

Classifying the topology of finite chiral structures using complete matchings

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(Received 15 May 2025; revised 16 April 2026; accepted 10 June 2026; published 28 July 2026)

We present the theory and experimental demonstration of a topological classification of finite tight binding Hamiltonians with chiral symmetry. Our classification can be applied to randomly connected chiral networks, and regular lattices, giving a significant generalization of the known classification. Using the graph-theoretic notion of complete matchings, we show that many chiral tight binding structures can be divided into a number of sections, each of which has independent topological phases. Hence, the overall classification is $N\mathbb{Z}_2$, corresponding to 2^N distinct phases, where N is the number of sections with a nontrivial $\mathbb{Z}_2$ classification. In our classification, distinct topological phases are separated by exact closures in the energy spectrum of the Hamiltonian, with degenerate pairs of zero-energy states. We show that these zero-energy states have an unusual localization across distinct regions of the structure, determined by the manner in which the sections are connected together. We use this localization to provide an experimental demonstration of the validity of the classification, through radio-frequency measurements on a coaxial cable network, which maps onto a tight-binding system. The structure we investigate is a cable analog of an ideal graphene ribbon, which divides into four sections and has a $4\mathbb{Z}_2$ topological classification.

DOI: [10.1103/hpvh-plhc](https://doi.org/10.1103/hpvh-plhc)

I. INTRODUCTION

Topology has become ubiquitous in modern physics, with many lattice structures and materials shown to have nontrivial topological attributes. A topological material is characterized by properties which remain unchanged during adiabatic evolution, making them robust in the presence of disorder. For instance, boundary states in one-dimensional topological lattices may have energies which are resistant to disorder [1], while in two-dimensions they may provide directional transport that is protected against backscattering [2–5]. In order to investigate these potentially useful properties, it is necessary to be able to classify the different topological phases of a given Hamiltonian.

Distinct topological phases are generally understood from two perspectives. The first is that two phases are distinct if, and only if, they cannot be related by (symmetry respecting) adiabatic evolution. The second is that a nontrivial topological phase has interesting boundary properties. The first perspective allows us to assign topological indices to distinct phases, and the second gives a physical significance to a classification. Both are connected by the famous bulk-boundary correspondence [6] which, since the discovery that the quantum Hall effect is topological [2,7] has proven to be powerful

tool for predicting topological boundary phenomena. Most approaches to classification make use of the bulk-boundary correspondence in some way.

Topological classification is most often rooted in the ten-fold way [8], in which K-theoretic methods are used to classify each of the Altland-Zirnbauer (AZ) symmetry classes of momentum-space Hamiltonians [9]. Another approach is topological quantum chemistry [10], where the classification is obtained by looking for topologically nontrivial states. In particular, the presence of states that cannot be adiabatically deformed to the atomic limit indicates a topologically nontrivial band in the bulk [11–13]. Such states may be found by considering the irreducible representations of a crystal's underlying space group, for which there have been a number of approaches [12–14]. These methods provide a classification of stable and fragile topological phases in a huge number of systems [15].

A useful approach for strong disorder relies on defining a noncommutative Brillouin zone [16–18]. Using noncommutative geometry, the Brillouin zone is modified to allow for variations in each unit cell of the structure, by defining a configuration space for the set of distinct unit cells. Translational invariance is no longer required, giving a disordered bulk-boundary correspondence.

For finite structures, a spectral localizer [19–21] can predict the presence of approximately zero-energy boundary states. Using the fact that such states cannot be localized to an atomic limit (similarly to topological quantum chemistry), topological nontriviality in a band structure may be inferred thereby classifying a structure via the converse of the bulk-boundary correspondence. Alternatively, local real-space markers may be used [22,23] whose average gives a

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topological index for the entire structure. While this averages to a quantized value in the thermodynamic limit, such local markers are not precisely quantized in finite structures.

By studying systems which are thermodynamically large in some dimensions, but finite in others, it has been shown that finite size effects may cause a rich sequence of topological phase transitions, corresponding to gap closures separating *topological bubbles* [24] in inversion symmetric structures. The number of approximately zero-energy boundary states in such a bubble takes an integer value corresponding to a $\mathbb{Z}$ classification in finite, inversion-symmetric Hamiltonians. These states are resistant to certain amounts of disorder. This approach has also been extended to time-reversal symmetric systems [25] and to semimetals [26]. Alternatively, the finite structure may be repeated as a periodic supercell [27,28] allowing the use of momentum-space methods to classify the topological phases.

These approaches leave open a problem: how may a finite structure be classified when we completely lose the bulk-boundary correspondence? That is, we no longer have any way to define a boundary or a bulk. This situation may occur in a small finite structure, particularly with strong disorder and/or no underlying lattice structure (for instance, a random finite network). Strongly disordered finite systems may have a nontrivial topological classification [19,29,30] although the number of distinct topological phases is not always clear, motivating a general approach to topologically classify finite structures that have lost the bulk-boundary correspondence. In this paper, we propose a partial solution to this problem. Using graph-theoretic methods, we give a topological classification for finite chiral symmetric structures. We achieve this by considering equivalence classes of finite real space Hamiltonians with arbitrary values for the hopping terms. A generalization of our classification to nonchiral structures will be the subject of upcoming work.

We stress that under our approach, we are calculating exact topological properties of finite real-space Hamiltonians, so any gap closures or change in topological indices are present in the finite structure itself (that is, without taking any thermodynamic limit). This is in contrast to conventional approaches to classification, where topological properties converge (exponentially fast) in the thermodynamic limit. As such, our classification is most relevant to thermodynamically small systems with tens to hundreds of sites. Many state-of-the-art experimental platforms—as well as coaxial cable networks [30–33] which are also considered herein—are of this scale. Examples include micropillar polariton devices [34], acoustic metamaterials [35,36], mechanical systems [37], topoelectrical circuits [38,39], and nanoscale devices [40–42].

In a finite structure at sufficient disorder, it may not be possible to define a bulk; however, nonadiabatic evolution may still be identified. For a finite system, a closing in the energy spectrum is analogous to a band gap closing in momentum space. In structures with chiral symmetry, and thus a symmetric energy spectrum, this corresponds to the appearance of pairs of zero-energy states. Determining the conditions on the hopping terms that lead to the presence of these states allows us to define equivalence classes of Hamiltonians. A similar approach to defining topological phase boundaries has been used in [19,29,30].

Using this definition of equivalence classes, we find that many structures have a rich classification resulting from the fact that they can be divided into a number of *sections*, each of which can be assigned an independent topological phase. These sections are identified as corresponding to factors of the Hamiltonian's determinant, and therefore play a role in defining the conditions for zero-energy states. Any hopping terms that do not appear in the determinant can be removed from the Hamiltonian without affecting the classification. This removal process results in the structure separating into disconnected pieces, each constituting a section. For each section we can determine an independent 0 or $\mathbb{Z}_2$ classification, using a sub-Hamiltonian defined on the section. The classification of the entire structure is then given by combining the independent classification of each section. Furthermore, under our definition of a topological phase boundary, our classification finds every distinct topological phase.

The existence of sections depends on the connectivity of the network representing the hopping terms in the Hamiltonian. They can occur in structures with some regularity, as in the graphene related examples that are considered in this work. However, they can also appear in more randomly connected structures. Checking 154 131 nonisomorphic graphs [43] that represent some of the connected chiral structures with 18 sites or fewer, and using randomized searches on larger finite chiral structures, we find that the classification of a structure is the most rich when each site has an average of three hopping terms. For an average of two, the underlying connectivity is generally too sparse to provide a rich classification, and at four or greater, the structure is often too constrained by its connectivity to allow the division into many sections.

Of the 154 131 graphs classified, 8% of structures had no sections, 68% had one section, and 24% had 2 to 9 sections (trivial sections can still affect physical properties, such as localization). As our classification is derived from the number of sections, our approach to classification applies only to structures with a nonzero number of sections. For all positive real hopping terms, this results in our classification predicting that 14% have no distinct topological phases, with the remaining structures being topologically nontrivial. 82% had two topological phases, and 4% had 4 to 16 distinct topological phases, with only 18 of the classified graphs having 16 phases. Note that the number of structures with any (integer) number of sections is infinite.

The infinite and finite classification of the sub-Hamiltonian corresponding to a section may be different. Considering a section as a one-dimensional supercell of an infinite lattice, a transfer matrix treatment leads to one trivial and one nontrivial topological phase, separated by a gap closure somewhere in the Brillouin zone. Hence, with this definition, every section has a $\mathbb{Z}_2$ classification. Furthermore, the classes AIII and BDI have a $\mathbb{Z}$ classification using the tenfold way. In this infinite interpretation, a section having a nontrivial topological phase results in boundary-localized topologically robust zero modes at one end of that section. This gives a connection to higher order topology arising through stacked chiral structures.

In this work, we seek to classify the actual finite structure, so we adopt the convention that only unavoidable gap closures that are observable (corresponding to a closure at

zero momentum) separate topological phases. This means that some sections will have a $\mathbb{Z}_2$ classification, while others will be topologically trivial. The complete structure behaves as a stack of sections, connected together in such a way that the topological phase of each section is independent. This gives a classification as the direct sum $\bigoplus_1^N \mathbb{Z}_2 \cong N\mathbb{Z}_2$ where N is the number of sections with a nontrivial classification, and depends on the underlying connectivity of the structure. For periodic materials, the stacking is not unlike that seen in [44]. Others have also shown that a rich classification can follow from the underlying connectivity of a structure [45,46].

We also show that the localization of the zero-energy states that accompany a topological phase transition is determined by the connections between the sections. We define a partial ordering of the sections based on this connectivity, such that each of the zero-energy states created by making one section topologically marginal spreads in only one direction through the remaining sections. This provides an experimentally accessible verification of the existence of the sections. We perform such an experiment using a coaxial cable network, which has been shown to map onto a tight-binding Hamiltonian [30–33]. The structure we consider is a zigzag “graphene” nanoribbon consisting of four rows of sites, each of which forms a separate section, leading to a $4\mathbb{Z}_2$ classification.

We begin by discussing the theory behind the classification in Sec. II before discussing experimental results in a small graphene structure in Sec. III, and we conclude in Sec. IV. Supplemental Material [47] is provided with more mathematical and experimental details.

II. CLASSIFYING FINITE CHIRAL STRUCTURES

Chiral symmetry is one of the fundamental symmetries of the Altland-Zirnbauer symmetry classes [9]. A chiral Hamiltonian H anticommutes with a unitary C , ensuring that nonzero eigenvalues come in $\pm\epsilon$ pairs. As a result of allowing algebraically independent hopping terms (thereby letting the structure have arbitrary hopping disorder) a Hamiltonian with chiral symmetry necessarily has two sublattices of sites, which we label “black” and “white” (a consequence that follows from applying the Hara-Sachs theorem [48,49]). Each black site is only connected (through nonzero hopping terms) to white sites, and each white site is only connected to black sites. An example is shown in Fig. 1.

For a structure with n_B black sites and n_W white sites, there are always at least $|n_B - n_W|$ zero-energy states. Since these exist independently of the values of the hopping terms in H , they are described as topologically protected states. Such states can also occur in structures with $n_B = n_W$. If a structure has such protected states, the determinant of the Hamiltonian is necessarily zero for all values of the hopping terms, so a classification based on the conditions for $|H| = 0$ is not possible. In this work, we consider only structures with equal numbers of black and white sites and no topologically protected states. We note that structures with topological protection may also have a nontrivial topological classification, and may additionally be assigned an $N\mathbb{Z}_2$ classification. The full subject of a topological classification of structures with

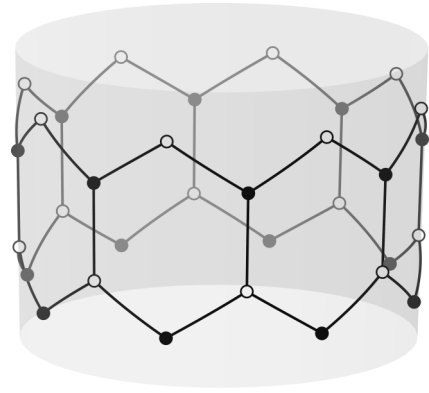


FIG. 1. A two-row ribbon zigzag graphene ribbon, looped to form a cylinder. The structure has chiral symmetry, with the two sublattices indicated by the black and white sites. The topological classification of structures like this can be found using the methods we describe: this one is $2\mathbb{Z}_2$.

topologically protected states will be detailed in upcoming work.

Two Hamiltonians, H_1 and H_2 , are considered to have different topological phase if they cannot be related under adiabatic evolution. That is, there exists no continuous way to evolve between H_1 and H_2 without a gap closing in the spectrum. For a chiral Hamiltonian with an even number of sites, zero-energy states necessarily occur in degenerate pairs. This degeneracy corresponds to a closed gap in the spectrum, indicating nonadiabatic evolution. Therefore, finding the conditions for a singular Hamiltonian determines the topological phase boundaries in a finite structure.

To explore distinct topological phases on a finite chiral structure, we consider a set of sites connected by nonzero hopping terms. Formally, this defines a graph G . On G , we define a tight-binding (TB) Hamiltonian H , where each hopping term may be continuously evolved, algebraically independently. To ensure that changes to G only include continuous evolution, we do not allow new connections to be introduced, or existing connections to be broken. In terms of H , this means that nonzero hopping terms must remain nonzero, and hopping terms that are zero cannot be modified. Otherwise, we allow complete freedom to continuously evolve hopping terms of H , giving access to strongly disordered Hamiltonians defined on G . We do, however, restrict the hopping terms to be real: any topological phase boundary in a finite structure can be avoided, by evolving the Hamiltonian through a path involving complex hopping terms. Although we can allow negative values, the requirement for nonzero real hopping terms means that they cannot change sign as they evolve.

The tuple of algebraically independent hopping terms on G , defines an affine space ξ . Each Hamiltonian H on G defines a point in ξ . Continuously evolving hopping terms in H corresponds to following a path in ξ . We refer to ξ as the *parameter space* of G . Topological phase boundaries in ξ correspond to boundary-free surfaces of one dimension lower than ξ . An example of a slice of a parameter space is given in Fig. 2.

The classification problem can be completely reduced to finding solutions to $|H| = 0$, which necessarily correspond to degenerate zero-energy states. Although degeneracies can

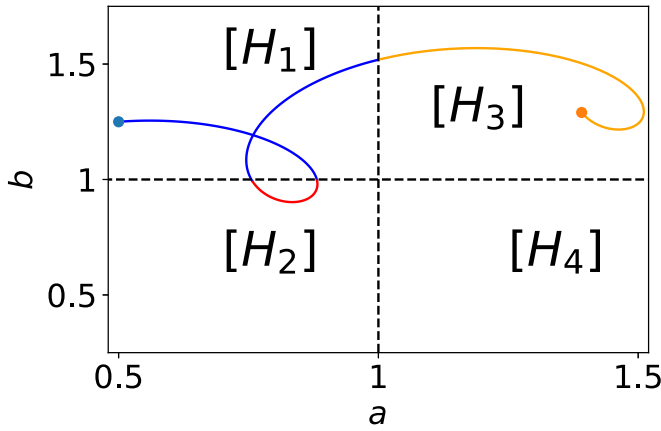


FIG. 2. A two-dimensional slice of the parameter space of all nonzero hopping terms for a two-row zigzag graphene ribbon, which has the classification $2\mathbb{Z}_2$. This slice is defined by setting all the hopping terms equal to 1, apart from one in each row, a and b , which are allowed to evolve independently. $[H_x]$ denotes a set of topologically equivalent Hamiltonians, and the dashed lines denote phase boundaries. The path shown undergoes three phase transitions between the phases $[H_1]$, $[H_2]$, and $[H_3]$, where each phase transition is identified by the appearance of a pair of unavoidable degenerate zero-energy states.

occur at nonzero energies, in a system with algebraically independent hopping terms, such a gap closure requires at least two constraints on the hopping terms (for which we give a proof in Sec. A.3 of the Supplemental Material [47]). Satisfying multiple such constraints is described by surfaces that are at least two dimensions lower than ξ , and therefore cannot divide it: they are always avoidable gap closures. This ensures that topological phase boundaries are only given by a collection of surfaces in ξ corresponding to $|H| = 0$.

The Hamiltonian for a chiral structure has an antidiagonal-block basis corresponding to ordering the sites by sublattice,

$$H = \begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix}, \quad (1)$$

so the determinant $|H| = -|Q||Q^\dagger|$. For equal numbers of black and white sites, Q is a square matrix, so $|Q| = |Q^\dagger|$, and we can use either to determine the classification.

Every term in the expansion of Q is algebraically independent, so $|Q|$ is a multilinear form of the hopping terms. That is, $|Q|$ varies linearly with respect to each hopping term in its expansion. For each hopping term in $|Q|$, solutions to $|Q| = 0$ follow from solving a linear equation, constraining exactly one hopping term. This ensures that every solution to $|Q| = 0$ corresponds to a surface in ξ , which is both unbounded, and one dimension lower than that of the parameter space, splitting ξ into distinct regions. This linear behavior also ensures that $|Q|$ changes sign as we go over a phase boundary, meaning $\text{Sign}[|Q|]$ is a topological invariant [19]. This is in contrast to $\text{Sign}[|H|]$, which does not change at a phase boundary.

To classify a structure, we must understand how to set $|Q|$ to zero by continuously evolving hopping terms, which may be done by finding the irreducible factors of $|Q|$. If a factorization of $|Q| = \prod |q_i|$ exists, and it is possible to solve $|q_i| = 0$, then the factor $|q_i|$ defines a pair of distinct phases,

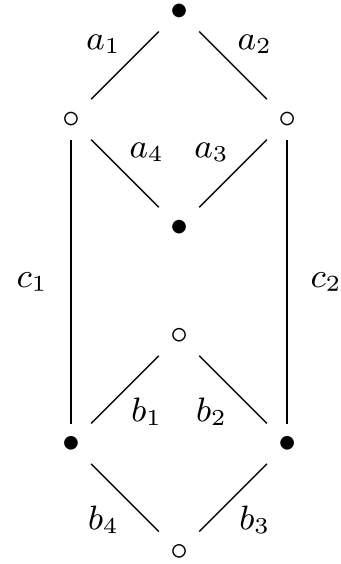


FIG. 3. A simple two-row graphene zigzag structure with the hopping terms labeled as in Eq. (2). This structure has two sections and four distinct topological phases, corresponding to a $2\mathbb{Z}_2$ classification.

where $\text{Sign}[|q_i|]$ is a $\mathbb{Z}_2$ topological invariant [50]. We say that such a factor is *nontrivial*. The classification of the structure is then given by $\bigoplus^N \mathbb{Z}_2 \cong N\mathbb{Z}_2$, where N is the number of nontrivial factors. This classification is unique under our definition of a topological phase boundary. This is because the determinant $|Q|$ has a unique irreducible factorization, ensuring that we detect every topologically distinct phase of the finite structure (further details are given in Sec. A of the Supplemental Material [47]). We note that trivial factors are also possible, where no solutions to $|q_i|$ exist for the specific domain of hopping terms. Our classification is based on the number nontrivial factors alone [51].

The determinant $|Q|$ defines a polynomial in a unique factorization domain (details are given in Sec. A of the Supplemental Material [47]) ensuring that the maximum number of nontrivial factors N of $|Q|$ is also a topological invariant, so the classification is well defined, unique, and robust to arbitrary continuous hopping disorder. It is only possible to change N by removing or introducing new hopping terms or sites, that is, under discontinuous evolution of H .

To show a simple example, we consider a two-row zigzag graphene structure, with four sites on each row, see Fig. 3. This is described by the matrix

$$Q = \begin{pmatrix} a_1 & a_4 & c_1 & 0 \\ a_2 & a_3 & 0 & c_2 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_4 & b_3 \end{pmatrix} \quad (2)$$

giving the determinant

$$|Q| = (a_1 a_3 - a_2 a_4)(b_1 b_3 - b_2 b_4) = |q_1||q_2|. \quad (3)$$

The condition $|Q| = 0$ can be satisfied by making either factor, $|q_1|$ or $|q_2|$, equal to zero. This gives four distinct topological phases corresponding to $(\text{Sign}[|q_1|], \text{Sign}[|q_2|]) \in \{(+1, +1), (+1, -1), (-1, +1), (-1, -1)\}$. Therefore, this

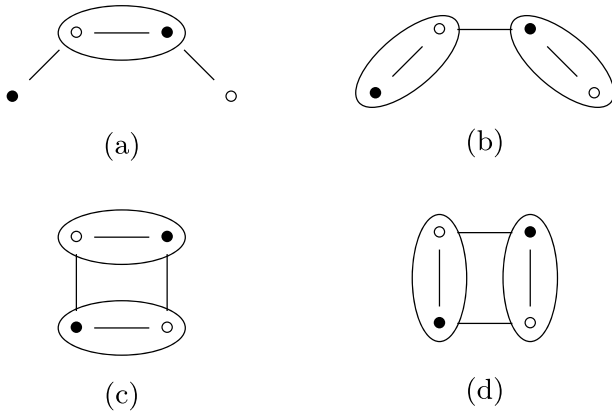


FIG. 4. (a) An example of a matching between the two internal sites of a simple four-site structure. (b) A complete matching of the same structure, where every site is included in an individual matching, with no sites left unmatched (in this structure, only one complete matching is possible). Panels (c) and (d) illustrate the two distinct complete matchings of a four-site looped structure. We note that the structure in (a) and (b) has two trivial sections, hence the classification 0, and the structure in (c) and (d) has one nontrivial section, having the classification $\mathbb{Z}_2$.

structure has the classification $2\mathbb{Z}_2$. The system can undergo a topological phase transition when the hopping terms are adjusted so that the sign of the determinant of one or both of the sections ($\text{Sign}[|q_1|]$ or $\text{Sign}[|q_2|]$ in this example) change.

It is also apparent from this simple calculation that some of the hopping terms— c_1 and c_2 —do not appear in the expansion of $|Q|$. This means that the topological classification will be the same as that of a structure in which these hopping terms have been set to zero. Physically, removing these links creates two completely separate lattices: the top and bottom loops in Fig. 3. We call these the two *sections* of the original structure.

The Hamiltonian, and thus Q , for a separated structure is block diagonal, so the determinant $|Q|$ is simply a product of the two diagonal-block determinants, as the expansion shows. The absence of the c terms from the expansion of $|Q|$ is a consequence of the block triangular form of Eq. (2). However, this pattern is only explicit if the sites are ordered correctly, which depends on finding the sections defining the blocks in the matrix. As we prove in Sec. A.1 of the Supplemental Material [47], the determinant is factored if and only if there exists a block-triangular form of Q , so finding an appropriate ordering of sites allows us to find the number of sections in a structure.

A graph-theoretic approach to this problem involves enumerating the *complete matchings* of a structure. On a graph, two sites are said to be in a matching if they are connected by a hopping term [for example, see Fig. 4(a)]. A complete matching is defined when each site is assigned to a single matching, without any site being left unmatched. Examples of complete matchings are given in Figs. 4(b)–4(d). An individual graph may have a very large number of distinct complete matchings. The complete matchings for the four-row graphene ribbon (for which we present an experimental investigation in Sec. III) are shown in Fig. 5.

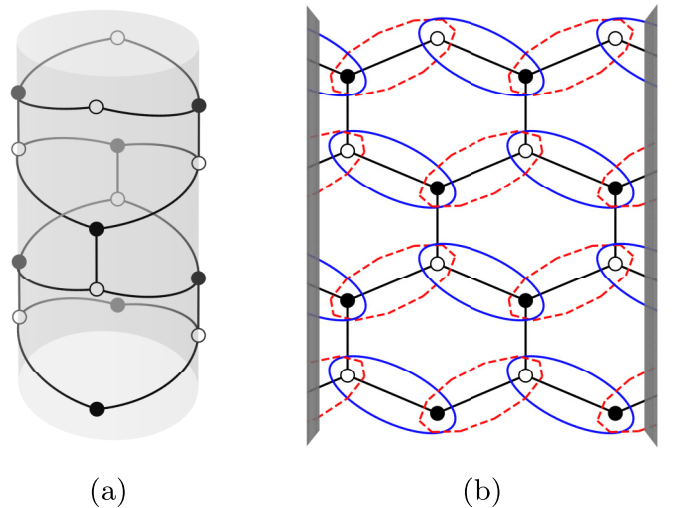


FIG. 5. (a) The four-row graphene ribbon structure looped to form a cylinder. (b) The complete matchings of this structure are formed by the drawn red dashed and blue solid matchings. A complete matching of the entire structure consists of one complete matching from each row (indicated with red dashed or blue solid matchings), where every black site is paired (matched) with an individual white site of each row. Each row has two distinct complete matchings, which may be chosen independently, so there are a total 2^4 unique complete matchings of the entire structure. The connections between the rows do not appear in any of the matchings, so each row forms a separate section. The topological classification for this structure is $4\mathbb{Z}_2$.

A complete matching has an algebraic interpretation: if we index separately the black and white sites such that each matched pair has the same index, the corresponding hopping terms appear along the diagonal of the matrix Q . The terms in the expansion of the determinant of a matrix correspond to the product of the diagonal elements for every possible permutation of the columns (or rows). Hence, finding the set of complete matchings for a structure gives all the terms in the expansion of the determinant (to get the signs, it is also necessary to keep track of the number of swaps required to go between complete matchings). This is the Harary-Sachs theorem [48,49] for weighted bipartite graphs.

A consequence of the relationship between complete matchings and the expansion of $|Q|$ is that any connection that is not included in any complete matching does not appear in the expansion, which is why removing all such connections reveals the sections. In the case of the structure in Fig. 5, all the connections between the rows can be removed, so the structure has four sections. For each section, there are two complete matchings, so each factor $|q_i|$ of $|Q|$ contains two terms. For this structure, one term in every factor has the opposite sign to the other, ensuring that every factor can be made to pass through zero, so the structure has the classification $4\mathbb{Z}_2$. Note that even small changes in connectivity can change this completely. If we add just one connection between a white site of top row and a black site of the bottom row of the structure, we find that there are complete matchings that include every connection, so there is one section and the classification is $\mathbb{Z}_2$.

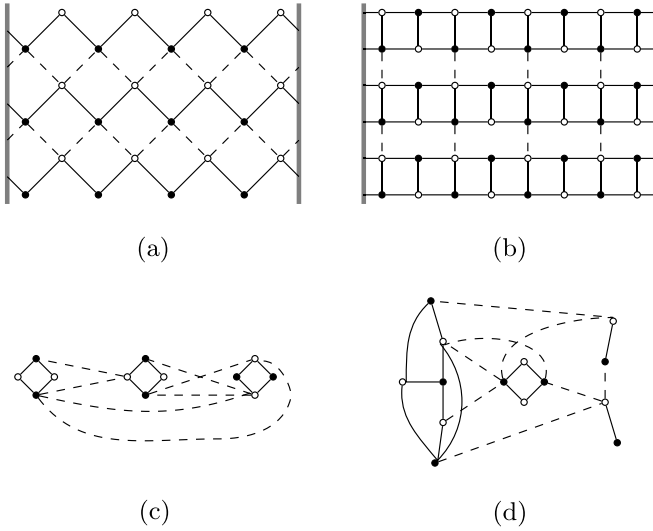


FIG. 6. Four structures where sections are depicted with regions connected with solid lines, and hopping terms that do not appear in sections depicted with dashed lines. Panels (a) and (b) both have three nontrivial sections, and hence a $3\mathbb{Z}_2$ classification. Our classification also applies more generally to bipartite graphs, with (c) having three nontrivial sections and thus a $3\mathbb{Z}_2$ classification. Panel (d) has two nontrivial sections (the left and central section), and two trivial sections (the two on the right-hand side of the figure) and thus a $2\mathbb{Z}_2$ classification.

The relationship between the number of sections of a structure and a block-triangular form of Q in Eq. (2) can be made rigorous. In Sec. A.1 of the Supplemental Material [47], we prove that the factorization of $|Q|$ according to sections, $|Q| = \prod_i |q_i|$, can be made if, and only if Q can be written in the block-triangular form

$$Q = \begin{pmatrix} q_j & \cdots & c_{i+1,j} & c_{i,j} \\ & \ddots & \vdots & \vdots \\ 0 & & q_{i+1} & c_{i,i+1} \\ & & & q_i \end{pmatrix}, \quad (4)$$

where the q_i and c_{ij} are matrix blocks. Therefore, by finding an appropriate order of the sites, the number of sections will be revealed in a structure.

In accordance with this observation, we can readily find a number of classifications of small-scale lattice structures. For instance, a Su–Schrieffer–Heeger [52] chain will have a $\mathbb{Z}_2$ classification for periodic boundary conditions, and an even number of sites (if every hopping term is real and of the same sign) [53]. Armchair ribbon graphene has an $N\mathbb{Z}_2$ classification, where N is the number of rows, as does a rotated square lattice depicted in Fig. 6(a), and the lattice in Fig. 6(b). Our approach may also be readily applied to structures that lack periodicity (so long as the underlying connectivity is known), such as strongly disordered finite amorphous or quasiperiodic systems, and, arbitrarily, to chiral lattices, for example, those in Figs. 6(c) and 6(d).

The diagonal blocks q_i in Eq. (4) define the sections g_i of the structure. They include the hopping terms that connect the black and white sites within the section. The off-diagonal blocks $c_{i,j}$ contain the connections between the sections. The

hopping terms in the off-diagonal blocks do not appear in the expansion of $|Q|$. Note that the connecting terms only appear within superdiagonal blocks—the corresponding sub-diagonal is necessarily zero. Thus, the connections between sections are always between the sublattices. For instance, if Q maps from the black to the white sublattice, then (with the ordering in Eq. (4)) only the white sites of g_j may connect to the black sites of g_i . We will make use of this shortly to define a partial ordering of the sections.

The relationship between the block structure of a matrix and the topological classification of a chiral structure has allowed us to find an algorithm to classify a random chiral network. Details of this algorithm are given in Sec. B of the Supplemental Material [47]. Of our classification algorithm, the most numerically expensive part is finding a basis with the largest number of triangular blocks. An alternative algorithm for this part of the classification is detailed in [54] although we have not compared the complexity of our algorithm with this alternative approach.

The factorization and triangular form of Q tells us more than just the topological classification of the Hamiltonian. Physically, the nature of localization at criticality (that is, at a topological phase boundary) is affected, with null states exactly restricted to a particular subset of sections. We consider a structure represented by the graph G , with a section g_i corresponding to the factor q_i . For a topologically marginal chiral structure, each null state can be localized to the black or white sublattice. If only one section is singular, the black state may be nonzero only on a subset G_b of noncritical sections, and the white state may be nonzero on a subset G_w of noncritical sections. The sets G_w , G_b and the critical section g_c are disjoint, so that the only section with support of both the black and white states is g_c . This nontrivial localization yields an experimental method to find the classification of a structure, discussed in Sec. III.

To understand this localization, we define a partial ordering of sections, $(g \in G, \leq)$, which provides the upper triangular form in Eq. (4). We say that $g_i < g_j$, or g_i is *lower* than g_j , if white sites of g_i connect to black sites of g_j , and $g_i > g_j$, or g_i is *higher* than g_j , if black sites of g_i connect to white sites of g_j . If the blocks q_i and q_j can be permuted in Q to swap places on the diagonal in such a way that maintains an upper triangular block matrix Q , then $g_i = g_j$. Otherwise, if g_i and g_j are not directly connected, we look to neighboring sections to define the partial ordering. As the structure is connected, we may always iterate to neighbors of neighbors until we have the partial ordering relationship between any two sections.

In order to demonstrate how the localization of null states in a critical section is altered by the partial ordering of sections, consider a four-section structure where Q has the form

$$Q = \begin{pmatrix} q_1 & c_{12} & c_{13} & 0 \\ 0 & q_2 & 0 & c_{24} \\ 0 & 0 & q_3 & c_{34} \\ 0 & 0 & 0 & q_4 \end{pmatrix} \quad (5)$$

where q_i and $c_{i,j}$ are block matrices. The sections have the partial ordering $g_1 < g_2 = g_3 < g_4$. Suppose that section g_3 is marginal, so $|q_3| = 0$ and $|q_1|, |q_2|, |q_4| \neq 0$. Then, there

exists a null eigenvector $|\phi\rangle$ such that $q_3|\phi\rangle = 0$. The solution over all of Q is then given by

$$\begin{pmatrix} q_1 & c_{12} & c_{13} & 0 \\ 0 & q_2 & 0 & c_{24} \\ 0 & 0 & q_3 & c_{34} \\ 0 & 0 & 0 & q_4 \end{pmatrix} \begin{pmatrix} -q_1^{-1}c_{13}|\phi\rangle \\ 0 \\ |\phi\rangle \\ 0 \end{pmatrix} = 0 \quad (6)$$

demonstrating the nullstate can have nonzero support on the black sites of g_1 and g_3 only. Similarly, $|q_3^\dagger| = 0$ so we have a similar solution for $Q^\dagger$ where

$$\begin{pmatrix} q_1^\dagger & 0 & 0 & 0 \\ c_{12}^\dagger & q_2^\dagger & 0 & 0 \\ c_{13}^\dagger & 0 & q_3^\dagger & 0 \\ 0 & c_{24}^\dagger & c_{34}^\dagger & q_4^\dagger \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ |\psi\rangle \\ -(q_4^\dagger)^{-1}c_{34}^\dagger|\psi\rangle \end{pmatrix} = 0 \quad (7)$$

and $q_3^\dagger|\psi\rangle = 0$. Hence, this null state has nonzero support on the white sites of g_3 and g_4 only. The localization of the support by sublattices is a direct consequence of the partial ordering, in this example $g_1 < g_2 = g_3 < g_4$. These rules generalize in a straightforward way when there are more sections: the black zero-energy state is localized on the marginal section and those lower in the partial ordering, while the white state is localized to the marginal section and those that are higher in the partial ordering.

It should be noted that within the critical section itself, for a randomly selected distribution of hopping terms that satisfy the condition for a section to be critical, the zero-energy states have, with probability one, support on every site of the section. Proofs of this, and of the general relationship between the factorization and localization, are given in Sec. A.1 of the Supplemental Material [47].

It is natural to ask what happens when a structure has more than one critical section. In some instances, it is possible to have more than two zero-energy states. However, for most structures, for almost all sets of hopping terms, there remain only two zero-energy states when there are multiple critical sections. This is because the null states originating from each critical section must be orthogonal to each other, which imposes additional constraints on the hopping terms, including those that connect the sections. Imposing the relevant additional constraints on the Hamiltonian to achieve a higher nullity can reveal an even richer classification, and is explored in our more recent work [55].

III. EXPERIMENTAL DEMONSTRATION OF CLASSIFICATION

To demonstrate the topological classification defined in Sec. III, we have performed experiments on a coaxial cable network that represents the four-row graphene ribbon structure of Fig. 5. A network in which all coaxial cables have the same transmission time, τ , maps onto a tight-binding model [30,31] where the sites are the junctions in the network. The “energy” ε is given by $\varepsilon = \cos \omega\tau$ where ω is the driving frequency. This yields a Schrödinger-type eigenvalue equation $H\psi = \varepsilon\psi$. Entries in ψ correspond to scaled voltages at the junctions. Up to this scaling factor, individual hopping terms are given by the reciprocal of the impedance of a cable connecting the sites. Swapping between cables of different

impedances thus allows us to traverse a structure’s parameter space ξ .

Experimental measurements are made with a vector network analyzer (VNA) and take two forms. From on-site reflectance measurements, we determine the structure’s impedance, the real part of which is proportional to the local density of states (LDOS) at that site [30]. We use this to demonstrate the localization of the null states. Transmittance measurements provide the relationship between the magnitude of a state on two sites. This gives a direct experimental determination of the block-triangular form of Q , Eq. (2), and thus the classification of the structure.

We also note that our experimental results demonstrate a robustness of our theoretical predictions regarding localization and transport. This is because our classification applies at the level of the structural connectivity of a system. However, because of imperfections in our network fabrication process, the physical cable networks unavoidably have small (but nonzero) and disordered on-site energies, giving the physical system a distinct structure from the one that we theoretically classify. However, as discussed in Secs. III A and III B, our predicted phenomena are still strongly corroborated, experimentally demonstrating a significant resilience of our predicted phenomena.

A. Measuring the local density of states

Measuring the local density of states on every site allows us to characterize the localization properties of the zero-energy states in a marginal structure with multiple sections, of which only one is critical. Figure 7 displays the LDOS measurements of the disordered looped four-row ribbon zigzag graphene in Fig. 5. According to the classification of Sec. II, this structure has four nontrivial sections, corresponding to the horizontal rows, so its topological classification is $4\mathbb{Z}_2$. Individual cables are chosen randomly from a binary distribution of $50\ \Omega$ and $93\ \Omega$ cables (see more details in Sec. C of the Supplemental Material [47]). One section is critical, the second row in the figure, while the other three are not. The localization of the nullstates is clear: the white state only lies on the top two rows, while the black state appears only on the lower three. This is in agreement with the predicted localization, given the ordering of the rows $g_1 > g_2 > g_3 > g_4$.

B. Topological classification through transmittance

Measurements of the transmittance can be used to verify the triangular form of Q , Eq. (2), and thus the division of the structure into sections. We describe here an experiment where we make a cut in each section of the structure, by disconnecting one end of a cable, and measure transmittance between various cuts. We show that transmittance at zero energy ($\cos \omega\tau = 0$) only occurs when the measurement is within one section: the transmittance between sections is zero.

In order to describe the transmittance of the coaxial cable network, we use a transfer matrix formalism to relate the voltages and currents at the output sites to those at the input sites. If we cut a site on every section (creating an input and output site on each section), the transfer matrix will have dimensions $2\tilde{N} \times 2\tilde{N}$, corresponding to the voltage and current variables

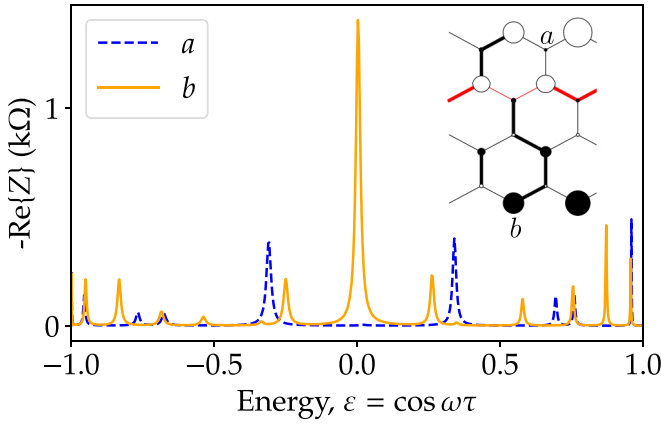


FIG. 7. Experimental measurement of the impedance spectra in a topologically marginal four-row ribbon graphene structure. The structure has four sections, corresponding to the rows of the structure. Spectra are shown for two sites on the black sublattice, labeled a and b in the structure diagram in the inset. In the plot of the structure, the diameter of the circle representing each site is proportional to the amplitude of the zero-energy state at that point, which is derived from the strength of the corresponding impedance peak. The widths of the lines showing the connections indicate the impedance of the corresponding cable, with wide lines being $93\ \Omega$ and narrow lines $50\ \Omega$ cables. The red lines pick out the second row that is the critical section. The localization of the zero-energy eigenstates onto distinct sublattices on the upper and lower sections relative to the critical row agrees with our theoretical prediction. Note that the structure has cylindrical boundary conditions as displayed in Fig. 5(b), with the hanging connections at either side linked together.

for each of the $\tilde{N}$ sections. Here, we use $\tilde{N}$ for the total number of sections, to distinguish from N , the number of topologically nontrivial sections defined earlier.

For a chiral structure at zero energy, all the variables can be colored black or white, according to the two sublattices, in such a way that the matrix consists of two $\tilde{N} \times \tilde{N}$ diagonal blocks. At zero energy, the voltage at a given site is determined entirely by the currents flowing out of the neighboring sites. The neighbors in a chiral structure are on the other sublattice, so, if we assign the voltage the same color as its site, and the currents flowing into and out of a site the opposite color to the site, the variables of the two colors are completely independent, giving the two blocks.

To see how the form of the transfer matrix for the cut structure is related to the sections, consider making a cut within a section, forming an input and output site. The cut adds a site to the structure, unbalancing the black and white sublattices in the section, and thus creating a new zero-energy state. For instance cutting on a white site creates one extra white site, resulting in a zero-energy state that only has support on the white sites of that section and of any sections higher in the partial ordering. This means that the block-triangular form of the matrix Q in Eq. (2) translates to a triangular form of each block of the transfer matrix. This triangular form persists regardless of the site at which the cut is made in each section; it can be shown to exist if, and only if, the uncut structure has at least $\tilde{N}$ sections.

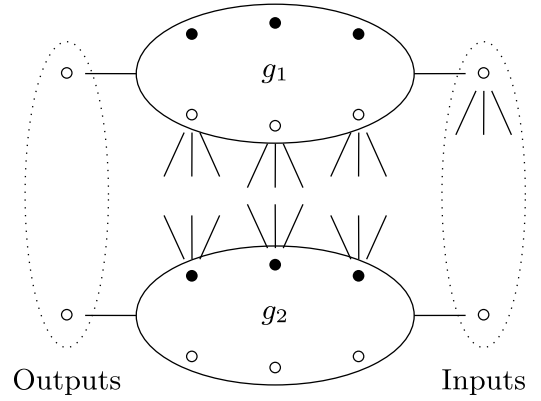


FIG. 8. A structure with two sections, g_1 and g_2 , with the partial ordering of $g_1 < g_2$ since white sites in g_1 connect to black sites in g_2 . The sections correspond to the ellipses, and contain arbitrary numbers of sites, which are not shown individually. The hopping terms connecting the two sections are denoted schematically by bunches of three lines, but they may join any white sites of g_1 to any black sites of g_2 . Each section has been cut to create input and output sites for transmittance measurements.

As a simple example, consider a structure with just two sections, g_1 and g_2 , as shown in Fig. 8. The white sites in g_1 connect to the black sites in g_2 , so in the partial ordering, $g_1 < g_2$. Each section is cut at a white site, so the white block of the transfer matrix relates the input and output voltages, while the black block connects the currents. The transfer matrix is thus

$$\begin{pmatrix} I_{\text{out}}^{g_1} \\ I_{\text{out}}^{g_2} \\ V_{\text{out}}^{g_1} \\ V_{\text{out}}^{g_2} \end{pmatrix} = \begin{pmatrix} \gamma & \delta & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & \nu & \mu \end{pmatrix} \begin{pmatrix} I_{\text{in}}^{g_1} \\ I_{\text{in}}^{g_2} \\ V_{\text{in}}^{g_1} \\ V_{\text{in}}^{g_2} \end{pmatrix}, \quad (8)$$

where the nonzero matrix entries represented by Greek letters are functions of the hopping amplitudes in the Hamiltonian, and depend on the details of the actual structure.

It is not possible to measure directly the form of such a transfer matrix using a two-port VNA, which can only give the transmittance between one input and one output site. The voltages and currents at the other input and output sites cannot simply be made zero: we can either leave them open-circuit, in which case there can be a voltage on the site, but no current flowing in or out, or they can be short-circuited, giving a current but no voltage. In our experiment, the output site is always open-circuit, and the choice of whether to short-circuit or leave open the input site is made according to the partial ordering.

To see how this works, consider the case where the input and output are connected to section g_1 and the input to g_2 is open-circuit. Then, $I_{\text{in}}^{g_2} = 0$ and $I_{\text{out}}^{g_2} = 0$. From Eq. (8), we get $I_{\text{out}}^{g_1} = \gamma I_{\text{in}}^{g_1}$ and $V_{\text{out}}^{g_1} = \alpha V_{\text{in}}^{g_1}$, so nonzero transmittance occurs. However, if the input to g_2 is short-circuited, the boundary conditions instead become $V_{\text{in}}^{g_2} = 0$ and $I_{\text{out}}^{g_2} = 0$. This gives $I_{\text{out}}^{g_1} = 0$ and $V_{\text{out}}^{g_1} = \alpha V_{\text{in}}^{g_1}$, so no transmittance can be seen, because this requires both the output voltage and current to be nonzero. Hence, for our classification experiment to work,

TABLE I. Conditions for short-circuiting or leaving open-circuit the inputs to the sections where the input site is not connected to the VNA. The requirements depend on the sublattice that the input belongs to, black or white, and its position in the partial ordering with respect to the section with an input site connected to the VNA. With these choices, nonzero transmittance occurs only when the output site is in the same section as the input.

Input site	Higher	Lower	Equal
Black	Short	Open	Open/Short
White	Open	Short	Open/Short

with intrasection transmittance nonzero, we need the g_2 input site to be open-circuit.

Proceeding in the same way, we find that if the input port is connected to the input site of g_1 and the output port to the output site of g_2 , we get nonzero transmittance if the input to g_2 is short-circuited, but not if it is open-circuit. Thus, to see no transmittance between sections, we again need g_2 to be open-circuit.

If we do these experiments with the input port connected to the input site of g_2 , the requirement is reversed: the g_1 input site has to be short-circuited to get nonzero transmittance within the section g_2 , and no transmittance between sections g_1 and g_2 . The reason for this difference is the partial ordering of the sections $g_1 < g_2$. However, these rules only work because we have cut the sections on the white sites. Going through the different cases, we find that the requirement for

short-circuiting or leaving open the unused input sites depends on the sublattice of the site and the position of the section in the partial ordering relative to the input site connected to the VNA. These requirements are summarised in Table I. When these rules are satisfied, there is nonzero transmittance between the input and output sites when they are on the same section, but not when they are on different sections. This pattern is a direct consequence of the triangular blocks in the transfer matrix, and does not occur otherwise. For example, in a structure without sections but with two cuts on white sites, short-circuiting one of the input sites results in no transmittance for either output site. Having established these rules, we use transmittance measurements to verify the presence of the sections in the looped four-row zigzag graphene ribbon shown in Fig. 5 (which we have used as an example elsewhere in this paper). Recall that this structure has four sections, corresponding to the four rows of sites. Where the subscript of each row is sequentially labeled, the partial ordering of the sections is $g_1 > g_2 > g_3 > g_4$. A cut is made in each row, so the looped structure is transformed into a sheet, shown in Fig. 9(a), creating an extra site on each row. However, note that the sections we find correspond to the uncut loop rather than the sheet. In two of the sections, the cut is made at a white site, while in the other two, the cut is made at a black site. We perform transmittance measurements by going through the four output sites, and measuring a spectrum for every input sites in each case. The rules in Table I determine whether the unconnected input sites are left open-circuit or short-circuited. For example, when the VNA is connected to the input site on the second row, the first row, which is higher in the ordering

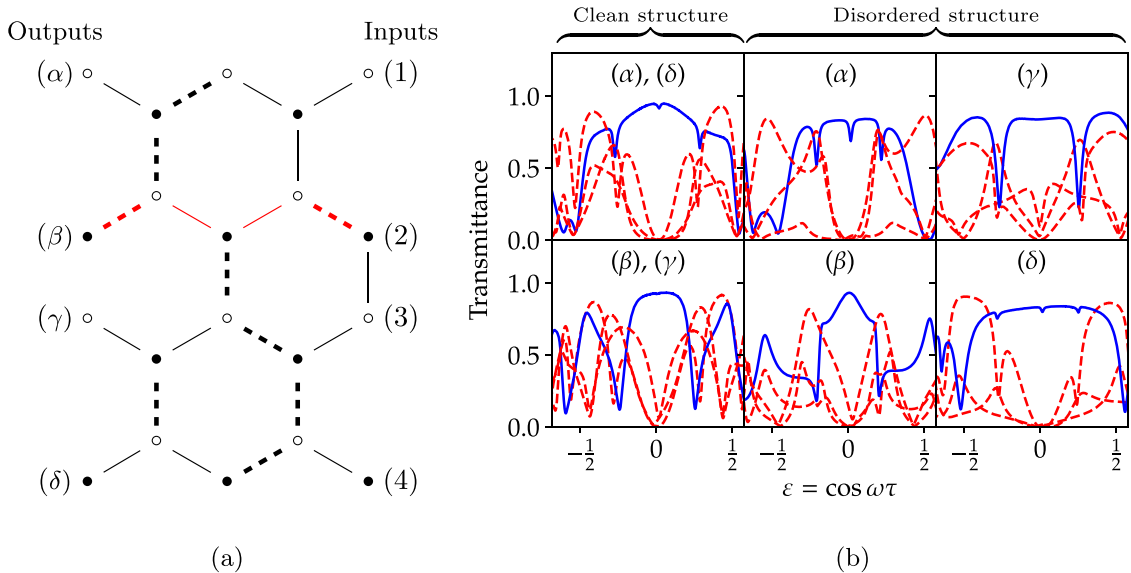


FIG. 9. (a) The looped four-row zigzag ribbon structure cut so it forms a flat sheet. The looped structure is predicted to consist of four sections, corresponding to the four rows of sites. Input sites for the transmittance measurements are labelled on the right-hand side, and outputs on the left-hand side. For the disordered structure, thick dashed lines indicate $93 \, \Omega$ RG62 cables, while thin solid lines indicate $50 \, \Omega$ RG58 cables. The red connections indicate the topologically marginal section. The clean structure had only $50 \, \Omega$ RG58 cables. (b) Experimental transmittance data. Note that $\varepsilon = \cos \omega\tau = 0$ corresponds to ≈ 114 MHz for the cable length used. Each subplot corresponds to a different output site according to the labels in (a), with spectra shown for all four inputs in each case. The blue spectrum is for the case where the input and output site are on the same section, with the dashed red spectra corresponding to transmittance between sections. The transmittance at zero energy is expected to be zero if the input and output sites are in different sections, but nonzero when they are in the same section. The experiment thus confirms that each row is a separate section.

and has a white input site, is left open, while the third and fourth rows are, respectively, short-circuited and open-circuit.

Using the transfer matrix for the clean structure, where all the cables have impedance $Z_0 = 50 \Omega$, this input gives

$$\begin{pmatrix} V_{\text{out}}^{g_1} \\ I_{\text{out}}^{g_2} \\ V_{\text{out}}^{g_3} \\ I_{\text{out}}^{g_4} \\ I_{\text{out}}^{g_1} \\ V_{\text{out}}^{g_2} \\ I_{\text{out}}^{g_3} \\ V_{\text{out}}^{g_4} \end{pmatrix} = \begin{pmatrix} 1 & Z_0 i & -1 & -Z_0 i \\ 0 & 1 & \frac{1}{Z_0} i & -1 \\ 0 & 0 & 1 & Z_0 i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ Z_0 i & 1 & 0 & 0 \\ -1 & \frac{1}{Z_0} i & 1 & 0 \\ -Z_0 i & -1 & Z_0 i & 1 \end{pmatrix} \begin{pmatrix} V_{\text{in}}^{g_1} \\ I_{\text{in}}^{g_2} \\ 0 \\ 0 \\ 0 \\ V_{\text{in}}^{g_2} \\ I_{\text{in}}^{g_3} \\ V_{\text{in}}^{g_4} \end{pmatrix} = \begin{pmatrix} V_{\text{in}}^{g_1} + Z_0 i I_{\text{in}}^{g_2} \\ I_{\text{in}}^{g_2} \\ 0 \\ 0 \\ 0 \\ V_{\text{in}}^{g_2} \\ I_{\text{in}}^{g_3} + \frac{1}{Z_0} i V_{\text{in}}^{g_2} \\ V_{\text{in}}^{g_4} + Z_0 i I_{\text{in}}^{g_3} - V_{\text{in}}^{g_2} \end{pmatrix}. \quad (9)$$

For the disordered structure, the blocks of the matrix are still triangular, but the expressions for the elements are more complicated. The numerical value for this matrix is given in Sec. C of the Supplemental Material [47].

To illustrate how the rules of Table I apply to our transmittance experiment, consider driving the input site of g_2 . If the output port of the VNA is not attached to the output site on the second row, the output site of g_2 is left open-circuit, that is $I_{\text{out}}^{g_2} = 0$. However, this forces $I_{\text{in}}^{g_2}$ to be zero, so no current enters the structure, and no transmittance occurs. However, for the output site on the same row, $I_{\text{out}}^{g_2} = I_{\text{in}}^{g_2}$ and $V_{\text{out}}^{g_2} = V_{\text{in}}^{g_2}$, so, in the absence of absorption, there is a perfect transmittance of 1.

The transmittance experiment was carried out twice, first using a “clean” structure with all 50Ω (RG58) cables, and then using a disordered structure where hopping terms were randomly selected from a binary distribution of 50Ω (RG58) and 93Ω (RG62) cables, identical to the one in Sec. III 1. The clean structure has reflection symmetry about a line between the second and third rows, so we need only to use two outputs, α and β [labelled in Fig. 9(a)], but in each case we measure transmittance for all four input sites. In the disordered structure this symmetry is broken, requiring measurements for every input and output site in order to complete the experiment.

The measured transmittance spectra are shown in Fig. 9(b). Each panel shows data for a particular output site, with the four spectra corresponding to the different input sites. As predicted, in both the clean and disordered structures, the transmittance at $\varepsilon = \cos \omega \tau = 0$ is nonzero only when the output site is on the same section as the input (blue curves). This confirms that the structure has four sections, experimentally verifying the $4\mathbb{Z}_2$ classification of this structure. The actual value of the zero-energy transmittance between the sites within a section is determined by how close the section is to being topologically marginal. For a marginal section, the ideal transmittance, in the absence of losses, would be expected to have a value of unity [30]. Although the differences are not great, it can be seen that the transmittance is highest in the clean structure, where all the sections are marginal, and for the second row section of the disordered structure [output (β)], which is also marginal.

As the structure has chiral symmetry, we expect a symmetry in the transmittance spectra around zero energy. The slight asymmetry in the data is mainly a result of loss in the cables, which has a greater effect at higher frequencies, although there is also some chiral symmetry breaking due to imperfections in the structure. This necessarily occurs in the disordered structure because the mapping of the coaxial cable structure to the tight-binding model requires that the entire connection (cable and SMA connectors) between two sites be of uniform impedance. However, the SMA connectors that form the junctions between the cables are 50Ω components, so for the connections with 93Ω cables this uniformity is unachievable. This imperfection results in a small shift in the on-site energies of the structure. As a result, the symmetry breaking is generally greater for the disordered structure than the clean one, where minor variations in the cable lengths are the likely primary cause of chiral symmetry breaking. Despite this source of disorder, our predicted phenomena are still strongly corroborated, indicating a robustness to small amounts of discontinuous structural disorder, as well as to arbitrary continuous disorder.

IV. DISCUSSION AND OUTLOOK

We have described an approach to determining exact topological phases in finite chiral structures, identifying the phase boundaries by the appearance of degenerate pairs of zero-energy states. This has been shown, in many cases, to lead to a rich topological classification, obtained by finding a division into sections that correspond to irreducible factors of the determinant of the Hamiltonian. The topological classification is then $N\mathbb{Z}_2$, where N is the number of topologically nontrivial sections or factors. The sections can be identified using the complete matchings of the underlying structure, relating the topological classification to the structure’s underlying connectivity. Each complete matching is related to a term in the expansion of the determinant, so a hopping term that does not appear in any matching can be omitted without changing the determinant. The sections correspond to parts of the structure that become separated when these connections are removed. We also give, in Sec. B of the Supplemental Material [47], a simple computational algorithm for finding the sections of a structure.

We have defined a partial ordering of the sections in a structure that gives rise to an unusual localization of the zero-energy states that are present when one of the sections is topologically marginal. The zero-energy states can be separated so that each state has support on a single sublattice. With our definition, the white state is confined to the marginal section and those that are higher in the partial ordering, while the black state appears on the marginal section and sections lower in the partial ordering. This localization can be seen as a property of finite structures that has some equivalence to the bulk boundary correspondence in infinite structures.

This localization provides a way in which the sections, and the corresponding topological classification can be demonstrated experimentally. We have performed such experiments on simple coaxial cable networks that map directly onto chiral tight-binding models. When we make one section marginal, we can use impedance measurements to map out the local density of states on each site. This provides a direct demonstration of the predicted localization related to the partial ordering. Even without a marginal section, we can use transmittance measurements to show that the structure separates into the expected sections. We have used this method to confirm the $4\mathbb{Z}_2$ classification that our theory predicts for a four-row ribbon graphene structure.

At a theoretical level, we classify a structure with a requirement of exactly zero-valued hopping terms. However, because of unavoidable structural disorder, our experiments demonstrate a robustness to the localization and transport phenomena our theory allows us to predict. Consequently, our predictions hold so long as the tight binding approximation used to describe a system is sufficiently accurate. To further illustrate this result and indicate the robustness of our conclusions, in Sec. A.2 of the Supplemental Material [47], we present numerical calculations for a nanographene, including second- and third-nearest-neighbor hopping terms (with values taken from [56]). Despite these terms changing the structure for which our classification applies, the localization phenomenon predicted by our theory is still maintained.

More concretely, introducing additional hopping terms between sites can have one of four effects on our classification. If the hopping term is above the block diagonal of each q_i in Q , then our classification is not altered at all. If the hopping term is within a section, then it can modify an individual section from being topologically trivial to being $\mathbb{Z}_2$, introducing additional topological phases to the structure (but it cannot change an individual section from nontrivial to trivial). If the hopping term is below the block diagonal in Q , then sections become merged, reducing the value of N in the resulting $N\mathbb{Z}_2$ classification. Finally, a hopping term can be outside of the block Q , and thus break chiral symmetry, in this case our classification no longer applies. Although in a physical system may not be possible to set all hopping terms that alter our classification to exactly zero, our predictions of transport and localization remain robust so long as such hopping terms are sufficiently small.

Although we focus on finite systems in this paper, we believe our work also has potential applications to the topological theory of infinite and thermodynamically large

systems, and may allow for a more complete understanding of topology in systems where the bulk-boundary correspondence applies. For instance, if we take a system with N sections and periodically repeat each section individually, then under open boundary conditions we would expect to see robust boundary states corresponding to an independent topological index of each section (resulting in a higher order topological insulator). That is, for a given unit cell we can identify sections that host independent topological indices, from which the structure's classification could be derived. Recent work has already found such a classification in rhombohedral graphene [57] that matches the number and distribution of sections identified in each unit cell.

Alternatively a structure could be made infinite or thermodynamically large by increasing the total number of sections. If one of these sections is made singular, our theory predicts two modes localized on this singular section, well within the bulk of the structure (as also displayed in our experimental results in Fig. 7). Such localized modes arise through fine-tuning just a single parameter, and display an unusual localization, where the two zero modes localize towards the singular section from opposite directions, each compactly localized on a distinct sublattice.

Our classification scheme may also be applied to non-Hermitian physics, where a doubled chiral Hamiltonian [58,59] can be used to classify distinct topological phases. Under such an approach, the blocks Q of our Hamiltonian would correspond to the non-Hermitian Hamiltonian being classified. Furthermore, while the focus of this work is on the fundamental classification of topological media, the localization of zero modes predicted by our classification scheme heralds important physical applications. For instance, many non-Hermitian optics-based sensors rely on changes in two-point transmission to observe a parametric change of the system [60,61]. Our scheme allows a simple way to read off such transmission properties from the underlying connectivity, allowing a rigorous graph-theoretic approach to using topology and connectivity as resource when designing optical and optomechanical sensors.

Finally, we note that while our classification applies specifically to systems with chiral symmetry, a generalization of our approach to classifying systems that lack chiral symmetry will be the subject of upcoming work.

ACKNOWLEDGMENTS

We are enormously grateful for help in the experimental work from Qingqing Duan, Ben Kinzig, and Elena Callus, and also to Belle Darling and Phillip Graham for huge help in finding space to run the experiments. Many thanks also to Patrick Fowler, Barry Pickup, Qingqing Duan, Ben Kinzig, and Elena Callus for many illuminating and insightful discussions while completing this work. M.M.M. wishes to thank the EPSRC for a studentship (Project No. 2482664).

DATA AVAILABILITY

The experimental data used to support the findings of this work, are publicly available [62].

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