

Spontaneous Chern-Euler duality transitions

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(Received 5 May 2025; revised 27 August 2026; accepted 1 September 2026; published 30 September 2026)

Topological phase transitions are typically characterized by abrupt changes in a quantized invariant. Here we report a contrasting paradigm in spontaneous parity-time symmetry-breaking transitions induced by non-Hermiticity. The topological invariant remains conserved, but its nature transitions between the Chern number, characteristic of chiral transport in complex bands, and the Euler number, which characterizes the number of nodal points in pairs of real bands. This conserved morphing of topological properties applies to all pairs of bands whose spectral gaps to other bands remain open during the transition. It features qualitative changes in the non-Abelian geometric phases, which quantize in different forms across the transition. Our findings establish a generic topological duality principle governing transitions across symmetry classes and reveal unique connections among topology, symmetry, and non-Abelian gauge structure in nonunitary processes.

DOI: [10.1103/g3pz-tmq7](https://doi.org/10.1103/g3pz-tmq7)

Introduction. Symmetry, geometry, and topology play fundamental roles in modern physics, from elementary particles and gravity to quantum materials. Knowledge of the global topology of energy bands in quantum materials provides insights into the nature of the quantum states and gives rise to precisely quantized observables [1]. The best known example of such a topological invariant is the Chern number, which describes the topology of complex vector bundles in two dimensions. The topological robustness of the Chern number underpins one of the most precisely measured physical constants, the Hall conductance in the quantum Hall effect. A transition between different quantized values requires the closing of a band gap, accompanied by the appearance of Dirac or Weyl points [2].

The advent of dissipative photonic or acoustic systems with parity-time (PT) symmetry has expanded the scope of topological physics [3–7]. PT-symmetric band structures are described by a real-valued “Hamiltonian” matrix $H(\mathbf{k})$, which is not Hermitian in the presence of loss and gain [8]. Eigenstates and eigenvalues of non-Hermitian real $H(\mathbf{k})$ are either real or appear in complex-conjugate pairs [9]. While $H(\mathbf{k})$ itself is PT-symmetric, each individual complex eigenstate that dominates the time evolution breaks PT symmetry, leading to spontaneous symmetry breaking in the system. Transitions between PT-symmetric and PT-breaking eigenstates take place

at exceptional points (EPs) in the spectrum of $H(\mathbf{k})$, at which two real eigenvalues of $H(\mathbf{k})$ meet and convert into a pair of complex-conjugate eigenvalues [10–18].

From a topological perspective, real and complex bands of eigenstates are sharply different: Complex bands exhibit chiral transport described by a Chern number, whereas the bands associated with real eigenstates always have a vanishing Chern number and chiral transport is ruled out. On the other hand, a pair of real bands may have nonchiral topology described by the integer-valued Euler number [19–26]. A nonzero Euler number signals an obstruction for opening a gap between the two bands [20,21,23]. The possibility to realize two different topological paradigms in the same physical system [27] sets non-Hermitian PT-symmetric systems apart from their counterparts without symmetries [28–33] as well as Hermitian systems. In this work, we show that the spontaneous real-to-complex transitions in PT-symmetric systems obey a universal Chern-Euler duality principle. This duality principle results in a *spontaneous Chern-Euler duality transition* that applies to all complete spontaneous PT-symmetry breaking between a pair of bands.

We illustrate this general duality principle by providing three PT-symmetric 3×3 band structures as minimal examples: (i) a Hermitian model $H_{\text{Euler}}(\mathbf{k})$, for which the lowest two bands are separated from the third band by a gap and have nontrivial topology with Euler number $|\chi| = 2$, (ii) a non-Hermitian band structure $H_{\text{Chern}}(\mathbf{k})$, which has two complex-conjugate bands with Chern number $C_{\pm} = \pm 2$ and one real band, and (iii) a Hermitian trivial model $H_{\text{trivial}}(\mathbf{k})$, which has three trivial bands separated by band gaps. Complex spectra corresponding to these models and their distinct transition mechanisms via nodal points (NPs), exceptional rings (ERs) [34,35] and third-order EPs are shown schematically in Fig. 1.

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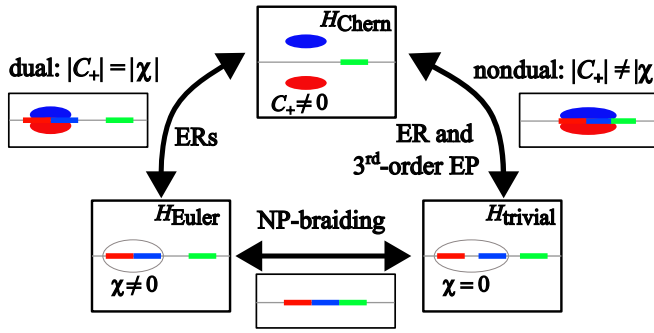


FIG. 1. Schematics of the transitions studied in this paper. The boxes schematically indicate the spectra at the three types of models H_{Euler} , H_{Chern} , and H_{trivial} and during the transitions between them. The Chern-Euler duality transition between bands 1 and 2 (shown in red and blue) takes place without involvement of the third real band (green). The nondual transition between bands with different Chern and Euler numbers forces a connection among all three bands, with a gap closing similar to conventional topological transitions. In the Hermitian regime, the Euler topology requires a minimal number of $2|\chi|$ NPs between bands 1 and 2 [20,21,23].

The real-to-complex transition between $H_{\text{Euler}}(\mathbf{k})$ and $H_{\text{Chern}}(\mathbf{k})$ takes place via spontaneous PT symmetry breaking. It starts with the formation of ERs in the lowest two bands. In the course of the transition, the ER sweeps over the Brillouin zone (BZ), so that eventually all eigenvalues and eigenstates in the initial two Euler bands become complex and an imaginary spectral gap opens up. The two real bands evolve into two complex bands without connecting to any other bands; the gap to the third (real) band remains open at all times. We prove that in general such a “dual transition” of a pair of bands can take place *if and only if* the absolute value of the Chern number of the complex band pair equals the absolute value of the Euler number of the real band pair,

$$|C_{\pm}| = |\chi|. \quad (1)$$

The “if” part may also be realized in explicit symmetry breaking in Hermitian regimes with fine-tuned perturbations [36,37], by extracting a chiral basis in the Euler bands [21,22,38]. The “only if” part, the necessary equality, is unique to spontaneous PT-symmetry breaking driven by loss and gain. It mandates two consequences: (I) Chern bands must appear after a spontaneous PT-symmetry breaking inside Euler bands. (II) If Eq. (1) is not met, including the case where no Euler number can be defined for the real band pair, a spontaneous PT-breaking transition involving only two bands is impossible. Instead, at least three bands must connect during the transition. Such a “nondual” transition may involve a third-order EP, as we observe for the transition $H_{\text{trivial}}(\mathbf{k}) \rightarrow H_{\text{Chern}}(\mathbf{k})$, see Fig. 1.

Models. The PT-symmetric three-band models are described by a three-component vector $|\psi(t, \mathbf{k})\rangle$ that evolves according to the Schrödinger-like equation $i\partial_t |\psi(t, \mathbf{k})\rangle = H(\mathbf{k})|\psi(t, \mathbf{k})\rangle$, where $H(\mathbf{k})$ is a real 3×3 matrix periodic in $\mathbf{k}$, as required by PT symmetry. The two variables $\mathbf{k} = (k_x, k_y)$, which may be quasi-momenta of photons or tuning parameters of the system [3–6], reside in a torus BZ. The eigenvalues $\omega_j(\mathbf{k})$ ($j = 1, 2, 3$) of $H(\mathbf{k})$ form three en-

ergy bands, which may be complex if $H(\mathbf{k})$ is non-Hermitian. A minimum of three bands is required for a nonvanishing Euler/Chern topology in PT-symmetric systems [27].

The three reference band structures $H_{\text{Euler}}(\mathbf{k})$, $H_{\text{Chern}}(\mathbf{k})$, and $H_{\text{trivial}}(\mathbf{k})$ are described by the 3×3 matrices

$$H_{\text{Euler}}(\mathbf{k}) = \begin{pmatrix} g_1(k_x, k_y) & f_1(\mathbf{k}) & f_2(k_x) \\ f_1(\mathbf{k}) & g_1(k_y, k_x) & f_2(k_y) \\ f_2(k_x) & f_2(k_y) & g_2(\mathbf{k}) \end{pmatrix}, \quad (2)$$

$$H_{\text{Chern}}(\mathbf{k}) = \begin{pmatrix} 0 & h(\mathbf{k}) & \sin k_y \\ -h(\mathbf{k}) & 0 & -\sin k_x \\ -\sin k_y & \sin k_x & 0 \end{pmatrix}, \quad (3)$$

$$H_{\text{trivial}}(\mathbf{k}) = \begin{pmatrix} 0 & 2 & \sin k_y \\ 2 & 0 & -\sin k_x \\ \sin k_y & -\sin k_x & 0 \end{pmatrix}, \quad (4)$$

where $g_1(k_x, k_y) = 1 + \cos k_y - \cos 2k_x$, $g_2(\mathbf{k}) = 2 - \gamma(\cos k_x + \cos k_y) + \cos 2k_x + \cos 2k_y$, $f_1(\mathbf{k}) = \cos(k_x - k_y) - \cos(k_x + k_y)$, $f_2(k) = -\gamma \sin k + \sin 2k$, and $h(\mathbf{k}) = 1 - \cos k_x - \cos k_y$. We set $\gamma = 0.6$ to maximize the real spectral gap between band 3 and bands 1,2 in $H_{\text{Euler}}(\mathbf{k})$. For the Hermitian models $H_{\text{Euler}}(\mathbf{k})$ and $H_{\text{trivial}}(\mathbf{k})$, all three eigenvalues and eigenstates are real. Both models have a spectral gap between the middle and upper band. We label the eigenvalues as $\omega_1(\mathbf{k}) \leq \omega_2(\mathbf{k}) < \omega_3(\mathbf{k})$. For $H_{\text{trivial}}(\mathbf{k})$ there is also a spectral gap between bands 1 and 2, and all the three bands are topologically trivial. For $H_{\text{Euler}}(\mathbf{k})$, the topology of the combined bands 1 and 2 is described by the Euler number $|\chi| = 2$. Note that the Euler number is even for a three-band periodic $H(\mathbf{k})$ on a torus BZ [38–41].

The Euler number χ can be calculated from the non-Abelian Berry curvature $B(\mathbf{k})$, a 2×2 matrix generalizing the Berry curvature to a pair of bands [42],

$$B(\mathbf{k}) = \partial_{k_x} A_y(\mathbf{k}) - \partial_{k_y} A_x(\mathbf{k}) - i[A_x(\mathbf{k}), A_y(\mathbf{k})]. \quad (5)$$

Here $A_{x/y}(\mathbf{k})_{ij} = i\langle \psi_i(\mathbf{k}) | \partial_{k_{x/y}} | \psi_j(\mathbf{k}) \rangle$ is the non-Abelian Berry connection, matrix indices $i, j \in \{1, 2\}$ correspond to bands indices, and $[\cdot, \cdot]$ denotes the commutator. If $H(\mathbf{k})$ is Hermitian, $B(\mathbf{k})$ is an antisymmetric 2×2 matrix and χ is given by the integral of the off-diagonal component $B_{12}(\mathbf{k})$ over the BZ [43],

$$\chi = \frac{1}{2\pi i} \int B_{12}(\mathbf{k}) d^2 k. \quad (6)$$

The expression also works for non-Hermitian real bands after applying an orthonormalization as described in Ref. [47]. The Euler number has a sign ambiguity from the gauge choice of the eigenstate basis $|\psi_{1,2}(\mathbf{k})\rangle$. Only its absolute value $|\chi|$ appears in physical expressions.

The continuous transition mechanism of the two Hermitian models $H_{\text{trivial}}(\mathbf{k})$ and $H_{\text{Euler}}(\mathbf{k})$ has previously been studied in the literature [20,21,23,44]. The nonzero Euler number obstructs a direct gap opening between bands 1 and 2 of $H_{\text{Euler}}(\mathbf{k})$. Instead, the gap opening between these two bands requires a closing of the gap with the third band first. After this gap closing, NPs connecting bands 2 and 3 appear. Winding of these “2-3 nodes” relative to the “1-2 nodes” between bands 1 and 2 changes the topology of the latter [20,21,44], allowing for the opening of a gap between bands 1 and 2 after the Euler topology is trivialized (see Fig. 1).

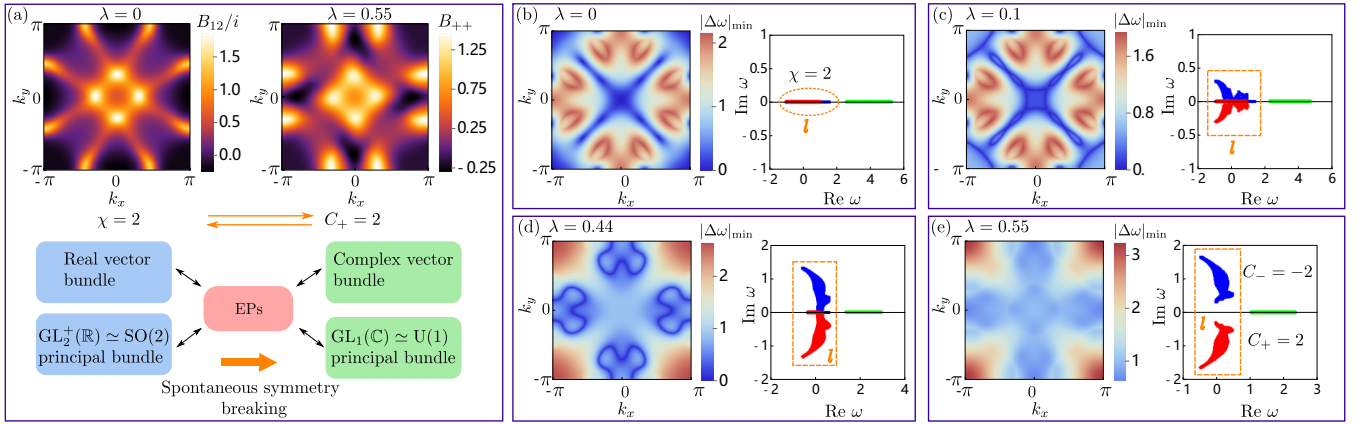


FIG. 2. The spontaneous Chern-Euler duality transition. (a) Top: The off-diagonal Berry curvature B_{12} at $\lambda = 0$ (left) and the diagonal Berry curvature B_{++} at $\lambda = 0.55$ (right) over the two-dimensional BZ. Their integrals are equal to $\chi = 2$ and $C_+ = 2$, respectively. Bottom: In all PT-breaking transitions without remote band participation, through EPs, the real bundle spanned by the two real bands with $\text{GL}_2^+(\mathbb{R})$ basis transformation continuously rotates to the realification of the complex conjugate bundles corresponding to the complex bands with $\text{GL}_1(\mathbb{C})$ basis transformation, obeying a general conservation $|\chi| = |C_\pm|$. [(b)–(e)] The minimum of the local eigenvalue gaps $|\Delta\omega(\mathbf{k})|_{\min} \equiv \text{Min}_{i,j} |\omega_i(\mathbf{k}) - \omega_j(\mathbf{k})|$ of H_λ at $\lambda = 0$ (b), 0.1 (c), 0.44 (d), and 0.55 (e), reflecting the shapes of level crossings in the BZ. The corresponding complex spectra are shown in the right panel of each figure, with the three bands labeled by different colors.

In the remainder of this article, we illustrate transitions obtained by adding the anti-Hermitian model H_{Chern} to the Hermitian models, which induces complex eigenstates that spontaneously break the PT symmetry. The Hamiltonian H_{Chern} has two complex bands that are complex conjugates of each other, and one real band. The complex bands are separated by an *imaginary* spectral gap, indicating a complete spontaneous PT-symmetry breaking [9]. We use $|\phi_+(\mathbf{k})\rangle$ to represent the complex eigenstate with eigenvalue $\omega(\mathbf{k})$ in the lower half of the complex plane, and $|\phi_-(\mathbf{k})\rangle \equiv |\phi_+(\mathbf{k})\rangle^*$ to represent the conjugate partner with eigenvalue $\omega^*(\mathbf{k})$. The topology of the complex bands is characterized by the Chern number $C_+ = -C_-$ [27], which is the integral of the diagonal Berry curvature $B_{jj}(\mathbf{k})$ associated with one of the complex bands over the BZ ($j \in \{+, -\}$),

$$C_j = \frac{1}{2\pi} \int B_{jj}(\mathbf{k}) d^2k. \quad (7)$$

Chern-Euler duality transition. We first look at a transition that satisfies Eq. (1). This is exemplified by interpolations $H_\lambda(\mathbf{k})$ between $H_{\text{Euler}}(\mathbf{k})$ and $H_{\text{Chern}}(\mathbf{k})$, $0 \leq \lambda \leq 1$,

$$H_\lambda(\mathbf{k}) = (1 - \lambda)H_{\text{Euler}}(\mathbf{k}) + \lambda H_{\text{Chern}}(\mathbf{k}). \quad (8)$$

Figure 2(a) compares the off-diagonal Berry curvature $B_{12}(\mathbf{k})$ in the Hermitian limit $\lambda = 0$ with the diagonal Berry curvature $B_{++}(\mathbf{k})$ for $\lambda = 0.55$, where the PT symmetry is spontaneously broken in the entire BZ. Initially at $\lambda = 0$, the integral of $B_{12}(\mathbf{k})$ takes the quantized value $\chi = 2$. In the fully PT-broken case at $\lambda = 0.55$, the integral of the diagonal Berry curvature $B_{++}(\mathbf{k})$ of the complex band is $C_+ = 2$ ($C_- = -2$ due to the PT symmetry). The transition from $H_{\text{Euler}}(\mathbf{k})$ to $H_{\text{Chern}}(\mathbf{k})$ takes place through the formation of ERs. As λ increases, these ERs grow out of the NPs of $H_{\text{Euler}}(\mathbf{k})$, then sweep over the BZ, see Figs. 2(b)–2(d), until all eigenvalues of bands 1 and 2 have gone through the real-to-complex transition and an imaginary spectral gap is opened between the resulting two complex bands [Fig. 2(e)]. A key observation

is that real band 3 remains detached from bands 1 and 2 at all times during the spontaneous PT-breaking transition, as shown in Figs. 2(b)–2(e). The opening of an imaginary gap between bands 1 and 2 without intermediate gap closings to any other band is a particular feature of the non-Hermitian regime, as the nontrivial Euler-number topology of bands 1 and 2 prevents the opening of a real gap between them in the Hermitian PT-symmetric regime [23,38].

Proof for generic cases. In the above transition without remote band touching, the topology $|\chi| = 2$ of real Euler bands 1 and 2 is passed on to that of the complex Chern bands carrying $C_\pm = \pm 2$. We prove that this conservation $|C_\pm| = |\chi|$ is not accidental to our example, but must hold generically for all transitions without participation of remote bands, regardless of the exact Euler/Chern numbers and the number of remote bands in the system. It is guaranteed by the band gaps to remote bands: *spontaneous PT-breaking transitions without involvement of remote bands can occur only between real and complex bands with $|\chi| = |C_\pm|$* [Eq. (1)]. This is the main result of this article. A direct consequence is that two-band models cannot host Chern bands after spontaneously breaking the PT symmetry [27,45], since the real band pair in a two-band model always has $\chi = 0$.

We consider a generic $N \times N$ transition Hamiltonian $H(\mathbf{k}, \lambda)$ parametrized by the “transition time” $0 \leq \lambda \leq 1$. At $\lambda = 0$, two real bands are band-gapped to remote bands, i.e., $\Delta(\mathbf{k}, \lambda = 0) = \text{Min}_i |\omega_i(\mathbf{k}, \lambda = 0) - \omega_j(\mathbf{k}, \lambda = 0)| > 0, \forall \mathbf{k}$, where $i \in \{1, 2\}, j \notin \{1, 2\}$. The two bands may be topological or trivial. At $\lambda = 1$, they eventually become two complex conjugate bands without touching remote bands during $\lambda \in [0, 1]$. For $(\mathbf{k}, \lambda)$ before the PT breaking, the real eigenstates $|\psi_1\rangle, |\psi_2\rangle$ span a 2D real vector space $V_\lambda^{(1)}(\mathbf{k})$. After the PT breaking, the two complex conjugate eigenstates can be “realified” by recombinations $(|\phi_+\rangle + |\phi_-\rangle)/2, (|\phi_+\rangle - |\phi_-\rangle)/2i$. These two realified vectors also span a 2D vector space $V_\lambda^{(3)}(\mathbf{k})$ for $\lambda, \mathbf{k}$ in a range different from that for $V^{(1)}$. The boundaries between the two

constructions in the $(\mathbf{k}, \lambda)$ -space are EPs, where $|\psi_1\rangle, |\psi_2\rangle$ coalesce into a single eigenstate $|\psi_*\rangle$ and then split into a complex conjugate pair. For an EP of eigenvalue ω_* , although there is only one eigenvector, we have another generalized eigenvector satisfying $(\omega_* - H)^2|\psi'_*\rangle = 0$ [46]. The real vectors $|\psi_*\rangle, |\psi'_*\rangle$ span a 2D real vector space $V_\lambda^{(2)}(\mathbf{k})$ in the $(\mathbf{k}, \lambda)$ -range of EPs. The most important step is that putting $V^{(1)}, V^{(2)}, V^{(3)}$ defined in mutually complementary $(\mathbf{k}, \lambda)$ -ranges together forms a continuous function of spaces $V_\lambda(\mathbf{k})$ for all $\lambda \in [0, 1]$, $\mathbf{k} \in \text{BZ}$. This observation comes from the connection between the projection operator P_{sum} to the two bands together and the Hamiltonian H [46]:

$$P_{\text{sum}}(\mathbf{k}, \lambda) = \frac{1}{2\pi i} \oint_l dz [z - H(\mathbf{k}, \lambda)]^{-1}, \quad (9)$$

where l can be any loop that encloses the spectrum of the two bands at $(\mathbf{k}, \lambda)$ [e.g., Figs. 2(b)–2(e)]. The right hand side is continuous whenever the two bands do not touch remote bands [27]. So $P_{\text{sum}}(\mathbf{k}, \lambda)$ is real and continuous for the whole transition. The images of P_{sum} at different $(\mathbf{k}, \lambda)$ are exactly the spaces $V^{(1)}, V^{(2)}, V^{(3)}$ [46,47]. This shows the continuity of $V_\lambda(\mathbf{k})$. Intuitively, generalized eigenvectors help $V^{(2)}$ glue $V^{(1)}$ and $V^{(3)}$ continuously together.

The continuity of $V_\lambda(\mathbf{k})$ imposes a rigid connection between the complex bands and the real bands across the transition. Before the transition, the spaces $V_{\lambda=0}(\mathbf{k}) \equiv V_{\lambda=0}^{(1)}(\mathbf{k})$ correspond to the rank-2 real vector bundle $F_{\mathbb{R}}$ formed by the eigenstates $|\psi_{1,2}(\mathbf{k}, \lambda = 0)\rangle$. After the transition, $|\phi_{\pm}(\mathbf{k})\rangle$ correspond to complex conjugate rank-1 vector bundles E_C, E_C^* , respectively. The two complex bundles admit a realification F^{RI} , given by $V_{\lambda=1}(\mathbf{k}) \equiv V_{\lambda=1}^{(3)}(\mathbf{k})$. The continuity of $V_\lambda(\mathbf{k})$ implies $F^{RI} \cong F_{\mathbb{R}}$ [48]. Note that F^{RI} and $F_{\mathbb{R}}$ are only isomorphic but usually different.

As a realification, F^{RI} satisfies two properties [43,49]. First, it has trivial first Stiefel-Whitney topology $w_1(F^{RI}) = 0$, also known as weak topology for real bands [39,50–52]. This means that adiabatic transport of a basis for F^{RI} around the k_x or k_y direction of the BZ-torus preserves its orientation. Second, its Euler number $\chi(F^{RI})$ is related to the Chern number of the complex band $|\chi(F^{RI})| = |C(E_C)|$. Combined with $F^{RI} \cong F_{\mathbb{R}}$, the first property enforces that if a transition not involving remote bands occurs, an Euler number can be assigned to the initial real band, as this is possible if and only if $w_1(F_{\mathbb{R}}) = 0$ [43]. The second property of F^{RI} dictates $|\chi(F_{\mathbb{R}})| = |C(E_C)|$. This concludes the generic proof. A physical illustration of the proof is given in Supplemental Material [47].

The necessary conservation $|C_{\pm}| = |\chi|$ is unique to spontaneous PT breaking, which realizes the realification of complex bundles in Chern's construction [53]. In comparison, transitions between real and complex bands in Hermitian regimes require explicitly breaking the symmetry in $H(\mathbf{k})$ [36–38]. In explicit symmetry breaking, the resulting complex bands are not PT-reversal complex conjugate partners. The realification cannot be constructed. The Chern numbers there are not forced to equal the initial Euler number $\chi(F_{\mathbb{R}})$. A typical example is the trivial bands $\chi(F_{\mathbb{R}}) = 0$. From knowledge of two-band models [1,54], two trivial bands can always be deformed into two Chern bands of arbitrary $\pm C \neq 0$ without

involving remote bands in explicit symmetry breaking. However, this is impossible in spontaneous PT breaking, as shown in the following example of nondual transitions.

Nondual transition. We contrast the Chern-Euler duality transition with the transition $H_{\text{trivial}}(\mathbf{k}) \rightarrow H_{\text{Chern}}(\mathbf{k})$, which is a spontaneous PT-breaking transition involving bands that violate Eq. (1). We consider the interpolating Hamiltonians

$$H_\mu(\mathbf{k}) = (1 - \mu)H_{\text{trivial}}(\mathbf{k}) + \mu H_{\text{Chern}}(\mathbf{k}). \quad (10)$$

Spectra of $H_\mu(\mathbf{k})$ at various stages in the transition are shown in Figs. 3(a)–3(d). As the real bands in the initial model $H_{\text{trivial}}(\mathbf{k})$ carry $\chi = 0$, the transition to the complex Chern bands of $H_{\text{Chern}}(\mathbf{k})$ can only take place after all three bands cross simultaneously. The eigenvalues in Figs. 3(b) and 3(c) indicate the emergence of ERs. The ERs can be put in two categories: those connecting bands 1 and 2 and those connecting bands 2 and 3. The two types of ERs can intersect at cusps of third-order EPs [Fig. 3(e)] from dimension counting [17,18], where all three bands become degenerate. The ERs expand and progressively sweep across the BZ as μ further increases to 0.55. The transition is complete at $\mu = 0.72$ [Fig. 3(d)], where two imaginary gaps to bands with Chern number $C_{\pm} = \pm 2$ are opened symmetrically around the real axis. The necessary connection of three bands during all nondual transitions provides fresh insights on higher-order EPs [14–16], and serves as a crucial distinction from explicit symmetry-breaking transitions.

Discussion. In this paper, we have extended the concept of topological transitions to the realm of topological duality. The spontaneous PT-symmetry breaking in non-Hermitian PT-symmetric systems corresponds to the natural duality between complex vector bundles and real vector bundles. As a result, the same topology manifests as two distinct quantized properties, depending on the symmetry regime. Dissipative photonic and acoustic systems give this paradigm in vector bundle theory a concrete physical context.

The Chern-Euler duality transition paves the way for overcoming symmetry constraints in engineering topological states. The dual and nondual transitions can be diagnosed by measuring different types of ERs in the BZ or closed surfaces of parameter spaces on a wide range of platforms [3–6,31–35]. Berry curvatures are geometric phases around small plaquettes in the $\mathbf{k}$ space [42,55]. Due to the imaginary spectral gap, $|\phi_{-}(\mathbf{k})\rangle$ dominates the dynamic evolution. The Berry curvature can be measured by the phase difference between the final states $|\phi_{-}(\mathbf{k})\rangle$ and $|\phi_{-}(\mathbf{k} + \delta\mathbf{k})\rangle$ at different $\mathbf{k}$ values in few-level-system simulations [56,57], by performing an evolution $\exp(-iHt)$ with $1/t$ smaller than the imaginary gap for any initial state. Another direct measurement is to match the number of Dirac nodes that reflects the Euler number before the transition and the spectral transport reflecting the Chern number after the transition [58]. The latter may be measured by parametric pumping [59].

The three models are the simplest settings for experimental verification. More illustrative examples are included in Ref. [47]. In addition to the spectral phenomena presented here, there are many other phenomena related to the topological bands, including various in-gap modes and Wannierization. These properties will be alternative signatures of the transitions that invite future research.

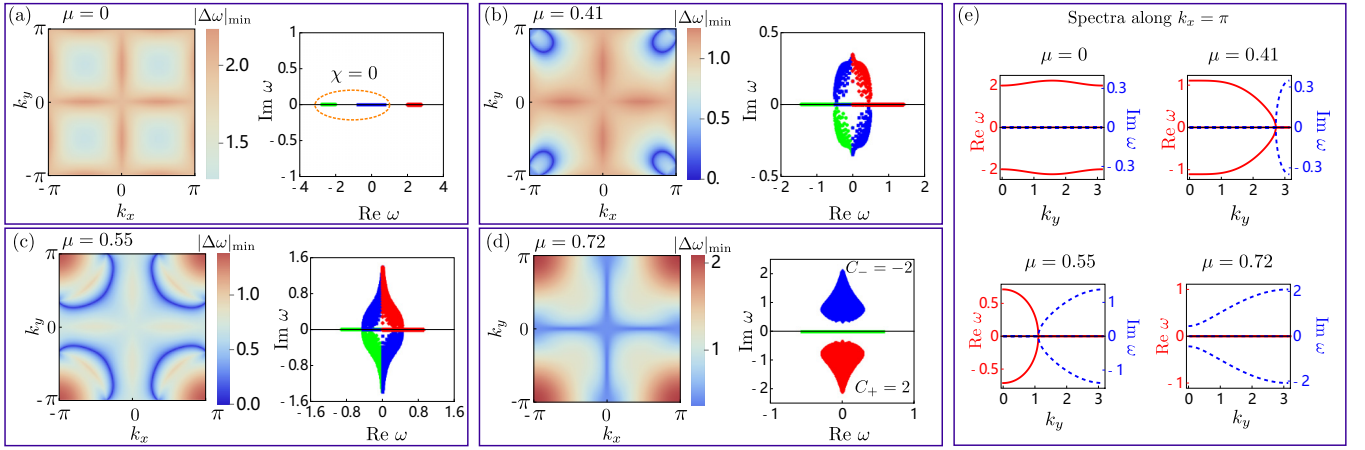


FIG. 3. The nondual transition from trivial bands to Chern bands. [(a)–(d)] The minimum eigenvalue gap $|\Delta\omega(\mathbf{k})|_{\min} = \text{Min}_{i,j} |\omega_i(\mathbf{k}) - \omega_j(\mathbf{k})|$ and the spectra of H_μ in the complex plane at $\mu = 0$ (a), 0.41 (b), 0.55 (c), and 0.72 (d). All three bands are connected together during the nondual transition, which is a universal feature of nondual transitions. (e) The real and imaginary parts of the spectra as a function of k_y at fixed $k_x = \pi$ for different μ . Third-order EPs are generated during this nondual transition.

Acknowledgments. The authors thank Vatsal Dwivedi and Felix von Oppen for fruitful discussions. K.Y. is supported by the ANR-DFG project (TWISTGRAPH). Z.L. is supported by Perimeter Institute; research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Colleges and Universities. P.X. is supported by the National Key R&D Program of China (Grant No. 2023YFA1406701)

and the National Natural Science Foundation of China (Grants No. 12025401 and No. 92265209). E.J.B. is supported by the Knut and Alice Wallenberg Foundation (Wallenberg Scholars Program, Grant No. 2023.0256) and the Göran Gustafsson Foundation for Research in Natural Sciences and Medicine.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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