

Nonlinear spin-Seebeck diode in f -wave magnets, third-order spin-Nernst effects in g -wave magnets, and spin-Nernst effects in i -wave altermagnets

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We analyze thermally induced spin transport in X -wave magnets with $X = p, d, f, g$, and i . A prominent feature of d -wave altermagnets is that spin current is generated by applying a temperature gradient, which is known as the spin-Nernst effect. We show in f -wave magnets that spin current is generated proportional to the square of the temperature gradient, which we call the nonlinear spin-Seebeck current. It can be used as a spin current diode. In addition, we show in g -wave altermagnets that spin current is generated in the third order of the temperature gradient. We also show in i -wave altermagnets that spin current is generated perpendicular to the temperature gradient, which is the spin-Nernst current. We have derived analytic formulas for these spin currents. On the other hand, we show in p -wave magnets that spin current is not generated by temperature gradient.

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Introduction. Spin current conveys spins without a net charge current. Spin current generation is a key concept in spintronics. It is generated by the spin-Hall effect perpendicular to the applied electric field in the presence of spin-orbit interactions [1–3]. In addition, spin current is generated by applying temperature gradient ∇T , which is known as the spin-Seebeck effect [4,5]. In addition, spin current flows perpendicular to the temperature gradient, which is known as the spin-Nernst effect [6,7]. These are linear responses. On the other hand, there are several studies on nonlinear responses by applying an electric field [8–25]. There are also a few studies on nonlinear responses to temperature gradient [26–34]. Nonlinear spin current generation [35–39] is interesting for spintronics applications. Especially, the second-order spin current generation can be used as a spin current diode [39]. It is an intriguing problem to search for nonlinear spin current generations induced by temperature gradient, such as the nonlinear spin-Seebeck effect and the nonlinear spin-Nernst effect.

Altermagnets attract much attention in the context of spintronics [40,41], which include d -wave, g -wave, and i -wave altermagnets from the viewpoint of the crystal symmetry. In addition, there are p -wave [42–44] and f -wave magnets [43,44]. They are antiferromagnets and have spin-split band structures characterized by the number N_X of nodes; see Table I. They are summarized in the notion of the X -wave magnets [39,45–47] with $X = p, d, f, g$, and i . A prominent feature of d -wave altermagnets is that spin current is generated perpendicular to the electric field without using spin-orbit interactions [48–52]. It is also known that spin current is generated perpendicular to the temperature gradient in d -wave

altermagnets, which is the spin-Nernst effect [45,48]. It is known [39] that spin current is generated proportional to the square of the applied electric field in f -wave magnets, which can be used for a perfect spin current diode. It is also known that the third-order and the fifth-order nonlinear spin currents are generated in g -wave and i -wave altermagnets, respectively.

In this paper, we investigate nonlinear spin current generation by temperature gradient in two dimensions. We derive the formula for the current generation as nonlinear responses to temperature gradient based on the Boltzmann equation, valid up to any order in temperature gradient. Then, we show in f -wave magnets that spin current is generated parallel to the square of the temperature gradient $(\nabla T)^2$, implying a nonreciprocity. It can be used as a spin current diode, which we call the nonlinear spin-Seebeck diode. It is interesting that this phenomenon occurs in the absence of spin-orbit interactions. In addition, linear spin current is generated perpendicular to the temperature gradient in i -wave altermagnets. It is contrasted to the spin current induced by an electric field, where there is no linear response. Furthermore, we show in g -wave altermagnets that there is a third-order response to the temperature gradient. Finally, we show that no linear nor nonlinear spin current is generated in p -wave magnets. They are summarized in Table I.

Nonlinear spin-Seebeck and spin-Nernst effects. We derive a current induced by temperature gradient ∇T based on the Boltzmann equation. By assuming the spatial dependence of temperature $T(\mathbf{r})$, the Fermi distribution function is given by $f(\mathbf{r}, \mathbf{k}) = 1/(\exp((\varepsilon(\mathbf{k}) - \mu)/(k_B T(\mathbf{r}))) + 1)$. The Boltzmann equation is given by

$$\frac{df(\mathbf{r}, \mathbf{k})}{dt} = -\frac{f(\mathbf{r}, \mathbf{k}) - f^{(0)}}{\tau}, \quad (1)$$

where τ is the relaxation time. For simplicity, we assume that the relaxation time is a constant. See Sec. I S1 of

TABLE I. Fermi-surface symmetry and current induced by temperature gradient in two dimensions. It is found that the second-order spin-Seebeck effect occurs only in f -wave magnets.

	s	p	d	f	g	i
Nodes N_X	0	1	2	3	4	6
Mirror M_y	Yes	Yes	No	Yes	No	No
First and third spin-Nernst	No	No	$j_{\text{spin}}^{(x;y)}$	No	$j_{\text{spin}}^{(x^3;y)}$	$j_{\text{spin}}^{(x;y)}$
Mirror M_x	Yes	No	No	No	No	No
Second spin-Seebeck	No	No	No	$j_{\text{spin}}^{(x^2;x)}$	No	No

Supplemental Material [53] for the case where the relaxation time has an energy dependence.

We assume that the temperature gradient is along the a direction, and we measure the induced current along the b direction. We expand the Fermi distribution in powers of $\partial_a T$, $f = f^{(0)} + f^{(1)} + \dots$. The current is correspondingly expanded as $j = j^{(a^0;b)} + j^{(a^1;b)} + \dots$, where

$$j^{(a^\ell;b)} = -e \int d^D k f_a^{(\ell)} v_b, \quad (2)$$

with the velocity $v_b = \partial \varepsilon / \partial (\hbar k_b)$ and dimension D . It is straightforward to solve the Boltzmann equation recursively to derive the formula

$$j^{(x^\ell;b)} = -e \int d^D k f^{(\ell)} v_b = e \left(-\frac{\tau}{\hbar} \partial_x T \right)^\ell \int d^D k \left(\frac{\partial \varepsilon}{\partial k_x} \right)^\ell \frac{\partial \varepsilon}{\partial k_b} \frac{\partial^\ell f^{(0)}}{\partial T^\ell}, \quad (3)$$

where we have assumed that the temperature gradient is along the x axis ($a = x$). See Supplemental Material Sec. I with respect to this formula [53].

At high temperature T , the derivative of the Fermi-distribution function is approximated as

$$\frac{\partial^\ell f^{(\ell)}}{\partial T^\ell} = (-1)^{\ell+1} \frac{\ell!}{4} \beta^{\ell+1} (\varepsilon - \mu), \quad (4)$$

where $\beta = 1/k_B T$.

We consider a Hamiltonian which is diagonal with respect to the spin, $H = H_{\text{kin}} \sigma_0 + H_J \sigma_z$, whose energy is given by $\varepsilon = \varepsilon_{\text{kin}} + s \varepsilon_J$ with $s = \pm 1$, representing the spin up and down. We define the ℓ -th order nonlinear charge current by

$$j_{\text{charge}}^{(x^\ell;b)} = j_{\uparrow}^{(x^\ell;b)} + j_{\downarrow}^{(x^\ell;b)}, \quad (5)$$

while the ℓ -th order nonlinear spin current by

$$j_{\text{spin}}^{(x^\ell;b)} = \frac{1}{2} (j_{\uparrow}^{(x^\ell;b)} - j_{\downarrow}^{(x^\ell;b)}), \quad (6)$$

where $b = x, y$. The diagonal charge (spin) current $j_{\text{charge}}^{(x^\ell;x)}$ ($j_{\text{spin}}^{(x^\ell;x)}$) induced by the ℓ -th power of the temperature gradient $(\partial_x T)^\ell$ is called the ℓ -th Seebeck (spin-Seebeck) current. On the other hand, off diagonal charge (spin) current $j_{\text{charge}}^{(x^\ell;y)}$ ($j_{\text{spin}}^{(x^\ell;y)}$) induced by the ℓ -th power of the temperature gradient $(\partial_x T)^\ell$ is called the ℓ -th Nernst (spin-Nernst) current. There is such a case as a ferromagnet where both charge and spin

currents flow, induced by the temperature gradient $\partial_x T$, which is called the spin-polarized Seebeck current.

Symmetry analysis. We first study the nonlinear spin-Seebeck current. If there is a mirror symmetry $x \rightarrow -x$ in the energy, $\varepsilon(-k_x) = \varepsilon(k_x)$, we have

$$\frac{\partial \varepsilon(-k_x)}{\partial k_x} = -\frac{\partial \varepsilon(k_x)}{\partial k_x}. \quad (7)$$

Then, it follows from (3) that the current is zero, $j_s^{(x^\ell;x)} = 0$ for even ℓ from Eq. (3).

Next, we study the nonlinear spin-Nernst current. If there is a mirror symmetry $y \rightarrow -y$ in the energy, $\varepsilon(-k_y) = \varepsilon(k_y)$, we have

$$\frac{\partial \varepsilon(-k_y)}{\partial k_x} = -\frac{\partial \varepsilon(k_y)}{\partial k_x}. \quad (8)$$

Then, it follows from (3) that the current is zero, $j_s^{(x^\ell;y)} = 0$. Hence, there are no spin-Nernst effects in f -wave magnets in two dimensions.

Even if there is no mirror symmetry $y \rightarrow -y$ in the energy, if there is a mirror symmetry $x \rightarrow -x$ in the energy, it follows from (3) that $j_s^{(x^\ell;x)} = 0$ for odd ℓ .

X-wave magnet. We study a metallic X -wave magnet with $X = p, d, f, g$ and i . The simplest model is given by the two-band Hamiltonian [39,45–47],

$$H_X(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + J f_X(\mathbf{k}) \sigma_z, \quad (9)$$

where the first term represents the kinetic term of electrons, while the second term represents the spin-dependent band splitting described by the function $f_X(\mathbf{k})$ with the coupling constant J and the Pauli matrix σ_z . The energy is given by

$$\varepsilon_s(\mathbf{k}) = \frac{\hbar^2 k^2}{2m} + s J f_X(\mathbf{k}), \quad (10)$$

with $s = \pm 1$, where $s = 1$ for up spin and $s = -1$ for down spin. We insert Eq. (10) into Eq. (3) to calculate the nonlinear spin current.

d-wave altermagnet. We revisit the spin-Nernst effect in d -wave altermagnets [45,48] in the present formalism. The d -wave symmetry is given by the function $f_X(\mathbf{k})$ with

$$f_d^{2D}(\mathbf{k}) = k_x k_y. \quad (11)$$

The Fermi surface is shown in Fig. 1(a). The shapes of the Fermi surfaces become identical between the up and down spins when we rotate the Fermi surface by 90 degrees and flip the spin. The band structure has fourfold rotational symmetry with spin inversion and four nodes. The Seebeck current is calculated as

$$\frac{j_{\text{charge}}^{(x;x)}}{\frac{e\tau}{\hbar} \partial_x T} = -\frac{\pi^2 \beta^2 (2 + J^2 m^2) (2 - m\mu)}{4m^3}. \quad (12)$$

The spin-Nernst current is calculated as

$$\frac{j_{\text{spin}}^{(x;y)}}{\frac{e\tau}{\hbar} \partial_x T} = -\frac{\beta^2 J \pi^2}{4m^2}. \quad (13)$$

It is shown in Fig. 1(b) as well as a numerically obtained result without using the high-temperature expansion. They agree for small $\beta \lesssim 1$. On the other hand, they behave in a different

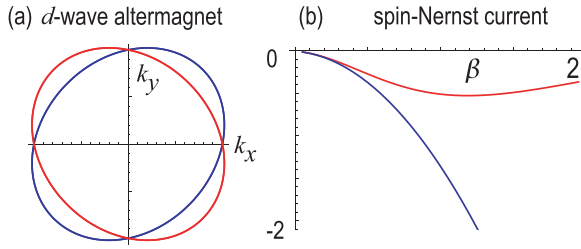


FIG. 1. d -wave altermagnet. (a) Fermi surfaces with two nodes. The red oval indicates the Fermi surface with up spin, and the blue oval indicates down spin. (b) Spin current $j_{\text{spin}}^{(x,y)}/\partial_x T$ in units of $e\tau/\hbar$ as a function of the inverse temperature $\beta = 1/k_B T$. The red curve represents a numerical result, while the blue curve represents an analytic result based on the high-temperature expansion. We have set $m = 1$, $J = 1/2$, $\mu = 1$, and $a = 1$.

way, where the result based on the high-temperature expansion monotonically becomes larger, while the result without using the high-temperature expansion saturates. It is understood as follows. The next order expansion of the derivative of the Fermi distribution reads

$$\frac{\partial^\ell f^{(\ell)}}{\partial T^\ell} = (-1)^{\ell+1} \left[\frac{\ell!}{4} \beta^{\ell+1} (\varepsilon - \mu) - \frac{(\ell+2)!}{96} \beta^{\ell+3} (\varepsilon - \mu)^3 \right], \quad (14)$$

where the sign of $\beta^{\ell+3}$ is opposite to that of $\beta^{\ell+1}$. Hence, the spin current is suppressed around $\beta \sim 1$.

f -wave magnet. The f -wave symmetry is given by

$$f_f^{2D}(\mathbf{k}) = k_x(k_x^2 - 3k_y^2). \quad (15)$$

The corresponding tight-binding Hamiltonian is given by [39,46,47]

$$H_f = 4J\sigma_z \sin ak_x \sin \frac{ak_x + \sqrt{3}ak_y}{2} \sin \frac{-ak_x + \sqrt{3}ak_y}{2}. \quad (16)$$

The kinetic term of the tight-binding model on the triangular lattice is given by

$$H_{\text{Kine, Tri}} = \frac{\hbar^2}{ma^2} \left(3 - \cos ak_x - \sum_{\pm} \cos a \frac{k_x \pm \sqrt{3}k_y}{2} \right). \quad (17)$$

The Fermi surface is shown in Fig. 2(a). The band structure has sixfold rotational symmetry with spin inversion and six

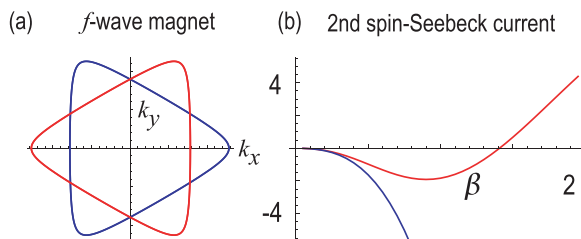


FIG. 2. f -wave magnet. (a) Fermi surfaces with three nodes. (b) Spin current $j_{\text{spin}}^{(x,y)}/(\partial_x T)^2$ in units of $e(\tau/\hbar)^2$ as a function of the inverse temperature $\beta = 1/k_B T$. See also the caption of Fig. 1.

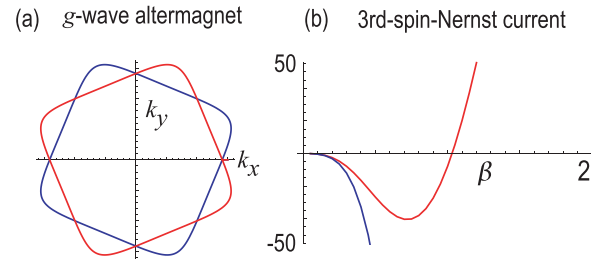


FIG. 3. g -wave altermagnet. (a) Fermi surfaces with four nodes. (b) Spin current $j_{\text{spin}}^{(x,y)}/\partial_x T$ in units of $e(\frac{\tau}{\hbar} \partial_x T)^3$ as a function of the inverse temperature $\beta = 1/k_B T$. See also the caption of Fig. 1.

nodes. The mirror symmetry $k_x \mapsto -k_x$ is broken, which enables a nonzero second-order nonlinear spin-Seebeck current.

The Seebeck current is calculated as

$$\frac{j_{\text{charge}}^{(x,x)}}{e\tau \partial_x T} = -\frac{2\beta^2 \pi (2 - 15\pi + m\mu(-2 + 9\pi))}{9\sqrt{3}m^3} + \frac{2\beta^2 \pi (J^2 m^2 (4 + 9\pi(-2 + m\mu)))}{9\sqrt{3}m^3}. \quad (18)$$

The second-order nonlinear spin-Seebeck current is induced as

$$\frac{j_{\text{spin}}^{(x^2,x)}}{e(\frac{\tau}{\hbar} \partial_x T)^2} = \frac{\beta^3 \pi (14 - 4m\mu + 9\pi + 9m\pi(-\mu + 6J^2 m(m\mu - 2)))}{3\sqrt{3}m^3} sJ. \quad (19)$$

It is shown in Fig. 2(b), together with a numerically obtained result without using the high-temperature expansion.

g -wave altermagnet. The g -wave symmetry is given by

$$f_g^{2D}(\mathbf{k}) = 4k_x k_y (k_x^2 - k_y^2). \quad (20)$$

The corresponding tight-binding Hamiltonian is given by [39,46,47]

$$H_g = 2J\sigma_z \sin ak_x \sin ak_y (\cos ak_y - \cos ak_z). \quad (21)$$

The kinetic term of the tight-binding Hamiltonian on the square lattice is given by

$$H_{\text{Kine, Sq}} = \frac{\hbar^2}{ma^2} (2 - \cos ak_x - \cos ak_y). \quad (22)$$

The Fermi surface is shown in Fig. 3(a). The band structure has eightfold rotational symmetry with spin inversion and eight nodes. The mirror symmetry $k_y \mapsto -k_y$ is broken, which enables a nonzero, nonlinear spin-Nernst current.

The Seebeck current is calculated as

$$\frac{j_{\text{charge}}^{(x,x)}}{e\tau \partial_x T} = -\frac{\beta^2 \pi^2 (40J^2 m^2 + 1)(m\mu - 2)}{m^3}. \quad (23)$$

The third-order nonlinear spin-Nernst effect is calculated as

$$\frac{j_{\text{spin}}^{(x^3,y)}}{e(\frac{\tau}{\hbar} \partial_x T)^3} = \frac{18\beta^4 \pi^2 (40J^2 m^2 - 1)(m\mu - 2)}{m^4} sJ. \quad (24)$$

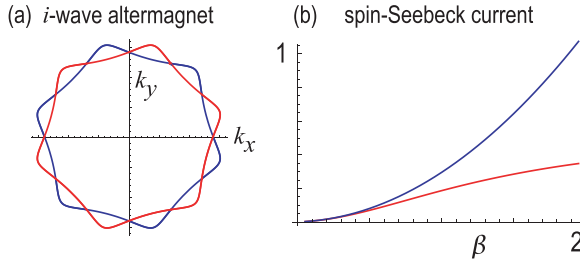


FIG. 4. *i*-wave altermagnet. (a) Fermi surfaces with six nodes. (b) Spin current $j_{\text{spin}}^{(x,y)} / (\partial_x T)^3$ in units of $e(\tau/\hbar)^3$ as a function of the inverse temperature $\beta = 1/k_B T$. See also the caption of Fig. 1.

It is shown in Fig. 3(b), together with a numerically obtained result without using the high-temperature expansion.

i-wave altermagnet. The *i*-wave symmetry is given by

$$f_i^{2D}(\mathbf{k}) = 2k_x k_y (3k_x^2 - k_y^2)(k_x^2 - 3k_y^2). \quad (25)$$

The corresponding tight-binding Hamiltonian is given by [39,46,47]

$$\begin{aligned} H_i = & -\frac{16}{3\sqrt{3}} J \sigma_z \\ & \times \sin ak_x \sin \frac{ak_x + \sqrt{3}ak_y}{2} \sin \frac{-ak_x + \sqrt{3}ak_y}{2} \\ & \times \sin \sqrt{3}ak_y \sin \frac{3ak_x + \sqrt{3}ak_y}{2} \sin \frac{-3ak_x + \sqrt{3}ak_y}{2}. \end{aligned} \quad (26)$$

The Fermi surface is shown in Fig. 4(a). The band structure has twelvefold rotational symmetry with spin inversion and twelve nodes. The mirror symmetry $k_y \mapsto -k_y$ is broken, which enables a nonzero, nonlinear spin-Nernst current.

The Seebeck current is calculated as

$$\begin{aligned} \frac{j_{\text{charge}}^{(x;x)}}{\frac{e\tau}{\hbar} \partial_x T} = & -\frac{\beta^2 \pi}{243\sqrt{3}m^3} [(27(2 - 15\pi + m\mu(9\pi - 2))) \\ & + (2J^2 m^2 (160 + 567\pi(m\mu - 2)))]. \end{aligned} \quad (27)$$

The spin-Nernst current is calculated as

$$\frac{j_s^{(x;y)}}{\frac{e\tau}{\hbar} \partial_x T} = \frac{-4\beta^2 \pi (3m\mu - 5)}{27\sqrt{3}m^2} sJ. \quad (28)$$

It is shown in Fig. 4(b), together with a numerically obtained result without using the high-temperature expansion.

Discussions. The second-order spin-Seebeck effect $j^{(x^2;x)} \propto (\partial_x T)^2$ occurs only in *f*-wave magnets, as in Table I, which is particularly interesting. There is a net spin current even if the temperatures are equal at the right- and left-hand sides, provided that there is a fluctuation of temperature, because spin current flows regardless of the direction of the temperature gradient. It acts as a spin current diode.

We have studied *X*-wave magnets in two dimensions. However, actual materials have three-dimensional structures.

There are two types of three-dimensional systems. One is a layered system made of two-dimensional *X*-wave magnets, where we may use the two-dimensional $f_X^{2D}(\mathbf{k})$. See Sec. IV, S7 to Sec. S10, of Supplemental Material [53]. The other is an intrinsic three-dimensional *X*-wave magnet [39–41,45–47] with the three-dimensional $f_X^{3D}(\mathbf{k})$. Explicit calculations show that there is no spin current generation in three dimensions except for *d*-wave altermagnets. Details are shown in Sec. IV, S11 to S14, of Supplemental Material [53].

We have derived analytic formulas for spin currents in the leading order of temperature gradient in a system where the spin is diagonal, and hence the single band is relevant. These formulas are not applicable in the presence of the spin-orbit interaction, such as the Rashba interaction. However, the order of J in these magnets is 100 meV, while the order of the Rashba interaction is 1 meV. Then, the J term is dominant compared to the Rashba interaction in actual materials. We have derived a general formula for nonlinear spin-Seebeck and spin-Nernst currents when the spin is not a good quantum number in Supplemental Material [53]: See Eq. (S81). It shows that the results shown in Table I with respect to the emergence of nonlinear spin currents are the same with and without the Rashba interaction, provided it is weak.

We have assumed that the relaxation time is constant. However, it has energy dependence [54–59] due to acoustic phonons, optical phonons, ions, and electron-electron interactions. These effects are studied in Sec. I S2 of Supplemental Material [53]. The acoustic phonon contribution dominates the scattering processes at high temperature because the number of phonons increases with the increase in temperature. Moreover, its effect is accurately captured within the constant relaxation time approximation as shown in Fig. S1. Consequently, treating the relaxation time as an energy-independent constant is justified.

We mention materialization of *f*-wave magnets, and *g*-wave and *i*-wave altermagnets in two dimensions. An *f*-wave magnet is theoretically proposed in $\text{Ba}_3\text{MnNb}_2\text{O}_9$ [60], FePO_4 [61], and in graphene by introducing spin nematic order [62]. *g*-wave and *i*-wave altermagnets are theoretically proposed in twisted magnetic van der Waals bilayers [63]. An *i*-wave altermagnet is also theoretically proposed in $\text{MnP}(\text{S,Se})_3$ [64].

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Data availability. The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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