

Beam-induced instabilities of Josephson plasma waves in a cylindrical beam-waveguide system with a layered superconducting shell

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We theoretically investigate the eigenspectrum and excitation of terahertz (THz) electromagnetic waves in a cylindrical beam-waveguide system containing a layered superconducting shell. The system is driven by a nonrelativistic tubular electron beam, which is assumed to be charge-neutralized by a stationary ion background. This charge compensation enables rectilinear beam propagation in the absence of an external focusing magnetic field and allows the superconductor to remain in the Meissner state. Starting from a hybrid electrodynamic boundary-value formulation, we derive the general dispersion relation for coupled beam-Josephson plasma waves (JPWs). The eigenmode spectrum and beam-induced instabilities are then analyzed in detail in the quasioleostatic slow-wave approximation. It is shown that bulk-surface JPWs can exhibit negative dispersion, which enables the development of absolute instability during their interaction with the electron beam. For surface JPWs, the propagating branch has positive dispersion, so its beam-induced excitation is associated with convective amplification rather than absolute instability. An analytical expression for the instability growth rate is derived for the absolute-instability regime. At large longitudinal wave numbers, the frequencies of the bulk-surface waves asymptotically approach the Josephson plasma frequency ω_J . The anisotropy coefficient γ is found to play a crucial role in the surface-JPW spectrum: the azimuthally symmetric surface mode ($n = 0$) preserves its traveling-wave character even at high anisotropy, whereas azimuthally asymmetric modes ($n \geq 1$) degenerate into localized oscillations with frequencies close to ω_J due to the shunting effect caused by the high in-plane conductivity. Numerical analysis of the absolute instability of bulk-surface JPWs shows that, among the modes considered, the maximum growth rate is attained for the $n = 1$ mode. A crossover is revealed in the dependence of the growth rate on the thickness of the superconducting shell. For low-azimuthal-index modes, thinner shells enhance the instability by reducing the modal energy stored in the superconducting material and increasing the effective coupling impedance, whereas for higher-order modes ($n \geq 9$), the instability is enhanced when the shell thickness increases because thicker shells support whispering-gallery-like field distributions. These properties indicate that the proposed structure can serve as a flexible platform for tunable THz radiation sources.

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I. INTRODUCTION

The rapid development of modern technologies, particularly beyond-5G telecommunications, radar systems, security screening, spectroscopy, and medical diagnostics, requires efficient sources of millimeter, submillimeter, and terahertz (THz) electromagnetic radiation [1–5]. Consequently, the generation of high-power THz radiation remains an important and actively developing research field. Among the different approaches to THz generation, electron-beam-driven electrodynamic systems are of particular interest because they

can provide high output power and can be scaled to short-wavelength frequency ranges.

Vacuum electronic devices, such as backward-wave oscillators (BWOs) and surface-wave oscillators (SWOs), remain classical but still actively evolving solutions for high-power radiation generation. Recent theoretical and experimental advances have demonstrated the possibility of driving such devices by weakly relativistic electron beams at increasingly high frequencies. For example, significant progress has been achieved in the development of SWOs operating in the W band (75–110 GHz) [6] and high-power oversized SWOs in the Y band, near 0.3 THz [7]. Similarly, BWOs are being extended toward the THz range for applications from high-power radiation generation to plasma diagnostics [8,9]. Further improvement of such sources is also stimulated by the development of novel cathode systems, including dual-gridded carbon-nanotube cathodes that enhance the emission current density [10].

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The efficiency and spectral properties of beam-driven sources can be strongly modified by using complex electrodynamic geometries. Recent studies have considered Cherenkov radiation in cylindrical metallic waveguides with rippled walls containing plasma or dielectric inserts [11,12]. In addition, instabilities of spoof surface plasmon modes excited by magnetized electron beams in corrugated structures have been analyzed as a mechanism for controlling the radiation characteristics [13]. Beyond traditional vacuum structures, beam-plasma interaction mechanisms also provide promising routes for THz generation. These include THz emission associated with the ion-focus regime [14], beam-plasma beat-wave interactions [15], and wakefields generated by rotating electron beams in magnetized plasma columns [16]. Accelerator-based schemes offer another powerful class of THz sources; for instance, high-repetition-rate THz pulses for pump-probe experiments can be generated using electron bunches in wigglers [17], while strong-field THz sources based on coherent transition radiation have also been demonstrated [18].

In parallel with electron-beam sources, photonic and material-based THz techniques are rapidly developing. Important examples include high-power THz sources based on optical rectification in nonlinear crystals [19] and ultrastable photomixing sources for precision spectroscopy [20]. Metamaterials provide additional possibilities for spectral and spatial control of THz radiation [21,22], while quantum-interference mechanisms in Mott insulators have also been proposed for THz generation [23].

Layered superconductors, such as natural BiSrCaCuO compounds and artificial Josephson-junction stacks, represent a special class of materials for THz electrodynamics [24]. Their strong anisotropy enables the propagation of Josephson plasma waves (JPWs) in the THz frequency range [25,26]. The spectrum and excitation of these waves exhibit a rich phenomenology, including oblique surface modes [27], resonant transparency effects [28], and nonlinear phenomena associated with the intrinsic nonlinearity of the Josephson effect [29–31]. The interaction of charged particles with layered superconductors is of particular interest for the development of new THz radiation sources. For example, the generation of bulk and surface transition radiation caused by an electron crossing the interface between a layered superconductor and an isotropic dielectric was studied in Ref. [32] in the THz range.

Layered cuprates can have either one or several superconducting layers in a crystallographic unit cell. In the present work we use the effective electrodynamic description of a single-layer layered superconductor, i.e., a material with one superconducting layer per unit cell, such as $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$ (LSCO) or $\text{Bi}_2\text{Sr}_2\text{CuO}_{6+\delta}$ (Bi-2201). This case should be distinguished from bilayer cuprates, such as $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$ (YBCO) and $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$ (Bi-2212), where the presence of two superconducting layers per unit cell and inequivalent Josephson junctions gives rise to additional Josephson plasma modes. In particular, two longitudinal Josephson-plasma resonances and one transverse-optical Josephson plasma resonance can occur in bilayer systems for electric-field polarization perpendicular to the superconducting layers; in optical geometries, these resonances can

hybridize with electromagnetic waves into Josephson plasma-polariton branches, as discussed in Refs. [33,34]. More recent work has also emphasized the richness of the THz Josephson-plasmon response in bilayer cuprates [35,36]. The extension of the present beam-wave instability theory to such bilayer systems is therefore a natural but separate problem.

Recently, the excitation of eigenwaves of a cylindrical layered superconductor by a tubular electron beam was studied in Ref. [37]. In that work, a solid cylindrical layered superconductor was considered, and the electron beam was treated in the thin-beam approximation, i.e., as radially infinitesimal and propagating in vacuum near the cylindrical surface. This model made it possible to analyze the hybrid structure of the eigenmodes, to predict hybrid pseudosurface Josephson plasma waves, and to study the beam-induced instabilities associated with the Vavilov-Cherenkov and anomalous Doppler mechanisms.

The present work extends that study to a different and more experimentally oriented beam-waveguide configuration. Here, the layered superconductor forms a cylindrical shell rather than a solid cylinder. The inner dielectric region, $0 < \rho < \rho_0$, can serve as a coolant channel, for example for liquid nitrogen, which is important for maintaining the superconducting state during beam operation. At the same time, this dielectric core is not merely a technical element of the structure. It enters the boundary-value problem and participates in the formation of the eigenmodes of the whole electrodynamic system. Therefore, its radius and permittivity affect the eigenmode spectrum, the field localization, and, consequently, the beam-wave coupling.

Another essential difference from Ref. [37] is the beam model. Instead of a radially infinitesimal tubular beam, we consider a finite-thickness charge-neutralized electron beam occupying the annular region $\rho_b < \rho < \rho_w$ between the outer surface of the superconducting shell and the perfectly conducting cylindrical wall. This formulation allows us to describe the interaction of the beam with the electromagnetic field distributed over a finite beam cross section and to relate the equilibrium beam density to the total beam current. As a result, the present model provides a more realistic description of a cylindrical superconducting slow-wave structure and makes it possible to study how the coolant-channel radius, the shell thickness, and the beam-channel geometry control the beam-induced instability of Josephson plasma waves.

The paper is organized as follows. In Sec. II, we formulate the problem and introduce the hybrid electrodynamic boundary-value formulation for the cylindrical beam-waveguide structure with a layered superconducting shell. In Sec. III, starting from the general dispersion relation, we derive and analyze the *quasielectrostatic slow-wave dispersion relations for coupled beam waves and Josephson plasma eigenmodes* of the structure. The numerical analysis in this section is focused on the intrinsic bulk-surface and surface JPW eigenmodes in the absence of their interaction with the electron beam. In Sec. IV, we include the interaction of these eigenmodes with the tubular electron beam. We analyze the absolute instability of bulk-surface JPWs and discuss the convective amplification of surface JPWs. Appendix A gives the reduced determinant of the *general dispersion relation for coupled beam-JPWs*, whereas Appendix B presents the

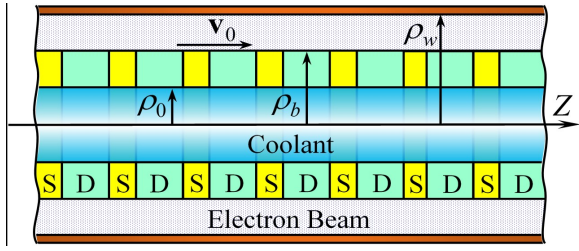


FIG. 1. Schematic longitudinal cross section of the cylindrical beam-waveguide structure in a plane containing the symmetry axis z . The dielectric core, which can serve as a coolant channel, occupies the region $0 < \rho < \rho_0$, the layered superconducting shell occupies $\rho_0 < \rho < \rho_b$, and the annular region $\rho_b < \rho < \rho_w$ contains the tubular electron beam. The outer boundary at $\rho = \rho_w$ is a perfectly conducting cylindrical wall. The structure is open along the z direction; no end wall is shown. Here, S and D denote the superconducting and dielectric layers within the layered superconducting shell, respectively.

supporting analysis of the field distributions and effective coupling-impedance estimates. Finally, Sec. V summarizes the main results.

II. PROBLEM STATEMENT

We consider an electrodynamic structure comprising a cylindrical layered superconducting shell with an inner radius ρ_0 and an outer radius ρ_b , located coaxially within a perfectly conducting cylindrical waveguide of radius ρ_w (see Fig. 1). The crystallographic axis $\mathbf{c}$ of the superconductor is aligned with the system's symmetry axis Oz . The inner region, $0 < \rho < \rho_0$, is filled with a dielectric material (e.g., liquid nitrogen) characterized by a permittivity ϵ_d . In the annular gap $\rho_b < \rho < \rho_w$, a charge-neutralized nonrelativistic electron beam propagates along the z -axis with an equilibrium density N_0 and mean velocity $v_0 \ll c$ (where c is the speed of light). The system is assumed to be infinitely long along the Oz axis (i.e., $-\infty < z < \infty$).

To describe the electromagnetic fields in this system, we first formulate a general hybrid boundary-value problem. In the dielectric core, $0 < \rho < \rho_0$, and in the layered superconducting shell, $\rho_0 < \rho < \rho_b$, the field representations can be written in the full electrodynamic form, with retardation effects retained. In the beam region, $\rho_b < \rho < \rho_w$, however, the electric field is described in the electrostatic approximation, which is appropriate for a nonrelativistic charge-neutralized electron beam [38]. A detailed justification of the validity of this electrostatic description of the fields in the beam region for low beam velocities v_0 is provided in Sec. III.

This hybrid formulation is used to construct the general boundary-value problem for the cylindrical beam-waveguide structure. Substitution of the field representations into the boundary conditions leads to a homogeneous system of algebraic equations for the expansion coefficients. The condition that the determinant of this system vanishes gives the *general dispersion relation for coupled beam-JPWs*. The explicit construction of the corresponding reduced determinant is given in Appendix A.

Although the electric field in the beam region is treated in the electrostatic approximation, the time-dependent magnetic field in this region is not set to zero. The magnetic field required to satisfy the boundary conditions at the material interfaces and at the perfectly conducting wall is retained; in the electrostatic limit used for the beam region, it is described by Laplace's equation.

In the electrostatic approximation, the electric field in the beam region is described by a self-consistent system of equations, which includes Maxwell's equations along with the linearized continuity and motion equations for the beam electrons:

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = 0, \quad (1)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = 0, \quad (2)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 4\pi eN(\mathbf{r}, t), \quad (3)$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0, \quad (4)$$

$$e \frac{\partial N(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r}, t) = 0, \quad (5)$$

$$\frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + v_0 \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial z} = \frac{e}{m_0} \mathbf{E}(\mathbf{r}, t), \quad (6)$$

where $e = -|e|$ is the electron charge, m_0 is the free-electron mass, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ are the electric and magnetic field vectors, respectively, and $\mathbf{D}(\mathbf{r}, t)$ is the electric displacement, $N(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ are the variable components of the beam density and velocity, $\mathbf{j}(\mathbf{r}, t) = eN_0\mathbf{v}(\mathbf{r}, t) + e\mathbf{v}_0N(\mathbf{r}, t)$ is the linearized perturbed current density component. The electromagnetic fields in the dielectric and superconductor regions are described by the following Maxwell's equations:

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t}, \quad (7)$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t}, \quad (8)$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = 0, \quad (9)$$

$$\nabla \cdot \mathbf{H}(\mathbf{r}, t) = 0. \quad (10)$$

All perturbed quantities are expanded in spatiotemporal harmonics as follows:

$$f(\mathbf{r}, t) = \sum_{n=-\infty}^{\infty} \iint_{-\infty}^{\infty} f_n(\rho, q_z, \omega) \times \exp[i(q_z z + n\varphi - \omega t)] dq_z d\omega, \quad (11)$$

where ω denotes the angular frequency, q_z is the longitudinal wave number, and n is the azimuthal mode index.

The constitutive relation between the electric displacement vector $\mathbf{D}(\mathbf{r}, t)$ and the electric field vector $\mathbf{E}(\mathbf{r}, t)$, which accounts for the temporal dispersion of the medium, is written as the following general nonlocal integral relation:

$$D_i(\mathbf{r}, t) = \int_{-\infty}^t \hat{\epsilon}_{ij}(t - t') E_j(\mathbf{r}, t') dt', \quad (12)$$

where the indices i, j run over $\{\rho, \varphi, z\}$, and summation is performed on the repeating index j . The kernel $\hat{\epsilon}_{ij}(t - t')$

represents the dielectric response tensor of the medium. Its dependence solely on the time difference $t - t'$ implies the temporal homogeneity of the structure's electrodynamic properties.

After transforming Eq. (12) to the frequency domain, the integral constitutive relation reduces to the local algebraic relation

$$D_i(\rho, q_z, \omega) = \varepsilon_{ij}(\omega) E_j(\rho, q_z, \omega). \quad (13)$$

Here, $\varepsilon_{ij}(\omega)$ is the frequency-dependent dielectric permittivity tensor, defined as the Fourier transform of the temporal response function $\hat{\varepsilon}_{ij}(\tau)$:

$$\varepsilon_{ij}(\omega) = \int_0^\infty \hat{\varepsilon}_{ij}(\tau) \exp(i\omega\tau) d\tau. \quad (14)$$

In the dielectric region ($0 < \rho < \rho_0$), we have $\hat{\varepsilon}_{ij}(\tau) = \varepsilon_d \delta_{ij} \delta(\tau)$, where δ_{ij} is the Kronecker δ symbol and $\delta(\tau)$ is the Dirac δ function. Then the constitutive relation (13) takes the form $\mathbf{D}(\rho, q_z, \omega) = \varepsilon_d \mathbf{E}(\rho, q_z, \omega)$.

The electron beam is assumed to be charge-neutralized by a stationary positive ion background. This charge compensation suppresses the radial Coulomb repulsion and enables rectilinear beam propagation without an external focusing magnetic field [38]. Accordingly, the external static magnetic field is absent, so that the layered superconductor remains in the Meissner state. The estimated self-magnetic field of the beam, $H_b < 1 \text{ Oe} \simeq 80 \text{ A/m}$, is several orders of magnitude smaller than the lower critical field of the superconductor, $H_{c1} \simeq 170 \text{ Oe} \simeq 13.6 \text{ kA/m}$ [39]. Therefore, the influence of the beam self-magnetic field on the superconducting state can be neglected. The analysis of beam behavior is carried out in the approximation of one-dimensional electron motion, in which the beam-wave interaction is assumed to be governed mainly by the longitudinal electric-field component.

The electrodynamic response of the layered superconductor is described by the diagonal permittivity tensor [40]

$$\hat{\varepsilon}_s = \begin{pmatrix} \varepsilon_{ab} & 0 & 0 \\ 0 & \varepsilon_{ab} & 0 \\ 0 & 0 & \varepsilon_c \end{pmatrix}, \quad (15)$$

with the components

$$\varepsilon_{ab} = \varepsilon \left(1 - \frac{\omega_J^2 \gamma^2}{\omega^2} \right), \quad \varepsilon_c = \varepsilon \left(1 - \frac{\omega_J^2}{\omega^2} \right), \quad (16)$$

where $\omega_J = (8\pi|e|DJ_c/\hbar\varepsilon)^{1/2}$ is the Josephson plasma frequency, D is the structure period, J_c is the maximum Josephson current density, $\hbar$ is the reduced Planck constant, ε is the interlayer dielectric constant, $\gamma = \lambda_c/\lambda_{ab}$ is the anisotropy coefficient, λ_{ab} characterizes the screening response due to currents flowing in the superconducting ab planes, whereas λ_c characterizes the response along the crystallographic c axis, i.e., across the layers [24].

Equations (15) and (16) correspond to the standard effective-medium model of a single-layer layered superconductor, where a single Josephson plasma frequency characterizes the interlayer electrodynamics. The formal limit $\gamma = 1$ reduces this tensor to an isotropic plasma-like response and will be used below only as a reference limiting case. The main numerical results for the layered-superconductor regime

are obtained for the strongly anisotropic case $\gamma = 200$, unless explicitly stated otherwise.

In Eqs. (16), dissipative losses in the superconductor are neglected. This approximation is appropriate for analyzing the onset and initial linear stage of the beam-wave instability at cryogenic temperatures. In layered cuprate superconductors, the Josephson plasma resonance is directly related to the c -axis superfluid response, while its linewidth is governed by the c -axis quasiparticle conductivity [41]. At low temperatures, the quasiparticle contribution to the c -axis response is reduced compared with the normal state, so that the internal damping of Josephson plasma oscillations can be small. Therefore, neglecting dissipative losses provides a reasonable first approximation for determining the instability threshold and the maximum growth rate. A quantitative analysis of damping effects would require introducing finite quasiparticle conductivities into ε_{ab} and ε_c and is beyond the scope of the present work.

The longitudinal permittivity of the electron beam is given by

$$\varepsilon_b = 1 - \frac{\omega_p^2}{(\omega - q_z v_0)^2}, \quad (17)$$

where $\omega_p = \sqrt{4\pi e^2 N_0/m_0}$ is the beam plasma frequency [38].

To derive the dispersion relation for coupled beam-JPWs, we impose the boundary conditions at the material interfaces $\rho = \rho_0$ and $\rho = \rho_b$, as well as at the perfectly conducting wall $\rho = \rho_w$. At the material interfaces, the standard electromagnetic boundary conditions require the continuity of the tangential components E_z , E_φ , H_z , and H_φ , together with the continuity of the normal components D_ρ and B_ρ , in the absence of singular surface charges and current sheets. Since the transverse field components can be expressed through the longitudinal field components E_{zn} and H_{zn} , it is sufficient to impose an independent set of four conditions at each material interface.

In what follows, we use the continuity of E_{zn} , H_{zn} , $E_{\varphi n}$, and $D_{\rho n}$ at $\rho = \rho_0$ and $\rho = \rho_b$. The remaining standard boundary conditions are not independent and are automatically satisfied once the field representations and the above independent boundary conditions are imposed. At the perfectly conducting wall $\rho = \rho_w$, the tangential electric field vanishes. In the cylindrical field representation used here, this condition gives

$$E_{zn}(\rho_w, q_z, \omega) = 0, \quad \left. \frac{\partial H_{zn}(\rho, q_z, \omega)}{\partial \rho} \right|_{\rho=\rho_w} = 0. \quad (18)$$

This yields a set of ten boundary conditions. Using the field expansion (11) and standard methods for solving Maxwell's equations in cylindrical coordinates [42–44], we determine the field expressions in each region of the structure.

In the dielectric core, $0 < \rho < \rho_0$, the longitudinal field components are written as

$$H_{zn}(\rho, q_z, \omega) = \begin{cases} A_n^H J_n(\kappa_d \rho), & \kappa_d^2 > 0; \\ A_n^H I_n(\chi_d \rho), & \kappa_d^2 < 0; \end{cases} \quad (19)$$

$$E_{zn}(\rho, q_z, \omega) = \begin{cases} A_n^E J_n(\kappa_d \rho), & \kappa_d^2 > 0; \\ A_n^E I_n(\chi_d \rho), & \kappa_d^2 < 0. \end{cases} \quad (20)$$

In the layered superconducting shell, $\rho_0 < \rho < \rho_b$, the corresponding solutions are

$$H_{zn}(\rho, q_z, \omega) = \begin{cases} B_n^H(q_z, \omega)J_n(\kappa_H \rho) \\ + C_n^H(q_z, \omega)Y_n(\kappa_H \rho), & \kappa_H^2 > 0; \\ B_n^H(q_z, \omega)I_n(\chi_H \rho) \\ + C_n^H(q_z, \omega)K_n(\chi_H \rho), & \kappa_H^2 < 0; \end{cases} \quad (21)$$

$$E_{zn}(\rho, q_z, \omega) = \begin{cases} B_n^E(q_z, \omega)J_n(\kappa_E \rho) \\ + C_n^E(q_z, \omega)Y_n(\kappa_E \rho), & \kappa_E^2 > 0; \\ B_n^E(q_z, \omega)I_n(\chi_E \rho) \\ + C_n^E(q_z, \omega)K_n(\chi_E \rho), & \kappa_E^2 < 0. \end{cases} \quad (22)$$

Here, $\kappa_d^2 = \varepsilon_d \omega^2 / c^2 - q_z^2$ is the squared transverse wave number in the dielectric core, and $\chi_d^2 = -\kappa_d^2$. In the layered superconductor, $\kappa_H^2 = \varepsilon_{ab} \omega^2 / c^2 - q_z^2$ and $\kappa_E^2 = \kappa_H^2 \varepsilon_c / \varepsilon_{ab}$ are the squared radial wave numbers for the H - and E -type waves, respectively. We also use $\chi_H^2 = -\kappa_H^2$ and $\chi_E^2 = -\kappa_E^2$.

In the electron-beam region, $\rho_b < \rho < \rho_w$, the longitudinal components are

$$H_{zn}(\rho, q_z, \omega) = F_n^H(q_z, \omega)I_n(|q_z|\rho) \\ + G_n^H(q_z, \omega)K_n(|q_z|\rho). \quad (23)$$

$$E_{zn}(\rho, q_z, \omega) = F_n^E(q_z, \omega)I_n(|q_z|\rho) \\ + G_n^E(q_z, \omega)K_n(|q_z|\rho), \quad (24)$$

In Eqs. (19)–(24), $J_n(u)$ and $Y_n(u)$ are Bessel functions of the first and second kind, respectively, whereas $I_n(u)$ and $K_n(u)$ are modified Bessel functions of the first and second kind, respectively. The function $K_n(u)$ is also known as the Macdonald function [45]. The coefficients $A_n^{E,H}(q_z, \omega)$, $B_n^{E,H}(q_z, \omega)$, $C_n^{E,H}(q_z, \omega)$, $F_n^{E,H}(q_z, \omega)$, and $G_n^{E,H}(q_z, \omega)$ are arbitrary constants. Their number is equal to the number of boundary conditions. The form of the solutions is chosen so that $E_{zn}(\rho, q_z, \omega)$ and $H_{zn}(\rho, q_z, \omega)$ remain finite at $\rho = 0$ and satisfy the boundary conditions at the perfectly conducting wall $\rho = \rho_w$.

Substituting the field representations (19)–(24) into the boundary conditions gives a homogeneous system of ten linear algebraic equations for the ten unknown expansion coefficients $A_n^{E,H}$, $B_n^{E,H}$, $C_n^{E,H}$, $F_n^{E,H}$, and $G_n^{E,H}$. In this procedure, the transverse field components entering the boundary conditions, in particular $E_{\varphi n}$ and $D_{\rho n}$, are expressed through the longitudinal field components E_{zn} and H_{zn} by means of Maxwell's equations.

A nontrivial solution of the resulting homogeneous system exists only when its determinant vanishes. In practice, six boundary conditions are used to eliminate six expansion coefficients, reducing the problem to a compact 4×4 determinant. The rows of this reduced determinant correspond to the continuity conditions for $E_{\varphi n}$ and $D_{\rho n}$ at $\rho = \rho_0$ and $\rho = \rho_b$. The explicit construction of this determinant is rather cumbersome and is therefore given in Appendix A.

III. ANALYSIS OF THE DISPERSION RELATIONS

In this section, we analyze the dispersion relations for coupled beam-JPWs obtained within the hybrid approach

described above. Starting from the general dispersion relation derived in Appendix A, we focus here on its electrostatic limit, which is most relevant for the nonrelativistic, charge-neutralized electron beam considered in this work. In this limit, we separately consider the coupling of the beam waves to bulk-surface and surface JPWs. Intrinsic bulk-surface JPWs are understood here as eigenmodes whose radial field profiles are oscillatory inside the layered superconducting shell, $\rho_0 < \rho < \rho_b$, and evanescent in the dielectric core, $0 < \rho < \rho_0$, and in the outer vacuum gap, $\rho_b < \rho < \rho_w$, in the absence of the electron beam.

In the following analytical and numerical analysis, we use the quasiolestatic slow-wave limit of the general hybrid formulation. This approximation is justified by the character of the beam-wave interaction considered in this work. The instability is associated with the Vavilov-Cherenkov synchronism condition (see Ref. [38])

$$v_{ph} = \frac{\omega}{q_z} = v_0,$$

where v_{ph} is the phase velocity of the wave excited by the electron beam. Since the beam is nonrelativistic, $v_0 \ll c$, the resonantly excited waves also satisfy $v_{ph} \ll c$. Therefore, for the beam-synchronous branches,

$$\frac{\omega^2}{c^2} \ll q_z^2.$$

Thus, retardation terms are small compared with the longitudinal wave-number terms. In this slow-wave limit, which is equivalent to taking $c \rightarrow \infty$, the terms in the general determinant (A1) that contain explicit factors of $1/c$ and $1/c^2$ vanish. At the same time, the retardation contributions proportional to ω^2/c^2 are neglected in the radial wave parameters. Consequently, the *general dispersion relation for coupled beam-JPWs* reduces to the *electrostatic forms* analyzed in Secs. III A and III C.

Accordingly, in the dielectric region we use

$$\kappa_d^2 = -q_z^2, \quad \chi_d^2 = q_z^2.$$

In the layered superconductor, the corresponding magnetic-type radial parameters become

$$\kappa_H^2 = -q_z^2, \quad \chi_H^2 = q_z^2.$$

For the electric-type radial parameter in the layered superconductor, the electrostatic limit gives

$$\kappa_E^2 = -q_z^2 \frac{\varepsilon_c}{\varepsilon_{ab}}, \quad \chi_E^2 = -\kappa_E^2 = q_z^2 \frac{\varepsilon_c}{\varepsilon_{ab}}.$$

The appropriate sign of the radial parameter is chosen according to the radial behavior of the considered mode: oscillatory for bulk-surface JPWs and evanescent for surface JPWs. Thus, the dispersion curves shown below describe the electrostatic Josephson-plasma branches relevant for Cherenkov synchronism with the nonrelativistic tubular electron beam, and are not intended to represent the full retarded spectrum near the vacuum or dielectric light lines.

A. Dispersion relation for coupled beam waves and bulk-surface JPWs in the electrostatic approximation

After taking the electrostatic slow-wave limit described above, only the determinant elements $\mathcal{M}_{21}$, $\mathcal{M}_{22}$, $\mathcal{M}_{41}$, and $\mathcal{M}_{42}$ in Eq. (A1) remain nonzero. As a result, the dispersion relation for coupled beam waves and bulk-surface JPWs takes the form

$$\frac{Y'_{nsb}}{Y_{nsb}} + \frac{\xi_2}{\xi_1} \frac{2Y_{nso}}{\pi \kappa_E \rho_b J_{nsb} (Y_{nsb})^2} = \frac{|q_z| \varepsilon_b}{\kappa_E \varepsilon_{ab}} R_n, \quad (25)$$

where

$$R_n = \frac{I'_{nbb}/I_{nbw} - K'_{nbb}/K_{nbw}}{I_{nbb}/I_{nbw} - K_{nbb}/K_{nbw}}, \quad (26)$$

$$\xi_1 = -\frac{\varepsilon_d}{|q_z \rho_b|} \left(\frac{J_{nso}}{J_{nsb}} - \frac{Y_{nso}}{Y_{nsb}} \right) \frac{I'_{ndo}}{I_{ndo}} + \frac{\varepsilon_{ab} \kappa_E \rho_b}{(q_z \rho_b)^2} \left(\frac{J'_{nso}}{J_{nsb}} - \frac{Y'_{nso}}{Y_{nsb}} \right), \quad (27)$$

$$\xi_2 = -\frac{\varepsilon_d}{|q_z \rho_b|} \frac{I'_{ndo}}{I_{ndo}} + \frac{\varepsilon_{ab} \kappa_E \rho_b}{(q_z \rho_b)^2} \frac{Y'_{nso}}{Y_{nso}}. \quad (28)$$

The following notations are used in formulas (25)–(28): $I_{ndo} = I_n(|q_z| \rho_0)$, $J_{nso} = J_n(\kappa_E \rho_0)$, $J_{nsb} = J_n(\kappa_E \rho_b)$, $Y_{nso} = Y_n(\kappa_E \rho_0)$, $Y_{nsb} = Y_n(\kappa_E \rho_b)$, $I_{nbb} = I_n(|q_z| \rho_b)$, $K_{nbb} = K_n(|q_z| \rho_b)$, $I_{nbw} = I_n(|q_z| \rho_w)$, $K_{nbw} = K_n(|q_z| \rho_w)$. Throughout the paper, primes attached to special functions denote differentiation with respect to the full argument of the corresponding function. For example, $J'_{nso} \equiv dJ_n(x)/dx|_{x=\kappa_E \rho_0}$, $Y'_{nso} \equiv dY_n(x)/dx|_{x=\kappa_E \rho_0}$, and $I'_{nbb} \equiv dI_n(x)/dx|_{x=|q_z| \rho_b}$. The same convention is used below for all Bessel and modified Bessel functions.

In the absence of the electron beam, i.e., for $\omega_p = 0$, the dispersion relation (25) describes the bulk-surface JPW eigenmodes of the structure and reduces to

$$\Delta_n^{\text{BSW}}(q_z, \omega) = 0, \quad (29)$$

where

$$\Delta_n^{\text{BSW}}(q_z, \omega) = \frac{Y'_{nsb}}{Y_{nsb}} + \frac{\xi_2}{\xi_1} \frac{2Y_{nso}}{\pi \kappa_E \rho_b J_{nsb} (Y_{nsb})^2} - \frac{|q_z| \rho_b}{\kappa_E \rho_b \varepsilon_{ab}} R_n, \quad (30)$$

$$R_n = \frac{I'_{nbb}/I_{nbw} - K'_{nbb}/K_{nbw}}{I_{nbb}/I_{nbw} - K_{nbb}/K_{nbw}}, \quad (31)$$

$I_{nbb} = I_n(|q_z| \rho_b)$, $K_{nbb} = K_n(|q_z| \rho_b)$, $I_{nbw} = I_n(|q_z| \rho_w)$, and $K_{nbw} = K_n(|q_z| \rho_w)$. In the open-system limit, $\rho_w \rightarrow \infty$, the conducting wall is removed and the physically acceptable solution in the outer region is the wave with the amplitude decaying at $\rho \rightarrow \infty$. Then the expression for R_n reduces to

$$R_n = \frac{K'_{nbb}}{K_{nbb}}. \quad (32)$$

In this limit, Eq. (29) describes bulk-surface JPWs in an open electrodynamic system.

B. Numerical analysis of the dispersion relation for bulk-surface JPW eigenmodes

In this subsection, we analyze the dispersion curves for the intrinsic bulk-surface JPW eigenmodes of the structure in the

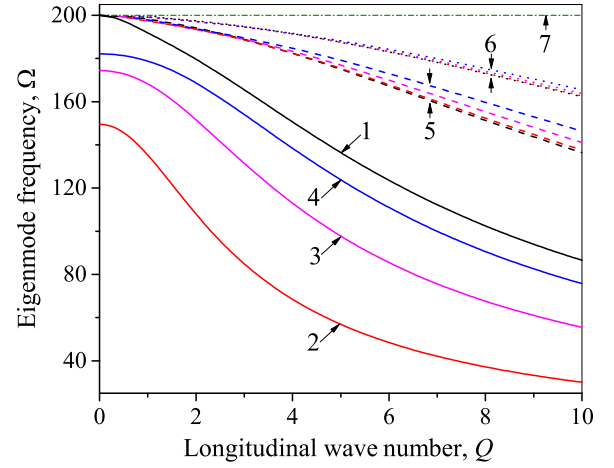


FIG. 2. Dispersion curves for bulk-surface JPWs for azimuthal mode indices $n = 0, 1, 2, 3$ and radial mode indices $s = 1, 2, 3$. Here, $\Omega = \omega/\omega_J$ and $Q = q_z \rho_b$ are the dimensionless frequency and longitudinal wave number, respectively. Curves 1–4 correspond to $n = 0, 1, 2, 3$ for $s = 1$; the groups of curves labeled 5 and 6 correspond to $s = 2$ and $s = 3$, respectively; within each group, n increases from 0 to 3 from bottom to top. Line 7 indicates $\Omega = \gamma$.

absence of the electron beam. These curves are obtained by solving the eigenmode dispersion equation $\Delta_n^{\text{BSW}}(q_z, \omega) = 0$, given by Eq. (29). The interaction of these intrinsic eigenmodes with the electron beam, as well as the resulting beam-induced instability, is considered below in Sec. IV. For the numerical analysis, we select the following structure parameters: $\rho_w = 5 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, $\rho_0 = 1.5 \times 10^{-3}$ m, $\varepsilon_d = 1.45$ (liquid nitrogen), $\varepsilon = 16$, $\gamma = 200$, $\omega_J = 4\pi \times 10^{12}$ s $^{-1}$ for BiSrCaCuO [24], and $v_0/c = 0.1$. Thus, unless otherwise specified, the numerical examples correspond to a strongly anisotropic single-layer layered superconductor.

Hereafter, we use the dimensionless frequency and longitudinal wave number

$$\Omega = \frac{\omega}{\omega_J}, \quad Q = q_z \rho_b. \quad (33)$$

Figure 2 shows the dispersion curves for bulk-surface JPWs. As seen from Fig. 2, dispersion curve 1, as well as the groups of curves 5 and 6, start at $Q = 0$ and $\Omega = \gamma$, whereas curves 2–4 start at $Q = 0$ and $\Omega < \gamma$. When the longitudinal wave number Q increases, the frequencies of all branches decrease, which indicates their negative dispersion. In the limit $Q \rightarrow \infty$, all branches asymptotically approach the Josephson plasma frequency, $\Omega = 1$. The negative dispersion of these intrinsic modes implies that their group velocity is directed opposite to the electron-beam velocity. When such modes are subsequently coupled to the electron beam, this counterpropagating beam-wave interaction can lead to the development of absolute instability (see Ref. [38]), which is analyzed in Sec. IV.

The numerical analysis also shows that changing the thickness of the layered superconducting shell affects different branches in different ways. When the shell thickness decreases, i.e., when ρ_0 increases while ρ_b and ρ_w remain fixed, the frequencies of branches 1, 5, and 6 decrease, whereas

the frequencies of branches 2–4 increase. For fixed ρ_0 and ρ_b , increasing ρ_w , which corresponds to a transition toward an open electrodynamic system, shifts all dispersion branches upward in frequency. A decrease in the anisotropy coefficient γ preserves the qualitative structure of the dispersion curves in Fig. 2, but lowers their frequencies because the existence interval $1 < \Omega < \gamma$ for bulk-surface JPWs becomes narrower. In the limit $\gamma \rightarrow 1$, the interval $\Delta\Omega = \gamma - 1$ vanishes, and bulk-surface JPWs disappear from the spectrum of the corresponding isotropic plasma-like medium.

C. Dispersion relation for coupled beam waves and surface JPWs in the electrostatic approximation

For surface waves, applying the same electrostatic slow-wave reduction to Eq. (A1) gives

$$\frac{K'_{nsb}}{K_{nsb}} - \frac{\xi_2}{\xi_1} \frac{K_{nso}}{\chi_E \rho_b I_{nsb} (K_{nsb})^2} = \frac{|q_z| \varepsilon_b R_n}{\chi_E \varepsilon_{ab}}, \quad (34)$$

$$\xi_1 = -\frac{\varepsilon_d}{|q_z \rho_b|} \left(\frac{I_{nso}}{I_{nsb}} - \frac{K_{nso}}{K_{nsb}} \right) \frac{I'_{ndo}}{I_{ndo}} + \frac{\varepsilon_{ab} \chi_E \rho_b}{(q_z \rho_b)^2} \left(\frac{I'_{nso}}{I_{nsb}} - \frac{K'_{nso}}{K_{nsb}} \right), \quad (35)$$

$$\xi_2 = -\frac{\varepsilon_d}{|q_z \rho_b|} \frac{I'_{ndo}}{I_{ndo}} + \frac{\varepsilon_{ab} \chi_E \rho_b}{(q_z \rho_b)^2} \frac{K'_{nso}}{K_{nsb}}. \quad (36)$$

In Eqs. (34)–(36), we use the notations $I_{nso} = I_n(\chi_E \rho_0)$, $I_{nsb} = I_n(\chi_E \rho_b)$, $K_{nso} = K_n(\chi_E \rho_0)$, $K_{nsb} = K_n(\chi_E \rho_b)$.

The dispersion relation for surface JPW eigenmodes in the absence of the beam in the electrostatic approximation is given by

$$\Delta_n^{SW}(q_z, \omega) = \frac{K'_{nsb}}{K_{nsb}} - \frac{\xi_2}{\xi_1} \frac{K_{nso}}{\chi_E \rho_b I_{nsb} (K_{nsb})^2} - \frac{|q_z| \rho_b}{\chi_E \rho_b \varepsilon_{ab}} R_n = 0, \quad (37)$$

In the open-system limit, $\rho_w \rightarrow \infty$, Eq. (37) describes surface JPWs provided that R_n is replaced by its asymptotic expression (32).

D. Numerical analysis of the dispersion relation for surface JPW eigenmodes

In this subsection, we analyze the dispersion curves for the intrinsic surface JPW eigenmodes of the structure in the absence of the electron beam. These curves are obtained by solving the eigenmode dispersion equation $\Delta_n^{SW}(q_z, \omega) = 0$, given by Eq. (37), using the electrodynamic parameters of the superconductor and dielectric layers specified above. The interaction of surface JPWs with the electron beam is taken into account below in Sec. IV in the analysis of the beam-induced instability.

Figure 3 shows the dispersion curves for azimuthally symmetric surface JPWs, $n = 0$, for several geometries. Curve 1 corresponds to the open-system limit, $\rho_w \rightarrow \infty$, whereas curves 2 and 3 correspond to a finite perfectly conducting waveguide. The transition from the open system to the screened waveguide, with ρ_b and ρ_0 kept fixed, reduces the eigenfrequency of the surface JPW. A reduction in the shell

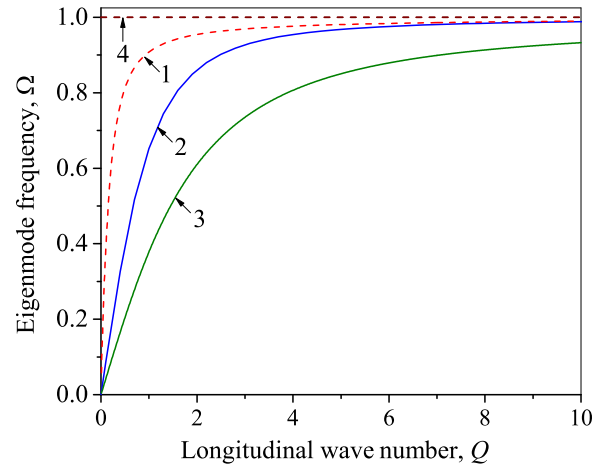


FIG. 3. Dispersion curves for azimuthally symmetric surface JPWs ($n = 0$) for different waveguide and shell geometries. Curve 1 corresponds to an open electrodynamic system, $\rho_w \rightarrow \infty$, with $\rho_b = 4.5 \times 10^{-3}$ m and $\rho_0 = 1.5 \times 10^{-3}$ m. Curve 2 corresponds to $\rho_w = 5 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, and $\rho_0 = 1.5 \times 10^{-3}$ m; curve 3 corresponds to $\rho_w = 5 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, and $\rho_0 = 4 \times 10^{-3}$ m. Line 4 denotes the Josephson plasma frequency, $\Omega = 1$.

thickness, i.e., an increase in ρ_0 at fixed ρ_w and ρ_b , also shifts the surface-mode frequency downward.

It should be noted that for the selected anisotropy coefficient $\gamma = 200$, the dispersion curves for azimuthally asymmetric surface JPWs ($n \geq 1$) practically coincide with the normalized Josephson plasma frequency, $\Omega = 1$, over the entire range of Q . Figure 4 illustrates the transformation of the surface JPW spectrum as the anisotropy coefficient γ varies. As seen from Fig. 4, in the isotropic case $\gamma = 1$, the spectrum of surface JPWs contains both the azimuthally

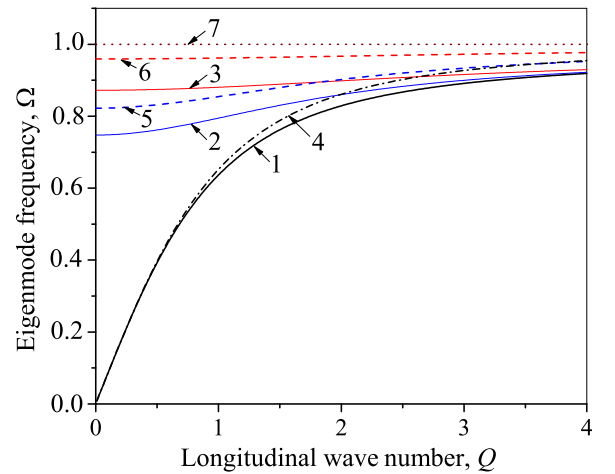


FIG. 4. Dispersion curves for surface JPWs for azimuthal mode indices $n = 0, 1, 2$ and different anisotropy coefficients γ . The structure parameters are $\rho_w = 5 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, and $\rho_0 = 1.5 \times 10^{-3}$ m; other material parameters are the same as in Fig. 3. Curves 1–3 correspond to $\gamma = 1$ for $n = 0, 1, 2$, respectively. Curve 4 corresponds to $n = 0$, $\gamma = 200$, whereas curves 5 and 6 correspond to $n = 1, 2$ at $\gamma = 1.1$. Line 7 denotes $\Omega = 1$.

symmetric mode ($n = 0$) and azimuthally asymmetric modes ($n = 1, 2$). In this case, the asymmetric modes have finite cut-off frequencies at $Q = 0$. Even a weak anisotropy, for example $\gamma = 1.1$, shifts the dispersion curves for the azimuthally asymmetric modes toward the normalized Josephson plasma frequency $\Omega = 1$. At the same time, the dispersion curve of the azimuthally symmetric mode remains close to its isotropic counterpart.

With a further increase in the anisotropy coefficient γ , the dispersion curves of the azimuthally asymmetric modes asymptotically approach $\Omega = 1$, whereas the frequency of the azimuthally symmetric mode also increases but its dispersion curve preserves its qualitative shape. Numerical calculations show that already at $\gamma = 1.3$ the dispersion curves for the azimuthally asymmetric modes practically coincide with the normalized Josephson plasma frequency. For example, for the asymmetric mode with $n = 1$ at $\gamma = 1.3$ and $Q = 0$, one obtains $\Omega - 1 \simeq 0.03 \ll 1$. For modes with $n > 1$, this difference is even smaller. Thus, the anisotropy strongly narrows the frequency range in which azimuthally asymmetric surface JPWs can exist as traveling waves. At $\gamma \gg 1$, these modes degenerate into localized surface Josephson plasma oscillations. By contrast, the azimuthally symmetric surface JPW retains its traveling-wave character even in the strongly anisotropic limit.

The physical interpretation of this behavior is as follows. The degeneration of azimuthally asymmetric surface JPWs ($n \geq 1$) into localized oscillations at $\gamma \gg 1$ is associated with the vanishing of their group velocity. As their dispersion curves flatten and approach the normalized Josephson plasma frequency $\Omega = 1$ for all values of Q , the dimensionless slope $d\Omega/dQ$ tends to zero. Consequently, the group velocity $v_g = d\omega/dq_z = \omega_J \rho_b d\Omega/dQ$ also vanishes. Physically, this means that energy transport along the waveguide is suppressed and the traveling surface waves transform into localized electrostatic oscillations near the Josephson plasma resonance. This effect is caused by the high in-plane conductivity of the layered superconductor, which shunts the tangential electric-field components required to sustain the propagation of modes with azimuthal variation.

For the azimuthally symmetric mode ($n = 0$), the situation is different. In this case, the azimuthal electric-field component is identically zero, $E_\varphi \equiv 0$. Therefore, the shunting effect associated with the high in-plane conductivity does not suppress the field structure responsible for the propagation of this mode. As a result, the corresponding dispersion curve retains a finite slope, $v_g \neq 0$, and the azimuthally symmetric surface JPW continues to propagate as a traveling wave even in the limit of strong anisotropy.

IV. ANALYSIS OF THE INSTABILITY GROWTH RATE

A. Analytical expression for the instability growth rate

To determine the growth rate of the beam-induced instability, we use a unified notation applicable to both bulk-surface and surface JPWs. Let $\alpha = \text{BSW}$ denote a bulk-surface Josephson plasma wave and $\alpha = \text{SW}$ denote a surface Josephson plasma wave. We introduce the eigenmode dispersion

function

$$\Delta_n^{(\alpha)}(q_z, \omega) = 0, \quad (38)$$

where

$$\Delta_n^{(\alpha)} = \begin{cases} \Delta_n^{\text{BSW}}, & \alpha = \text{BSW}, \\ \Delta_n^{\text{SW}}, & \alpha = \text{SW}. \end{cases} \quad (39)$$

Correspondingly, the radial wave parameter entering the beam-coupling term is written as

$$\mathcal{K}_\alpha = \begin{cases} \kappa_E, & \alpha = \text{BSW}, \\ \chi_E, & \alpha = \text{SW}. \end{cases} \quad (40)$$

Using the standard Vavilov-Cherenkov resonance condition [38], we write

$$\omega = \omega_{\text{res}}^{(\alpha)} = q_z v_0, \quad \Delta_n^{(\alpha)}(q_z, \omega_{\text{res}}^{(\alpha)}) = 0. \quad (41)$$

Introducing a small complex frequency shift

$$\delta\omega = \omega - \omega_{\text{res}}^{(\alpha)}, \quad |\delta\omega| \ll \omega_{\text{res}}^{(\alpha)}, \quad (42)$$

and expanding the coupled beam-JPW dispersion relation near the resonant point, we obtain

$$\delta\omega^3 = - \frac{\omega_p^2 |q_z| R_n}{\varepsilon_{ab} \mathcal{K}_\alpha \partial \Delta_n^{(\alpha)}(q_z, \omega) / \partial \omega|_{\omega=\omega_{\text{res}}^{(\alpha)}}}. \quad (43)$$

The instability growth rate is the positive imaginary part of the corresponding root,

$$\eta_\alpha = \frac{\sqrt{3}}{2} \left| \frac{\omega_p^2 |q_z| R_n}{\varepsilon_{ab} \mathcal{K}_\alpha \partial \Delta_n^{(\alpha)}(q_z, \omega) / \partial \omega|_{\omega=\omega_{\text{res}}^{(\alpha)}}} \right|^{1/3}. \quad (44)$$

Thus, Eqs. (43) and (44) apply to both types of eigenmodes. For bulk-surface JPWs one should set $\Delta_n^{(\alpha)} = \Delta_n^{\text{BSW}}$ and $\mathcal{K}_\alpha = \kappa_E$, whereas for surface JPWs one should set $\Delta_n^{(\alpha)} = \Delta_n^{\text{SW}}$ and $\mathcal{K}_\alpha = \chi_E$. As evident from Eq. (44), the growth rate scales proportionally to the cube root of the beam particle density ($N_0^{1/3}$), which is characteristic of the hydrodynamic regime of the beam-induced Cherenkov instability.

B. Numerical analysis of the instability growth rate

In this subsection, we analyze the beam-induced instability separately for two types of Josephson plasma eigenmodes of the structure: bulk-surface JPWs and surface JPWs. These two classes of eigenmodes have different dispersion properties and, therefore, lead to different types of beam-induced instability.

1. Bulk-surface JPWs: Absolute instability

In this part, we consider the excitation of bulk-surface JPWs by the tubular electron beam. As shown in Fig. 2, the dispersion curves for the intrinsic bulk-surface JPW eigenmodes have negative dispersion: their frequency decreases as the longitudinal wave number increases. Consequently, the group velocity of these waves is directed opposite to the electron-beam velocity. Under the Vavilov-Cherenkov resonance condition, this counterpropagating beam-wave interaction can give rise to an absolute instability.

An absolute instability means that the amplitude of a perturbation grows with time at a fixed spatial position. In other

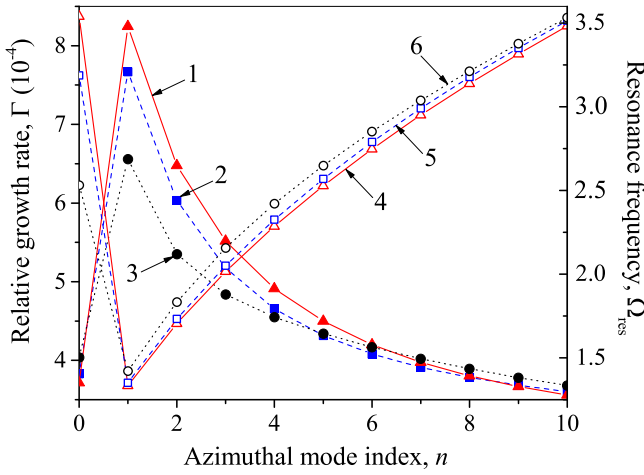


FIG. 5. Relative instability growth rate $\Gamma = \eta/\omega_{\text{res}}$ (curves 1–3) and normalized resonant frequency $\Omega_{\text{res}} = \omega_{\text{res}}/\omega_J$ (curves 4–6) of excited bulk-surface JPWs as functions of the azimuthal mode index n . The radial mode index is $s = 1$. Curves 1 and 4 correspond to $\rho_0 = 3.3 \times 10^{-3}$ m, curves 2 and 5 to $\rho_0 = 3 \times 10^{-3}$ m, and curves 3 and 6 to $\rho_0 = 2 \times 10^{-3}$ m.

words, the instability is not merely carried away from the excitation region by the propagating wave; instead, the field grows locally. This is the basic mechanism by which backward-wave oscillators and similar electron-beam radiation sources operate.

The field-distribution analysis below leads to a simple physical picture: the beam-wave coupling is governed primarily by the penetration of the longitudinal electric field into the beam region and by the fraction of the modal energy stored inside the superconducting shell. For low-order modes, these factors favor thinner shells and lead to the maximum growth rate for the $n = 1$ bulk-surface mode. For higher azimuthal indices, the field distribution becomes whispering-gallery-like, and a thicker shell becomes more favorable. These conclusions can be seen directly from the field profiles shown in Figs. 6–8.

For the numerical analysis of the instability growth rate, we use the following structure parameters: $\rho_w = 5 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, and $v_0/c = 0.1$. The beam current is $I_b = 0.1$ A. Assuming that the tubular beam uniformly fills the annular region $\rho_b < \rho < \rho_w$, the equilibrium beam density is calculated as

$$N_0 = \frac{I_b}{|e|v_0\pi(\rho_w^2 - \rho_b^2)}.$$

The material parameters of the layered superconductor and the dielectric are the same as those used above. The inner radius takes the values $\rho_0 = 2 \times 10^{-3}$ m, 3×10^{-3} m, and 3.3×10^{-3} m. For these beam parameters, the self-magnetic field is only 0.044 Oe (≈ 3.56 A/m), which is several orders of magnitude smaller than the lower critical field of the superconductor, $H_{c1} \approx 13.6$ kA/m. Thus, the Meissner-state approximation remains valid.

Figure 5 shows the relative instability growth rate $\Gamma = \eta/\omega_{\text{res}}$ and the normalized resonant frequency $\Omega_{\text{res}} = \omega_{\text{res}}/\omega_J$ of excited bulk-surface JPWs as functions of the azimuthal

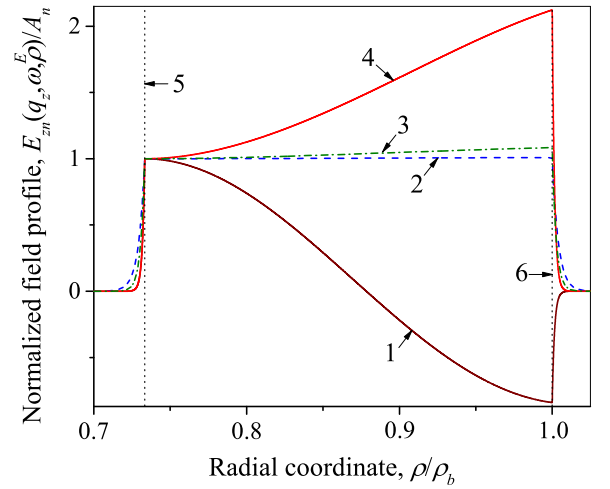


FIG. 6. Radial distributions of the normalized field profile $E_{zn}(q_z, \omega, \rho)/A_n^E$ of the longitudinal electric-field component for bulk-surface JPWs. The distributions are calculated for $\rho_0 = 3.3 \times 10^{-3}$ m, $\rho_b = 4.5 \times 10^{-3}$ m, $\rho_w = 5 \times 10^{-3}$ m, and $s = 1$. Curves 1–4 correspond to $n = 0, 1, 2, 10$, respectively. Lines 5 and 6 mark the inner and outer boundaries of the layered superconducting shell, $\rho_0/\rho_b \approx 0.73$ and $\rho/\rho_b = 1$, respectively.

mode index n . The calculations are performed for the radial mode index $s = 1$ and three different values of the inner radius ρ_0 . As seen from Fig. 5, the bulk-surface eigenmode with the azimuthal index $n = 1$ gives the maximum instability growth rate for all considered values of ρ_0 . For low azimuthal indices, when the thickness of the cylindrical superconducting shell, $w_s = \rho_b - \rho_0$, increases from $w_s = 1.2 \times 10^{-3}$ m for

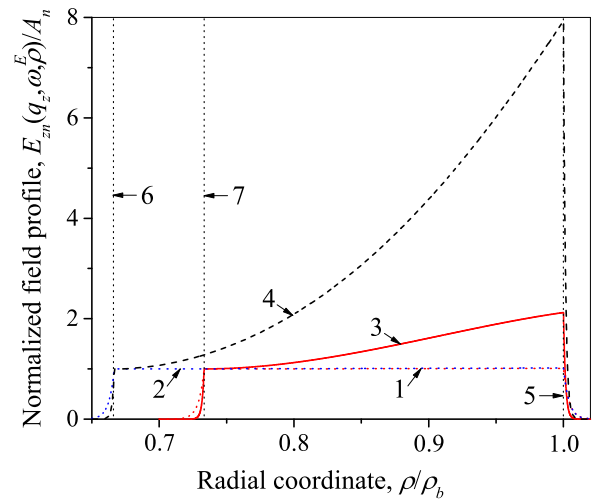


FIG. 7. Radial distributions of the normalized field profile $E_{zn}(q_z, \omega, \rho)/A_n^E$ for bulk-surface JPWs with $s = 1$ at different azimuthal indices and shell thicknesses. The fixed radii are $\rho_b = 4.5 \times 10^{-3}$ m and $\rho_w = 5 \times 10^{-3}$ m. Curves 1 and 2 correspond to $n = 1$ for $\rho_0 = 3.3 \times 10^{-3}$ m and $\rho_0 = 3 \times 10^{-3}$ m, respectively; curves 3 and 4 correspond to $n = 10$ for the same two values of ρ_0 . Lines 5, 6, and 7 denote the outer shell boundary $\rho/\rho_b = 1$ and the inner shell boundaries for $\rho_0 = 3 \times 10^{-3}$ m and $\rho_0 = 3.3 \times 10^{-3}$ m, respectively.

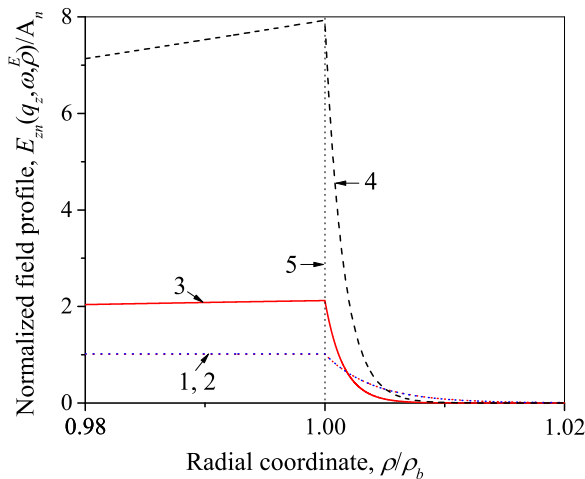


FIG. 8. Enlarged view of the field distributions shown in Fig. 7 near the outer boundary of the superconducting shell, $\rho/\rho_b = 1$. The curve numbers are the same as in Fig. 7.

curve 1 to $w_s = 2.5 \times 10^{-3}$ m for curve 3, the peak growth rate decreases. Thus, in this range of mode indices, a thinner superconducting shell provides more efficient beam-wave coupling. For higher azimuthal indices, $n \geq 9$, the trend is reversed: when the shell thickness w_s increases, the growth rate also increases. This behavior indicates a crossover in the role of the shell thickness in the beam-wave coupling. The resonant frequency also exhibits two distinct regimes. For the azimuthally symmetric mode ($n = 0$), thinner superconducting shells correspond to higher resonant frequencies. In contrast, for azimuthally asymmetric modes ($n \geq 1$), when the shell thickness w_s increases, the resonant frequency also increases.

The qualitatively different behavior of the growth rate as a function of the superconducting-shell thickness for small and large azimuthal mode indices n , shown in Fig. 5, originates from the competition between field penetration into the beam region and the modal energy stored inside the superconducting shell. For low-order bulk-surface modes, the field structure is close to quasiplanar. In this regime, the mode with $n = 1$ provides the most favorable overlap with the tubular electron beam and therefore gives the maximum instability growth rate. A thinner superconducting shell further enhances the coupling of these low-order modes because it reduces the fraction of electromagnetic energy stored inside the superconducting material and increases the effective coupling impedance.

For high azimuthal indices, $n \geq 9$, the situation is different. The field distribution becomes strongly localized near the outer cylindrical surface of the superconducting shell and acquires a whispering-gallery-like character. Such modes require a sufficiently thick superconducting shell to support and confine their transverse field structure. Therefore, in this high-order regime, increasing the shell thickness enhances the coupling to the electron beam. Thus, the reversal in the ordering of curves 1–3 in Fig. 5 reflects a crossover from quasiplanar low-order modes to whispering-gallery-like high-order modes. The supporting field profiles and the corresponding estimates of the effective coupling impedance

are given in Appendix B. The behavior of the resonant frequency in Fig. 5 also requires a separate comment. It is worth noting that the influence of the superconducting-shell thickness $w_s = \rho_b - \rho_0$ on the eigenfrequency spectrum of excited bulk-surface JPWs is not universal and depends on the range of the dimensionless longitudinal wave number Q .

As can be seen from Fig. 2, in the long-wavelength limit the eigenfrequencies of bulk-surface JPW eigenmodes in the absence of the electron beam decrease as the thickness of the superconducting shell increases. However, the resonant frequencies shown in Fig. 5 correspond to the Vavilov-Cherenkov interaction between the electron beam and bulk-surface JPWs. In this case, the resonance condition selects specific values of Q and Ω for each azimuthal mode index.

The opposite tendency observed for the resonant frequencies of azimuthally asymmetric bulk-surface JPWs can be interpreted in terms of the effective surface kinetic inductance of the layered superconducting shell and the shunting effect caused by the high in-plane conductivity. For modes with azimuthal variation, the tangential electric-field components are strongly affected by currents flowing along the superconducting layers. An increase in the shell thickness effectively provides a larger transverse conducting region, which enhances the shunting of these tangential field components and shifts the resonant conditions selected by the beam-wave interaction. As a result, the frequency shift observed at the resonant points in Fig. 5 does not necessarily follow the long-wavelength trend of the dispersion curves shown in Fig. 2.

Thus, the trends observed for the resonant frequencies of excited bulk-surface JPWs in Fig. 5 should not be directly identified with the long-wavelength behavior of the dispersion curves for bulk-surface JPW eigenmodes in the absence of the electron beam. The influence of the superconducting-shell thickness on the resonant frequency and instability growth rate is determined by the combined effect of the intrinsic eigenmode spectrum, field localization, effective surface kinetic inductance, and beam-wave coupling conditions.

2. Surface JPWs: Convective instability

We now consider the excitation of surface JPWs by the tubular electron beam. In contrast to the bulk-surface JPWs discussed above, the dispersion curves for surface JPWs shown in Figs. 3 and 4 have positive dispersion for the propagating surface branch. Therefore, the corresponding group velocity is directed along the electron-beam propagation direction. In this case, the beam-wave interaction can lead to a convective instability rather than an absolute one.

A convective instability is characterized by amplification of a perturbation as it propagates through the interaction region. Thus, the disturbance grows along the direction of energy transport but is carried away from a fixed spatial point. In an open system without external feedback, such an instability corresponds to spatial amplification rather than to self-sustained temporal growth at a fixed position.

As established in Sec. III, the strong anisotropy of the layered superconductor, $\gamma = 200$, leaves only the azimuthally symmetric surface mode ($n = 0$) with a pronounced traveling-wave character and a finite group velocity. The azimuthally

asymmetric surface modes ($n \geq 1$) are shifted close to the Josephson plasma frequency and become localized surface Josephson plasma oscillations. Therefore, under the conditions considered here, only the $n = 0$ surface JPW can be efficiently excited by the electron beam as a propagating surface wave.

For comparison, numerical calculations were also performed for the surface JPW with $n = 0$, using the same material and geometrical parameters as for the bulk-surface modes and the same definition of the effective coupling impedance as in Appendix B. The calculated value of $\Gamma = \eta/\omega_{\text{res}}$ for this surface mode is approximately three times larger than that for the corresponding bulk-surface JPW with the same azimuthal index $n = 0$ and radial index $s = 1$.

This enhancement is caused by the stronger localization of the surface-wave field near the outer boundary of the superconducting shell, $\rho = \rho_b$, where the electron beam propagates. As a result, the surface JPW has a larger effective coupling impedance than the corresponding bulk-surface JPW. For example, at $\rho_0 = 3.3 \times 10^{-3}$ m, the surface JPW with $n = 0$ gives $\Gamma \simeq 1.1 \times 10^{-3}$ and $Z_c(0) \simeq 3 \times 10^5 \Omega/\text{m}^2$, whereas the corresponding bulk-surface JPW gives $\Gamma \simeq 3.7 \times 10^{-4}$ and $Z_c(0) \simeq 1 \times 10^5 \Omega/\text{m}^2$.

V. CONCLUSIONS

In this work, we have presented a theoretical study of the eigenspectrum and beam-wave interaction in a cylindrical waveguide containing a layered superconducting shell. We first formulated a general hybrid electrodynamic boundary-value problem, in which the electromagnetic fields in the dielectric and superconducting regions are described in the full electrodynamic form, while the beam region is treated in the electrostatic approximation. The dispersion curves and instability growth rates were then analyzed in the quasioleostatic slow-wave limit, which is justified by the Vavilov-Cherenkov synchronism with a nonrelativistic, charge-neutralized tubular electron beam. The charge compensation enables rectilinear beam propagation without an external focusing magnetic field and allows the layered superconductor to remain in the Meissner state. The main results of the study can be summarized as follows.

First, we derived and analyzed the dispersion relations for both bulk-surface and surface JPWs. It was shown that bulk-surface JPWs exhibit negative dispersion. This property enables the development of absolute instability during their interaction with the electron beam. Accordingly, the analytical expression derived for the instability growth rate was used to analyze this absolute-instability regime. By contrast, the propagating surface-JPW branch has positive dispersion, so its interaction with the electron beam leads to convective amplification rather than to absolute instability in an open system.

For surface JPWs, the strong anisotropy of the layered superconductor, $\gamma \gg 1$, leads to a qualitative difference between azimuthally symmetric and azimuthally asymmetric modes. The azimuthally symmetric mode ($n = 0$) retains its traveling-wave character and propagates with a finite group velocity. In contrast, azimuthally asymmetric modes ($n \geq 1$) are strongly suppressed by the high in-plane conductivity and transform into localized oscillations with frequencies

close to the Josephson plasma frequency. Therefore, under the strongly anisotropic conditions considered here, only the $n = 0$ surface JPW can be efficiently excited by the electron beam as a propagating surface mode.

Second, the transition from an open electrodynamic system to a screened waveguide reduces the eigenfrequencies of surface JPWs. The resonant frequencies of excited bulk-surface JPWs also show a strong dependence on the superconducting-shell thickness. This dependence is governed by the hyperbolic character of the layered superconductor, for which $\varepsilon_{ab} < 0$ and $\varepsilon_c > 0$. In particular, for azimuthally asymmetric bulk-surface modes, a reduction in the shell thickness can decrease the resonant frequency, which is opposite to the behavior expected for conventional resonators.

Third, the numerical analysis of the absolute instability of bulk-surface JPWs shows that, among the modes considered, the strongest interaction with the electron beam is achieved for the $n = 1$ bulk-surface eigenmode. We also identified a crossover in the dependence of the growth rate on the superconducting-shell thickness. For low azimuthal indices, $n < 9$, thinner shells enhance the instability growth rate by reducing the modal electromagnetic energy stored in the superconducting material and increasing the effective coupling impedance. In contrast, for high-order modes, $n \geq 9$, thicker shells are more favorable because they support whispering-gallery-like field distributions and improve the penetration of the field into the beam region.

These findings provide a consistent physical picture of wave processes in cylindrical single-layer layered superconductors and demonstrate the potential of such structures as frequency-tunable slow-wave systems for THz radiation sources.

The present analysis was restricted to a single-layer layered superconductor, where one superconducting layer per unit cell leads, in the effective-medium approximation used here, to one Josephson plasma frequency in the c -axis response. A natural direction for future research is to generalize the same beam-wave instability analysis to bilayer cuprates, such as YBCO and Bi-2212. In these materials, two superconducting layers per unit cell and inequivalent intrinsic Josephson junctions produce a richer c -axis Josephson-plasma electrodynamics, including two longitudinal Josephson-plasma resonances, a transverse-optical Josephson plasma resonance, and Josephson plasma-polariton branches [33–36]. It would therefore be of interest, in future work, to determine how the richer bilayer JPW spectrum modifies both the absolute instability of bulk-surface waves and the convective amplification of surface waves in cylindrical beam-waveguide structures.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DISPERSION RELATION FOR COUPLED BEAM-JPWs

Using the field representations (19)–(24) and imposing the boundary conditions described above, we obtain the general dispersion relation for coupled beam waves and Josephson plasma eigenmodes, including bulk-surface and surface branches. It is written as the condition that the following determinant vanishes:

$$\begin{vmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} & \mathcal{M}_{14} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} & \mathcal{M}_{34} \\ \mathcal{M}_{41} & \mathcal{M}_{42} & \mathcal{M}_{43} & \mathcal{M}_{44} \end{vmatrix} = 0. \quad (\text{A1})$$

Here, $\mathcal{M}_{ij}$ denote the elements of the reduced 4×4 determinant obtained after eliminating six expansion coefficients from the original system of ten boundary-condition equations. The first and second rows of this determinant correspond to the continuity conditions for the φ component of the electric field and the ρ component of the electric displacement at the boundary $\rho = \rho_0$. The third and fourth rows correspond to the analogous continuity conditions at the boundary $\rho = \rho_b$. The elements $\mathcal{M}_{ij}$ of the determinant are given by

$$\begin{aligned} \mathcal{M}_{11} &= \beta_0 \alpha_3 \frac{nq_z}{\rho_0}; & \mathcal{M}_{12} &= \beta_0 \alpha_1 \frac{nq_z}{\rho_0} \frac{Y_{ns0}^E}{J_{nd0}}; \\ \mathcal{M}_{13} &= \frac{i\omega}{c} \theta_2; & \mathcal{M}_{14} &= \frac{i\omega}{c} \alpha_0 \frac{Y_{ns0}^H}{J_{nd0}} \theta_1; \\ \mathcal{M}_{21} &= iq_z \theta_3; & \mathcal{M}_{22} &= iq_z \alpha_1 \theta_4; \\ \mathcal{M}_{23} &= \beta_1 \frac{\omega n}{c \rho_0} \alpha_2; & \mathcal{M}_{24} &= \beta_1 \frac{\omega n}{c \rho_0} \alpha_0 \frac{Y_{ns0}^H}{J_{nd0}}; \\ \mathcal{M}_{31} &= 0; & \mathcal{M}_{32} &= -\frac{n\omega^2 \varepsilon_{ab} \alpha_1 Y_{nsb}^E}{c^2 \rho_b q_z \kappa_H^2}; \\ \mathcal{M}_{33} &= -\frac{i\omega}{c} \frac{\alpha_6}{\kappa_H} J_{nd0}; & \mathcal{M}_{34} &= -\frac{i\omega}{c} \frac{\alpha_0}{\kappa_H} Y_{nsb}^H; \\ \mathcal{M}_{41} &= iq_z \varepsilon_{ab} \alpha_7 \frac{\kappa_E}{\kappa_H^2} J_{nd0}; & \mathcal{M}_{42} &= iq_z \theta_5; \\ \mathcal{M}_{43} &= 0; & \mathcal{M}_{44} &= -\frac{\omega n}{c \rho_b} \alpha_0 \frac{\varepsilon_{ab}}{\kappa_H^2} Y_{nsb}^H. \\ \theta_1 &= \frac{1}{\kappa_H} \frac{Y_{ns0}^H}{Y_{ns0}^E} - \frac{1}{\kappa_d} \frac{J'_{nd0}}{J_{nd0}}; & \theta_2 &= \frac{\alpha_4}{\kappa_H} - \frac{\alpha_2}{\kappa_d} \frac{J'_{nd0}}{J_{nd0}}; \\ \theta_3 &= \frac{\varepsilon_d \alpha_3}{\kappa_d} \frac{J'_{nd0}}{J_{nd0}} - \frac{\varepsilon_{ab} \kappa_E \alpha_5}{\kappa_H^2}; \\ \theta_4 &= \frac{Y_{ns0}^E}{J_{nd0}} \left(\frac{\varepsilon_d}{\kappa_d} \frac{J'_{nd0}}{J_{nd0}} - \frac{\varepsilon_{ab} \kappa_E}{\kappa_H^2} \frac{Y_{ns0}^E}{Y_{ns0}^H} \right); \\ \theta_5 &= \frac{i}{q_z} \varepsilon_b \alpha_8 J_{nd0} + \frac{\alpha_1 \kappa_E \varepsilon_{ab}}{\kappa_H^2} Y_{nsb}^E. \\ \alpha_0 &= \frac{I'_{nbw}}{Y_{nsb}^E} \left(\frac{I_{nbb}}{I'_{nbw}} - \frac{K_{nbb}}{K'_{nbw}} \right); \\ \alpha_1 &= \frac{I_{nbw}}{Y_{nsb}^E} \left(\frac{I_{nbb}}{I_{nbw}} - \frac{K_{nbb}}{K_{nbw}} \right); \end{aligned}$$

$$\begin{aligned} \alpha_2 &= \frac{J_{nsb}^H}{J_{nd0}} \left(\frac{J_{ns0}^H}{J_{nsb}^H} - \frac{Y_{ns0}^H}{Y_{nsb}^H} \right); \\ \alpha_3 &= \frac{J_{nsb}^E}{J_{nd0}} \left(\frac{J_{ns0}^E}{J_{nsb}^E} - \frac{Y_{ns0}^E}{Y_{nsb}^E} \right); \\ \alpha_4 &= \frac{J_{nsb}^H}{J_{nd0}} \left(\frac{J_{ns0}^H}{J_{nsb}^H} - \frac{Y_{ns0}^H}{Y_{nsb}^H} \right); \\ \alpha_5 &= \frac{J_{nsb}^E}{J_{nd0}} \left(\frac{J_{ns0}^E}{J_{nsb}^E} - \frac{Y_{ns0}^E}{Y_{nsb}^E} \right); \\ \alpha_6 &= \frac{J_{nsb}^H}{J_{nd0}} \left(\frac{J_{ns0}^H}{J_{nsb}^H} - \frac{Y_{ns0}^H}{Y_{nsb}^H} \right); \\ \alpha_7 &= \frac{J_{nsb}^E}{J_{nd0}} \left(\frac{J_{ns0}^E}{J_{nsb}^E} - \frac{Y_{ns0}^E}{Y_{nsb}^E} \right); \\ \alpha_8 &= \frac{I_{nbw}}{J_{nd0}} \left(\frac{I'_{nbb}}{I_{nbw}} - \frac{K'_{nbb}}{K_{nbw}} \right). \\ \beta_0 &= \frac{1}{\kappa_H^2} - \frac{1}{\kappa_d^2}; \\ \beta_1 &= \frac{\varepsilon_{ab}}{\kappa_H^2} - \frac{\varepsilon_d}{\kappa_d^2}, \end{aligned}$$

where

$$\begin{aligned} J_{nd0} &= J_n(\kappa_d \rho_0); \\ J_{ns0}^E &= J_n(\kappa_E \rho_0); & J_{nsb}^E &= J_n(\kappa_E \rho_b); \\ J_{ns0}^H &= J_n(\kappa_H \rho_0); & J_{nsb}^H &= J_n(\kappa_H \rho_b); \\ Y_{ns0}^E &= Y_n(\kappa_E \rho_0); & Y_{nsb}^E &= Y_n(\kappa_E \rho_b); \\ Y_{ns0}^H &= Y_n(\kappa_H \rho_0); & Y_{nsb}^H &= Y_n(\kappa_H \rho_b); \\ I_{nbb} &= I_n(|q_z| \rho_b); & I_{nbw} &= I_n(|q_z| \rho_w); \\ K_{nbb} &= K_n(|q_z| \rho_b); & K_{nbw} &= K_n(|q_z| \rho_w). \end{aligned}$$

The prime notation follows the convention introduced after Eqs. (25)–(28). The determinant in Eq. (A1) is written explicitly for the bulk-surface branch. The surface-wave branch is obtained from the same boundary-condition determinant by replacing the oscillatory radial functions in the superconducting shell by the corresponding evanescent solutions, i.e., by using I_n , K_n , χ_E , and χ_H instead of J_n , Y_n , κ_E , and κ_H , as in Eqs. (21) and (22).

APPENDIX B: FIELD DISTRIBUTIONS AND EFFECTIVE COUPLING IMPEDANCE

This Appendix provides the field-distribution and coupling-impedance analysis that supports the physical interpretation of the qualitatively different behavior of the growth rate as a function of the superconducting-shell thickness for small and large azimuthal mode indices n , as discussed in Sec. IV.

Figure 6 shows the radial distributions of the normalized longitudinal electric-field profile $E_{zn}(q_z, \omega, \rho)/A_n^E$ for bulk-surface JPWs with different azimuthal mode indices. The distributions are calculated at the resonant frequencies and wave numbers satisfying the Vavilov-Cherenkov condition,

using the same material and geometrical parameters as in Fig. 5 for $\rho_0 = 3.3 \times 10^{-3}$ m and $s = 1$.

As seen from Fig. 6, the field of the bulk-surface mode with $n = 1$ penetrates most deeply into the beam region, $\rho/\rho_b > 1$. When the azimuthal mode index n increases, this penetration depth decreases. For the mode with $n = 10$, the penetration depth into the beam region becomes comparable to that for the azimuthally symmetric mode $n = 0$. This explains why the mode with $n = 1$ interacts most efficiently with the electron beam and gives the largest instability growth rate in Fig. 5. When the field penetration into the beam region becomes smaller for higher n , the beam-wave coupling becomes weaker. Therefore, when the azimuthal mode index n increases, the growth rate decreases, as observed in Fig. 5. At the same time, Fig. 6 shows that when the azimuthal mode index n increases, the field penetration into the dielectric core, $\rho < \rho_0$, also decreases. Therefore, the effective resonant volume of the electromagnetic mode becomes smaller, which is accompanied by an increase in the resonant frequency for higher-order azimuthal modes.

The qualitative change in the behavior of curves 1–3 in Fig. 5 with variation of the thickness of the cylindrical superconducting shell is explained by the field distributions shown in Figs. 7 and 8. Curves 1 and 2 in these figures correspond to the radial distributions of the longitudinal electric-field component for the mode with $s = 1$ and $n = 1$ at $\rho_0 = 3.3 \times 10^{-3}$ m and $\rho_0 = 3 \times 10^{-3}$ m, respectively. In the beam region, $\rho/\rho_b > 1$, these two field profiles are nearly identical; in particular, their normalized field values coincide at the outer surface of the superconducting shell, $\rho = \rho_b$. However, inside the superconducting shell, the corresponding fields occupy substantially different volumes. At the same time, as seen from Fig. 5, the growth rate for the mode corresponding to curve 1 in Figs. 7 and 8 is noticeably larger than that for the mode corresponding to curve 2.

This difference is caused by the different values of the effective coupling impedance per unit area,

$$Z_c(n) = \frac{\langle |E_{zn}(q_z, \omega, \rho)|^2 \rangle}{2\mathcal{W}_{\text{total}}|v_{gr}|}, \quad (\text{B1})$$

where

$$\langle |E_{zn}(q_z, \omega, \rho)|^2 \rangle = \frac{\int_{\rho_b}^{\rho_w} |E_{zn}(\rho)|^2 \rho d\rho}{\int_{\rho_b}^{\rho_w} \rho d\rho} \quad (\text{B2})$$

is the squared longitudinal electric field averaged over the beam cross section, and

$$\mathcal{W}_{\text{total}} = \mathcal{W}_{\text{diel}} + \mathcal{W}_{\text{SC}} + \mathcal{W}_{\text{beam}} \quad (\text{B3})$$

is the total stored energy of the mode per unit length in the dielectric ($\mathcal{W}_{\text{diel}}$), superconductor ($\mathcal{W}_{\text{SC}}$), and beam ($\mathcal{W}_{\text{beam}}$) regions, respectively. These contributions are

$$\mathcal{W}_{\text{diel}} = \frac{1}{16\pi} \int_0^{\rho_0} \varepsilon_d |E(q_z, \omega, \rho)|^2 2\pi \rho d\rho, \quad (\text{B4})$$

$$\mathcal{W}_{\text{SC}} = \frac{1}{16\pi} \int_{\rho_0}^{\rho_b} \left[\frac{\partial(\omega \varepsilon_{ab})}{\partial \omega} (|E_\rho(q_z, \omega, \rho)|^2 + |E_\varphi(q_z, \omega, \rho)|^2) + \frac{\partial(\omega \varepsilon_c)}{\partial \omega} |E_{zn}(q_z, \omega, \rho)|^2 \right] 2\pi \rho d\rho, \quad (\text{B5})$$

$$\mathcal{W}_{\text{beam}} = \frac{1}{16\pi} \int_{\rho_b}^{\rho_w} |E(q_z, \omega, \rho)|^2 2\pi \rho d\rho. \quad (\text{B6})$$

With this normalization, Z_c has the dimension Ω/m^2 in SI units.

The group velocity of the bulk-surface JPWs under consideration is negative, $v_g < 0$. For the mode corresponding to field distribution 1 in Figs. 7 and 8, the effective coupling impedance is $Z_c \simeq 1 \times 10^5 \Omega/\text{m}^2$, whereas for the mode corresponding to field distribution 2, it is $Z_c \simeq 8 \times 10^4 \Omega/\text{m}^2$. Therefore, as seen from Fig. 5, the growth rate for the mode with $n = 1$ and $\rho_0 = 3.3 \times 10^{-3}$ m, which corresponds to curve 1, is larger than that for the mode with $\rho_0 = 3.0 \times 10^{-3}$ m, which corresponds to curve 2. For the same mode with $\rho_0 = 2.0 \times 10^{-3}$ m, corresponding to curve 3 in Fig. 5, the coupling impedance decreases to $Z_c \simeq 2 \times 10^4 \Omega/\text{m}^2$.

Thus, for low azimuthal indices, $n < 9$, the reduction of the superconducting-shell thickness $w_s = \rho_b - \rho_0$ enhances the beam-wave coupling. The physical reason is that these eigenmodes have a quasiparallel field structure. In this regime, a thinner superconducting shell reduces the part of the modal electromagnetic energy stored inside the superconducting material. This energy does not directly enhance the beam-wave interaction but contributes to the total modal energy $\mathcal{W}_{\text{total}}$. Since the effective coupling impedance is inversely proportional to the total modal energy, $Z_c(n) \propto \langle |E_z|^2 \rangle / \mathcal{W}_{\text{total}}$, the reduction of $\mathcal{W}_{\text{total}}$ leads to an increase in $Z_c(n)$ and, consequently, enhances the instability growth rate.

For higher azimuthal indices, $n \geq 9$, the situation is reversed: the effective coupling impedance becomes larger for thicker superconducting shells. This is explained, on the one hand, by the strong field localization near the outer boundary of the superconducting shell, $\rho/\rho_b = 1$, and, on the other hand, by the larger penetration depth of the fields into the beam region for thicker shells. In this high-order regime, the field distribution acquires a whispering-gallery-like character. Such modes require a sufficiently thick superconducting shell to support and confine their transverse field structure. In very thin shells, the whispering-gallery-like field profile cannot be formed efficiently. Therefore, increasing w_s enhances the coupling of these high-order modes to the electron beam.

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