we studied dynamical manifestations of nonstabilizerlessness, further results on the nonstabilizerlessness of PXP eigenstates can be found in Appendix D. Utilizing a quantum circuit construction, we provided an intuitive understanding of the origin of nonstabilizerlessness of an MPS in terms of the number of nonstabilizer unitaries used to generate it. While we expect this analysis to have broader applicability beyond the PXP model, to make it more rigorous, it would be necessary to better understand the impact of the MPS gauge degree of freedom on nonstabilizerlessness.

Our work highlights the fact that some low bond-dimension MPSs can exhibit high levels of nonstabilizerlessness. Since all 1D area-law entangled states—including the ground states of local, gapped Hamiltonians—can be arbitrarily well approximated by MPS [65,66], they can be efficiently generated using classical algorithms such as density matrix renormalization group. Most of these wave functions possess intrinsic nonstabilizerlessness, which implies that there may be certain ground states which are easy to generate classically but difficult to create in a fault-tolerant quantum computer, in particular, the points of maximum SRE identified for the Rydberg model above. Finally, an intriguing question is which many-body systems saturate the upper bound of nonstabilizerlessness. In this context, Rydberg atom arrays exhibit a variety of exotic phases [67,68] and it would be interesting to study the nonstabilizerlessness in their ground states to see if any of them further approach the upper bound.

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APPENDIX A: CALCULATING SRE BY PAULI BASIS CONVERSION

Calculating the SRE for an MPS state by the replica trick becomes infeasible for states with large bond dimensions due to the replicated state scaling as χ^{2n} . Instead, this method can be reformulated to allow truncation of the bond dimension during the calculation.

For a normalized state $|\psi\rangle$ which is represented by an MPS of physical dimension d and bond dimension χ , we can define its density matrix as $\rho = |\psi\rangle\langle\psi|$, an MPO of bond dimension χ^2 and two physical dimensions d and d' . We then introduce a three-legged Pauli tensor $P_{i,i'}^\alpha = \frac{1}{\sqrt{2}}(\sigma^\alpha)_{i,i'}$ with bond dimension $\chi = d$ and physical dimension d^2 . This Pauli tensor is then contracted on both physical bonds on each site of ρ , thereby converting it into the Pauli basis and leaving an MPS $|P(\psi)\rangle$ with bond dimension χ^2 and physical dimension d^2 . This now grants us the ability to truncate the bond dimensions to some χ_P if required.

Now that the MPS has been converted to the Pauli basis, we can perform the replica trick at a reduced cost. To do this, we define a diagonal operator W whose diagonal elements are just the components of the Pauli MPS at each site. This MPO is built by contracting a three-legged delta tensor of dimensions d^2 . This delta tensor is contracted onto each physical leg of the Pauli MPS to produce the MPO W . For some stabilizer Renyi index n , we apply the MPO W , $n - 1$ times onto $|P(\psi)\rangle$ to obtain $|P^{(n)}(\psi)\rangle = W^{n-1}|P(\psi)\rangle$, while also truncating the bond dimension after each application of W . Therefore, the n th order SRE is given by

$$M^{(n)} = \frac{1}{1-n} \log(\langle P^{(n)}(\psi) | P^{(n)}(\psi) \rangle) - N, \quad (\text{A1})$$

and, similarly, the SRE density is $m^{(n)} = M^{(n)}/N$.

APPENDIX B: PARENT HAMILTONIAN FOR MPS ANSATZ

Any matrix-product state $|\psi(A)\rangle$ is the ground state of a so-called parent Hamiltonian which is a local, frustration-free Hamiltonian constructed as the sum of projectors

$$H^{\text{parent}} = \sum_j \mathbb{P}_j, \quad \mathbb{P}_j^2 = \mathbb{P}_j. \quad (\text{B1})$$

The MPS of interest is in the null space of the projector $\mathbb{P}_j$ and therefore has energy exactly $E = 0$ with respect to this parent Hamiltonian. We would like to find a parent Hamiltonian for the MPS used in Eq. (15) the main text. In this section, we will work in the thermodynamic limit and assume a single-site unit cell, as we are interested in the dynamics of $|0\rangle$ state.

The projector which defines the parent Hamiltonian for the MPS in Eq. (15) is found by constructing the reduced density ρ_n of $|\psi(A)\rangle$ over n sites. The value of n should be chosen such that ρ_n has at least one eigenvalue zero. This is typically the case when $d^n > \chi^2$. For the PXP model, the Hilbert space obeys the Rydberg blockade constraint which excludes certain

states, hence ρ_n does not grow exactly like d^n . Nevertheless, $n = 3$ is suitable since ρ_n has five eigenvalues, which is larger than the $\chi^2 = 4$.

We choose the basis $\{|000\rangle, |001\rangle, |010\rangle, |100\rangle, |101\rangle\}$ for ρ_3 . The leading left/right eigenvalues of the MPS transfer matrix are

$$(\mathbb{L}) = (1, 0, 0, 1), \quad (\mathbb{R}) = \frac{1}{1+s^2}(1, cs, cs, s^2), \quad (\text{B2})$$

where we have introduced a shorthand notation $c \equiv \cos(\theta/2)$ and $s \equiv \sin(\theta/2)$. Furthermore, ρ_3 can be shown to be

$$\rho_3 = \frac{1}{1+s^2} \begin{pmatrix} c^4 & -ic^4se^{i\phi} & -ic^2se^{i\phi} & -ic^4se^{i\phi} & -c^4s^2e^{2i\phi} \\ ic^4se^{-i\phi} & c^2s^2 & c^2s^2 & c^4s^2 & -ic^2s^3e^{i\phi} \\ ic^2se^{-i\phi} & c^2s^2 & s^2 & c^2s^2 & -ic^2s^3e^{i\phi} \\ ic^4se^{-i\phi} & c^4s^2 & c^2s^2 & c^2s^2 & -ic^2s^3e^{i\phi} \\ -c^4s^2e^{2i\phi} & ic^2s^3e^{-i\phi} & ic^2s^3e^{-i\phi} & ic^2s^3e^{-i\phi} & s^4 \end{pmatrix}. \quad (\text{B3})$$

Thus, ρ_3 has a single eigenvalue zero with eigenvector

$$|\phi\rangle = \frac{1}{\sqrt{1+s^2/c^4}}(ie^{i\phi}s/c^2, 0, 1, 0, 0)^T. \quad (\text{B4})$$

If we define $z = s/c^2$, then the parent Hamiltonian of the MPS Ansatz is $\mathbb{P}_j = |\phi\rangle\langle\phi|$, where

$$\begin{aligned} \mathbb{P}_j &= \frac{1}{1+z^2}(z^2|000\rangle\langle 000| + |010\rangle\langle 010| \\ &\quad -ize^{-i\phi}|000\rangle\langle 010| + i ze^{i\phi}|010\rangle\langle 000|)_{j-1,j,j+1} \\ &= \frac{z}{1+z^2}P_{j-1}(zP_j + z^{-1}n_j + \cos(\phi)\sigma_j^y + \sin(\phi)\sigma_j^x)P_{j+1}. \end{aligned} \quad (\text{B5})$$

When $\phi = 0$, this is the model introduced in Ref. [61] and discussed in the main text, up to an overall prefactor. Although considerably more tedious, it is possible to generalize this to the case of two-site unit cell which would be needed for describing the dynamics of the $|\mathbb{Z}_2\rangle$ state. We have not pursued this, however, since the nonstabilizerness along the $|\mathbb{Z}_2\rangle$ orbit is smaller than for the $|0\rangle$ state.

APPENDIX C: ROBUSTNESS OF MAGIC

In Sec. II, we introduced the mixed state SRE for a two-site density matrix as an experimental witness for calculating SREs. However, there are other potential magic monotones that could be considered. In addition to the mixed state SRE, we also studied a different measure for local nonstabilizerness, the RoM [18,32,69], which quantifies the minimal weight of stabilizer states that yields a stabilizer state when mixed with a density matrix ρ . Since the set of stabilizer states is overcomplete, it is possible to decompose ρ into a weighted sum of stabilizer states, $\rho = \sum_i X_i S_i$, where S_i is a stabilizer state, while satisfying the normalization constraint $\sum_i X_i = 1$. For ρ to have nonstabilizerness, at least one of the weights must be negative. Therefore, the ROM is expressed as

a convex optimization problem:

$$\mathcal{R} = \inf_{\{X_i\}} \left\{ \sum_i |X_i| - 1 : AX = B \right\}, \quad (\text{C1})$$

where $A_{\alpha\beta} = \text{tr}(\sigma^\alpha S_\beta)$ and $B_\alpha = \text{tr}(\sigma^\alpha \rho)$, with σ^α as a *two-qubit* Pauli string. However, the RoM is known to be multiplicative under tensor product, therefore we instead define the log-free RoM $\ln(\mathcal{R} + 1)$ [21] which is additive under the tensor product and allows for a better comparison with SREs.

Figure 4(a) shows the RoM for the two-site unit cell MPS in Eq. (15). While the RoM captures some of the arrowhead-like structures of the SRE, it diverges dramatically from the SRE in the long-range nonstabilizerness regions we identified in Fig. 1(d). Noticeably, a large area of stabilizerness is created at the center of the unit cell where the points of the arrowheads should be located. In Fig. 4(b), we plot the phase diagram of the log-free robustness of magic, $\ln(\mathcal{R} + 1)$, for the ground state of the Rydberg model. This measure produces

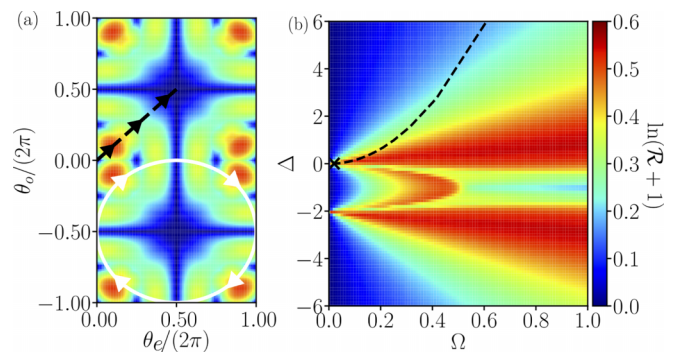


FIG. 4. (a) Phase diagram for the log-free RoM $\ln(\mathcal{R} + 1)$ across the MPS manifold $\mathcal{M}$. (b) Phase diagram of the log-free RoM for the ground state of the Rydberg model in Eq. (20). Simulations were performed for $N = 51$ atoms at $\chi = 8$ using iTensor [60] with fixed $V = 1$ and interactions truncated beyond next-nearest neighbors.

more qualitative accuracy with the original phase diagram in Fig. 3(a) compared to the two-site PXP phase diagram in Fig. 1(c). Therefore, the RoM could also be used as a measure of local nonstabilizerlessness similar to mixed state SREs. This metric is able to approximately capture the W-shaped band of nonstabilizerlessness that emanates from $\Delta \approx 0$ and extends to larger Ω .

APPENDIX D: NONSTABILIZERNESS OF PXP EIGENSTATES

In the main text, we focused on the PXP model and explored the nonstabilizerlessness generated under its quantum dynamics. Here we turn our attention to the nonstabilizerlessness of energy eigenstates of the PXP model. One of the remarkable properties of this model is that it has a few eigenstates in the middle of the energy spectrum that can be written as exact low bond-dimension MPSs [52]. This will allow us to directly evaluate their SRE with the replica MPS approach outlined above and in the main text.

First, we analyze two exact scarred eigenstates with energy $E = 0$ exactly in the middle of the spectrum. Assuming even chain lengths and periodic boundary conditions (PBCs), these states were written in Ref. [52] using the following MPS Ansatz:

$$|\psi_1\rangle = \sum_{\{\sigma_i\}} \text{tr}(B_1^{\sigma_1} C_2^{\sigma_2} \dots B_1^{\sigma_{N-1}} C_N^{\sigma_N}) |\sigma_1 \dots \sigma_N\rangle, \quad (\text{D1})$$

$$|\psi_2\rangle = \sum_{\{\sigma_i\}} \text{tr}(C_1^{\sigma_1} B_2^{\sigma_2} \dots C_1^{\sigma_{N-1}} B_N^{\sigma_N}) |\sigma_1 \dots \sigma_N\rangle, \quad (\text{D2})$$

with the MPS matrices given by

$$B^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B^1 = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad (\text{D3})$$

$$C^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C^1 = \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}. \quad (\text{D4})$$

For $|\psi_1\rangle$ and $|\psi_2\rangle$, we find that $m^{(2)} \approx 0.376$, which is significantly larger than, e.g., the SRE at the critical point of the Ising model. However, we find that the SRE of these eigenstates is slightly smaller than maximal SRE we found in the main text.

Apart from the MPS states in Eqs. (D1)–(D4), the majority of PXP eigenstates are believed to be volume-law entangled [41]. Hence, we will rely on Markov chain Monte Carlo sampling to evaluate their SRE. This method overcomes the 4^N scaling of the exact evaluation of the SRE, but it comes at the cost of generally poor convergence with the number of samples required for accurate convergence being $N_s \approx 1 \times 10^6$.

At energy $E = 0$ in the middle of the spectrum, similar to the MPS states in Eqs. (D1)–(D4) above, there are also some volume-law entangled states that can be written in analytic form. For example, for a spin-1/2 PXP chain with PBCs and size $N = 2L$, where L is the size of the half chain, an exact volume-law entangled eigenstate at energy $E = 0$ is given by the following rainbowlike state [70]:

$$|E\rangle = \frac{1}{\sqrt{|F_L|}} \sum_{f \in F_L} (-1)^{|f|} |f\rangle_{1,\dots,L} \otimes |f\rangle_{L+1,\dots,2L}, \quad (\text{D5})$$

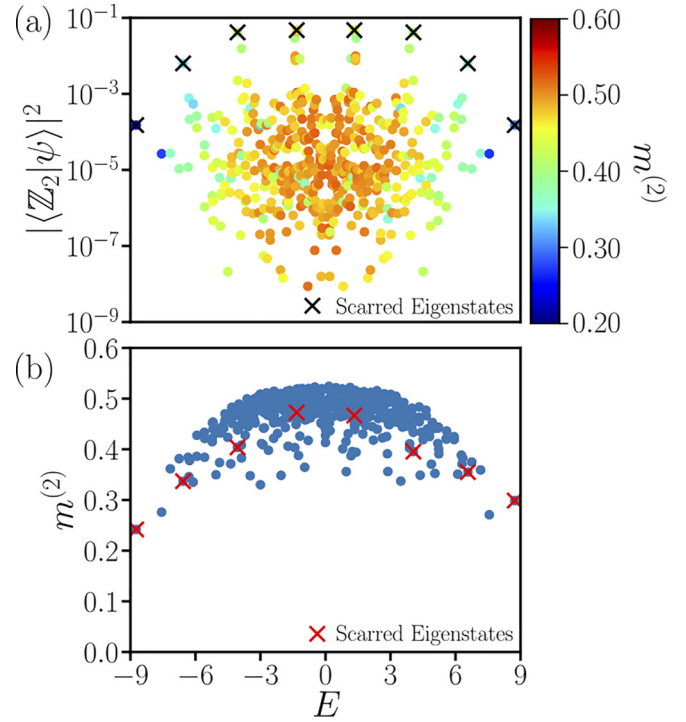


FIG. 5. (a) Overlap between the $|\mathbb{Z}_2\rangle$ and all eigenstates of the PXP model plotted as a function of their energy. Each eigenstate is colored by its $m^{(2)}$ value (color bar) obtained by Monte Carlo sampling. (b) SRE of the PXP eigenstates as a function of their energy E . In both panels, the crosses indicate the scarred eigenstates. The simulations were performed for the PXP model with $N = 14$ spins with OBCs and in the $P = +1$ parity sector, using 300 000 Monte Carlo samples.

where F_L is the set of bitstrings for a chain of L spins with PBCs and respecting the Rydberg blockade constraint. The normalization $|F_L| = \varphi_{L-1} + \varphi_{L+1}$ is given in terms of Fibonacci numbers, φ_n , while $|f|$ denotes the parity of a given bitstring. Performing the Monte Carlo sampling, we computed the SRE of the $|E\rangle$ state for several system sizes L , obtaining the $L \rightarrow \infty$ extrapolated SRE of $m^{(2)} = 0.180 \pm 0.008$. This implies that, in the thermodynamic limit, the $|E\rangle$ state is closer to a stabilizer state compared, e.g., to the peak in the SRE of the critical Ising model. Intuitively, this may be expected due to the sparse structure of the $|E\rangle$ state when we consider the Hilbert space of the entire $2L$ chain. Additionally, the nonzero amplitudes on basis states in $|E\rangle$ are uniform, requiring fewer non-Clifford gates to construct the state.

Finally, in Fig. 5 we study the SRE of all eigenstates in the PXP model with $N = 14$ spins. We use ED to extract the eigenstates and then apply Monte Carlo sampling to obtain their SRE. Here we assume OBCs while also resolving parity symmetry. We found that OBCs result in a much better convergence of the Monte Carlo sampler compared to PBCs. Figure 5(a) shows the overlaps of eigenstates with the $|\mathbb{Z}_2\rangle$ state as a function of their energy E , with the color bar showing $m^{(2)}$ of each eigenstate. We also highlight the scarred eigenstates using crosses. The bulk of the spectrum contains a large density of thermal eigenstates with large nonstabilizerlessness. Toward the edges of the spectrum, we see a noticeable

drop in $m^{(2)}$, indicating that the ground state and low-energy excitations are closer to stabilizer states.

To isolate the nonstabilizerness of the scarred states, we plot the SRE of the eigenstate spectrum in Fig. 5(b) and also label the scarred states. The ground state of the spectrum has the lowest SRE of the spectrum with a value comparable to

the SRE of the 1D critical Ising model [24,36]. The scarred eigenstates exhibit a roughly linear increase of SRE until we reach the midspectrum eigenstates with the largest overlap on $|\mathbb{Z}_2\rangle$ state. We note that any asymmetry in the plots in Fig. 5 around $E = 0$ is due to accidental degeneracies in the spectrum or convergence issues in Monte Carlo sampling.

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