

Ordering transition of the three-dimensional four-state random-field Potts model

Manoj Kumar^{1,2} and Martin Weigel^{3,*}

¹*School of Physical Sciences, National Institute of Science Education and Research Bhubaneswar, Jatni 752050, India*

²*Homi Bhabha National Institute, Training School Complex, Anushakti Nagar, Mumbai 400094, India*

³*Institut für Physik, Technische Universität Chemnitz, 09107 Chemnitz, Germany*



(Received 8 April 2025; accepted 2 June 2025; published 24 June 2025)

Spin systems exposed to the influence of random magnetic fields are paradigmatic examples for studying the effect of quenched disorder on condensed-matter systems. In this context, previous studies have almost exclusively focused on systems with Ising or continuous symmetries, while the Potts symmetry, albeit being of fundamental importance also for the description of realistic physical systems, has received very little attention. In the present study, we use a recently developed quasiaexact method for determining ground states in the random-field Potts model to study the problem with four states. Extending the protocol applied for the three-state model, we use extensive finite-size scaling analyses of the magnetization, Binder parameter, energy cumulant, specific heat, and the connected as well as disconnected susceptibilities to study the magnetic ordering transition of the model. In contrast to the system in the absence of disorder, we find compelling evidence for a continuous transition, and we precisely determine the critical point as well as the critical exponents, which are found to differ from the exponents of the three-state system as well as from those of the random-field Ising model.

DOI: [10.1103/PhysRevB.111.214434](https://doi.org/10.1103/PhysRevB.111.214434)

I. INTRODUCTION

Random disorder is nearly unavoidable in laboratory samples of condensed-matter systems. Understanding the effect of random doping, lattice defects, and similar impurities is hence a problem of fundamental interest for a modern theory of condensed-matter systems [1]. The most prominent simplified models for such problems are systems with bond and site dilution, random-bond ferromagnets, random-field models and spin glasses [2]. The random-field problem, in particular, has received much attention in the past decades, and it exhibits a rich phenomenology with a violation of hyperscaling [3] and the curiosity of dimensional reduction [4,5] that is present in dimensions $d \geq 6$ (or maybe $d \geq d_c \approx 5$, see Refs. [6,7]), but broken in lower dimensions. Most effort in this direction was spent on the random-field Ising model (RFIM), for which there is now an overall good understanding of the phase diagram for continuous field distributions, see, e.g., Ref. [7]. For systems with continuous symmetries such as the $O(n)$ model, on the other hand, random fields destroy long-range order in dimensions $d < 4$ [8,9]. An interesting and relevant intermediate case are hence discrete spin systems with more than two states, such as the Potts model [10].

The q -state Potts model occupies a special place in statistical physics due to its numerous physical realizations [10,11], including soap froths, cellular tissues, grain growth,

nucleation as well as static and dynamic recrystallization, but also for being a generalization of several special cases such as the Ising model and random percolation. Regarding quenched disorder, the effects of bond and site dilution as well as more general bond randomness have been relatively well studied [12–16]. On the contrary, randomness coupling to the order parameter, i.e., the random-field Potts model (RFPM) that is particularly applicable to describing magnetic grains, anisotropic orientational glasses, randomly diluted molecular crystals [17,18], structural transitions in SrTiO_3 crystals [19], and phase transitions in type I antiferromagnets (such as NdSb , NdAs , CeAs) in a uniform field [20], has hardly been considered. Until recently, there were only a few studies of this system in the literature [21–26], where the analytical approaches have generally used mean-field techniques, while the computational studies are based on Monte Carlo methods.

To illustrate the effect of the quenched random fields, consider the behavior for the pure system. In two dimensions ($d = 2$), it exhibits a continuous phase transition for $q \leq 4$ and a first-order one of $q > 4$ [10]. Hence, $q_c = 4$ forms a tricritical point there. For the case of three dimensions as considered in the present paper, the transition is only continuous for $q = 2$, while the three-state model already undergoes a weak first-order transition [27]. The tricritical point has been argued to take the noninteger value $q_c \approx 2.35$ there [28]. Subjecting this system to quenched disorder in the form of random fields, there arises the possibility of a softening of the transitions in the first-order regime to continuous ones. As is rigorously known [9], in two dimensions this should already occur for the smallest amount of disorder. On the contrary, in three and higher dimensions a finite amount of disorder is in general required [29]. Also, as the random fields yield the critical behavior of an effectively reduced

*Contact author: martin.weigel@physik.tu-chemnitz.de

dimensionality [4,5], one would expect the tricritical point $q_c(d)$ above which the transition becomes discontinuous to move to larger values when random fields are included. Goldschmidt and Xu conducted a $1/q$ expansion and predict first-order transitions for $q \geq 3$ [30], whereas Blankschtein *et al.* argue that both $q = 3$ and $q = 4$ might still remain continuous [21]. Numerically, on the other hand, early results by Reed [26] were interpreted as indicative of a first-order transition for $q = 3$, while later Eichhorn and Binder [23–25] found some evidence for a continuous transition for this case (however with a different distribution of random fields as compared to our setup). Since these numerical studies were not yet able to make use of advanced simulation techniques to accelerate relaxation, the rough free-energy landscape observed for the RFPM restricted such numerical studies to very small system sizes with the resulting strong scaling corrections. Overall the picture of the phase diagram of the RFPM at this stage remained rather speculative.

Inspired by the substantial progress achieved for the RFIM through the use of combinatorial optimization methods to find exact ground states [31–34], we recently developed an algorithmic approach suitable for finding quasiexact ground states of samples of the RFPM for useful system sizes [35]. For the random-field problem, the renormalization-group fixed point is located at $T = 0$, such that the relevant critical behavior can be studied via the ground states. In contrast to the RFIM, however, the ground-state problem for the RFPM with $q > 2$ is nondeterministic polynomial (NP) hard [31,36]. Hence, an efficient exact algorithm is very unlikely to exist. To compensate for this, we explored in Ref. [37] how repeated runs of the method for n different initial conditions can be used to systematically extrapolate the approximate results in the quasiexact limit for $n \rightarrow \infty$. In [38], we conducted an in-depth study of the three-state RFPM in three dimensions using the quasiexact estimates to determine its critical behavior, finding a very clearly continuous transition with critical exponents that are very similar to, but likely different from those of the three-dimensional RFIM. In the present work, we extend this line of study to the case of the $q = 4$ RFPM. We run the ground-state method for $n = 100$ initial conditions and extrapolate all of the disorder-averaged physical quantities to the limit of an infinite number of initial conditions, $n \rightarrow \infty$, where the approach becomes exact. The corresponding critical exponents are calculated using finite-size scaling techniques.

The remainder of this paper is organized as follows. Section II describes the model and the numerical details of our simulations. In Sec. III, we present detailed numerical results of our study and investigate the critical behavior of the four-state RFPM from the approximate ground states. Finally, in Sec. IV, we conclude this paper with a summary and discussion.

II. MODEL AND METHODS

A. Random field Potts model

Depending on how the random fields couple to the spins, the Hamiltonian for the q -state RFPM can take a variety of forms [21,23–25,30]. We consider a model where one employs a symmetric coupling of continuous fields to each of

the possible orientations of the Potts spins [21], as follows:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j} - \sum_i \sum_{\alpha=0}^{q-1} h_i^\alpha \delta_{s_i, \alpha}. \quad (1)$$

Here, $\delta_{x,y}$ is the Kronecker delta function, and each spin s_i takes one of q orientations, *viz.* $s_i = \{0, 1, \dots, q-1\}$. The sum across $\langle i, j \rangle$ is over nearest neighbors i and j on the chosen lattice, which is taken to be simple cubic for the present work. Periodic boundary conditions are applied. The variables $\{h_i^\alpha\}$ denote quenched, q -component random fields at site i , each drawn from a symmetric normal distribution, *i.e.*,

$$P(h_i^\alpha) = \frac{1}{\sqrt{2\pi}\Delta} \exp\left[-\frac{(h_i^\alpha)^2}{2\Delta^2}\right]. \quad (2)$$

The width Δ of the distribution is a measure of the disorder strength in this system. For the special case $q = 2$, it can be easily seen that the Hamiltonian Eq. (1) corresponds to the RFIM at coupling $J/2$ and field strength $\Delta/\sqrt{2}$ (plus an irrelevant constant shift) [35].

An alternative model Hamiltonian for the RFPM with discrete distribution of the disorder was used in Refs. [23,30],

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j} - \Delta \sum_i \delta_{s_i, h_i}, \quad (3)$$

where the quenched random variables h_i are chosen uniformly from the set $\{0, 1, \dots, q-1\}$. In this model variant, Δ no longer corresponds to the width of a Gaussian distribution, but it still represents the effective disorder strength. At zero temperature, one has a unique ground state for the continuous system (1) [39], while the alternative (3) might admit (extensive) degeneracies. For the RFIM, some differences in the behavior are observed for discrete and continuous field distributions [40] and the same might be expected here. In order to avoid the possible subtleties associated with the existence of degeneracies, we will focus on the form (1) of the interactions for the purpose of the present work.

B. Quasi-exact ground-state calculations

Our study is focused on the four-state ($q = 4$) RFPM in $d = 3$, using simple cubic lattices of L^3 sites. In order to approximate the ground states of the considered random-field configurations, we resort to a recently developed quasiexact ground-state scheme for the RFPM [35,37]. The basic ingredient is a technique originally developed in computer vision for the purpose of segmenting an image with q colors [36]. Based on the observation that multilabel problems of this type are NP hard and hence cannot be solved for large instances while two-color labeling can be performed efficiently using graph-cut (or, equivalently, maximum-flow) techniques [31], the full q -color labeling is approached by randomly picking one color, and then proposing exchanges from all other colors to the selected one and vice versa, thus effectively embedding a solvable two-label problem into the hard q -color one. This approach, known as α -expansion [36], usually yields low-lying metastable states. Another approach is the α - β swap, which involves randomly selecting any two distinct colors, say α and β , and then proposing a swap between them while

TABLE I. Minimum number N_{samp} of disorder samples used per lattice size. In the critical regime, approximately 50% more samples were employed. The third row shows the values $H(L)$ of the nonzero uniform field for each system size L that is used for explicitly breaking the symmetry for the calculation of the susceptibility as discussed in Sec. III F.

L	16	20	24	32	40	48	64	80	96
$N_{\text{samp}}/10^3$	50	40	35	30	20	10	8	5	3
$H(L) \times 10^2$	8	5.72	4.36	2.83	2.02	1.54	1	0.72	0.54

freezing all other colors, thus again effectively reducing the q -label problem to a two-label one. Both approaches iterate until a local or global minimum is reached. However, a full cycle of the α - β swap takes about q^2 steps, whereas an α -expansion cycle requires only q steps. Therefore, we adopt the α -expansion technique and combine it with a systematic use of repeated runs from random initial states in order to enable an extrapolation of the calculation to the exact ground state [37]. We provide the code for computing and analyzing RFPM ground states via α -expansion in a public repository [41]. For more details regarding this algorithm we refer the reader to Refs. [35–38].

C. Analysis

We performed simulations for various systems of linear size L and many disorder realizations, the exact specifications are summarized in Table I. For each disorder configuration, we conduct simulations for $n = 100$ different random initial conditions and pick the run resulting in the lowest energy as the ground-state estimate. For each disorder sample, we determine two basic observables: the order parameter m and the bond-energy per spin e_J . The former is defined as [42]

$$m(L, \Delta, n) = \frac{q\rho - 1}{q - 1}, \quad (4)$$

where

$$\rho = \frac{1}{L^3} \max_{\alpha} \left(\sum_i \delta_{s_i, \alpha} \right) \quad (5)$$

denotes the fraction of spins attaining the most common orientation, while the latter is given by

$$e_J(L, \Delta, n) = -\frac{1}{L^3} \sum_{\langle ij \rangle} \delta_{s_i, s_j}. \quad (6)$$

In a second step, by taking a disorder average $[\dots]_{\text{av}}$, we arrive at the order parameter $[m]_{\text{av}}$, the average bond-energy per spin $[e_J]_{\text{av}}$, and several other physical quantities associated with these, namely the disconnected susceptibility χ_{dis} , the Binder-cumulant U_4 , the specific heat C , and the energy-cumulant V_4 , which, in turn, are defined by the following expressions:

$$\chi_{\text{dis}}(L, \Delta, n) = L^3 [m^2]_{\text{av}}, \quad (7)$$

$$U_4(L, \Delta, n) = 1 - \frac{[m^4]_{\text{av}}}{3[m^2]_{\text{av}}^2}, \quad (8)$$

$$C(L, \Delta, n) = \frac{\partial [e_J]_{\text{av}}}{\partial \Delta}, \quad (9)$$

$$V_4(L, \Delta, n) = 1 - \frac{[e_J^4]_{\text{av}}}{3[e_J^2]_{\text{av}}^2}. \quad (10)$$

These quantities predict the critical behavior of the system near the transition, see, e.g., Refs. [42,43]. We note that the expression (9) corresponds to a zero-temperature proxy for the specific heat, see the discussion in Sec. III E.

The error bars for each observable are calculated via the jackknife method applied over the set of disorder samples [44,45]. For determining the critical exponents, in some cases we use scaling collapses performed using the tool `autoscale.py`, developed by O. Melchert [46]. This program uses a minimization procedure to optimize the scaling parameters via a downhill simplex algorithm [47]. We also determine the goodness-of-fit parameter Q to quantify the quality of fit. This is defined as the incomplete gamma function of χ^2 and the number f of degrees-of-freedom [47]:

$$Q = \Gamma\left(\frac{\chi^2}{2}, \frac{f}{2}\right). \quad (11)$$

Q determines the probability that a value of

$$\chi^2 = \sum_i^N \left(\frac{y_i - g(x_i)}{\sigma_i} \right)^2, \quad (12)$$

with N data points $(x_i, y_i \pm \sigma_i)$ from a fit of the function g to the data identical to or worse than the observed value should occur *by chance* if one assumes that the model is correct [47]. According to common practice, if $Q \gtrsim 0.1$ the goodness-of-fit is believable. If $Q \gtrsim 0.001$, the fit may be acceptable if the errors are non-normal or have been moderately underestimated. If $Q < 0.001$, then the fit is not acceptable, see, e.g., Ref. [47].

III. SIMULATION RESULTS

A. Extrapolation of the physical quantities

As outlined above, in order to improve on the results of the α -expansion minimization of a sample, we repeat such calculations for n distinct initial conditions of the spins and pick the run resulting in the lowest energy. To extrapolate the results, for each disorder realization we run the simulations for different values of n up to $n_{\text{max}} = 100$ and inspect the functional form of the dependence of numerical averages of observables on n .

Figure 1 shows a typical plot of different disorder-averaged quantities (magnetization $[m]_{\text{av}}$, disconnected susceptibility χ_{dis} , Binder-cumulant U_4 , and the bond-energy per spin $[e_J]_{\text{av}}$) as a function of n at a fixed disorder strength $\Delta = 1.63$ and for a lattice size of $L = 64$. Typically, we observe a two-stage pattern for the convergence of these estimated zero-temperature averages. Initially, there is a fast relaxation followed by a much slower convergence as n increases. This behavior is effectively described by the sum of two power laws [37],

$$O(L, \Delta, n) = an^{-b}(1 + cn^{-e}) + O^*(L, \Delta), \quad (13)$$

where $b < e$ is the *asymptotic* slow exponent, e represents the initial fast decay of finite- n corrections, and O^* denotes the

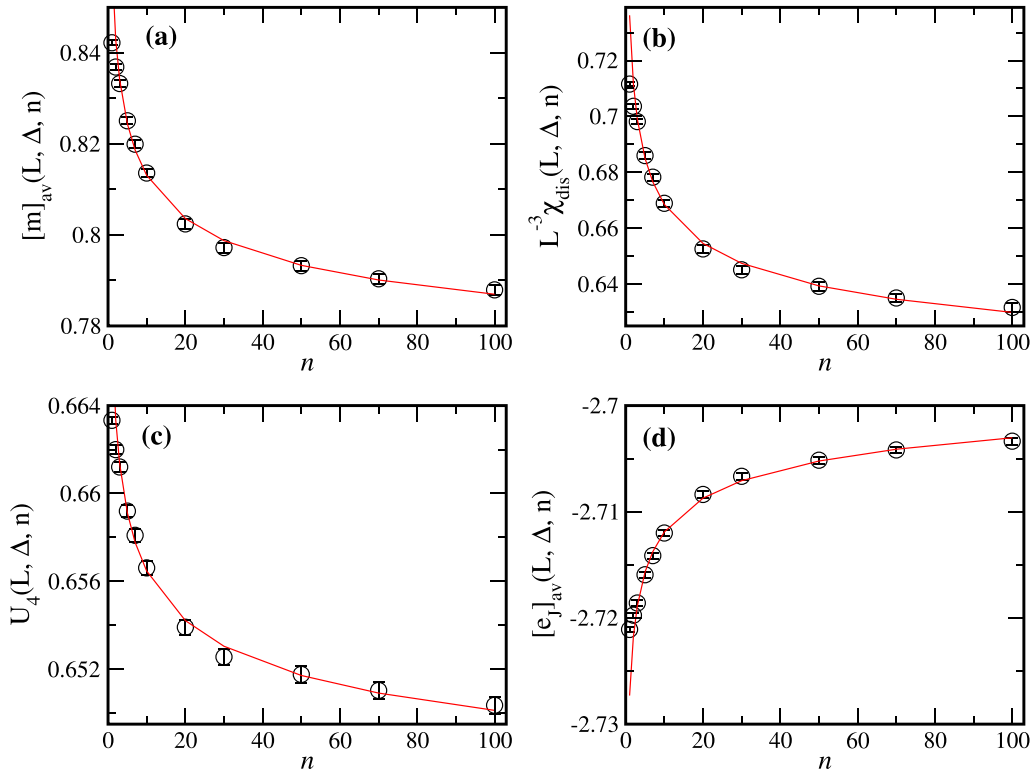


FIG. 1. Disorder-averaged estimates of the magnetization $[m]_{\text{av}}$, the disconnected susceptibility χ_{dis} , the Binder-cumulant U_4 , and the bond-energy per spin $[e_J]_{\text{av}}$ as a function of the number of initial conditions n at a fixed value of $\Delta = 1.63$ and for lattice size $L = 64$. The red lines show simultaneous fits to Eqs. (14)–(17) with $b = 0.02$ and $e = 0.242 \pm 0.023$.

limiting value for $n \rightarrow \infty$. As we have previously shown in Ref. [37], this form is quite generic and applies well to a specific subset of samples with known exact ground states, which are very useful for benchmarking the approach. For these cases we extended our study to much larger $n_{\text{max}} = 10\,000$ and found that the residuals with respect to the exact results, i.e., $O(n) - O_{\text{ex}}$, for all considered quantities scale as $an^{-b}(1 + cn^{-e})$, with $b \simeq 0.02$ and $e \simeq 0.5$. This behavior extends even to the cases where the exact results are not available [37,38]. The value of b is found to be very stable, such that we fix it to be 0.02 for the subsequent fits of the present study, for which $n \leq 100$. In order to arrive at stable results, we perform joint fits for the different quantities with a common value of e , i.e.,

$$[m]_{\text{av}}(n) = a_0 n^{-b}(1 + c_0 n^{-e}) + m^*, \quad (14)$$

$$\chi_{\text{dis}}(n) = a_1 n^{-b}(1 + c_1 n^{-e}) + \chi_{\text{dis}}^*, \quad (15)$$

$$U_4(n) = a_2 n^{-b}(1 + c_2 n^{-e}) + U^*, \quad (16)$$

$$-[e_J]_{\text{av}}(n) = a_3 n^{-b}(1 + c_3 n^{-e}) - e^*, \quad (17)$$

where a_i and c_i , $i = 0, \dots, 3$ are amplitude parameters and m^* , χ_{dis}^* , U^* , and e^* are the asymptotic values of $[m]_{\text{av}}$, χ_{dis} , U_4 , and $[e_J]_{\text{av}}$, respectively, in the limit $n \rightarrow \infty$.

In Fig. 2 we show the behavior of the residuals for the different quantities considered. According to the functional form (13), if one plots $n^b[O(n) - O^*]$ as a function of n^{-e} , the data should fall on a straight line of intercept a and slope ac . Plotting the data in this way is hence useful to identify any

deviations from the scaling form (13), especially in the asymptotic limit $n \rightarrow \infty$. This is what we show in Fig. 2 for different (L, Δ) as specified, where the solid lines are fits according to the forms (14)–(17) which yield fit qualities $Q = 0.43, 0.81$, and 0.74 for panels (a), (b), and (c), respectively. Overall, this analysis lends credibility to the claim that the results presented below are indeed representative of the true zero-temperature limit of the RFPM.

In the rest of this paper, we use the extrapolated estimates together with the estimates for finite n to investigate the critical behavior via a comprehensive finite-size scaling (FSS) analysis.

B. Energy cumulant and phase transition order

In view of the first-order nature of the transition of the $q = 4$ Potts model in the absence of disorder and the lack of clarity from previous studies about the shift in the tricritical point q_c , [21,23–25,30] our first task is to determine the order of the transition. One useful tool for this purpose is the fourth order energy cumulant $V_4(L, \Delta)$ as defined in Eq. (10). It's general behavior was discussed in detail in Ref. [42]. Away from a phase transition, the probability distribution of the energy always tends to a (single) Gaussian in the thermodynamic limit, centered at the expected energy $\langle E \rangle$. Also, the relative width of this Gaussian shrinks to zero, so the distribution turns into a δ function. In this case, it is easily seen [48] that $V_4(L) \rightarrow 2/3$. The same is the case for $T = T_c$ at a continuous transition. In this case, the shape of the distribution is no longer Gaussian, but its relative width still shrinks to zero as $L \rightarrow \infty$. On the

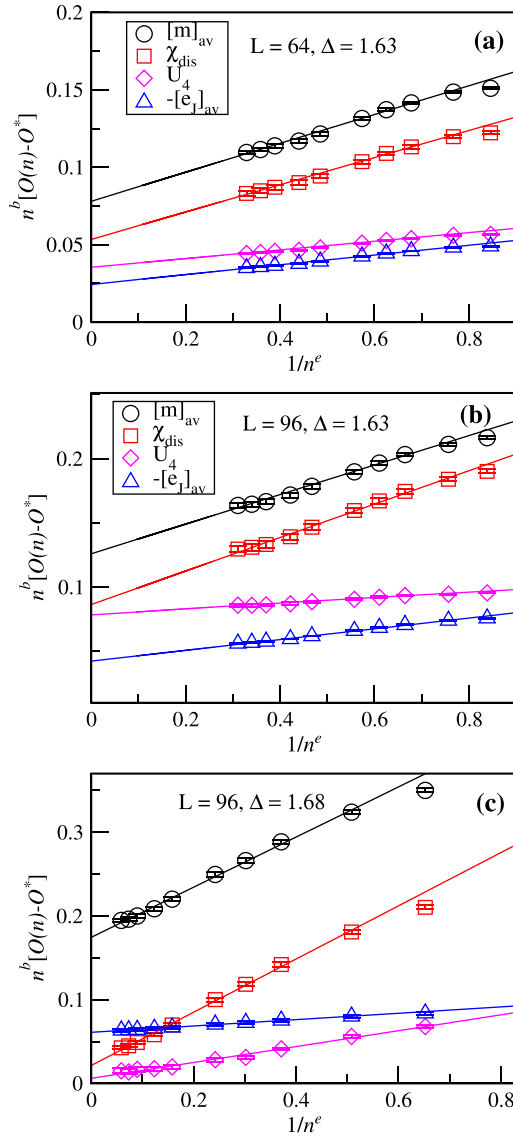


FIG. 2. Residuals $n^b[O(n) - O^*]$ as a function of $1/n^e$ according to the scaling form (13) for various quantities O : magnetization $[m]_{av}$, disconnected susceptibility χ_{dis} , Binder-cumulant U_4 , and bond-energy per spin $[e_l]_{av}$. Panel (a) is for $L = 64$ and $\Delta = 1.63$, corresponding to the data in Fig. 1. Panel (b) is for $L = 96$ and $\Delta = 1.63$, and panel (c) is for $L = 96$ and $\Delta = 1.68$. The solid lines are joint fits according to Eqs. (14)–(17), which produce exponent estimates $e = 0.242(23)$ with fit quality $Q = 0.43$ for the top panel (a), $e = 0.253(28)$ with $Q = 0.81$ for panel (b), and $e = 0.60(25)$ with $Q = 0.74$ for the bottom panel (c).

contrary, for a first-order transition the limit of $V_4(L)$ is related to the distance of the ordered and disordered peaks that stays finite in the thermodynamic limit. Hence, $V_4(L)$ approaches a nontrivial value in this case [42,49].

Figure 3(a) shows the behavior of $V_4(L, \Delta)$ as a function of Δ at $n = 100$ and for different L . It is clearly visible that $V_4(L, \Delta)$ displays a minimum at a certain Δ , say Δ_{min} , and the depth of the minimum strongly decreases with increasing system size L . To analyze this size dependence, we performed parabolic fits near the minimum: $V_4(\Delta) = a_0(\Delta - \Delta_{min})^2 +$

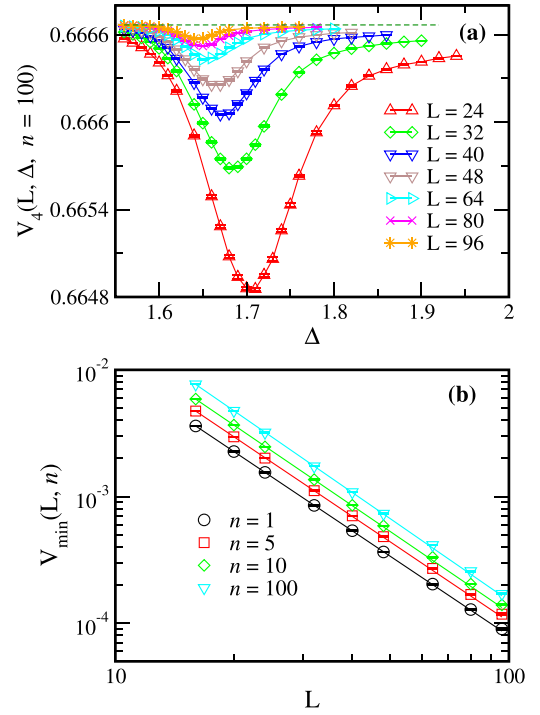


FIG. 3. (a) The energetic cumulant $V_4(L, \Delta)$ for $n = 100$ as a function of Δ for various system sizes L . The dashed line indicates the trivial limit $2/3$ of $V_4(L, \Delta)$. (b) $V_{min}(L) = 2/3 - V_4(L, \Delta_{min})$ as a function of L on a log-log scale. For a better view, the data for different n are shifted relative to each other. The solid lines are power-law fits $V_{min}(L) \sim L^a$ with a value of a in the range 2.058 to 2.085.

$V_4(\Delta_{min})$ and obtained $V_4(\Delta_{min})$ as the depth of the minimum of V_4 . In Fig. 3(b), we plot (on a doubly-logarithmic scale) the depth of the minima after subtraction from $2/3$ as a function of L and for different n , i.e.,

$$V_{min}(L, n) = 2/3 - V_4(L, \Delta_{min}, n). \quad (18)$$

Apparently, this dependence is well described by a power law,

$$V_{min}(L) \sim L^{-a},$$

as indicated by the fits drawn as solid lines. The decay exponent is found to be $a = 2.058(8)$ for $n = 1$ and $a = 2.085(9)$ for $n = 100$, clearly different from the expected values $a = d = 3$ for a first-order transition [42]. From this nontrivial value of a and the convergence of V_4 to $2/3$ we hence conclude that the $q = 4$ RFPM undergoes a continuous transition at $T = 0$.

C. The order parameter

We now turn to an analysis of the critical behavior of the system at its continuous transition. We start by considering the order parameter. Figure 4(a) shows the disorder-averaged magnetization $[m]_{av}$ as a function of Δ for system size $L = 96$ and a range of different numbers n of initial conditions as well as the extrapolated estimate for $n \rightarrow \infty$. As is clearly visible, $[m]_{av}$ approaches the limit $n \rightarrow \infty$ rather smoothly, and we hence do not expect strong corrections from the extrapolation procedure. In Fig. 4(b), we show the lattice size

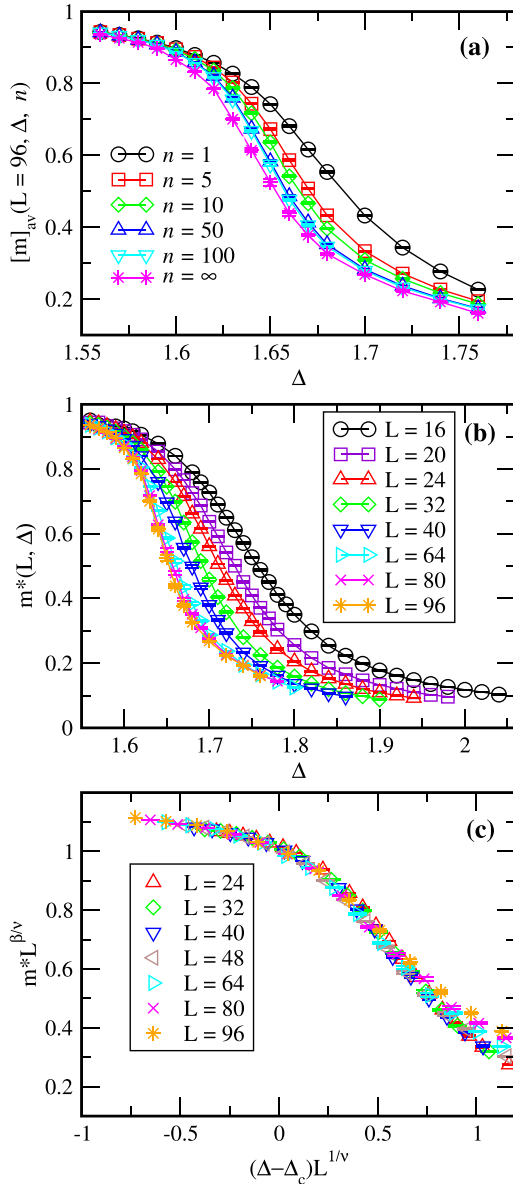


FIG. 4. (a) Average magnetization $[m]_{av}$ as a function of Δ for $L = 96$ and different numbers n of initial conditions employed. (b) Extrapolated estimate m^* as a function of Δ for various system sizes L . (c) Scaling plot of $m^*(L, \Delta)L^{\beta/\nu}$ as a function of $(\Delta - \Delta_c)L^{1/\nu}$ with $\Delta_c = 1.607(2)$, $1/\nu = 0.645(8)$, and $\beta/\nu = 0.0393(12)$.

dependence of the extrapolated magnetizations m^* as a function of disorder Δ . The expected finite-size scaling (FSS) form of m^* is [50]

$$m^*(\Delta, L) = L^{-\beta/\nu} \tilde{\mathcal{M}}[(\Delta - \Delta_c)L^{1/\nu}]. \quad (19)$$

Hence, when plotting $m^*(L, \Delta)L^{\beta/\nu}$ against $(\Delta - \Delta_c)L^{1/\nu}$ with the correct values of the parameters Δ_c , ν and β/ν , the data for different L should collapse onto a single master curve near the critical region $\Delta \approx \Delta_c$, corresponding to the universal scaling function $\tilde{\mathcal{M}}$. A correspondingly rescaled representation of the data is shown in Fig. 4(c). Here, the parameters are determined using `autoscale.py` [46] with

TABLE II. Estimates of Δ_c , ν , and β/ν according to Eq. (19) as well as $\tilde{\gamma}/\nu$ according to Eq. (28) extracted from scaling collapses of the data for different n as well as the extrapolated data for $n \rightarrow \infty$ ($L_{\min} = 24$). S_1 and S_2 are the qualities of the collapses according to (19) and (28), respectively ($S \approx 1$ for perfect collapses).

n	Δ_c	$1/\nu$	β/ν	$\tilde{\gamma}/\nu$	S_1	S_2
1	1.637(4)	0.734(6)	0.0556(7)	2.8902(14)	2.82	2.88
5	1.625(3)	0.747(5)	0.0452(8)	2.9092(13)	1.62	2.69
10	1.621(5)	0.740(6)	0.0438(8)	2.9123(15)	1.58	1.20
50	1.617(7)	0.726(4)	0.0422(9)	2.9153(17)	1.45	1.08
100	1.615(4)	0.702(5)	0.0413(8)	2.9182(17)	1.40	0.96
∞	1.607(2)	0.645(8)	0.0393(12)	2.9215(25)	1.22	0.77

the initial choice of parameters $\Delta_c = 1.62$, $1/\nu = 0.7$, $\beta/\nu = 0.05$ and excluding the data for $L < L_{\min} = 24$ to effectively take scaling corrections into account. The best collapse is obtained with the scaling parameters $\Delta_c = 1.607(2)$, $1/\nu = 0.645(8)$, and $\beta/\nu = 0.0393(12)$ for a fitting range $-1 \leq (\Delta - \Delta_c)L^{1/\nu} \leq 1$. We also performed a collapse analysis for the magnetization for finite $n = 1, 5, 10, 50$, and 100 . In Table II, we summarize the exponents from the scaling of the magnetization for finite as well as infinite n along with the qualities S_1 of the scaling collapses [46]. We also attempted to extrapolate the exponent estimates for $1/\nu$ themselves for $n \rightarrow \infty$ and found these data to be consistent with the behavior of the extrapolated magnetizations, see Appendix A.

D. Binder cumulant

Next we conduct an analysis of the Binder parameter according to Eq. (8). As is well known, the intersections of the cumulant curves $U_4(\Delta, L)$ predict the location of the critical point Δ_c at which they become independent of lattice size L [51,52]. In Fig. 5(a) we show the Binder cumulants $U_4(\Delta, L)$ against Δ for $n = 100$ and different linear lattice sizes L . The cumulant curves intersect in the range $\Delta \in (1.60, 1.62)$ for $L \geq 32$, hence suggesting a corresponding location of the critical point Δ_c for $n = 100$ in this area.

In order to conduct the extrapolation for $n \rightarrow \infty$, we considered the extrapolated results of $U_4(L, \Delta)$ in this limit, cf. Fig. 5(b). In this plot, however, we do not see a consistent crossing of the cumulant curves, although a maximum crossing can be seen in the range of $\Delta \in (1.6, 1.61)$. To check whether this lack of consistency arises from instabilities in the extrapolations across different values of L and Δ , we attempted to simultaneously extrapolate the cumulants for all L and Δ within the range of 1.58 to 1.64 by using the extrapolated form (16) with a common exponent e . The result of this analysis is presented in Fig. 5(c), showing a clearer crossing of the cumulant curves in the range $\Delta \in (1.59, 1.61)$. In the inset, an enlarged view of the crossing of $U^*(\Delta, L)$ is presented for selected values of L .

The expected finite-size scaling form for $U^*(\Delta, L) = U_4(\Delta, L, n \rightarrow \infty)$ is given by [50]

$$U^*(\Delta, L) = \tilde{\mathcal{U}}[(\Delta - \Delta_c)L^{1/\nu}], \quad (20)$$

where $\tilde{\mathcal{U}}$ is a universal scaling function. We again performed a scaling analysis using `autoscale.py` to check for consistency with this form. The resulting collapses are

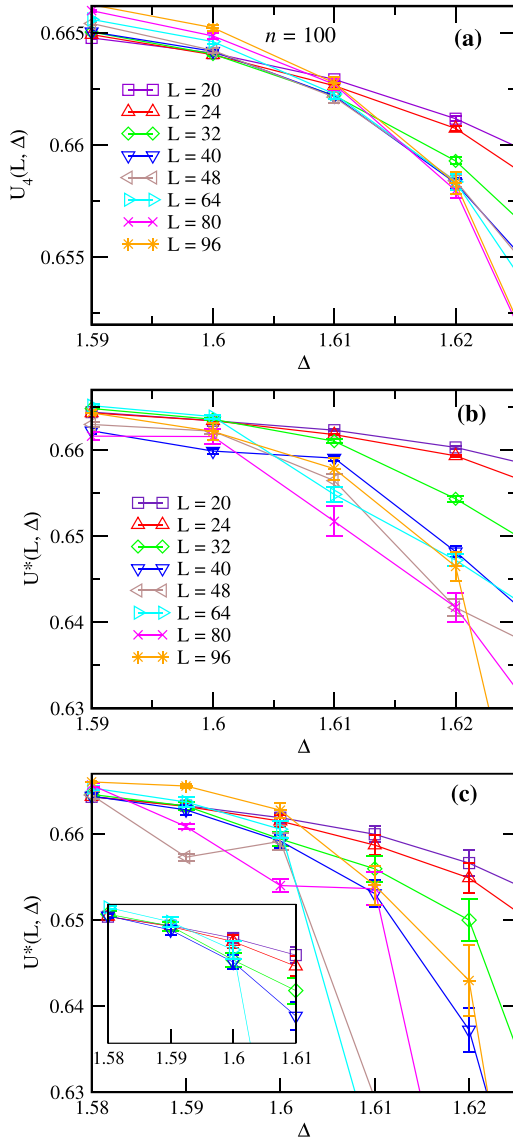


FIG. 5. (a) Binder cumulant $U_4(L, \Delta)$ for $n = 100$ as a function of Δ for various system sizes L as specified. (b) Extrapolated estimates U^* of the Binder cumulant as a function of Δ for various system sizes L . (c) Extrapolated U^* as determined from joint fits of $U_4(L, \Delta, n)$ for all L and Δ in the range 1.58 to 1.64. The joint fits are performed according to the extrapolated form (16) with the shared exponent e and $n \geq 2$. The inset is an enlarged view of the crossing region shown for selected L .

displayed in Fig. 6, where panel (a) shows the best scaling collapse of the cumulants $U_4(\Delta, L)$ for $n = 100$, which is achieved for $\Delta_c = 1.6124(3)$ and $1/\nu = 0.7026(16)$ with a quality of collapse parameter $S = 0.74$. Panel (b) shows the collapse of the extrapolated data $U^*(\Delta, L)$ from the joint fits, yielding estimates $\Delta_c = 1.593(3)$ and $1/\nu = 0.655(44)$ with a quality of $S = 2.34$. Comparing to the data for the order parameter, we find consistent estimates for Δ_c and $1/\nu$ for $n = 100$. For the extrapolated data $1/\nu$ is also consistent, but Δ_c is slightly shifted—an effect that we attribute to the observed difficulties with extrapolations for this observable.

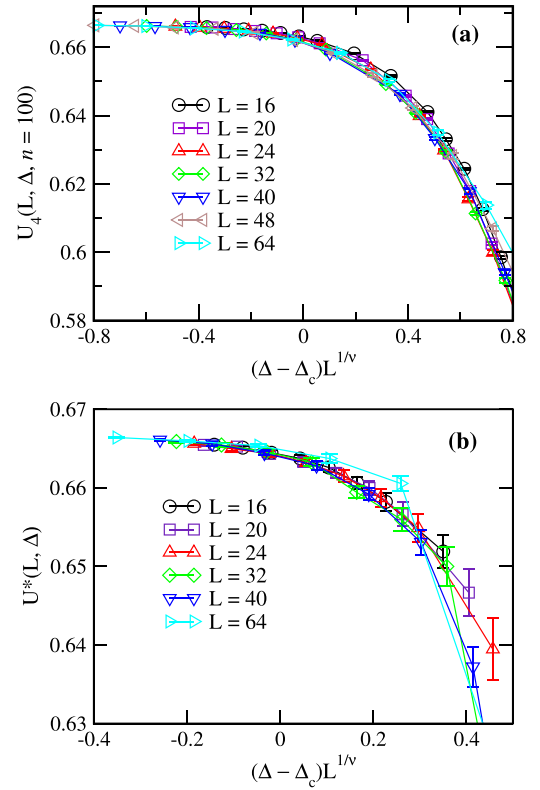


FIG. 6. (a) Scaling collapse of the Binder cumulant $U_4(L, \Delta)$ for $n = 100$ initial conditions as a function of $(\Delta - \Delta_c)L^{1/\nu}$ for system sizes in the range $L = 16 - 64$, which yields $\Delta_c = 1.6124(3)$ and $1/\nu = 0.7026(16)$ with the quality of collapse parameter $S = 0.74$. (b) Collapse of $U^*(L, \Delta)$ versus $(\Delta - \Delta_c)L^{1/\nu}$ with $\Delta_c = 1.593(3)$, $1/\nu = 0.655(44)$, and $S = 2.34$.

E. Specific heat

A crucial quantity in the energetic sector is the specific heat. In numerical calculations this is usually computed from the standard fluctuation-dissipation relation or from a temperature derivative of the internal energy. Since we operate at zero temperature, however, these approaches are not viable here. Instead, a specific-heat-like quantity can be obtained by differentiating the disorder-averaged internal energy e_J with respect to Δ as indicated in Eq. (9) (see Ref. [53] for details).

Numerically, we determine it using a three-point formula by taking the derivative at an intermediate point. Since our Δ values are not equally spaced (a finer grid is used in the vicinity of the critical point), the usual symmetric difference formulas are not suitable. Instead, we use a three-point formula based on the Lagrange interpolating polynomial [47]. If Δ_1 , Δ_2 , and Δ_3 are three different consecutive values of Δ , an estimate of the specific-heat C at Δ_2 can thus be computed as

$$C(\Delta_2) = \frac{(\Delta_2 - \Delta_3)}{(\Delta_1 - \Delta_2)(\Delta_1 - \Delta_3)} [e_J(\Delta_1)]_{\text{av}} + \frac{(2\Delta_2 - \Delta_1 - \Delta_3)}{(\Delta_2 - \Delta_1)(\Delta_2 - \Delta_3)} [e_J(\Delta_2)]_{\text{av}} + \frac{(\Delta_2 - \Delta_1)}{(\Delta_3 - \Delta_1)(\Delta_3 - \Delta_2)} [e_J(\Delta_3)]_{\text{av}}. \quad (21)$$

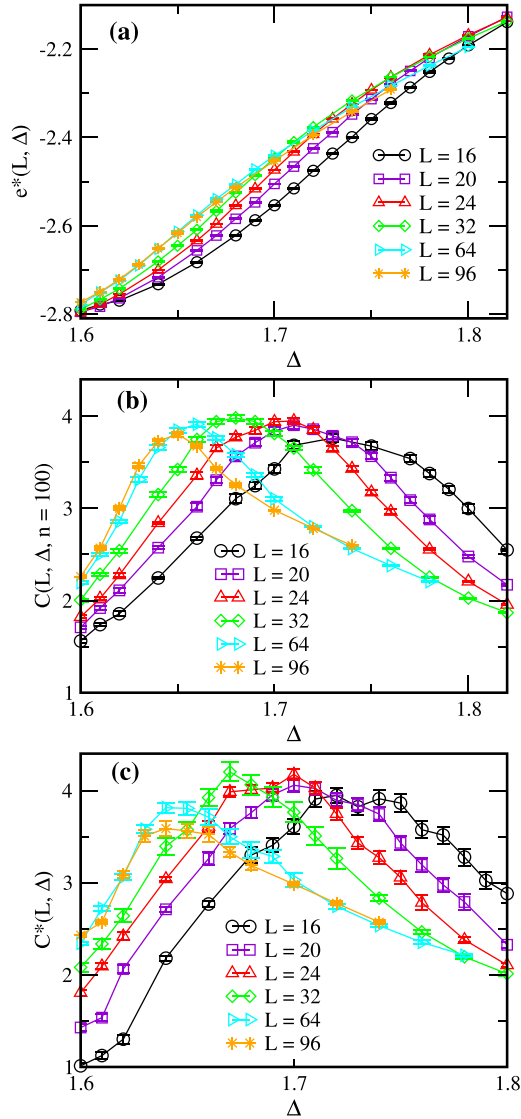


FIG. 7. (a) Extrapolated estimate $e^*(\Delta, L)$ of the bond energy as a function of Δ for various system sizes L . (b) Specific heat $C(L, \Delta)$ for $n = 100$ according to Eq. (21). (c) Extrapolated estimate $C^*(L, \Delta)$ of the specific heat derived by differentiating e^* .

Using this formula, we determine the specific heat C for different L , Δ and n . In panel (a) of Fig. 7, we first present the extrapolated estimates of the disorder-averaged bond energy per spin $e^*(L, \Delta)$ against Δ and for different values of L . Figures 7(b) and 7(c) display the behavior of the specific heat as a function of Δ for different L , where panel (b) shows $C(L, \Delta)$ for $n = 100$, whereas panel (c) displays the extrapolated specific heat $C^*(L, \Delta)$ after a numerical differentiation of $e^*(L, \Delta)$ with respect to Δ . In these plots, a clear peak in the specific heat can be seen, which moves towards lower Δ with increasing L , and the height of the peak initially grows with an increase in L . Eventually, it decreases with L , indicating a negative specific heat exponent α . To determine the peak positions and heights, for every L we performed simulations for additional Δ values near the peak and used a parabolic fit to the peak region of the form $C(L, \Delta) = a_0(\Delta - \Delta_{\max, C})^2 + C_{\max}$ in order to obtain the peak positions $\Delta_{\max, C}(L)$ and the peak

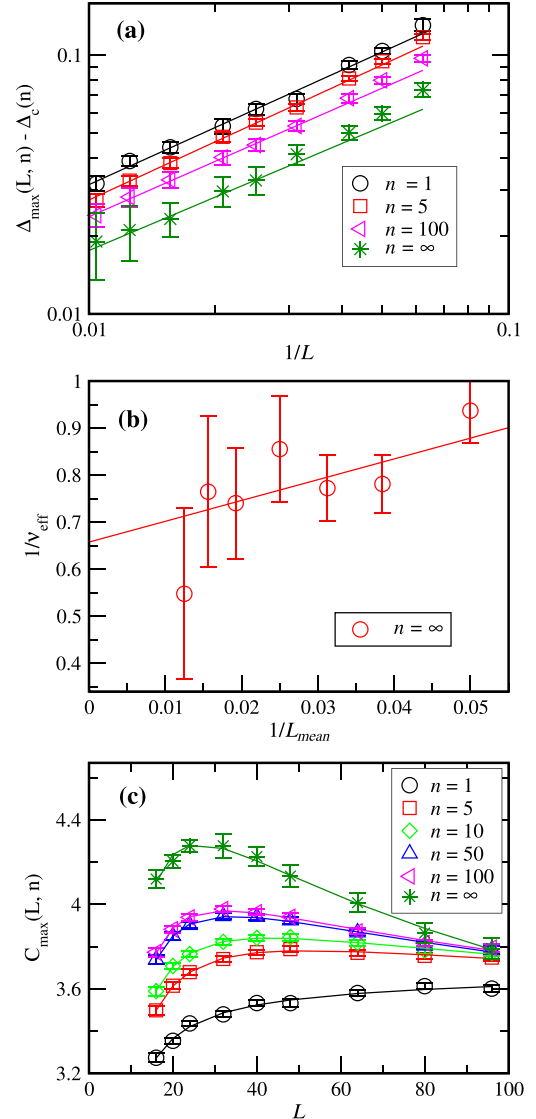


FIG. 8. (a) Residual peak locations $\Delta_{\max}(L, n) - \Delta_c(n)$ of the specific heat against $1/L$ (on a double log scale) for different n , where $\Delta_c(n)$ are determined from fits of the form of Eq. (23). Solid lines are power-law fits, whose slope give the exponent $1/\nu$, see Table III. For increased clarity, the data for different n are slightly shifted relative to each other. (b) Effective exponent $1/\nu_{\text{eff}}$ as a function of $1/L_{\text{mean}}$ for $n = \infty$. The solid line is a linear fit, yielding an extrapolated estimate $1/\nu = 0.658(49)$, consistent with the corresponding estimate from the FSS of m^* , see Table II. (c) Scaling of the maxima $C_{\max}(L)$ as a function of L for different n . The solid curves are fits of Eq. (27). The corresponding fit parameters α/ν and ω are collected in Table III.

heights $C_{\max}(L)$. The point symbols in panels (a) and (c) in Fig. 8, respectively, show $\Delta_{\max, C}(L)$ and $C_{\max}(L)$ as a function of system size L for finite as well as infinite n .

To analyze the peak positions and heights, we note that the singular part of the specific-heat is expected to scale as

$$C_s \sim L^{\alpha/\nu} \tilde{C}[(\Delta - \Delta_c)L^{1/\nu}], \quad (22)$$

where ν is the correlation-length exponent and α is the specific-heat exponent. At the peak, if the argument $x = (\Delta - \Delta_c)L^{1/\nu}$ of the scaling function $\tilde{C}$ takes some value, say x_0 ,

TABLE III. Parameters of fits of the forms (23) and (27) to the locations $\Delta_{\max}(L, n)$ and heights $C_{\max}(L, n)$, respectively, of the specific-heat peaks. Q_1 is the quality of the fit for $\Delta_{\max}(L, n)$, and Q_2 is the quality of the fit for $C_{\max}(L, n)$.

n	Δ_c	$1/\nu$	α/ν	ω	Q_1	Q_2
1	1.639(10)	0.734(92)	-0.004(56)	1.07(97)	0.03	0.42
5	1.633(5)	0.747(71)	-0.058(51)	1.07(74)	0.19	0.98
10	1.628(4)	0.773(64)	-0.075(52)	1.08(73)	0.14	0.97
50	1.624(3)	0.742(54)	-0.099(45)	1.18(66)	0.26	0.91
100	1.621(3)	0.721(51)	-0.088(30)	1.47(63)	0.21	0.83
∞	1.612(6)	0.686(69)	-0.201(75)	1.35(66)	0.58	0.95

then the peak position $\Delta_{\max,C}(L)$ should vary as

$$\Delta_{\max,C}(L) \approx \Delta_c + x_0 L^{-1/\nu}, \quad (23)$$

and the maximum value of the singular part of the specific heat scales as

$$C_{s,\max}(L) \sim L^{\alpha/\nu}. \quad (24)$$

We use Eq. (23) to estimate the infinite-size critical disorder Δ_c and the correlation length exponent $1/\nu$ from the data for $\Delta_{\max,C}(L)$. The corresponding fits are shown together with the data in Fig. 8(a). The resulting fit parameters for Δ_c and the exponent $1/\nu$ for different values of n are summarized in Table III, which also contains the overall satisfactory values of the fit qualities Q_1 . We note that the estimate of $1/\nu$ is slightly larger than the one extracted from the order parameter, cf. Table II. In order to check whether this is an effect of additional scaling corrections that we did not account for, we also considered an analysis of *effective exponents*. To this end, we fixed the estimate $\Delta_c = 1.612$ and performed fits of the functional form (23) to the data for three consecutive lattice sizes $L_1 < L_2 < L_3$ for increasing values of $L_{\text{mean}} = (L_1 + L_3)/2$. The corresponding effective estimates $1/\nu_{\text{eff}}$ are shown as a function of L_{mean} in Fig. 8(b). Based on general arguments of finite-size scaling, one expects [54]

$$1/\nu_{\text{eff}}(L) \approx 1/\nu + bL^{-\omega}, \quad (25)$$

where ω is the Wegner correction exponent. Our data for $1/\nu_{\text{eff}}$ are not accurate enough to allow for a determination of ω , but a simple linear extrapolation (that is consistent with the values of ω extracted from the peak heights below, cf. the data in Table III) yields an estimate $1/\nu = 0.658(49)$ that is noticeably closer to the value extracted from the scaling of the order parameter than the uncorrected variant. This fit is also shown in Fig. 8(b). We also again considered an extrapolation of $\nu^{-1}(n)$ for $n \rightarrow \infty$, for details see Appendix A.

Let us now investigate $C_{\max}(L)$ as shown in Fig. 8(c), which for larger values of n exhibits a strong curvature (also on a logarithmic scale), suggesting that α is negative. To determine α , we initially tried to fit the functional form

$$C_{\max}(L) = C_0 + aL^{\alpha/\nu}, \quad (26)$$

to the data, where C_0 represents a nonsingular background term and $L^{\alpha/\nu}$ is the singular term as given in Eq. (24). This form works for smaller values of n , but it is unable to represent the nonmonotonic behavior seen for $n \geq 10$, cf. Fig. 8(c).

To describe such effects, we need to include corrections to scaling, implementing fits of the form

$$C_{\max}(L) = C_0 + c_1 L^{\alpha/\nu} (1 + c_2 L^{-\omega}), \quad (27)$$

where ω corresponds to the Wegner exponent and c_2 is some constant. Due to the nonlinearity, we are not able to reliably fit this five-parameter form to the data and, instead, we find that the results are heavily dependent on the initial parameter values. For a negative α the maxima $C_{\max}(L)$ will approach C_0 as $L \rightarrow \infty$, but a saturation to a nonzero C_0 is not visible for larger n as well as for $n \rightarrow \infty$ within the range of available system sizes, cf. Fig. 8(c). In order to stabilize the fits, we might hence assume that $C_0 = 0$, and we indeed find such fits to work quite reliably. The best fits of this form are shown as solid curves in Fig. 8(c) and the resulting estimates for the exponents α/ν and ω are collected in Table III together with the fit qualities Q_2 , which are found to be excellent. Finally, concerning the possibility that $\alpha \simeq 0$, from the form (27) we would conclude that $C_{\max}(L) \simeq C'_0 + c'_2 L^{-\omega}$, which would again imply saturation of $C_{\max}$ as $L \rightarrow \infty$. Since our data for larger n do not show any sign of saturation of $C_{\max}$, we rule out this possibility and conclude that α is negative as given in Table III.

F. Susceptibility

We finally considered the connected and disconnected susceptibilities. The *disconnected* susceptibility is defined in Eq. (7). Panel (a) of Fig. 9 illustrates its extrapolated estimates $\chi_{\text{dis}}^*(L, \Delta)$ as a function of Δ for different $L \geq 24$. The expected FSS form is

$$\chi_{\text{dis}}^*(L, \Delta) = L^{\bar{\gamma}/\nu} \tilde{\chi}_{\text{dis}}[(\Delta - \Delta_c) L^{1/\nu}]. \quad (28)$$

Based on this scaling form, we arrive at a clean scaling collapse for $\bar{\gamma}/\nu = 2.9215(25)$, cf. Fig. 9(b). The values of Δ_c and the exponent $1/\nu$ agree with those from the magnetization scaling. We also performed FSS of the disconnected susceptibility for finite n and obtained excellent scaling collapses. As an example, in Fig. 9(c) we show the best scaling collapse for $n = 100$. The values of the exponent $\bar{\gamma}/\nu$ for different n , including the one for extrapolated $\chi_{\text{dis}}^*(L, \Delta)$, are listed in Table II, along with the qualities S_2 of the data collapse. We also arrive at exponent estimates for $1/\nu$, but we find them to be nearly indistinguishable from the values found from the magnetization (cf. Table II), so we do not list separate values here.

The *connected* susceptibility $\chi(L, \Delta)$ is the response of the system to the presence of a small uniform external field h . Since this study is conducted at temperature $T = 0$, we cannot use the usual fluctuation-dissipation relation to determine χ . Instead, we generalize arguments for the RFIM by Schwartz and Soffer [55] to express the magnetic susceptibility for the RFPM in a different form. Consider the Hamiltonian of the RFPM in a uniform external magnetic field H^α ,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \delta_{s_i, s_j} - \sum_i \sum_{\alpha=0}^{q-1} (h_i^\alpha + H^\alpha) \delta_{s_i, \alpha}. \quad (29)$$

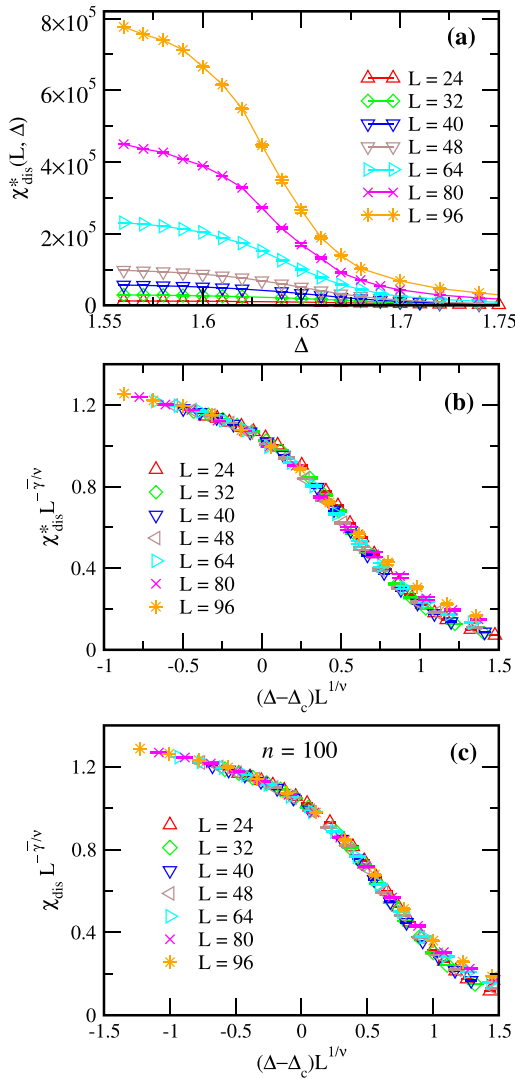


FIG. 9. (a) Extrapolated estimate $\chi_{\text{dis}}^*(L, \Delta)$ of the disconnected susceptibility as a function of Δ for various system sizes L . Error bars are also shown, but they are much smaller than the symbol size. (b) Scaling plot of $\chi_{\text{dis}}^*(L, \Delta)L^{-\tilde{\gamma}/\nu}$ as a function of $(\Delta - \Delta_c)L^{1/\nu}$ with the values Δ_c , $1/\nu$, and $\tilde{\gamma}/\nu$ in Table II. (c) Same as (b) but for $n = 100$.

Then the (q -component) magnetic susceptibility is defined as

$$\chi^\mu = \lim_{H^\mu \rightarrow 0} \frac{1}{N} \left[\frac{\partial \langle M^\mu \rangle}{\partial H^\mu} \right]_{\text{av}}, \quad (30)$$

where $\langle M^\mu \rangle$ denotes the thermal average of the total magnetization, and N is the number of spins. Defining $\bar{h}_i^\alpha = h_i^\alpha + H^\alpha$, the disorder average $[\dots]_{\text{av}}$ in the above equation can be expressed as

$$\left[\frac{\partial \langle M^\mu \rangle}{\partial H^\mu} \right]_{\text{av}} = \int d\{\bar{h}_i^\alpha\} P(\{\bar{h}_i^\alpha\}) \frac{\partial \langle M^\mu \rangle_{\bar{h}_i^\alpha}}{\partial H^\mu}. \quad (31)$$

Given that

$$\frac{\partial \langle M^\mu \rangle}{\partial H^\mu} = \sum_{i,\alpha} \frac{\partial \langle M^\mu \rangle}{\partial \bar{h}_i^\alpha} \frac{\partial \bar{h}_i^\alpha}{\partial H^\mu} = \sum_i \frac{\partial \langle M^\mu \rangle}{\partial \bar{h}_i^\mu},$$

partial integration applied to Eq. (31) yields

$$\left[\frac{\partial \langle M^\mu \rangle}{\partial H^\mu} \right]_{\text{av}} = - \sum_i \int d\{\bar{h}_i^\alpha\} \frac{\partial P(\{\bar{h}_i^\alpha\})}{\partial \bar{h}_i^\mu} \langle M^\mu \rangle_{\{\bar{h}_i^\alpha\}}. \quad (32)$$

Since $P(\bar{h}_i^\alpha)$ is a normal distribution of mean H^α and variance Δ , we have

$$\frac{\partial P(\bar{h}_i^\mu)}{\partial \bar{h}_i^\mu} = - \frac{\bar{h}_i^\mu - H^\mu}{\Delta^2} P(\bar{h}_i^\mu).$$

Using this in (32), we hence find for the susceptibility,

$$\chi^\mu = \lim_{H^\mu \rightarrow 0} \frac{1}{N} \left[\frac{\partial \langle M^\mu \rangle}{\partial H^\mu} \right]_{\text{av}} = \frac{1}{\Delta^2} \left[\langle m^\mu \rangle \sum_i h_i^\mu \right]_{\text{av}}, \quad (33)$$

where $m^\mu = M^\mu/N$ is the magnetization per spin in the state μ . This form for χ^μ provides the correct susceptibility as long as spontaneous symmetry breaking occurs between the q different states. While this is the case in the thermodynamic limit, a suitably modified approach is necessary for finite systems. An explicit symmetry breaking in this case can be achieved by applying a small external field that must be, however, sufficiently strong to actually break the symmetry. From the Hamiltonian (29), if we look at the typical scale of energy contributions due to a constant external field H and the random fields of strength Δ , we see that an external field $H \gtrsim \Delta N^{-1/2}$ is sufficient to break the symmetry (note $N = L^3$). In Table I, we present the values of the constant external field H for different lattice sizes L chosen to break the symmetry, such that the susceptibility χ exhibits a maximum. These fields are applied to spin state 1 (i.e., $\mu = 1$) and χ is computed from Eq. (33) for different L and Δ after averaging over a large number of disorder realizations N_{samp} as listed in Table I. Since χ , as defined in (33), amounts to a correlation measure among the magnetization and the total random field over the distribution of the latter, we find that no reliable extrapolation in n is possible for this quantity. We hence study it for our largest finite $n = 100$.

Figure 10(a) shows $\chi(L, \Delta)$ as a function of Δ for $n = 100$ and all lattice sizes $16 \leq L \leq 96$. A clear maximum in $\chi(L, \Delta)$ can be seen, and the maxima move to lower Δ with growing L ; further, the height of the maximum continuously grows with L , signaling that χ has a divergent behavior. To analyze this divergence, we fit a parabola near the maximum in order to obtain the location $\Delta_{\text{max},\chi}$ and the height χ_{max} of the susceptibility maximum. Figures 10(b) and 10(c), respectively, show $\Delta_{\text{max},\chi}(L)$ and $\chi_{\text{max}}(L)$ as a function of L and for different n . Note that $\chi_{\text{max}}(L)$ is plotted on a log-log scale. From the FSS ansatz

$$\chi(L, \Delta) = L^{\gamma/\nu} \tilde{\chi}[(\Delta - \Delta_c)L^{1/\nu}] \quad (34)$$

we deduce that

$$\Delta_{\text{max},\chi}(L) \approx \Delta_c + a_1 L^{-1/\nu}, \quad (35)$$

and

$$\chi_{\text{max}}(L) \sim L^{\gamma/\nu}. \quad (36)$$

Fits of the form (35) to our data for $\Delta_{\text{max},\chi}(L)$ work marginally well and lead to the parameters Δ_c and $1/\nu$ that are compiled in Table IV. If we compare these values to

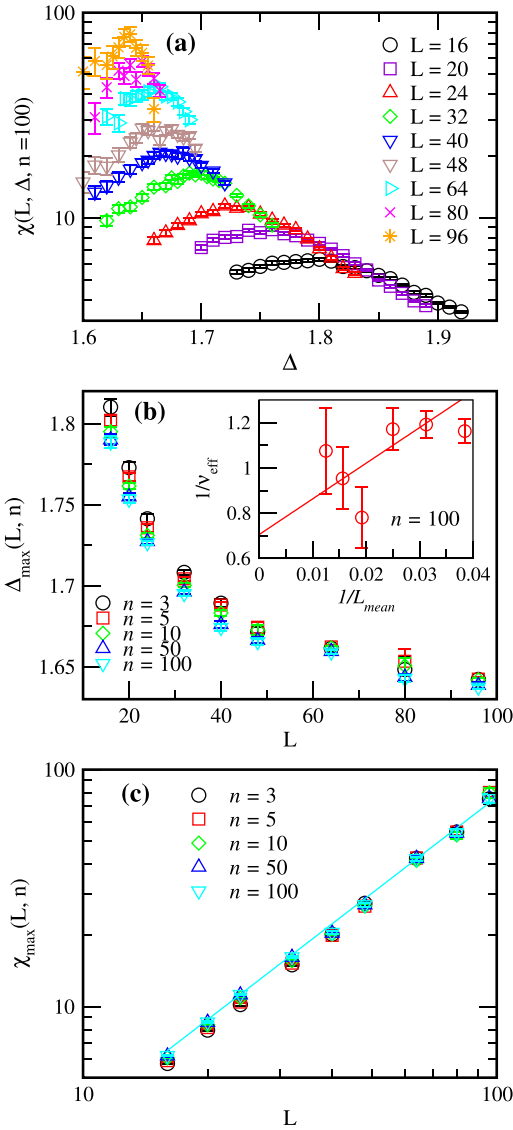


FIG. 10. (a) $\chi(L, \Delta)$ (on a semi log scale) for $n = 100$ as a function of Δ and for all system sizes L as specified. (b) $\Delta_{\max}$, the location of the maxima of χ , as a function of L and for different n . Inset: effective exponent $1/\nu_{\text{eff}}$ as a function of $1/L_{\text{mean}}$ for $n = 100$. The solid line is a linear fit, yielding an extrapolated estimate $1/\nu = 0.704(43)$ of the exponent, consistent with the corresponding estimate from the FSS of $[m]_{\text{av}}(L, \Delta, n = 100)$, see Table II. (c) Maximum $\chi_{\max}(L)$ of the susceptibility against L (on a log-log scale) for different n . The line corresponds to a power-law fit of the form $\chi_{\max}(L) \sim L^{\gamma/\nu}$ with an estimate of $\gamma/\nu = 1.343(9)$ to the data for $\chi_{\max}(L)$ at $n = 100$.

TABLE IV. Estimates of Δ_c , $1/\nu$, and γ/ν from fits of the locations $\Delta_{\max}(L, n)$ and the heights $\chi_{\max}(L, n)$ of the susceptibility. Q_1 denotes the quality of the fit for the data of $\Delta_{\max}(L, n)$, whereas Q_2 is the quality of the fit for the data of $\chi_{\max}(L, n)$.

n	Δ_c	$1/\nu$	γ/ν	Q_1	Q_2
3	1.625(6)	1.058(99)	1.401(10)	0.08	0.03
5	1.622(5)	1.065(85)	1.375(10)	0.04	0.06
10	1.620(4)	1.064(77)	1.349(10)	0.04	0.02
50	1.618(4)	1.054(89)	1.343(9)	0.05	0.05
100	1.617(4)	0.963(67)	1.343(9)	0.05	0.03

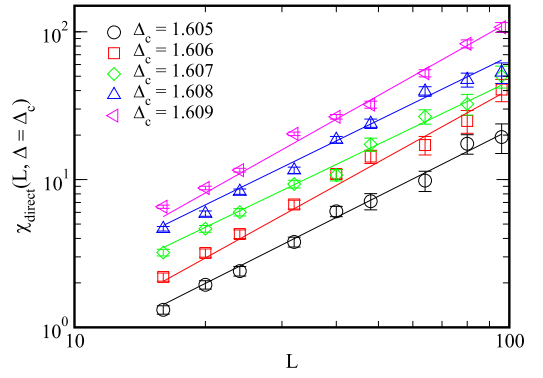


FIG. 11. FSS behavior of χ_{direct} (on a double-log scale), evaluated at several Δ_c for the largest $n = 100$. For a better view, the data for different Δ_c are shifted relative to each other through multiplication by a constant factor. The lines are fits of the power-law $\chi_{\text{direct}}(L, \Delta_c) \sim L^{\gamma/\nu}$ for all $L \geq L_{\min}$ to the data. The resulting exponents γ/ν from different choices of $L_{\min}$ are collected in the Table V.

those from magnetization (Table II) and specific heat (Table III), we see that the values of Δ_c are consistent but the values of $1/\nu$ are appreciably larger than the previous estimates, strongly suggesting the presence of corrections to scaling. Unfortunately, including a correction-to-scaling term, i.e., $\Delta_{\max, \chi}(L) = \Delta_c + a_1 L^{-1/\nu}(1 + a_2 L^{-\omega})$ leads to very unstable fits and hence a reliable analysis is not possible. Instead, we again reverted to the concept of effective exponents analogous to Eq. (25). A linear extrapolation of the estimates $1/\nu_{\text{eff}}(L)$ for $n = 100$ as shown in the inset of Fig. 10(b) for this case yields $1/\nu = 0.704(43)$, which is consistent with the corresponding $n = 100$ estimate for the order parameter scaling.

Turning to $\chi_{\max}(L)$, we fit the power-law $\chi_{\max}(L) \sim L^{\gamma/\nu}$ to the data of Fig. 10(c) and collect the results for the exponent γ/ν along with the (rather marginal) fit qualities Q_2 in Table IV. Here, we do not find any clean signature of a correction to scaling; the fit results vary only slightly on excluding small L , and within error bars they are consistent. Given the experience with the peak locations we were nevertheless skeptical about the apparent absence of corrections, and we hence conducted additional analyses of the susceptibility χ at fixed Δ and *without* the explicit symmetry breaking through a small external field. As our best estimate for the critical field strength is $\Delta_c = 1.607(2)$, we conducted such simulations in the range $\Delta_c \in [1.605, 1.609]$ with steps of size 0.001. The resulting quantity, which we denote as χ_{direct} , is shown at different estimates for Δ_c in Fig. 11. (The data have been shifted vertically by multiplying a constant factor so that χ_{direct} at different Δ_c can be seen distinctly.) Power-law fits according to Eq. (34) work quite well. To identify any corrections to scaling, the power law fits are performed on various ranges of $L \geq L_{\min}$. The fit results for the exponent γ/ν together with the fit qualities Q are collected in Table V. Clearly, there is very little systematic variation in the resulting estimate of γ/ν on variation of $L_{\min}$. We hence put forward the fits for the complete range of L , i.e., for $L_{\min} = 16$. Such fits at various Δ_c are shown as solid lines in Fig. 11, and the corresponding γ/ν estimates for different Δ_c as found in Table V are consistent with each other. We quote $\gamma/\nu = 1.467(53)$ corresponding

TABLE V. Estimates of the exponent γ/ν after fitting the form $\chi(L, \Delta_c) \sim L^{\gamma/\nu}$ for all five Δ_c to the corresponding data. The fits are performed to all $L \geq L_{\min}$ with the cut-offs $L_{\min} = 16, 20$, and 24 , respectively.

Δ_c	$L_{\min} = 16$		$L_{\min} = 20$		$L_{\min} = 24$	
	γ/ν	Q	γ/ν	Q	γ/ν	Q
1.605	1.542(52)	0.62	1.529(64)	0.52	1.557(80)	0.43
1.606	1.562(51)	0.22	1.560(62)	0.16	1.559(78)	0.11
1.607	1.467(53)	0.63	1.447(66)	0.54	1.463(84)	0.42
1.608	1.488(50)	0.45	1.531(62)	0.49	1.499(79)	0.42
1.609	1.556(48)	0.72	1.577(59)	0.66	1.558(74)	0.57

to the value at $\Delta_c = 1.607$ and for $L_{\min} = 16$ as our final estimate. Clearly, there is a noticeable difference in the values of γ/ν estimated through this direct approach from the ones of the χ estimates including the symmetry-breaking field, but we believe that the unbroken estimates are the more reliable ones. This is in broad agreement with other experiences concerning the connected susceptibility in critical phenomena, which is often found to suffer from significant corrections, see, e.g., Refs. [56,57].

Finally, in Table VI, we present all our extrapolated results for the critical exponents $\nu, \alpha, \beta, \gamma$, and $\bar{\gamma}$. The exponents $\eta, \bar{\eta}$, and θ are obtained using the relations $\gamma = \nu(2 - \eta)$, $\bar{\gamma} = \nu(4 - \bar{\eta})$, and $\theta = 2 - \bar{\eta} + \eta$. With these estimates, one can check the validity of the Rushbrooke equality $\alpha + 2\beta + \gamma = 2$ and the *modified* hyperscaling relation $2 - \alpha = \nu(d - \theta)$. [3,58,59]. It can be easily inspected from Table VI that both relations are well satisfied (within error bars). The so-called two-exponent scaling scenario predicts $\bar{\eta} = 2\eta$ [60,61]. This is also compatible with our results for $q = 4$ as we find a very marginal value of the difference $2\eta - \bar{\eta} = 0.013(53)$, indicating that there are only two independent critical exponents. In Table VI, we also show the exponents from our previous study for $q = 3$ RFPM [38] and the recent estimates for the three-dimensional RFIM [7].

IV. SUMMARY AND DISCUSSION

Based on a recently developed tool-chain of quasixact ground-state calculations for Potts models with random fields,

TABLE VI. Critical exponents of the $q = 4$ RFPM as compared to those of the $q = 3$ RFPM [38] and the RFIM (the $q = 2$ case) [7]. All exponents are for dimension $d = 3$ and the numbers in the parentheses denote the error bars in the last significant digits.

	RFIM	$q = 3$ RFPM	$q = 4$ RFPM
ν	1.38(10)	1.383(8)	1.55(2)
α	-0.16(35)	-0.082(28)	-0.31(12)
β	0.019(4)	0.0423(32)	0.061(2)
γ	2.05(15)	2.089(84)	2.274(87)
$\bar{\gamma}$	4.10(6)	4.07(3)	4.53(6)
η	0.5139(9)	0.49(6)	0.533(53)
$\bar{\eta}$	1.028(2)	1.060(3)	1.080(3)
θ	1.487(1)	1.43(6)	1.45(6)
$\alpha + 2\beta + \gamma$	2.00(31)	2.08(9)	2.09(15)

we have investigated the four-state random-field Potts model on the cubic lattice with a focus on its zero-temperature critical behavior. Our approach is based on the application of graph-cut methods, which are known to be exact for two-state (Ising-like) systems, to an embedding of two-state variables into the q states of the Potts system [35,36]. Combining this method with a systematic extrapolation technique based on n repetitions of such approximate ground-state calculations provides access to zero-temperature behavior of the system while avoiding the exponentially growing effort for any exact approach applied to this NP hard optimization problem [37].

Akin to the situation for other random-field systems, the location of the disordered fixed point at temperature $T = 0$ implies that such ground-state calculations also describe the critical behavior at nonzero temperatures. While for the pure Potts system discontinuous transitions occur for all $q \geq 3$, the transition order for the random-field problem is not known. In a previous study we showed that for $q = 3$ states the RFPM retains a continuous transition [38]. In the present work we generalized these considerations to the four-states problem. We have calculated the physical quantities, such as the magnetization, Binder cumulant, bond energy, specific heat, as well as the connected and disconnected susceptibilities, on various system sizes up to 96^3 . To average these quantities, the simulations were performed for a large number of disorder samples and each disorder sample is further simulated for $n = 100$ initial conditions to facilitate the extrapolation to the quasixact limit $n \rightarrow \infty$. Employing a comprehensive finite-size scaling analysis, we find that the first-order transition of the pure system is softened through the random fields to become continuous. Since the studied case of $T = 0$ corresponds to the strongest disorder along the transition line, and at the weakest disorder close to the pure case the transition is of first order, one should expect a tricritical point at some intermediate disorder strength which, however, could only be discovered using finite-temperature methods such as Monte Carlo simulations. This structure of the phase diagram is rather different from the ones proposed in Refs. [23,26,30] that had anticipated first-order transitions at zero temperature.

Studying the finite-size scaling of the magnetic and energetic quantities, we extract the full set of critical exponents of the model. Our estimates of various exponents are broadly presented in Tables II to V. The last Table VI lists the extrapolated estimate of all exponents. As a sanity check, we test these exponent estimates against the Rushbrooke scaling law and find it satisfied. It is clear that the $q = 4$ estimates are distinct from those of the $q = 3$ RFPM as well as the RFIM, indicating that the $q = 4$ RFPM lies in a different universality class. In particular, the correlation length exponent as well as the magnetic exponents appear to be clearly distinct from the cases $q = 2$ and $q = 3$. The hyperscaling-violation exponent θ , on the other hand, is consistent within error bars between all three models, so it is conceivable that it is independent of q in this class of models [62]. The most subtle determination for random-field problems concerns the specific-heat exponent α . Carefully taking scaling corrections into account, we here find a clearly negative value $\alpha = -0.31(12)$, whereas the three-state model was found to be closer to marginal with $\alpha = -0.082(28)$.

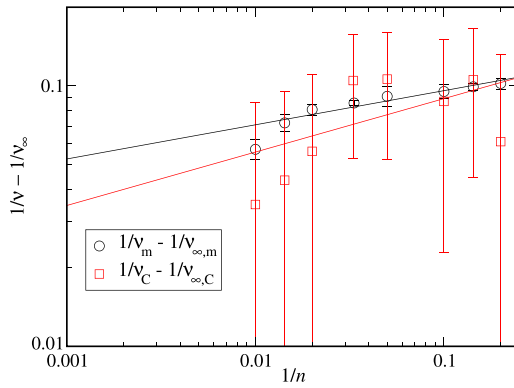


FIG. 12. Extrapolation of the exponent estimates for ν^{-1} to the limit $n \rightarrow \infty$. The lines show the corresponding fits of the functional form (A1) to the data.

While these results imply a significant step forward in the understanding of the RFPM in three dimensions, a lot remains to be desired. Is there a multicritical value q_c such that the transitions become first order even at $T = 0$ for $q > q_c$? How does the finite-temperature phase-diagram look like? Does the behavior of the RFPM depend on the particular form of coupling the random fields to the spins and on the random-field distribution? These and related questions form interesting avenues for further studies.

ACKNOWLEDGMENTS

M.K. and M.W. acknowledge the support of the Royal Society through a Royal Society-SERB Newton International Fellowship (NIF\R1\180386) and the Newton International Fellowships Alumni Grants (No. AL\221025, No. AL\231045, and No. AL\24100026). M.K. acknowledges the support of ANRF via a Ramanujan Fellowship (RJF/2022/00149). M.W. thanks DAAD (project ID 57622781) and DST (DST/INT/DAAD/P-02/2022) for support through an Indo-German exchange scheme. The bulk of numerical simulations were done

on the parallel compute cluster *Zeus* of Coventry University.

DATA AVAILABILITY

The code used to produce the data that support the findings of this article is openly available [41].

APPENDIX: DIRECT EXTRAPOLATION OF THE CORRELATION-LENGTH EXPONENT

In the main text we compared the values of critical exponents determined from the data for finite n with those extracted from the data extrapolated for $n \rightarrow \infty$. Here, we investigate the consistency of the two approaches for the example of the disorder-averaged magnetization $[m]_{av}(L, \Delta, n)$ and the peak positions of specific-heat, viz. $\Delta_{max,C}(L, n)$ (see Tables II and III). To this end, we considered the possibility of a power-law form for the finite- n deviations of the (inverse) correlation length exponent $1/\nu$,

$$\nu^{-1}(n) = \nu_{\infty}^{-1} + b n^{-c}, \quad (\text{A1})$$

where ν_{∞}^{-1} is the extrapolated value of ν^{-1} in the limit $n \rightarrow \infty$. Attempting such fits for the estimates $\nu_m^{-1}(n)$ from the magnetization data collapse and ν_c^{-1} for the scaling of the specific-heat peak locations, we are not able to arrive at stable results if leaving all three parameters to vary. The fit routine produces negative extrapolated values for both $1/\nu_{\infty,m}$ and $1/\nu_{\infty,c}$ with a small power-law exponent c , which are nonphysical. If, on the other hand, we fix $\nu_{\infty,m}^{-1} = 0.645$ and $\nu_{\infty,c}^{-1} = 0.686$ from Tables II and III, respectively, we find acceptable fits with $b = 0.1294(66)$, $c = 0.131(17)$ with $\chi^2/\text{d.o.f} = 1.86$ from $1/\nu_m$, and $b = 0.143(55)$, $c = 0.206(26)$ with $\chi^2/\text{d.o.f} = 0.23$ from $1/\nu_c$. As the residuals $\nu^{-1}(n) - \nu_{\infty}^{-1}$ shown in Fig. 12 reveal, the signal-to-noise ratio in these corrections is too small for a reliable nonlinear three-parameter fit. We hence conclude that the extrapolations of the exponent estimates for finite n are consistent with the $n \rightarrow \infty$ result, but an extrapolation of the observable quantities to $n \rightarrow \infty$ is the more reliable strategy as compared to an extrapolation of the exponent estimates themselves.

- [1] V. Dotsenko, *Introduction to the Replica Theory of Disordered Statistical Systems* (Cambridge University Press, Cambridge, 2001).
- [2] A. P. Young, ed., *Spin Glasses and Random Fields* (World Scientific, Singapore, 1997).
- [3] D. S. Fisher, Scaling and critical slowing down in random-field Ising systems, *Phys. Rev. Lett.* **56**, 416 (1986).
- [4] A. Aharony, Y. Imry, and S.-K. Ma, Lowering of dimensionality in phase transitions with random fields, *Phys. Rev. Lett.* **37**, 1364 (1976).
- [5] G. Parisi and N. Sourlas, Random magnetic fields, supersymmetry, and negative dimensions, *Phys. Rev. Lett.* **43**, 744 (1979).
- [6] M. Tissier and G. Tarjus, Supersymmetry and its spontaneous breaking in the random field Ising model, *Phys. Rev. Lett.* **107**, 041601 (2011).
- [7] N. G. Fytas, V. Martín-Mayor, M. Picco, and N. Sourlas, Restoration of dimensional reduction in the random-field Ising model at five dimensions, *Phys. Rev. E* **95**, 042117 (2017).
- [8] Y. Imry and S.-K. Ma, Random-field instability of the ordered state of continuous symmetry, *Phys. Rev. Lett.* **35**, 1399 (1975).
- [9] M. Aizenman and J. Wehr, Rounding of first-order phase transitions in systems with quenched disorder, *Phys. Rev. Lett.* **62**, 2503 (1989).
- [10] F.-Y. Wu, The Potts model, *Rev. Mod. Phys.* **54**, 235 (1982).
- [11] R. B. Potts, Some generalized order-disorder transformations, in *Mathematical Proceedings of the Cambridge Philosophical Society* (Cambridge University Press, Cambridge, 1952), Vol. 48, pp. 106–109.
- [12] M. Picco, Weak randomness for large q -state Potts models in two dimensions, *Phys. Rev. Lett.* **79**, 2998 (1997).

- [13] H. G. Ballesteros, L. A. Fernández, V. Martín-Mayor, A. Muñoz Sudupe, G. Parisi, and J. J. Ruiz-Lorenzo, Effect of dilution on first order transitions: The three dimensional three states Potts model, *Phys. Rev. B* **61**, 3215 (2000).
- [14] C. Chatelain, P. E. Berche, B. Berche, and W. Janke, Influence of dilution on the strong first-order phase transition of the 3D 4-state Potts model, *Comput. Phys. Commun.* **147**, 431 (2002).
- [15] B. Berche and C. Chatelain, Phase transitions in two-dimensional random Potts models, in *Order, Disorder and Criticality*, edited by Y. Holovatch (World Scientific, Singapore, 2004), p. 147.
- [16] G. Delfino, Exact results for quenched bond randomness at criticality, *Phys. Rev. Lett.* **118**, 250601 (2017).
- [17] K. Binder and J. Reger, Theory of orientational glasses models, concepts, simulations, *Adv. Phys.* **41**, 547 (1992).
- [18] K. Michel, Theory of the orientational glass state in $M(\text{CN})_x\text{X}_{1-x}$ mixed crystals, *Phys. Rev. Lett.* **57**, 2188 (1986); Theory of the orientational glass state in mixed crystals $M(\text{CN})_x\text{X}_{1-x}$. II. Dynamics, *Phys. Rev. B* **35**, 1414 (1987).
- [19] A. Aharony, K. Müller, and W. Berlinger, Trigonal-to-tetragonal transition in stressed SrTiO_3 : A realization of the three-state Potts model, *Phys. Rev. Lett.* **38**, 33 (1977).
- [20] E. Domany, Y. Shnidman, and D. Mukamel, Type I FCC anti-ferromagnets in a magnetic field: a realisation of the $q = 3$ - and $q = 4$ -state Potts models, *J. Phys. C* **15**, L495 (1982).
- [21] D. Blankshtein, Y. Shapir, and A. Aharony, Potts models in random fields, *Phys. Rev. B* **29**, 1263 (1984).
- [22] H. Nishimori, Potts model in random fields, *Phys. Rev. B* **28**, 4011 (1983).
- [23] K. Eichhorn and K. Binder, Monte Carlo investigation of the three-dimensional random-field three-state Potts model, *J. Phys.: Condens. Matter* **8**, 5209 (1996).
- [24] K. Eichhorn and K. Binder, The three-dimensional three-state Potts ferromagnet exposed to random fields: Evidence for a second order transition, *Z. Phys. B* **99**, 413 (1995).
- [25] K. Eichhorn and K. Binder, Finite-size scaling study of the three-state Potts model in random fields: Evidence for a second-order transition, *Europhys. Lett.* **30**, 331 (1995).
- [26] P. Reed, The Potts model in a random field: A Monte Carlo study, *J. Phys. C* **18**, L615 (1985).
- [27] W. Janke and R. Villanova, Three-dimensional 3-state Potts model revisited with new techniques, *Nucl. Phys. B* **489**, 679 (1997).
- [28] A. K. Hartmann, Calculation of partition functions by measuring component distributions, *Phys. Rev. Lett.* **94**, 050601 (2005).
- [29] J. Cardy, Quenched randomness at first-order transitions, *Physica A* **263**, 215 (1999).
- [30] Y. Y. Goldschmidt and G. Xu, Phase diagrams of the random-field Potts model in three dimensions, *Phys. Rev. B* **32**, 1876 (1985); The random-field Potts model in three dimensions, *Nucl. Phys. B* **265**, 1 (1986).
- [31] J. A. d'Auriac, M. Preissmann, and R. Rammal, The random field Ising model: Algorithmic complexity and phase transition, *J. Phys. Lett.* **46**, 173 (1985).
- [32] L. R. Ford Jr. and D. R. Fulkerson, *Flows in Networks* (Princeton University Press, Princeton, 2015).
- [33] A. V. Goldberg and R. E. Tarjan, A new approach to the maximum-flow problem, *J. ACM* **35**, 921 (1988).
- [34] Y. Boykov and V. Kolmogorov, An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision, *IEEE Trans. Pattern Anal. Machine Intell.* **26**, 1124 (2004).
- [35] M. Kumar, R. Kumar, M. Weigel, V. Banerjee, W. Janke, and S. Puri, Approximate ground states of the random-field Potts model from graph cuts, *Phys. Rev. E* **97**, 053307 (2018).
- [36] Y. Boykov, O. Veksler, and R. Zabih, Fast approximate energy minimization via graph cuts, *IEEE Trans. Pattern Anal. Machine Intell.* **23**, 1222 (2001).
- [37] M. Kumar and M. Weigel, Quasi-exact ground-state extrapolation for the random-field Potts model, *Comput. Phys. Commun.* **286**, 108685 (2023).
- [38] M. Kumar, V. Banerjee, S. Puri, and M. Weigel, Critical behavior of the three-state random-field Potts model in three dimensions, *Phys. Rev. Res.* **4**, L042041 (2022).
- [39] M.-S. Vaezi, G. Ortiz, M. Weigel, and Z. Nussinov, Binomial spin glass, *Phys. Rev. Lett.* **121**, 080601 (2018).
- [40] T. Nattermann, Theory of the random field Ising model, in *Spin Glasses and Random Fields* (World Scientific, Singapore, 1998), pp. 277–298.
- [41] <https://github.com/manojkmr8788/potts>.
- [42] M. S. Challa, D. P. Landau, and K. Binder, Finite-size effects at temperature-driven first-order transitions, *Phys. Rev. B* **34**, 1841 (1986).
- [43] S. Chen, A. M. Ferrenberg, and D. Landau, Monte Carlo simulation of phase transitions in a two-dimensional random-bond Potts model, *Phys. Rev. E* **52**, 1377 (1995).
- [44] B. Efron, *The Jackknife, the Bootstrap and Other Resampling Plans* (SIAM, Philadelphia, 1982).
- [45] R. G. Miller, The jackknife—a review, *Biometrika* **61**, 1 (1974).
- [46] O. Melchert, AutoScale.py—A program for automatic finite-size scaling analyses: A user's guide, [arXiv:0910.5403](https://arxiv.org/abs/0910.5403).
- [47] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C* (Cambridge University Press Cambridge, 1996).
- [48] Note that the cumulant was *not* defined using central moments.
- [49] J. Lee and J. Kosterlitz, Finite-size scaling and Monte Carlo simulations of first-order phase transitions, *Phys. Rev. B* **43**, 3265 (1991).
- [50] V. Privman, Finite-size scaling theory, in *Finite Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990), pp. 1–98.
- [51] K. Binder, Finite size scaling analysis of Ising model block distribution functions, *Z. Phys. B* **43**, 119 (1981).
- [52] K. Binder and D. W. Heermann, *Monte Carlo Simulation in Statistical Physics*, 5th ed. (Springer, Berlin, Heidelberg, 2010).
- [53] A. Hartmann and A. Young, Specific-heat exponent of random-field systems via ground-state calculations, *Phys. Rev. B* **64**, 214419 (2001).
- [54] D. Amit and V. Martín-Mayor, *Field Theory, The Renormalization Group, and Critical Phenomena: Graphs to Computers*, 3rd ed. (World Scientific, Singapore, 2005).
- [55] M. Schwartz and A. Soffer, Exact inequality for random systems: Application to random fields, *Phys. Rev. Lett.* **55**, 2499 (1985).
- [56] W. Janke and A. M. J. Schakel, Fractal structure of spin clusters and domain walls in the two-dimensional Ising model, *Phys. Rev. E* **71**, 036703 (2005).

- [57] M. Akritidis, N. G. Fytas, and M. Weigel, Geometric clusters in the overlap of the Ising model, *Phys. Rev. E* **108**, 044145 (2023).
- [58] M. Schwartz, Breakdown of hyperscaling in random systems—an inequality, *Europhys. Lett.* **15**, 777 (1991).
- [59] G. Grinstein, Ferromagnetic phase transitions in random fields: The breakdown of scaling laws, *Phys. Rev. Lett.* **37**, 944 (1976).
- [60] M. Schwartz, The random-field puzzle. I. Solution by equivalent annealing, *J. Phys. C* **18**, 135 (1985).
- [61] M. Schwartz, M. Gofman, and T. Natterman, On the missing scaling relation in random field systems, *Physica A* **178**, 6 (1991).
- [62] This is in contrast to the RFIM for different lattice dimensions, where θ is found to increase up to $\theta = 2$ in $d = 5$, see Ref. [7].