

Purely electrical detection of the spin-splitting vector in p -wave magnets based on linear and nonlinear conductivities

Motohiko Ezawa 

Department of Applied Physics, *The University of Tokyo*, 7-3-1 Hongo, Tokyo 113-8656, Japan



(Received 27 March 2025; revised 15 July 2025; accepted 21 August 2025; published 9 September 2025)

A p -wave magnet has zero-net magnetization and induces a momentum-dependent spin-split band structure just as in the case of an altermagnet. It will be useful for high-density and ultrafast memory, where the direction of the spin-splitting vector may be used as a bit. However, it is a nontrivial problem to detect the spin-splitting vector in a p -wave magnet because time-reversal symmetry is preserved, while this is not a problem in an altermagnet because the anomalous Hall conductivity is present due to the breaking of time-reversal symmetry. Here, we show that it is possible to detect the in-plane component of the spin-splitting vector in the p -wave magnet by measuring the linear transverse and longitudinal Drude conductivity. Remarkably, this measurement is possible without using magnetization. Furthermore, we study the nonlinear Drude conductivity, the quantum-metric and the Berry curvature dipole induced nonlinear conductivity in the presence of tiny magnetization along the z axis. It is possible to detect the z -component of the spin-splitting vector by measuring the above nonlinear conductivities. We obtain analytic formulas for them in the first-order perturbation theory, which agree quite well with numerical results without perturbation.

DOI: [10.1103/jf1w-wdg8](https://doi.org/10.1103/jf1w-wdg8)

I. INTRODUCTION

Ferromagnets are useful for nonvolatile memories, where the up and down spins act as a bit. On the other hand, antiferromagnets are expected to act as more efficient memories owing to their zero net magnetization, where high-density and quick-switchable memories may be possible. However, it is very hard to read out the direction of the Néel vector of antiferromagnet [1–7]. Altermagnets are attractive [8–10] because they have both the merits of ferromagnets and antiferromagnets. Altermagnets have zero net magnetization and they break time-reversal symmetry. Accordingly, the Néel vector is observable by means of the anomalous Hall conductivity [11–14], and hence its direction can be used as a bit. A characteristic feature of altermagnets which is absent in both ferromagnets and antiferromagnets is the momentum-dependent spin splitting [8–10,15–17], which includes the d -wave, g -wave and i -wave band structures. Indeed, momentum-dependent band structures have been observed by angle-resolved photoemission spectroscopy (ARPES) [18–22]. Furthermore, spin current is generated in d -wave altermagnets [17,23–25].

In general, we may think of the higher symmetric X -wave magnets [26], where $X = p, d, f, g, i$. They have zero net magnetization as in the case of antiferromagnets. The spin texture is collinear and the band structure breaks time-reversal symmetry for $X = d, g, i$. They are altermagnets. On the other

hand, the spin texture is noncollinear and the band structure preserves time-reversal symmetry for $X = p, f$, which leads to the zero anomalous Hall conductivity. The effective Hamiltonian [26] for electrons associated with the X -wave magnet is concisely described by a two-band model containing the term $(\mathbf{s} \cdot \boldsymbol{\sigma})f(\mathbf{k})$, where $\mathbf{s}$ is a three-dimensional constant unit vector representing the spin-splitting direction, and $f(\mathbf{k})$ is a symmetric function characterizing the X -wave magnet. We may call $\mathbf{s}$ the spin-splitting vector.

Recently, p -wave magnets attract growing interests [26–37]. The p -wave symmetry is concisely described by the term $(\mathbf{s} \cdot \boldsymbol{\sigma})k_x$ in the Hamiltonian. This term is actually derived from the four-band Hamiltonian [27,36] constructed for the p -wave magnet CeNiAsO based on the density-functional theory, as we show in this work. The direction of the spin-splitting vector can be used as a bit. However, there is so far no known method to detect the spin-splitting vector, which makes difficult to use it for spintronics. Experiments on the p -wave magnets were reported [38] in $\text{Gd}_3\text{Ru}_4\text{Al}_{12}$ and reported [39] in NiI_2 . Moreover, an experiment on an electrical switch of the spin-splitting vector was reported [39] in the p -wave magnet NiI_2 .

The linear conductivity $\sigma^{a;b}$ is defined by $j^{a;b} = \sigma^{a;b}E^a$, where E^a is an applied electric field along the a direction and $j^{a;b}$ is the current along the b direction. They include the longitudinal Drude conductivity $\sigma^{x;x}$ and the transverse Drude conductivity $\sigma^{y;x}$. On the other hand, there are several studies on nonlinear conductivity [40–52]. Especially, the second-order nonlinear conductivity $\sigma^{ab;c}$ is defined by $j^{ab;c} = \sigma^{ab;c}E^aE^b$, where E^a is an applied electric field along the a direction and $j^{ab;c}$ is the current along the c direction. Among them, the nonlinear Drude conductivity is proportional to τ , where τ is the electron relaxation time.

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

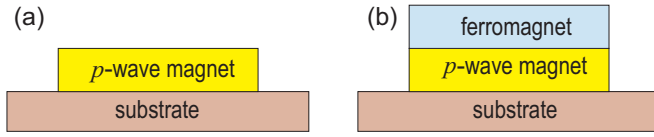


FIG. 1. Illustration of the system made of (a) p -wave-magnet-substrate structure and (b) ferromagnet- p -wave-magnet-substrate structure.

It is an extrinsic nonlinear conductivity, while the quantum-metric induced nonlinear conductivity is intrinsic because it is irrelevant to τ .

In this paper, we show that the in-plane component of the spin-splitting vector of p -wave magnets is detectable by measuring the transverse and longitudinal linear Drude conductivities. Remarkably, this is possible without using magnetization. In addition, the z component of the spin-splitting vector of p -wave magnets is detectable by measuring the nonlinear conductivity with the aid of the magnetization along the z axis.

This paper is composed as follows: In Sec. II, we introduce the minimal two-band Hamiltonian describing p -wave magnets. We also derive this minimal Hamiltonian from the four-band Hamiltonian [27,36] constructed based on the density-functional theory. In Sec. III, we study linear conductivities including the Drude conductivity and the anomalous Hall conductivity. We show that the in-plane component of the spin-splitting vector is observable without applying a magnetic field. In Sec. IV, we study nonlinear conductivities including the nonlinear Drude conductivity, the quantum-metric induced nonlinear conductivity and the Berry curvature dipole induced nonlinear conductivity. The out-of-plane component of the spin-splitting vector is observable in the presence of magnetic field. Section V is devoted to discussions.

II. MODEL

A. Two-band Hamiltonian

We analyze a system where a p -wave magnet is placed on a substrate as illustrated in Fig. 1(a). The substrate breaks inversion symmetry and induces the Rashba interaction. Then, the minimal two-band Hamiltonian is given by

$$H = H_0 + H_\lambda + H_p + H_B. \quad (1)$$

The first term represents the kinetic energy of free electrons present in the interface,

$$H_0 = \frac{\hbar^2(k_x^2 + k_y^2)}{2m} \sigma_0, \quad (2)$$

where m is the free-fermion mass, and σ_0 is 2×2 identity matrix.

The second term represents the Rashba interaction,

$$H_\lambda = \lambda(-k_y \sigma_x + k_x \sigma_y), \quad (3)$$

where λ is the magnitude of the spin-orbit interaction, and σ_x and σ_y are the Pauli matrices for the spin. It is introduced by making an interface between the p -wave magnet and the substrate, breaking inversion symmetry.

The third term

$$H_p(\mathbf{k}) = J(\mathbf{s} \cdot \boldsymbol{\sigma})k_x \quad (4)$$

describes the effect of the p -wave magnet on electrons in the interface [26,27,34,35,35,37,37,53], where J is the s - d coupling constant between the localized spin of the p -wave magnet and free electrons in the interface, and $\mathbf{s}$ is the three-dimensional constant unit vector parametrized as

$$\mathbf{s} = (\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta), \quad (5)$$

with constants Θ and Φ . It is called the spin-splitting vector. We set

$$J_x = J \sin \Theta \cos \Phi, \quad J_y = J \sin \Theta \sin \Phi, \quad J_z = J \cos \Theta. \quad (6)$$

A comment is in order on the spin-splitting vector. When the noncollinear helical spin texture leads to the p -wave spin-splitting band structure, the spin-splitting vector $\mathbf{s}$ is orthogonal to the helix rotation direction. Indeed, the generator of the helix is given by

$$\exp[i(\mathbf{s} \cdot \boldsymbol{\sigma})k_x x], \quad (7)$$

where the rotation angle of the helix is orthogonal to the spin-splitting vector. For example, if the spin rotates in the x - y plane, the spin-splitting vector is along the z axis.

The last term,

$$H_B = B\sigma_z, \quad (8)$$

describes the Zeeman term, which breaks time-reversal symmetry. This term may be introduced either by attaching a ferromagnet as illustrated in Fig. 1(b), or by applying external magnetic field. The breaking of time-reversal symmetry is necessary for nonzero nonlinear conductivities [50].

The energy spectrum of the Hamiltonian (1) is given by

$$\varepsilon_\pm = \frac{\hbar^2 k^2}{2m} \pm \sqrt{h_x^2 + h_y^2 + h_z^2}, \quad (9)$$

where

$$\begin{aligned} h_x &= -\lambda k_y + Jk_x \sin \Theta \cos \Phi, \\ h_y &= \lambda k_x + Jk_x \sin \Theta \sin \Phi, \\ h_z &= B + Jk_x \cos \Theta. \end{aligned} \quad (10)$$

We introduce the polar coordinate $k_x = k \cos \phi$ and $k_y = k \sin \phi$. In the first order of J and B , the energy is expanded as

$$\begin{aligned} \varepsilon_\pm &= \frac{\hbar^2 k^2}{2m} \\ &\pm \left(\frac{BJ}{\lambda} \cos \Theta \cos \phi + k\lambda - Jk \cos \phi \sin \Theta \sin(\phi - \Phi) \right). \end{aligned} \quad (11)$$

The band structure is shown in Fig. 2, where the spin-splitting vector is taken along the x axis. A Dirac cone exists at the momentum $k_x = k_y = 0$ as shown in Figs. 2(a) and 2(c), which is formed by the Rashba interaction. We set $\bar{m} \equiv m/\hbar^2$ in the following.

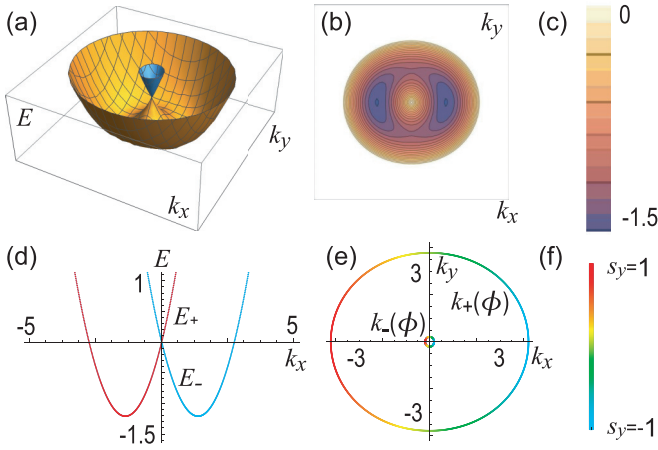


FIG. 2. (a) Bird's eye view of the band structure, where the spin-splitting vector is taken along the y axis with $\Theta = \pi/2$ and $\Phi = \pi/2$. (b) Its contour plot with the color palette (c). (d) Its cross section along the k_x axis. The horizontal axis is k_x in units of k_0 . (e) Fermi surfaces at $\mu = -0.2\epsilon_0$. Two ellipses represent p -wave symmetric Fermi surfaces $k_{\pm}(\phi)$, with $k_x = k \cos \phi$ and $k_y = k \sin \phi$. (f) The color palette for panels (d) and (e), where s_y is shown in the color. We have set $\hbar^2 k_0^2 / (2m) = \epsilon_0/2$, $\lambda = \epsilon_0/k_0$, and $J = 0.5\epsilon_0/k_0$ with $\epsilon_0 \equiv m\lambda^2/2\hbar^2$ and $k_0 = m\lambda/2\hbar^2$.

There are two Fermi surfaces characterized by $\eta = \pm$ for the energy ϵ_χ with $\chi = \pm$, which are obtained as

$$k_\eta^x(\phi) = -\chi \bar{m}\lambda + \eta \bar{m}J \cos \phi \sin \Theta \sin(\phi - \Phi) + \eta \left\{ \frac{2(\mu\lambda - \chi BJ \cos \Theta \cos \phi)}{\bar{m}\lambda} + [\lambda - J \cos \phi \sin \Theta \sin(\phi - \Phi)]^2 \right\}^{1/2}, \quad (12)$$

where $\eta = 1$ represents the outer Fermi surface and $\eta = -1$ represents the inner Fermi surface. The Fermi surfaces have the p -wave symmetry as shown in Fig. 2(b). There are two Fermi surfaces $k_{\pm}(\phi)$ as shown in Fig. 2(d), where $k_x = k \cos \phi$ and $k_y = k \sin \phi$.

B. Derivation of the two-band Hamiltonian

We discuss the relation between the minimal two-band Hamiltonian used in this work and the four-band Hamiltonian [27,36] derived for the p -wave magnet CeNiAsO based on the density-functional theory. The four-band Hamiltonian reads

$$H = 2t \left(\tau_x \cos \frac{ak_x}{2} + \cos ak_y \right) + 2t_J \left(\sigma_x \tau_y \sin \frac{ak_x}{2} + \sigma_y \tau_z \cos ak_y \right), \quad (13)$$

where σ and τ correspond to the spin and site degrees of freedom, $t = 0.8$ eV is the magnitude of the spin-independent hopping term, $t_J = 0.2$ eV is the magnitude of the exchange-dependent hopping term, and $a = 8.1$ Å is the lattice constant. The band structure is shown in Fig. 3(a).

We first diagonalize the Hamiltonian (13) at the Γ point,

$$U_0 H(k_x = 0, k_y = 0) U_0^{-1} = \text{diag}\{E_-, E_-, E_+, E_+\}, \quad (14)$$

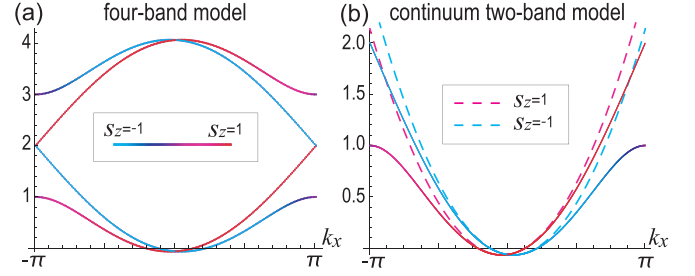


FIG. 3. (a) The band structure of the four-band model (13). The color shows the s_z component, where the color palette is shown in the inset. (b) Comparison of the band structures of the lowest two bands between the four-band model (13) and the two-band continuum model (22). The solid curves are based on the four-band Hamiltonian (13). Dashed curves are based on the two-band continuum Hamiltonian (22), where $s_z = 1$ is colored in magenta and $s_z = -1$ is colored in cyan. We have set $t = 1$ and $t_J = 0.25$.

where the eigenenergies are

$$E_{\pm} = 2(t \pm \sqrt{t^2 + t_J^2}), \quad (15)$$

with twofold degeneracy. The eigenvectors are given by

$$\psi_{\pm}^a = c_a \left\{ \pm \frac{it_J}{t}, \pm \frac{\sqrt{t^2 + t_J^2}}{t}, 0, 1 \right\}^t, \quad (16)$$

$$\psi_{\pm}^b = c_b \left\{ \pm \frac{\sqrt{t^2 + t_J^2}}{t}, \pm \frac{it_J}{t}, 1, 0 \right\}^t, \quad (17)$$

where c_a and c_b are normalization constants. The unitary transformation U_0 in Eq. (14) is given by

$$U_0 = \{\psi_-^a, \psi_-^b, \psi_+^a, \psi_+^b\}, \quad (18)$$

with the use of the eigenvectors.

Then, we make a unitary transformation with the use of the matrix U_0 in the vicinity of the Γ point,

$$U_0 H(k_x, k_y) U_0^{-1}. \quad (19)$$

Taking the lowest two bands, we obtain

$$H_{\text{eff}} = -2|t| \cos \frac{ak_x}{2} + \frac{t_J^2}{|t|} \left(\cos \frac{ak_x}{2} - 2 \cos ak_y \right) + \frac{2t_J^2}{t} \sin \frac{ak_x}{2} \sigma_z, \quad (20)$$

up to the second order in J . It is expanded in k_x and k_y as

$$H_{\text{eff}} = -2|t| \left(1 - \frac{(ak_x)^2}{8} \right) + \frac{t_J^2}{|t|} \left[\left(1 - \frac{(ak_x)^2}{8} \right) - 2 \left(1 - \frac{(ak_y)^2}{2} \right) \right] + \frac{t_J^2}{t} ak_x \sigma_z, \quad (21)$$

which is summarized as

$$H_{\text{eff}} = \frac{\hbar^2 k_x^2}{2m_x} + \frac{\hbar^2 k_y^2}{2m_y} + Jk_x \sigma_z + \text{const}, \quad (22)$$

with

$$\frac{\hbar^2}{2m_x} \equiv \frac{a^2}{4} + \frac{t_J^2 a^2}{4|t|}, \quad \frac{\hbar^2}{2m_y} \equiv \frac{t_J^2 a^2}{|t|}, \quad (23)$$

$$J \equiv \frac{t_J^2}{t} a. \quad (24)$$

It is essentially identical to the term (4) of the minimal two-band Hamiltonian with $s = (0, 0, 1)$ except for the mass anisotropy for $m_x \neq m_y$. The mass anisotropy does not play any role in the p -wave magnet by scaling $k_y/\sqrt{m_y} \mapsto k'_y/\sqrt{m_x}$. Note that there is no Rashba interaction in this model. The two-band model (22) reproduces well the lowest two bands of the four-band model (13) in a wide range of k_x around the Γ point, as shown in Fig. 3(b).

C. Material parameters

We summarize material parameters in the Hamiltonian (1). The mass is given by $m = \hbar^2/(ta^2)$, where it is estimated [27,36,37] that the hopping amplitude $t = 1$ eV and the lattice constant $a = 8.1$ Å. The magnitude of the band splitting is estimated [27,36] as 200 meV. The strength of the Rashba interaction is $\lambda = 0.03$ eV Å at the Ag(111) surface [54], $\lambda = 0.07$ eV Å at the InGaAs/InAlAs surface [55], and $\lambda = 0.33$ eV Å at the Au(111) surface [56]. A typical momentum is $k = \bar{m}\lambda$ from Eq. (12). We may argue that the energy of this Hamiltonian is estimated as $\bar{m}\lambda^2$, which is of the order of 1 meV by using $\lambda = 0.33$ eV Å.

III. LINEAR CONDUCTIVITY

A. Drude conductivity

We first study linear conductivities. We set $B = 0$ in the Hamiltonian (8) in this section. The Drude conductivity is given by the formula [57]

$$\sigma^{a:b} = \frac{e^2 \tau}{\hbar^2} \sum_n \int d^2 k f_n \frac{\partial^2 \varepsilon_n}{\partial k_a \partial k_b}, \quad (25)$$

where $f_n = 1/\{\exp[(\varepsilon_n - \mu)/k_B T] + 1\}$ is the Fermi distribution function for the band n , ε_n is the energy of the band n , μ is the chemical potential, k_B is the Boltzmann constant, and T is the temperature. We calculate the Drude conductivity numerically for a wide range of J/λ in Figs. 4(b) and 4(d). To understand the results analytically, we use a perturbation theory in J/λ assuming $|J| \ll \hbar^2 k_0/(2\bar{m})$ with $k_0 \equiv \bar{m}\lambda/2\hbar^2$.

The longitudinal Drude conductivity is calculated up to the second order in J/λ as

$$\begin{aligned} \sigma_{\text{Drude}}^{x:x} = \frac{e^2 \tau}{\hbar^2} & \left[2\pi \lambda \bar{m} \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}} + \frac{\pi J_y (\bar{m}\lambda^2 - \mu)}{\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}} \right. \\ & \left. + \frac{5\pi \lambda J_y^2 (\bar{m}\lambda^2 + 3\mu)}{(\lambda^2 + \frac{2\mu}{\bar{m}})^{3/2}} \right] \end{aligned} \quad (26)$$

for $-\bar{m}\lambda^2/2 < \mu < 0$, and

$$\begin{aligned} \sigma_{\text{Drude}}^{x:x} = \frac{e^2 \tau}{\hbar^2} \pi & \left\{ 4\mu + 2\bar{m}\lambda^2 + \bar{m}\lambda J_y \right. \\ & \left. + \frac{\bar{m}}{2} [J_z^2 + 3(J_x^2 + J_y^2)] \right\} \end{aligned} \quad (27)$$

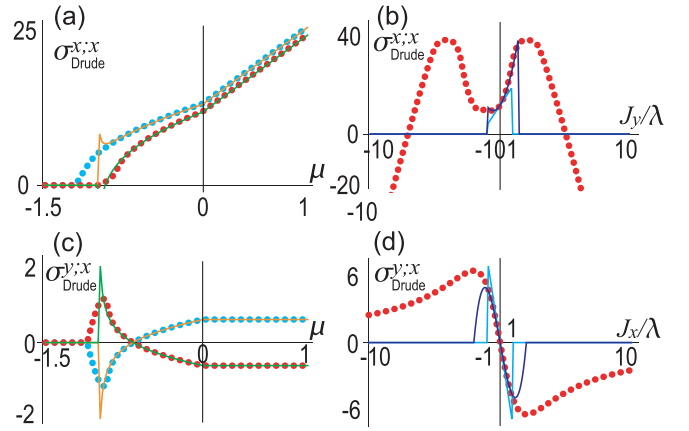


FIG. 4. (a), (b) Longitudinal Drude conductivity $\sigma_{\text{Drude}}^{x:x}$ in units of $e^2 \tau / \hbar^2$, where the spin-splitting vector is taken along the y axis. (c), (d) Transverse Drude conductivity $\sigma_{\text{Drude}}^{y:x}$ in the unit of $e^2 \tau / \hbar^2$, where the spin-splitting vector is taken along the x axis. (a), (c) The horizontal axis is the chemical potential μ in units of ε_0 . (b) The horizontal axis is J_y/λ . (d) The horizontal axis is J_x/λ . (a), (c) Red (cyan) dots indicate numerically obtained results by setting $J = 0.1\varepsilon_0/k_0$ ($J = -0.1\varepsilon_0/k_0$). Green (orange) curves indicate analytically obtained results up to linear order in J/λ by setting $J = 0.1\varepsilon_0/k_0$ ($J = -0.1\varepsilon_0/k_0$). (b), (d) Red dots (cyan lines) indicate numerically (analytically up to linear order in J/λ) obtained results by setting $\mu = -0.2\varepsilon_0$. Blue curves represent analytic results up to second order in J/λ in panel (b) and up to the third order in J/λ in panel (d). We set $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$ and $\lambda = \varepsilon_0/k_0$.

for $\mu > 0$. It is shown as a function of the chemical potential μ in Fig. 4(a) and as a function of J_y in Fig. 4(b). Hence, J_y is detectable by means of the longitudinal linear Drude conductivity. The sign change occurs in $\sigma_{\text{Drude}}^{x:x}$ around at $J_y \simeq 5\lambda$ as shown in Fig. 4(b).

The transverse Drude conductivity is calculated as

$$\sigma_{\text{Drude}}^{y:x} = -\frac{e^2 \tau}{\hbar^2} \pi J_x \left(\frac{3\mu + \bar{m}\lambda^2}{\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}} + J_y \frac{\mu(3\mu + 2\bar{m}\lambda^2)}{\bar{m}\lambda(\lambda^2 + \frac{2\mu}{\bar{m}})^{3/2}} \right) \quad (28)$$

up to the second order in J/λ for $-\bar{m}\lambda^2/2\mu < 0$, and

$$\sigma_{\text{Drude}}^{y:x} = -\frac{e^2 \tau}{\hbar^2} \bar{m} \pi J_x \left(\lambda - \frac{J_x^2 + J_y^2}{4\lambda} \right) \quad (29)$$

up to the third order in J/λ for $\mu > 0$. It is proportional to J_x . It is shown as a function of the chemical potential μ in Fig. 4(c) and as a function of J_x in Fig. 4(d). Hence, J_x is also detectable by means of the transverse linear Drude conductivity. There is no sign change in $\sigma_{\text{Drude}}^{y:x}$ as a function of J_x as shown in Fig. 4(d) in contrast with $\sigma_{\text{Drude}}^{x:x}$.

We note that the sign change occurs at $\mu = -\bar{m}\lambda^2/3$ in Fig. 4(c), which is understood analytically in Eq. (28).

Quantitative differences between the analytic results based on the perturbation theory and numerical results exist in the vicinity of the band bottom. It is because we assume two Fermi surfaces surrounding the Γ point ($k_x = k_y = 0$) as in Eq. (12) in analytic calculations. However, actually, there are

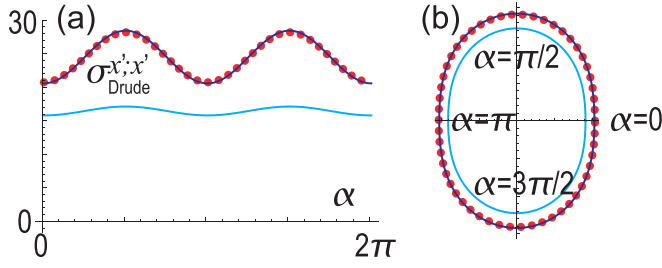


FIG. 5. (a) $\sigma_{\text{Drude}}^{x';x'}$ as a function of angle α . The vertical axis is $\sigma_{\text{Drude}}^{x';x'}$ in units of $e^2\tau/\hbar^2$. (b) Corresponding polar plot. The cyan curves represent analytic results Eq. (35) up to first order in J/λ , while the dark blue curves represent analytic results Eq. (35) up to second order in J/λ . Red dots indicate numerically obtained results. We have set $J = 0.5\varepsilon_0/k_0$, $\Theta = \pi/2$, $\Phi = \pi/2$, $\hbar^2k_0^2/(2m) = \varepsilon_0/2$, $\mu = 0.2\varepsilon_0$, and $\lambda = \varepsilon_0/k_0$.

two Fermi surfaces, which do not surround the Γ point as shown in Fig. 2(b). This is the reason for the discrepancy.

Next, we study the angle dependence of the Drude conductivity, where the applied electric field and the lead to read out the conductivity are rotated through the angle α ,

$$k'_x = k_x \cos \alpha - k_y \sin \alpha, \quad (30)$$

$$k'_y = k_x \sin \alpha + k_y \cos \alpha. \quad (31)$$

The derivatives are

$$\frac{\partial}{\partial k'_x} = \cos \alpha \frac{\partial}{\partial k_x} - \sin \alpha \frac{\partial}{\partial k_y}, \quad (32)$$

$$\frac{\partial}{\partial k'_y} = \sin \alpha \frac{\partial}{\partial k_x} + \cos \alpha \frac{\partial}{\partial k_y}. \quad (33)$$

The Drude conductivity along the x' direction is calculated as

$$\sigma_{\text{Drude}}^{x';x'} = \sigma_{\text{Drude}}^{xx} \cos^2 \alpha - \sigma_{\text{Drude}}^{yx} \sin 2\alpha + \sigma_{\text{Drude}}^{yy} \sin^2 \alpha, \quad (34)$$

or

$$\begin{aligned} \sigma_{\text{Drude}}^{x';x'} &= -\frac{e^2\tau}{\hbar^2} \pi \left\{ 2\bar{m}\lambda^2 + 4\mu - \bar{m}\lambda[J_x \sin 2\alpha + J_y(\cos 2\alpha - 2)] \right. \\ &\quad \left. - \frac{\bar{m}}{2}[(\cos 2\alpha - 4)(J_x^2 + J_y^2) + (\cos 2\alpha - 2)J_z^2] \right\} \end{aligned} \quad (35)$$

up to second order in J/λ for $\mu > 0$. The angle dependence of the Drude conductivity is shown in Fig. 5. It has p -wave symmetry, as shown in Fig. 5(b).

B. Anomalous Hall conductivity

In the following, we consider the case $B \neq 0$ in the Hamiltonian (8).

The anomalous Hall conductivity is given by the formula

$$\sigma_{\text{AHE}}^{y;x} = \frac{e^2}{\hbar} \sum_n \int d^2k f_n \Omega_n, \quad (36)$$

where the Berry curvature $\Omega_{\pm}$ is defined by

$$\Omega_{\pm}(\mathbf{k}) \equiv \partial_x A_{\pm}^y(\mathbf{k}) - \partial_y A_{\pm}^x(\mathbf{k}), \quad (37)$$

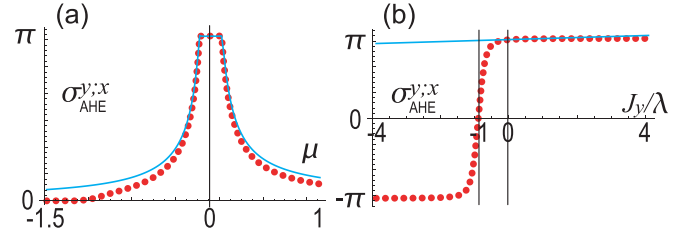


FIG. 6. (a) μ dependence of anomalous Hall conductivities $\sigma_{\text{AHE}}^{y;x}$, where the spin-splitting vector is taken along the y axis. The horizontal axis is the chemical potential μ in units of ε_0 . We set $J = -0.1\varepsilon_0/k_0$. (b) J_y/λ dependence of anomalous Hall conductivities $\sigma_{\text{AHE}}^{y;x}$, where the spin-splitting vector is taken along the y axis. The horizontal axis is J_y/λ . We set $\mu = 0$. Colored dots and curves are explained in the caption of Fig. 4. We set $\hbar^2k_0^2/(2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, and $B = 0.1\varepsilon_0$.

with the Berry connection

$$A_{\pm}^{\mu}(\mathbf{k}) \equiv i\langle\psi_{\pm}(\mathbf{k})|\partial_{k_{\mu}}|\psi_{\pm}(\mathbf{k})\rangle. \quad (38)$$

The Berry curvature $\Omega_{\pm}$ for the band $\varepsilon_{\pm}$ is calculated as

$$\Omega_{\pm} = \pm \frac{B}{4\lambda^2 k^3} [2\lambda + 3J_x \sin 2\phi - J_y(3 \cos 2\phi - 1)] \quad (39)$$

up to first order in $B/\bar{m}\lambda^2$ and J/λ . The anomalous Hall conductivity is obtained as

$$\sigma_{\text{AHE}}^{y;x} = \frac{e^2 B \pi}{\hbar \lambda} \left(\frac{\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}}{\mu} - \frac{J_y}{m\lambda\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}} \right)$$

for $\mu < -|B|$,

$$\sigma_{\text{AHE}}^{y;x} = \frac{e^2 B \pi}{\hbar \mu} \quad (40)$$

for $\mu > |B|$,

$$\begin{aligned} \sigma_{\text{AHE}}^{y;x} &= \pi \text{sgn}[B] - \frac{B}{\sqrt{B^2 + 2\bar{m}\lambda^2(\mu + \bar{m}\lambda\Lambda)}} \\ &\quad + J_y \frac{\pi B \bar{m}^2 \lambda^2 \Lambda (2\mu + \bar{m}\lambda\Lambda)}{2(2\mu + \bar{m}\lambda^2)(B^2 + 2\bar{m}\lambda^2(\mu + \bar{m}\lambda\Lambda))} \end{aligned} \quad (41)$$

for $|\mu| < |B|$ with

$$\Lambda \equiv \lambda + \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}. \quad (42)$$

The anomalous Hall conductivity is almost quantized to be $\pm\pi$ but slightly deviated from its quantized value. It is because there is a Fermi surface for $|\mu| < |B|$. Note that exact quantization occurs in the absence of a Fermi surface between the Dirac gap.

The anomalous Hall conductivity is shown as a function of μ in Fig. 6(a) and as a function of J_y/λ in Fig. 6(b).

IV. NONLINEAR CONDUCTIVITY

The second-order nonlinear conductivity $\sigma^{ab;c}$ is expanded in terms of the electron relaxation time τ as [50]

$$\sigma^{ab;c} = \sigma_{\text{Metric}}^{ab;c} + \sigma_{\text{Dipole}}^{ab;c} + \sigma_{\text{NLDrude}}^{ab;c}, \quad (43)$$

where

$$\sigma_{\text{Metric}}^{ab;c} \propto \tau^0, \quad \sigma_{\text{Dipole}}^{ab;c} \propto \tau, \quad \sigma_{\text{NLDrude}}^{ab;c} \propto \tau^2. \quad (44)$$

First, only the term $\sigma_{\text{Metric}}^{ab;c}$ survives in the dirty limit $\tau \rightarrow 0$, which is the intrinsic nonlinear conductivity. It is the quantum-metric induced nonlinear conductivity given by [50]

$$\begin{aligned} \sigma_{\text{Metric}}^{ab;c} &= -\frac{e^3}{\hbar} \sum_n \int d^2k f_n \left[2 \frac{\partial G_n^{ab}}{\partial k_c} - \frac{1}{2} \left(\frac{\partial G_n^{bc}}{\partial k_a} + \frac{\partial G_n^{ac}}{\partial k_b} \right) \right], \end{aligned} \quad (45)$$

where G_n^{ab} is the band-energy normalized quantum metric or the Berry connection polarizability. It is given as [40,42,45,49,50,58]

$$G_n^{ab} = 2\text{Re} \sum_{m \neq n} \frac{A_{nm}^a(\mathbf{k}) A_{mn}^b(\mathbf{k})}{\varepsilon_n(\mathbf{k}) - \varepsilon_m(\mathbf{k})}, \quad (46)$$

with ε_n being the energy of band n , and A_{nm}^a being the inter-band Berry connection

$$A_{nm}^a(\mathbf{k}) = i \langle \psi_n(\mathbf{k}) | \partial_{k_a} | \psi_m(\mathbf{k}) \rangle. \quad (47)$$

Second, $\sigma_{\text{Dipole}}^{ab;c}$ is the nonlinear transverse conductivity induced by the Berry curvature dipole (BCD) [41],

$$\sigma_{\text{Dipole}}^{ab;c} = -\frac{e^3 \tau}{\hbar^2} \sum_n \int d^2k f_n \left(\frac{\partial \Omega_n^{bc}}{\partial k_a} + \frac{\partial \Omega_n^{ac}}{\partial k_b} \right), \quad (48)$$

with the Berry curvature

$$\Omega_n^{ab} \equiv \partial_a A_{nn}^b(\mathbf{k}) - \partial_b A_{nn}^a(\mathbf{k}). \quad (49)$$

It is an extrinsic nonlinear conductivity, since it vanishes as $\tau \rightarrow 0$.

Third, $\sigma_{\text{NLDrude}}^{ab;c}$ is the nonlinear Drude conductivity [59],

$$\sigma_{\text{NLDrude}}^{ab;c} = -\frac{e^3 \tau^2}{\hbar^3} \sum_n \int d^2k f_n \frac{\partial^3 \varepsilon_n}{\partial k_a \partial k_b \partial k_c}. \quad (50)$$

It is also an extrinsic nonlinear conductivity.

A. Nonlinear longitudinal conductivity

We proceed to study nonlinear conductivities. The band structure is shown in Fig. 7, where the spin-splitting vector is taken along the z axis. A Dirac cone at the momentum $k_x = k_y = 0$ has a gap with $2|B|$ as shown in Figs. 7(a) and 7(c). There are two Fermi surfaces as in Fig. 7(d) formed by the lower band ε_- for $\mu < -|B|$, and formed by the lower band ε_- and the upper band ε_+ for $\mu > |B|$, while there is a Fermi surface formed by the lower band E_- for $|\mu| < |B|$, as shown in Fig. 7(c).

We first study nonlinear longitudinal conductivity. There are two contributions. One is the nonlinear Drude conductivity [60] and the other is the quantum-metric induced nonlinear conductivity. We use a perturbation theory in J/λ and $B/\bar{m}\lambda^2$ assuming $|J| \ll \hbar^2 k_0/(2\bar{m})$ and $|B| \ll \hbar^2 k_0^2/(2\bar{m})$.

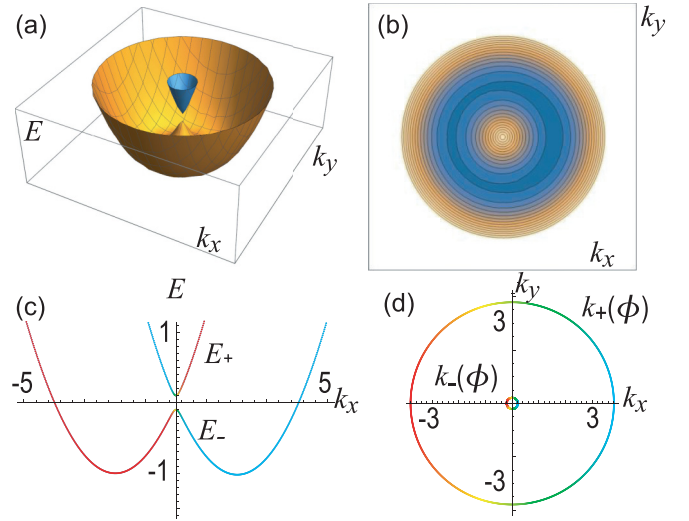


FIG. 7. (a) Bird's eye view of the band structure, where the spin-splitting vector is taken along the z axis with $\Theta = 0$ and $\Phi = 0$. (b) Its contour plot. The color palette is given in Fig. 2(c). (c) Its cross section along the k_x axis. The horizontal axis is k_x in units of k_0 . (d) Fermi surfaces at $\mu = -0.2\varepsilon_0$. The color palette for panels (c) and (d) is given in Fig. 2(f). We set $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, $J = 0.5\varepsilon_0/k_0$, $B = 0.1\varepsilon_0$, and $k_0 = \bar{m}\lambda/2\hbar^2$. Two Fermi surfaces are presented as $k_{\pm}(\phi)$, where $k_x = k \cos \phi$ and $k_y = k \sin \phi$.

The nonlinear Drude conductivity at the chemical potential μ is analytically calculated based on Eq. (50) as

$$\sigma_{\text{NLDrude}}^{xx;x} = \frac{e^3 \tau^2}{\hbar^3} \frac{3\pi \xi(\mu)}{2\bar{m}\lambda \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}} B J_z, \quad (51)$$

where $\xi(\mu) = 1$ for $|\mu| > |B|$, and $\xi(\mu) = 1/2$ for $|\mu| < |B|$. It is valid for $\mu > -\bar{m}\lambda^2/2$. It is proportional to J_z .

On the other hand, the quantum-metric induced nonlinear conductivity is analytically obtained based on Eq. (45) as

$$\sigma_{\text{Metric}}^{xx;x} = \frac{e^3}{\hbar} \frac{5\pi(\mu - \bar{m}\lambda^2)}{8\mu^2 \bar{m}^2 \lambda^3 \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}} B J_z \quad (52)$$

for $|\mu| > |B|$, and

$$\sigma_{\text{Metric}}^{xx;x} = \frac{e^3}{\hbar} \frac{5\pi \left[\bar{m}\lambda \left(\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}} + \lambda \right) - 2\mu \right]}{8\bar{m}^3 \lambda^3 (\bar{m}\lambda^2 - 2\mu) \left(\sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}} + \lambda \right)^3} B J_z \quad (53)$$

for $|\mu| < |B|$. It is also proportional to J_z . Hence, J_z is observable. The leading order of the nonlinear conductivity is proportional to BJ , which means that magnetization B is necessary to detect J_z .

We show analytical results based on the perturbation theory and numerical results without using the perturbation theory. The nonlinear conductivity is shown as a function of the chemical potential μ in Fig. 8. They agree with each other very well, which assures the validity of the perturbation theory. The nonlinear Drude conductivity diverges at the band bottom $\mu = -\bar{m}\lambda^2/2$ and its value becomes half inside of the bulk gap $|\mu| < |B|$ comparing with the value outside of the

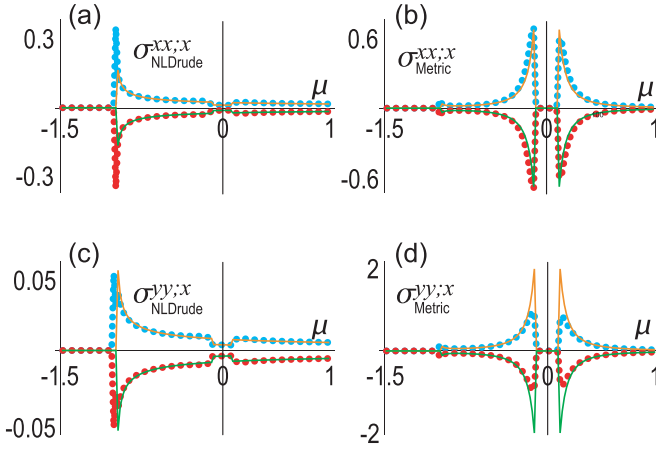


FIG. 8. μ dependence of nonlinear conductivities, where the spin-splitting vector is taken along the z axis. (a) Nonlinear longitudinal Drude conductivity $\sigma_{\text{NLDru}}^{\text{xxx}}$ in units of $e^3 \tau^2 \varepsilon_0 / (\hbar^3 k_0)$. (b) Nonlinear transverse Drude conductivity $\sigma_{\text{NLDru}}^{\text{yyx}}$ in units of $e^3 \tau^2 \varepsilon_0 / (\hbar^3 k_0)$. (c) Quantum-metric induced nonlinear longitudinal conductivity $\sigma_{\text{Metric}}^{\text{xxx}}$ in units of $e^3 / (\hbar \varepsilon_0 k_0)$. (d) Quantum-metric induced nonlinear transverse conductivity $\sigma_{\text{Metric}}^{\text{yyx}}$ in units of $e^3 / (\hbar \varepsilon_0 k_0)$. The horizontal axis is the chemical potential μ in units of ε_0 . Colored dots and curves are explained in the caption of Fig. 4. We set $\hbar^2 k_0^2 / (2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, $J = -0.1 \varepsilon_0/k_0$, and $B = 0.1 \varepsilon_0$.

bulk gap $|\mu| > |B|$. On the other hand, the quantum-metric induced nonlinear conductivity diverges at the band edge $|\mu| = |B|$ and takes a tiny value inside the band gap $|\mu| < |B|$. Therefore, the two contributions are differentiated although the explicit value of τ is unknown.

The nonlinear conductivities are shown as a function of J/λ in Figs. 9(a) and 9(b). The $\sigma_{\text{NLDru}}^{\text{xxx}}$ and $\sigma_{\text{Metric}}^{\text{xxx}}$ approach constant values.

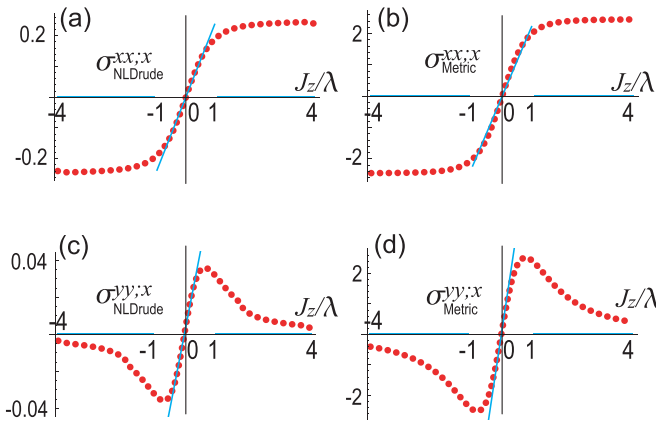


FIG. 9. J/λ dependence of nonlinear conductivities, where the spin-splitting vector is taken along the z axis. (a) Nonlinear longitudinal Drude conductivity $\sigma_{\text{NLDru}}^{\text{xxx}}$ in units of $e^3 \tau^2 E_0 / (\hbar^3 k_0)$. (b) Quantum-metric induced nonlinear longitudinal conductivity $\sigma_{\text{Metric}}^{\text{xxx}}$ in units of $e^3 / (\hbar E_0 k_0)$. (c) Nonlinear transverse Drude conductivity $\sigma_{\text{NLDru}}^{\text{yyx}}$ in units of $e^3 \tau^2 E_0 / (\hbar^3 k_0)$. (d) Quantum-metric induced nonlinear transverse conductivity $\sigma_{\text{Metric}}^{\text{yyx}}$ in units of $e^3 / (\hbar \varepsilon_0 k_0)$. The horizontal axis is the magnetization J_z/λ . Colored dots and curves are explained in the caption of Fig. 4. We have set $\hbar^2 k_0^2 / (2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, $B = -0.1 \varepsilon_0/k_0$, and $\mu = -0.2 \varepsilon_0$.

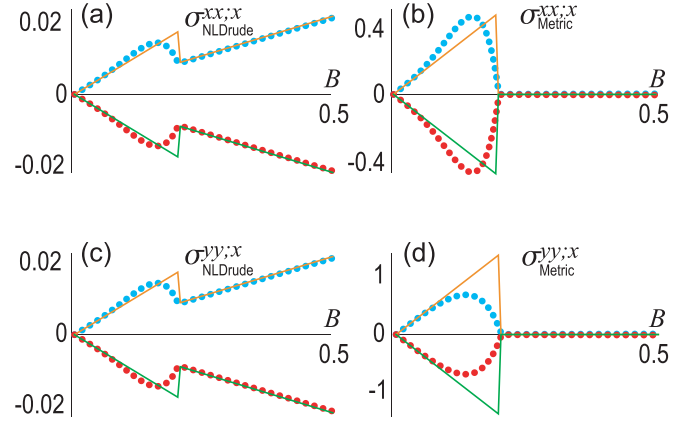


FIG. 10. B dependence of nonlinear conductivities, where the spin-splitting vector is taken along the z axis. (a) Nonlinear longitudinal Drude conductivity $\sigma_{\text{NLDru}}^{\text{xxx}}$ in units of $e^3 \tau^2 E_0 / (\hbar^3 k_0)$. (b) Quantum-metric induced nonlinear longitudinal conductivity $\sigma_{\text{Metric}}^{\text{xxx}}$ in units of $e^3 / (\hbar E_0 k_0)$. (c) Nonlinear transverse Drude conductivity $\sigma_{\text{NLDru}}^{\text{yyx}}$ in units of $e^3 \tau^2 E_0 / (\hbar^3 k_0)$. (d) Quantum-metric induced nonlinear transverse conductivity $\sigma_{\text{Metric}}^{\text{yyx}}$ in units of $e^3 / (\hbar \varepsilon_0 k_0)$. The horizontal axis is the magnetization B in units of ε_0 . Colored dots and curves are explained in the caption of Fig. 4. We set $\hbar^2 k_0^2 / (2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, $J = -0.1 \varepsilon_0/k_0$, and $\mu = -0.2 \varepsilon_0$.

The nonlinear conductivities are shown as a function of B in Figs. 10(a) and 10(b). They show nonmonotonic behavior and there is a sudden jump at $|B| = |\mu|$ because nonlinear conductivities depend on whether the chemical potential μ is outside the band gap $|B|$ or inside the band gap.

B. Nonlinear transverse conductivity

Next, we study nonlinear transverse conductivity. There are three contributions. Nonlinear Drude conductivity, the quantum-metric induced nonlinear conductivity, and the Berry curvature dipole induced nonlinear conductivity.

The nonlinear Drude conductivity at the chemical potential μ is analytically calculated based on Eq. (50) as

$$\sigma_{\text{NLDru}}^{\text{yyx}} = \frac{e^3 \tau^2}{\hbar^3} \frac{\pi}{2\tilde{m}\lambda \sqrt{\lambda^2 + \frac{2\mu}{\tilde{m}}}} B J_z \xi(\mu). \quad (54)$$

On the other hand, quantum-metric induced nonlinear conductivity is analytically obtained based on Eq. (45) as

$$\sigma_{\text{Metric}}^{\text{yyx}} = \frac{e^3}{\hbar} \frac{29\pi(\tilde{m}\lambda^2 + \mu)}{8\mu^2 \tilde{m}^2 \lambda^3 \sqrt{\lambda^2 + \frac{2\mu}{\tilde{m}}}} B J_z \quad (55)$$

for $|\mu| > |B|$, and

$$\sigma_{\text{Metric}}^{\text{yyx}} = \frac{e^3}{\hbar} \frac{29\pi \left[\tilde{m}\lambda \left(\sqrt{\lambda^2 + \frac{2\mu}{\tilde{m}}} + \lambda \right) - 2\mu \right]}{16m^3 \lambda^3 (\tilde{m}\lambda^2 - 2\mu) \left(\sqrt{\lambda^2 + \frac{2\mu}{\tilde{m}}} + \lambda \right)^3} B J_z \quad (56)$$

for $|\mu| < |B|$. Hence, we can also observe J_z by measuring the nonlinear transverse conductivity. The behavior of the nonlinear transverse conductivity as a function of μ is similar to that of the nonlinear longitudinal conductivity.

The nonlinear conductivities are shown as a function of J_z in Figs. 9(c) and 9(d). The $\sigma_{\text{NLDrude}}^{yy,x}$ and $\sigma_{\text{Metric}}^{yy,x}$ approach constant values. They are linear in J_z but decreases and approach zero for large J_z . However, there is no sign change in them.

The nonlinear conductivities are shown as a function of B in Figs. 10(c) and 10(d). They show nonmonotonic behavior and there is a sudden jump at $|B| = |\mu|$ as in the case of the nonlinear longitudinal conductivity. Quantitative differences exist in the vicinity of the gap $B = \mu$ in Fig. 10 because we made a perturbation with respect to B , which is invalid around the gap.

C. Berry curvature dipole

The Berry curvature dipole induced nonlinear conductivity is given by

$$\sigma_{\text{Dipole}}^{xx,y} = -2 \frac{e^3 \tau}{\hbar^2} \sum_n \int d^2 k f_n \frac{\partial \Omega_n^{yx}}{\partial k_x}. \quad (57)$$

By differentiating Eq. (39), we have

$$\begin{aligned} \frac{\partial \Omega_n^{yx}}{\partial k_x} = & - \frac{3Bk\lambda^4 k_x}{2(k^2\lambda^2 + B^2)^{5/2}} \\ & - J_z \frac{3B^2\lambda^2 \{2B^2 - \lambda^2[3k^2 + 5(k_x^2 - k_y^2)]\}}{4(k^2\lambda^2 + B^2)^{7/2}}, \end{aligned} \quad (58)$$

up to the first order in J/λ , when the spin-splitting vector is along the z direction.

The Berry curvature dipole induced nonlinear conductivity is analytically obtained as

$$\begin{aligned} \sigma_{\text{Dipole}}^{xx,y} = & \frac{J_z \bar{m} \pi}{2[B^2 + 2\bar{m}\lambda^2(\mu + \bar{m}\lambda\Lambda_{\pm})]^{5/2}} \\ & \times \left[\sum_{\pm} \frac{5\bar{m}^3\lambda^5\Lambda_{\pm}^4}{\Lambda_0} + \sum_{\pm} 6B^2\lambda^2(\mu + \bar{m}\lambda\Lambda_{\pm}) \right. \\ & \left. - \frac{\lambda^3\Lambda_{\pm}[3B^2 + 10\bar{m}\lambda^2(\mu + \bar{m}\lambda\Lambda_{\pm})]}{\Lambda_0} \right] \end{aligned} \quad (59)$$

for $|\mu| > |B|$ and

$$\sigma_{\text{Dipole}}^{xx,y} = 0 \quad (60)$$

for $|\mu| < |B|$, with

$$\Lambda_{\pm} \equiv \lambda \pm \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}, \quad (61)$$

$$\Lambda_0 \equiv \sqrt{\lambda^2 + \frac{2\mu}{\bar{m}}}. \quad (62)$$

The anomalous Hall conductivity is shown as a function of μ in Fig. 11(a) and as a function of J in Fig. 11(b).

V. DISCUSSION

p -wave magnets have zero-net magnetization, which may lead to a high-density and ultrafast memory by using the spin-splitting vector as the Ising variable as in the case of antiferromagnets and altermagnets. We found that J_x and J_y are detectable by the linear Drude conductivity without using magnetization. It contrasts with the d -wave altermagnet, where the Néel vector is not detectable by the linear Drude

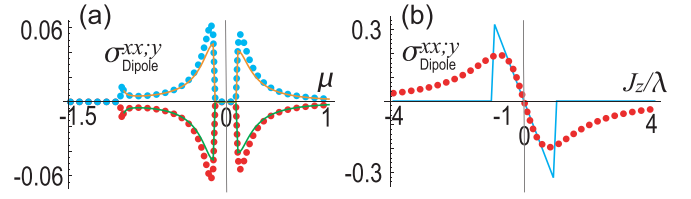


FIG. 11. (a) μ dependence of Berry curvature dipole induced nonlinear conductivities $\sigma_{\text{Dipole}}^{xx,y}$, where the spin-splitting vector is taken along the z axis. The horizontal axis is the chemical potential μ in units of ε_0 . We set $J = -0.1\varepsilon_0/k_0$. (b) J_z/λ dependence of Berry curvature dipole induced nonlinear conductivities $\sigma_{\text{Dipole}}^{xx,y}$, where the spin-splitting vector is taken along the z axis. The horizontal axis is J_z/λ . We set $\mu = 0$. Colored dots and curves are explained in the caption of Fig. 4. We set $\hbar^2 k_0^2/(2m) = \varepsilon_0/2$, $\lambda = \varepsilon_0/k_0$, and $B = 0.1\varepsilon_0$.

conductivity and the measurement of the nonlinear conductivity is necessary [61]. On the other hand, J_z is detectable by measuring the nonlinear conductivity with the aid of magnetization.

We have analyzed the Hamiltonian (1) numerically to obtain results for a wide range of J/λ , and perturbatively to obtain analytical results for the strong Rashba regime $|\lambda/J| \gg 1$. It is numerically checked that the sign of the conductivities are identical except for the longitudinal Drude conductivity. Hence, $J(\mathbf{s} \cdot \boldsymbol{\sigma})$ is detectable in a wider range of $|J/\lambda|$. In the current experimental stage, it is difficult to identify whether strong or weak Rashba limit is realized in a particular sample. We may expect the weak Rashba interaction if the p -wave magnet is metallic, while we may expect the strong Rashba interaction if the p -wave magnet is insulating and the free fermions exist at the interface.

We make a comment on the spin-orbit interaction. We determined the direction of the spin-splitting vector by comparing it with the direction of the Rashba term by purely electrical means. Indeed, the energy is independent of the direction of the spin-splitting vector when there is no spin-orbit interaction, where the linear and nonlinear Drude conductivities are independent of the direction of the spin-splitting vector. Namely, the spin-orbit interaction is necessary to read out the direction of the spin-splitting vector.

ACKNOWLEDGMENTS

The author is very much grateful to S. Okumura and M. Hirschberger for helpful discussions on the subject. This work is supported by CREST, JST (Grant No. JPMJCR20T2) and Grants-in-Aid for Scientific Research from MEXT KAKENHI (Grant No. 23H00171).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

- [1] T. Jungwirth, X. Marti, P. Wadley, and J. Wunderlich, Antiferromagnetic spintronics, *Nat. Nanotechnol.* **11**, 231 (2016).
- [2] V. Baltz, A. Manchon, M. Tsoi, T. Moriyama, T. Ono, and Y. Tserkovnyak, Antiferromagnetic spintronics, *Rev. Mod. Phys.* **90**, 015005 (2018).
- [3] J. Han, R. Cheng, L. Liu, H. Ohno, and S. Fukami, Coherent antiferromagnetic spintronics, *Nat. Mater.* **22**, 684 (2023).
- [4] Z. Ni, A. V. Haglund, H. Wang, B. Xu, C. Bernhard, D. G. Mandrus, X. Qian, E. J. Mele, C. L. Kane, and L. Wu, Imaging the Néel vector switching in the monolayer antiferromagnet MnPSe_3 with strain-controlled Ising order, *Nat. Nanotechnol.* **16**, 782 (2021).
- [5] J. Godinho, H. Reichlov, D. Kriegner, V. Novak, K. Olejnik, Z. Kašpar, Z. Šoban, P. Wadley, R. P. Campion, R. M. Otxoa, P. E. Roy, J. Železný, T. Jungwirth, and J. Wunderlich, Electrically induced and detected Néel vector reversal in a collinear antiferromagnet, *Nat. Commun.* **9**, 4686 (2018).
- [6] K. Kimura, Y. Otake, and T. Kimura, Visualizing rotation and reversal of the Néel vector through antiferromagnetic trichroism, *Nat. Commun.* **13**, 697 (2022).
- [7] Y.-H. Zhang, T.-C. Chuang, D. Qu, and S.-Y. Huang, Detection and manipulation of the antiferromagnetic Néel vector in Cr_2O_3 , *Phys. Rev. B* **105**, 094442 (2022).
- [8] L. Šmejkal, A. H. MacDonald, J. Sinova, S. Nakatsuji, and T. Jungwirth, Anomalous Hall antiferromagnets, *Nat. Rev. Mater.* **7**, 482 (2022).
- [9] L. Šmejkal, J. Sinova, and T. Jungwirth, Beyond conventional ferromagnetism and antiferromagnetism: A phase with nonrelativistic spin and crystal rotation symmetry, *Phys. Rev. X* **12**, 031042 (2022).
- [10] L. Šmejkal, J. Sinova, and T. Jungwirth, Emerging research landscape of altermagnetism, *Phys. Rev. X* **12**, 040501 (2022).
- [11] A. Fakhredine, R. M. Sattigeri, G. Cuono, and C. Autieri, Interplay between altermagnetism and nonsymmorphic symmetries generating large anomalous Hall conductivity by semi-Dirac points induced anticrossings, *Phys. Rev. B* **108**, 115138 (2023).
- [12] T. Tschirner, P. Keler, R. D. G. Betancourt, T. Kotte, D. Kriegner, B. Buechner, J. Dufouleur, M. Kamp, V. Jovic, L. Šmejkal, J. Sinova, R. Claessen, T. Jungwirth, S. Moser, H. Reichlova, and L. Veyrat, Saturation of the anomalous Hall effect at high magnetic fields in altermagnetic RuO_2 , *APL Mater.* **11**, 101103 (2023).
- [13] T. Sato, S. Haddad, I. C. Fulga, F. F. Assaad, and J. van den Brink, Altermagnetic anomalous Hall effect emerging from electronic correlations, *Phys. Rev. Lett.* **133**, 086503 (2024).
- [14] M. Leiviskä, J. Rial, A. Bad'ura, R. L. Seeger, I. Kounta, S. Beckert, D. Kriegner, I. Joumard, E. Schmoranzarová, J. Sinova, O. Gomonay, A. Thomas, S. T. B. Goennenwein, H. Reichlová, L. Šmejkal, L. Míchez, T. Jungwirth, and V. Baltz, Anisotropy of the anomalous Hall effect in the altermagnet candidate Mn_5Si_3 films, *Phys. Rev. B* **109**, 224430 (2024).
- [15] K.-H. Ahn, A. Hariki, K.-W. Lee, and J. Kunes, Antiferromagnetism in RuO_2 as d -wave Pomeranchuk instability, *Phys. Rev. B* **99**, 184432 (2019).
- [16] S. Hayami, Y. Yanagi, and H. Kusunose, Momentum-dependent spin splitting by collinear antiferromagnetic ordering, *J. Phys. Soc. Jpn.* **88**, 123702 (2019).
- [17] M. Naka, S. Hayami, H. Kusunose, Y. Yanagi, Y. Motome, and H. Seo, Spin current generation in organic antiferromagnets, *Nat. Commun.* **10**, 4305 (2019).
- [18] J. Krempaský, L. Šmejkal, S. W. D'Souza, M. Hajlaoui, G. Springholz, K. U. F. Alarab, P. C. Constantinou, V. Strocov, D. Usanov, W. R. Pudelko, R. Gonzalez-Hernandez, A. B. Hellenes, Z. Jansa, H. R. Z. Šob, R. D. G. Betancourt, P. Wadley, J. Sinova, D. Kriegner, J. Min, J. H. Dil, and T. Jungwirth, Altermagnetic lifting of Kramers spin degeneracy, *Nature (London)* **626**, 517 (2024).
- [19] S. Lee, S. Lee, S. Jung, J. Jung, D. Kim, Y. Lee, B. Seok, J. Kim, B. G. Park, L. Šmejkal, C.-J. Kang, and C. Kim, Broken Kramers degeneracy in altermagnetic MnTe , *Phys. Rev. Lett.* **132**, 036702 (2024).
- [20] O. Fedchenko, J. Minar, A. Akashdeep, S. W. D'Souza, D. Vasilyev, O. Tkach, L. Odenbreit, Q. L. Nguyen, D. Kutnyakhov, N. Wind, L. Wenthous, M. Scholz, K. Rossnagel, M. Hoesch, M. Aeschlimann, B. Stadtmueller, M. Klauui, G. Schoenhense, G. Jakob, T. Jungwirth, L. Šmejkal, J. Sinova, and H. J. Elmers, Observation of time-reversal symmetry breaking in the band structure of altermagnetic RuO_2 , *Sci. Adv.* **10**, eadj4883 (2024).
- [21] T. Osumi, S. Souma, T. Aoyama, K. Yamauchi, A. Honma, K. Nakayama, T. Takahashi, K. Ohgushi, and T. Sato, Observation of a giant band splitting in altermagnetic MnTe , *Phys. Rev. B* **109**, 115102 (2024).
- [22] Z. Lin, D. Chen, W. Lu, X. Liang, S. Feng, K. Yamagami, J. Osiecki, M. Leandersson, B. Thiagarajan, J. Liu, C. Felser, and J. Ma, Observation of giant spin splitting and d -wave spin texture in room temperature altermagnet RuO_2 , [arXiv:2402.04995](https://arxiv.org/abs/2402.04995).
- [23] R. Gonzalez-Hernandez, L. Šmejkal, K. Vborn, Y. Yahagi, J. Sinova, T. Jungwirth, and J. Železný, Efficient electrical spin splitter based on nonrelativistic collinear antiferromagnetism, *Phys. Rev. Lett.* **126**, 127701 (2021).
- [24] M. Naka, Y. Motome, and H. Seo, Perovskite as a spin current generator, *Phys. Rev. B* **103**, 125114 (2021).
- [25] A. Bose, N. J. Schreiber, R. Jain, D.-F. Shao, H. P. Nair, J. Sun, X. S. Zhang, D. A. Muller, E. Y. Tsybal, D. G. Schlom, and D. C. Ralph, Tilted spin current generated by the collinear antiferromagnet ruthenium dioxide, *Nat. Electron.* **5**, 267 (2022).
- [26] M. Ezawa, Third-order and fifth-order nonlinear spin-current generation in g -wave and i -wave altermagnets and perfectly nonreciprocal spin current in f -wave magnets, *Phys. Rev. B* **111**, 125420 (2025).
- [27] A. B. Hellenes, T. Jungwirth, R. Jaeschke-Ubiergo, A. Chakraborty, J. Sinova, and L. Šmejkal, p -wave magnets, [arXiv:2309.01607](https://arxiv.org/abs/2309.01607).
- [28] I. Martin and A. F. Morpurgo, Majorana fermions in superconducting helical magnets, *Phys. Rev. B* **85**, 144505 (2012).
- [29] S. Hayami, Y. Yanagi, and H. Kusunose, Spontaneous antisymmetric spin splitting in noncollinear antiferromagnets without spin-orbit coupling, *Phys. Rev. B* **101**, 220403(R) (2020).
- [30] S. Hayami, Y. Yanagi, and H. Kusunose, Bottom-up design of spin-split and reshaped electronic band structures in spin-orbit-coupling free antiferromagnets: Procedure on the basis of augmented multipoles, *Phys. Rev. B* **102**, 144441 (2020).

- [31] S. Hayami, Mechanism of antisymmetric spin polarization in centrosymmetric multiple- Q magnets based on bilinear and bi-quadratic spin cross products, *Phys. Rev. B* **105**, 024413 (2022).
- [32] S. Okumura, T. Morimoto, Y. Kato, and Y. Motome, Quadratic optical responses in a chiral magnet, *Phys. Rev. B* **104**, L180407 (2021).
- [33] Y. B. Kudasov, Topological band structure due to modified kramers degeneracy for electrons in a helical magnetic field, *Phys. Rev. B* **109**, L140402 (2024).
- [34] B. Brekke, P. Sukhachov, H. G. Giil, A. Brataas, and J. Linder, Minimal models and transport properties of unconventional p -wave magnets, *Phys. Rev. Lett.* **133**, 236703 (2024).
- [35] M. Ezawa, Topological insulators and superconductors based on p -wave magnets: Electrical control and detection of a domain wall, *Phys. Rev. B* **110**, 165429 (2024).
- [36] A. Chakraborty, A. B. Hellenes, R. Jaeschke-Ubiergo, T. Jungwirth, L. Šmejkal, and J. Sinova, Highly efficient non-relativistic Edelstein effect in nodal p -wave magnets, *Nat. Commun.* **16**, 7270 (2025).
- [37] M. Ezawa, Out-of-plane Edelstein effects: Electric-field induced magnetization in p -wave magnets, *Phys. Rev. B* **111**, L161301 (2025).
- [38] R. Yamada, M. T. Birch, P. R. Baral, S. Okumura, R. Nakano, S. Gao, Y. Ishihara, K. K. Kolincio, I. Belopolski, H. Sagayama, H. Nakao, K. Ohishi, T. Nakajima, Y. Tokura, T.-H. Arima, Y. Motome, M. M. Hirschmann, and M. Hirschberger, Gapping the spin-nodal planes of an anisotropic p -wave magnet to induce a large anomalous Hall effect, [arXiv:2502.10386](https://arxiv.org/abs/2502.10386) [Nature (to be published)].
- [39] Q. Song, S. Stavric, P. Barone, A. Droghetti, D. S. Antonenko, J. W. F. Venderbos, C. A. Occhialini, B. Ilyas, E. Ergenc, N. Gedik, S.-W. Cheong, R. M. Fernandes, S. Picozzi, and R. Comin, Electrical switching of a p -wave magnet, *Nature (London)* **642**, 64 (2025).
- [40] Y. Gao, S. A. Yang, and Q. Niu, Field induced positional shift of Bloch electrons and its dynamical implications, *Phys. Rev. Lett.* **112**, 166601 (2014).
- [41] I. Sodemann and L. Fu, Quantum nonlinear Hall effect induced by Berry curvature dipole in time-reversal invariant materials, *Phys. Rev. Lett.* **115**, 216806 (2015).
- [42] H. Liu, J. Zhao, Y.-X. Huang, W. Wu, X.-L. Sheng, C. Xiao, and S. A. Yang, Intrinsic second-order anomalous Hall effect and its application in compensated antiferromagnets, *Phys. Rev. Lett.* **127**, 277202 (2021).
- [43] Y. Michishita and N. Nagaosa, Dissipation and geometry in nonlinear quantum transports of multiband electronic systems, *Phys. Rev. B* **106**, 125114 (2022).
- [44] H. Watanabe and Y. Yanase, Nonlinear electric transport in odd-parity magnetic multipole systems: Application to Mn-based compounds, *Phys. Rev. Res.* **2**, 043081 (2020).
- [45] C. Wang, Y. Gao, and D. Xiao, Intrinsic nonlinear Hall effect in antiferromagnetic tetragonal CuMnAs, *Phys. Rev. Lett.* **127**, 277201 (2021).
- [46] R. Oiwa and H. Kusunose, Systematic analysis method for non-linear response tensors, *J. Phys. Soc. Jpn.* **91**, 014701 (2022).
- [47] A. Gao, Y.-F. Liu, J.-X. Qiu, B. Ghosh, T. V. Trevisan, Y. Onishi, C. Hu, T. Qian, H.-J. Tien, S.-W. Chen *et al.*, Quantum metric nonlinear Hall effect in a topological antiferromagnetic heterostructure, *Science* **381**, 181 (2023).
- [48] N. Wang, D. Kaplan, Z. Zhang, T. Holder, N. Cao, A. Wang, X. Zhou, F. Zhou, Z. Jiang, C. Zhang *et al.*, Quantum metric-induced nonlinear transport in a topological antiferromagnet, *Nature (London)* **621**, 487 (2023).
- [49] K. Das, S. Lahiri, R. B. Atencia, D. Culcer, and A. Agarwal, Intrinsic nonlinear conductivities induced by the quantum metric, *Phys. Rev. B* **108**, L201405 (2023).
- [50] D. Kaplan, T. Holder, and B. Yan, Unification of nonlinear anomalous Hall effect and nonreciprocal magnetoresistance in metals by the quantum geometry, *Phys. Rev. Lett.* **132**, 026301 (2024).
- [51] Y. D. Wang, Z. F. Zhang, Z.-G. Zhu, and G. Su, Intrinsic non-linear Ohmic current, *Phys. Rev. B* **109**, 085419 (2024).
- [52] L. Xiang, B. Wang, Y. Wei, Z. Qiao, and J. Wang, Linear displacement current solely driven by the quantum metric, *Phys. Rev. B* **109**, 115121 (2024).
- [53] K. Maeda, B. Lu, K. Yada, Y. Tanaka, Theory of tunneling spectroscopy in p -wave altermagnet-superconductor hybrid structures, *J. Phys. Soc. Jpn.* **93**, 114703 (2024).
- [54] H. Cercellier, C. Didiot, Y. Fagot-Revurat, B. Kierren, L. Moreau, and D. Malterre, Interplay between structural, chemical, and spectroscopic properties of Ag/Au 111 epitaxial ultrathin films: A way to tune the Rashba coupling, *Phys. Rev. B* **73**, 195413 (2006).
- [55] J. Nitta, T. Akazaki, H. Takayanagi, and T. Enoki, Gate control of spin-orbit interaction in an inverted InGaAs/InAs heterostructure, *Phys. Rev. Lett.* **78**, 1335 (1997).
- [56] S. LaShell, B. A. McDougall, and E. Jensen, Spin splitting of an Au(111) surface state band observed with angle resolved photoelectron spectroscopy, *Phys. Rev. Lett.* **77**, 3419 (1996).
- [57] Y. Fang, J. Cano, and S. A. A. Ghorashi, Quantum geometry induced nonlinear transport in altermagnets, *Phys. Rev. Lett.* **133**, 106701 (2024).
- [58] M. T. Mercaldo, M. Cuoco, and C. Ortix, Nonlinear planar magnetotransport as a probe of the quantum geometry of topological surface states, *Phys. Rev. B* **111**, 155442 (2025).
- [59] D. Kaplan, T. Holder, and B. Yan, Unifying semiclassics and quantum perturbation theory at nonlinear order, *SciPost Phys.* **14**, 082 (2023).
- [60] T. Ideue, K. Hamamoto, S. Koshikawa, M. Ezawa, S. Shimizu, Y. Kaneko, Y. Tokura, N. Nagaosa, and Y. Iwasa, Bulk rectification effect in a polar semiconductor, *Nat. Phys.* **13**, 578 (2017).
- [61] M. Ezawa, Detecting the Néel vector of altermagnets in heterostructures with a topological insulator and a crystalline valley-edge insulator, *Phys. Rev. B* **109**, 245306 (2024).