

Local density of states distribution and multifractal eigenvectors of weighted random networks via the cavity approach

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We study the local density of states (LDOS) distribution of a general class of weighted Erdős-Rényi graphs. Using the cavity method, we obtain a good approximation to the full LDOS distribution and compact expressions for its power-law tails, which we show to have exponent 3 in the extended phase. We deduce that the eigenvectors in the continuous part of the spectrum are extended but (weakly) multifractal, and we extract expressions for the associated fractal dimensions and the singularity spectrum. We also demonstrate that the inverse participation ratio in this multifractal phase exhibits an unusual logarithmic scaling with system size, which is neither fully extended nor localized by the usual definitions. Finally, we verify that some symmetry properties (derived from the nonlinear sigma model), which have been shown to hold for many systems exhibiting multifractality, also hold in our case, both for the LDOS distribution and the singularity spectrum.

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I. INTRODUCTION

Anderson localization remains one of the foremost examples of a disorder-induced phenomenon [1,2]. The classic Anderson tight-binding model exhibits a transition from a “conducting” phase to an “insulating” phase at sufficiently strong onsite disorder. That is, the quantum states of non-interacting spinless fermions in this model go from being distributed statistically evenly across sites to being exponentially localized around single sites. At the critical point between these phases, multifractality and nonergodic behavior can be observed [3–5]. Recently, evidence has emerged that these phenomena might also occur deep within the conducting phase. This means that extended or metallic states can have the curious property of occupying a vanishing fraction of the state-space volume in the thermodynamic limit while the absolute number of occupied sites diverges, and they can be characterized by a spectrum of fractal dimensions [2,6–8].

Nonergodicity of states is also a characteristic of many-body localization [9,10], in which interacting quantum systems exhibit an out-of-equilibrium transition from an ergodic metal at low disorder to a nonergodic insulator at strong disorder, interpreted as localization in Fock space [11,12]. With the advent of low-temperature experiments in recent decades, the existence of many-body localized states has now been well established theoretically in one-dimensional (1D) random spin chains [13,14], and experimental signatures have been observed in ultracold atomic gases [15–18]. An important correspondence between the many-body Fock space of

the interacting quantum system and the single-particle Anderson model was postulated in the seminal work of [19]. Nonergodic extended states (also known as “bad metal” behavior) in the many-body metallic regime have been hinted at, with characteristic subdiffusive behavior having been found in the extended phase of many-body systems [10,20,21]. Insight into the origin of multifractality in Anderson-type models, or random matrix models more generally, is therefore expected to reveal something of the possible mechanism for nonergodicity in the many-body analog.

Aside from the prominent applications of Anderson and many-body localization, the nature of the eigenvector statistics has become a central topic of interest in random matrix theory more generally [22]. This has become especially true with the invention of more exotic random matrix ensembles in various modern applications, which break rotational symmetry [23], and thus may deviate from the usual universal eigenvalue and eigenvector statistics [24–26]. Accordingly, the possibility of nonergodicity and multifractality has been investigated in many variations of the single-particle Anderson model on the Bethe lattice and random regular graphs [7,27–32]. It has also been investigated in critical random power-law banded matrices [33], generalized β ensembles [34], the adjacency and Laplacian matrices of complex networks built via the configuration model [35–37], and in the generalized Rosenzweig-Porter model (with both Gaussian and heavy-tailed off-diagonal elements) [11,38–42].

In this work, we examine analytically the origin of multifractality and nonergodicity in weighted Erdős-Rényi graphs, which were shown numerically in Ref. [43] to exhibit multifractal behavior. Starting from the cavity equations [44,45], we are able to derive simple approximation schemes for the distribution of the local density of states (LDOS), and then extract information about the localization and multifractality of the eigenvectors. We demonstrate that, as long as the squares of the edge weights have a probability density function (pdf) that is nonzero at the origin, the LDOS distribution has the

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same power-law tail behavior with universal exponent 3. This remains true in the presence of moderate onsite and diagonal disorder.

Using this result, we demonstrate that the eigenvectors in the continuous part of the eigenvalue spectrum are extended but (weakly) nonergodic and multifractal, with an inverse participation ratio (IPR) of $I_2 \sim N^{-1} \ln N$. This constitutes a middle ground between the usual extended ($I_2 \sim N^{-1}$) and localized ($I_2 \sim N^0$) behaviors. We also extract the so-called singularity spectrum and fractal dimensions, seeing there the signature of multifractality as well. These results apply generally to a variety of edge-weight distributions, and we verify this numerically.

The rest of the text is arranged as follows. In Sec. II, we introduce the class of random matrix ensembles at the center of our analysis, as well as important quantities like the local density of states, singularity spectrum, and fractal dimensions. In Secs. III and IV, we describe our general strategy for quantifying the behavior of the LDOS via the cavity approach. In Sec. V, we perform a rare-event analysis of our expression for the LDOS distribution to understand its power-law tail. Using this power-law behavior, we deduce the singularity spectrum and fractal dimensions in Secs. VI and VII, respectively, and we use this information to understand in what sense the eigenvectors are nonergodic. We also discuss how the behavior is modified at the approach to the edge of the continuous part of the eigenvalue spectrum in Sec. VIII, and finally in Sec. IX we conclude.

II. RANDOM MATRIX ENSEMBLES AND QUANTITIES OF INTEREST

We consider a general ensemble of weighted Erdős-Rényi (ER) graphs of size N whose weighted adjacency matrices we denote $\underline{J}$. The elements of this matrix can be decomposed as a direct product $J_{ij} = W_{ij}A_{ij}$, where $W_{ij} = W_{ji}$ are the weights and $A_{ij} \in \{0, 1\}$ are the elements of the unweighted ER graph.

As usual, $A_{ij} = A_{ji}$ is 1 with probability p/N and 0 otherwise independently of the existence of any of the other edges. To rule out any trivial effects coming from disconnected components of the graph, we examine only the giant component of the network in numerics. That is, the numerics are used to verify that the nonergodic and multifractal features of the theory are not due to isolated clusters of nodes.

Weights are drawn according to $W_{ij} = S_{ij}\sqrt{u_{ij}}$ where S_{ij} is a random sign ± 1 and the u_{ij} are positive and independent and identically distributed. For numerical results we will use uniform, half-Gaussian and half-exponential distributions each with average $\langle u_{ij} \rangle = 1/p$ so that the limit $p \rightarrow \infty$ yields the usual Wigner semicircle law. Additionally, for technical reasons that we elucidate in Sec. VD, we are restricted to considering distributions for u_{ij} with a positive probability density at zero.

The statistics of the weighted network manifest in our calculations via the nonvanishing higher-order statistics of the matrix elements. Specifically, we define

$$\langle J_{ij}^2 \rangle = \frac{1}{N}, \quad \langle J_{ij}^4 \rangle = \frac{\beta_4}{pN}, \quad \dots \quad \langle J_{ij}^{2k} \rangle = \frac{\beta_{2k}}{p^{k-1}N}. \quad (1)$$

Here $p > 1$ controls the speed of decay of higher-order moments, while the coefficients β_{2k} are all assumed to be order one and independent of N and p . We thus see that the random matrices $\underline{J}$ in this work have fourth-and-higher moments of the same order in N as the variance. That is, we should not expect the matrix $\underline{J}$ to have similar properties to a Gaussian random matrix for $p \sim N^0$. Indeed, the eigenvalue statistics of random matrices of this kind were previously considered in Ref. [23], where the deviation from Wigner's semicircle was characterized, as were the nontrivial long-range eigenvalue correlations and nonzero spectral compressibility. Ultimately, it will be the non-negligible higher-order statistics of J_{ij} that will give rise to a nontrivial LDOS distribution here. Later, we also consider the possibility of the diagonal elements J_{ii} being drawn from a uniform distribution, but for now we imagine that $J_{ii} = 0$.

Now that we have discussed the ensembles of interest, let us detail the targets of our investigation, which are all quantities derived from the eigenvectors and eigenvalues of the matrix $\underline{J}$. We denote the i th component of the eigenvector corresponding to the eigenvalue λ_v as $\psi_i^{(v)}$, and we employ the normalization condition $\sum_i |\psi_i^{(v)}|^2 = 1$. We can then define the global density of states and the eigenvector distribution (at a fixed location ω in the spectrum)

$$\rho(\omega) = \frac{1}{N} \left\langle \sum_v \delta(\omega - \lambda_v) \right\rangle, \quad P_\psi(y) = \frac{1}{N^2 \rho(\omega)} \left\langle \sum_{i,v} \delta(\omega - \lambda_v) \delta(y - N |\psi_i^{(v)}|^2) \right\rangle, \quad (2)$$

where we suppress the dependence of $P_\psi(y)$ on ω for notational convenience.

For extended eigenvectors, all of the eigenvector components have roughly the same magnitude of $|\psi_i^{(v)}|^2 \sim 1/N$. For example, the distribution of eigenvector components in the Gaussian orthogonal ensemble (GOE) takes the Porter-Thomas form $P_\psi(y) = e^{-y/2}/\sqrt{2\pi y}$. In contrast, we demonstrate that the eigenvectors in the extended phase of the weighted ER graphs considered here exhibit many different scales. More precisely, we can define the scaling exponent $\alpha_i^{(v)} = -\log(|\psi_i^{(v)}|^2)/\log(N)$, so that $|\psi_i^{(v)}|^2 = N^{-\alpha_i^{(v)}}$. One can then characterize the distribution of the different magnitudes of the eigenvector components using

$$\Omega(\alpha; \omega) = \frac{1}{N^2 \rho(\omega)} \sum_{i,v} \langle \delta(\alpha - \alpha_i^{(v)}) \delta(\omega - \lambda_v) \rangle. \quad (3)$$

The distribution $\Omega(\alpha; \omega)$ can be related to the distribution of eigenvector components $P_\psi(y)$ simply via a change of variables $y(\alpha) = N^{1-\alpha}$ so that $\Omega(\alpha) = \ln N y(\alpha) P_\psi(y(\alpha))$ (once again, we suppress the ω dependence). One can then define the so-called singularity spectrum $f(\alpha)$ [6–8] via the scaling relationship $\Omega(\alpha) = N^{f(\alpha)-1}$, which we expect to be valid in the limit $N \rightarrow \infty$. The singularity spectrum can thus be written

$$f(\alpha) = \lim_{N \rightarrow \infty} \frac{\log[N y(\alpha) P_\psi(y(\alpha))]}{\log(N)}. \quad (4)$$

We can also define the set of generalized inverse participation ratios [2,6]

$$I_q(\omega) = \frac{\left\langle \sum_v \delta(\omega - \lambda_v) \sum_i |\psi_i^{(v)}|^{2q} \right\rangle}{N\rho(\omega)} = \int_0^N dy y^q N^{1-q} P_\psi(y). \quad (5)$$

Positing the scaling behavior $I_q \sim N^{-\tau(q)}$ allows the extraction of the set of fractal dimensions $D(q) = \tau(q)/(q-1)$. In the case of the GOE, for which the eigenvectors are fully extended, and we have $D(q) = 1$ identically for $q > 1$. For Anderson localized states, in contrast, we have $I_q \sim N^0$. Multifractal states constitute a middle ground between these two extremes, having $0 < D(q) < 1$ with $D(q)$ varying as a function of q (i.e., a range of fractal dimensions). The fractal dimensions can be extracted from the singularity spectrum above in the limit $N \rightarrow \infty$ via the Legendre transform [35,43]

$$D(q) = -\frac{\max_\alpha [f(\alpha) - q\alpha]}{q-1}. \quad (6)$$

The quantity at the center of our analysis, which will permit us to extract the singularity spectrum and the fractal dimensions, is the local density of states (LDOS)

$$\rho_i(\omega) = \sum_v \delta(\omega - \lambda_v) |\psi_i^{(v)}|^2. \quad (7)$$

If the distribution $P_{\text{LDOS}}(x; \omega) = N^{-1} \langle \sum_i \delta[x - \rho_i(\omega)] \rangle$ has a power-law tail as a function of x , we expect the distribution of eigenvector components $P_\psi(y)$ also to have a power-law tail with the same exponent [35,43]. Such a power-law tail connotes a nontrivial set of fractal dimensions, and therefore multifractality and nonergodicity, which we will derive for the ensembles of matrices that satisfy Eq. (1).

III. LOCAL DENSITY OF STATES (LDOS)

A fixture in the analysis of the spectral properties of random matrices is the matrix resolvent (also known as the matrix Green's function)

$$G_{ij}(z) = ([z\mathbb{1} - \underline{J}]^{-1})_{ij} = \sum_v \frac{\psi_i^{(v)} \psi_j^{(v)*}}{z - \lambda_v}, \quad (8)$$

where z is in general complex and the last equality follows from spectral decomposition. The LDOS is obtained from the local resolvent $G_{ii}(z)$ by taking z close to the real line and using the identity $\delta(\omega) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \frac{\epsilon}{\epsilon^2 + \omega^2}$ to obtain

$$\rho_i(\omega) = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0^+} \text{Im } G_{ii}(\omega - i\epsilon), \quad (9)$$

where ω and ϵ here are real. The off-diagonal elements $G_{ij}(z)$ are known to be negligible for large N (see, e.g., [38,46–48]). The diagonal elements can be obtained using the cavity method [44]. Specifically, we have

$$G_{ii} = \frac{1}{z - \Sigma_i}, \quad (10)$$

where we have introduced the self-energy

$$\Sigma_i = \sum_j J_{ij}^2 G_{jj}^{(i)}. \quad (11)$$

Here, $\underline{G}^{(i)}$ is the resolvent matrix of the $N-1 \times N-1$ matrix $\underline{J}^{(i)}$, which is formed by removing the row and column i from the original matrix $\underline{J}$. These quantities can be determined self-consistently by solving the so-called cavity equations

$$G_{jj}^{(i)} = \frac{1}{z - \Sigma_j^{(i)}}, \quad (12)$$

where $\Sigma_j^{(i)} = \Sigma_j - J_{ij}^2 G_{ii}^{(j)}$. Equation (12) permits a closed solution procedure, and is valid in the large- N limit for a locally treelike network, of which the ER graph is an example. In [44] the cavity equations were solved numerically for finite single instances of random networks, and analytically in the large-connectivity limit where self-averaging results in considerable simplification. In [49], Bordenave and Lelarge treat the cavity equations in a distributional sense in the limit of large N . It is this latter philosophy that we employ for the remainder of the article: we treat the elements of the resolvent as random variables and seek an approximate solution for their distribution. Indeed, self-consistent determination of the self-energy distribution in this way is a well-established approach in the theory of localization [38,46,50,51].

IV. PERTURBATIVE APPROXIMATION FOR THE SELF-ENERGY DISTRIBUTION

A central object of our analysis is the joint distribution of the real and imaginary parts of the self-energy $\Sigma_i = X_i + iY_i$, which we denote $P_\Sigma(X, Y)$. We now seek to obtain an approximation for the self-energy distribution, which will later help us obtain the LDOS distribution. We do this by performing a perturbative $1/p$ expansion using the cavity equations. In the end, we obtain an expression for the self-energy distribution that is accurate up to order $O(p^{-2})$. As such, the approximation that we obtain captures the typical fluctuations of the self-energy. In the subsequent section, we discuss how we may also understand the tails of the LDOS distribution (i.e., the rare fluctuations).

The $1/p$ expansion that we employ here is particularly useful because it ignores the effects of the isolated clusters of nodes of the full ER graph. This is because the probability for a randomly selected node to belong to a cluster of size K on the ER graph follows a Borel distribution $P_K = \frac{(pK)^{K-1}}{K!} e^{-pK}$, so a $1/p$ expansion necessarily ignores nonperturbative ($\propto e^{-Kp}$) contributions coming from eigenvectors localized to isolated clusters. This is analogous to the way that Lifshitz tails in the eigenvalue spectrum of the ER graph are ignored by a $1/p$ expansion of the global density of states [52,53]. We therefore expect our theory, which is derived by performing a $1/p$ expansion for quantities on the full ER graph, to capture well the behavior of extended states on the giant component. Indeed, we test our theory against numerics for the giant component, finding excellent agreement. We compare with numerics for the full ER graph in Appendix F.

A. Statistics of Σ_i

Since the elements J_{ij} and $G_{ij}^{(i)}$ are statistically independent, we observe the following simple relations:

$$\begin{aligned}\langle X_i \rangle &= \sum_j \langle J_{ij}^2 \rangle \langle \text{Re} G_{jj}^{(i)} \rangle = \text{Re} \langle G \rangle, \\ \langle Y_i \rangle &= \sum_j \langle J_{ij}^2 \rangle \langle \text{Im} G_{jj}^{(i)} \rangle = \text{Im} \langle G \rangle,\end{aligned}\quad (13)$$

where we note that the statistics of the cavity resolvent elements are the same as those of the true resolvent due to the fact that A_{ij} are all independent random variables in the case of the ER graph. We are interested in the fluctuations about this average.

To this end, we consider the characteristic function of the centered variables $\delta_i^{(X)} = X_i - \text{Re} \langle G \rangle$ and $\delta_i^{(Y)} = Y_i - \text{Im} \langle G \rangle$:

$$\begin{aligned}\mathcal{F}_\Sigma(k_X, k_Y) &\equiv \int dX dY P_\Sigma(X, Y) \exp[i(k_X X + k_Y Y)] \\ &\quad \times \exp[-i(k_X \text{Re} \langle G \rangle + k_Y \text{Im} \langle G \rangle)] \\ &= \left\langle \exp \left[i J_{ij}^2 (k_X \text{Re} G_{jj}^{(i)} + k_Y \text{Im} G_{jj}^{(i)}) \right. \right. \\ &\quad \left. \left. - \frac{i}{N} (k_X \text{Re} \langle G \rangle + k_Y \text{Im} \langle G \rangle) \right] \right\rangle^N,\end{aligned}\quad (14)$$

where we note that the last expression applies for any $i \neq j$. If we then expand the exponential, we obtain

$$\begin{aligned}\langle \exp [i J_{ij}^2 (k_X \text{Re} G_{jj}^{(i)} + k_Y \text{Im} G_{jj}^{(i)})] \rangle &\approx 1 \\ &+ \frac{i}{N} (k_X \text{Re} \langle G \rangle - k_Y \text{Im} \langle G \rangle) \\ &- \frac{\beta_4}{2! p N} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^2 \rangle \\ &- \frac{i \beta_6}{3! p^2 N} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^3 \rangle,\end{aligned}\quad (15)$$

where we use the statistics in Eq. (1) and neglect terms proportional to the higher-order moments of the matrix entries. In the limit $N \rightarrow \infty$, we reexponentiate to obtain

$$\begin{aligned}\mathcal{F}_\Sigma(k_X, k_Y) &\approx \exp \left[-\frac{\beta_4}{2! p} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^2 \rangle \right. \\ &\quad \left. - \frac{i \beta_6}{3! p^2} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^3 \rangle \right].\end{aligned}\quad (16)$$

We comment on the fact that it is necessary to go up to at least $O(p^{-2})$ in our approximation scheme to correctly capture the tail of the LDOS distribution in Sec. V. We also argue that the power-law nature of the tail of the LDOS distribution is not affected by taking into account higher-order terms in $1/p$ in the exponent of Eq. (16); only the prefactor of the power-law tail would be corrected.

If we ignore terms $O(p^{-2})$, we see that $\mathcal{F}_\Sigma(k_X, k_Y)$ is just the Fourier transform of a 2-variable Gaussian distribution. A simple way, then, to take into account the higher-order moments systematically is to perform an Edgeworth expansion [54]. That is, we approximate $\exp[-\frac{i \beta_6}{3! p^2} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^3 \rangle] \approx 1 - \frac{i \beta_6}{3! p^2} \langle (k_X \text{Re} [G_{jj}] + k_Y \text{Im} [G_{jj}])^3 \rangle$, and

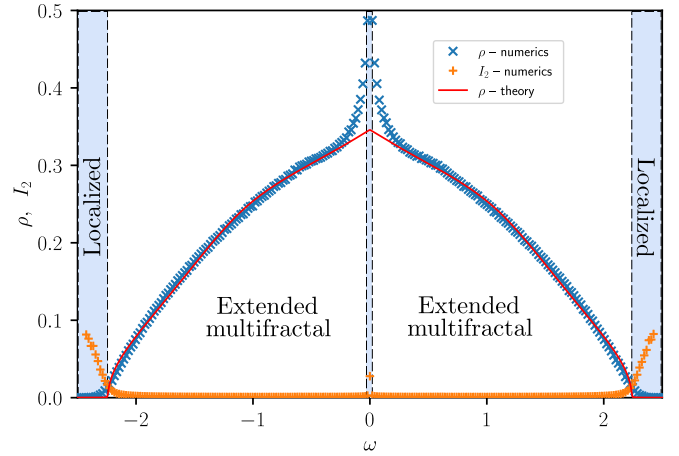


FIG. 1. Global density of states ρ and IPR I_2 for a sparse ER graph with uniformly distributed $u_{ij} = W_{ij}^2$. Other weight distributions produce qualitatively similar results. The theory line for $\rho(\omega)$ is found by solving Eq. (17) numerically with $\beta_4 = \frac{4}{3}$ and $\beta_6 = 2$ and $p = 5$ and using $\rho = \pi^{-1} \lim_{\epsilon \rightarrow 0} \text{Im} \langle G(\omega - i\epsilon) \rangle$. Numerics are the result of diagonalizing 500 matrices with size $N = 4000$.

perform the inverse Fourier transform, obtaining a lengthy but elementary expression for $P_\Sigma(X, Y)$ in terms of the aforementioned Gaussian and its derivatives.

B. Recursive evaluation of the statistics of G_{ii} using the statistics of Σ_i

Equation (16) for the characteristic function still contains the statistics of G_{ii} itself, implying the need to solve self-consistently. We approximate the statistics of G_{ii} up to the required order in p^{-1} so that the exponent in Eq. (16) is accurate to $O(p^{-2})$. That is, we must find $\langle G \rangle$ up to order $O(p^{-2})$, find the second moments accurately up to order p^{-1} , and the approximation $\langle (\text{Re} G_{ii})^3 \rangle \approx [\text{Re} \langle G \rangle]^3$ (and so on) will suffice for third-order moments.

Let us begin with the mean. This is accomplished by using the definition $G_{ii} = \frac{1}{\omega - X_i - iY_i}$, leveraging the knowledge that we have already obtained of the cumulants of X_i and Y_i . Ultimately, as we show in Appendix A, we arrive at the following expression, which is valid up to order $O(p^{-2})$ and can be solved numerically for $\langle G \rangle$:

$$\langle G \rangle \approx \frac{1}{\omega - \langle G \rangle} + \frac{\beta_4}{p} \langle G \rangle^5 + \frac{\beta_6}{p^2} \langle G \rangle^7 + \frac{\beta_4^2}{p^2} \langle G \rangle^9. \quad (17)$$

This expression was obtained previously using diagrammatic techniques [23,55,56], and it is verified in Fig. 1. We note that similar expressions have been obtained using the replica trick [52], and exact evaluation of the global density of states has been achieved previously using the population dynamics method [57]. Using a similar approach, we can also evaluate the higher moments, which enter into the expression for $\mathcal{F}_\Sigma(k_X, k_Y)$ in Eq. (16). We have

$$\begin{aligned}\langle (\text{Re} G_{ii})^2 \rangle &\approx \langle \text{Re} G_{ii} \rangle^2 + \frac{\beta_4}{p} [\text{Re} \langle (G_{ii})^3 \rangle]^2, \\ \langle (\text{Im} G_{ii})^2 \rangle &\approx \langle \text{Im} G_{ii} \rangle^2 + \frac{\beta_4}{p} [\text{Im} \langle (G_{ii})^3 \rangle]^2,\end{aligned}$$

$$\langle \text{Re} G_{ii} \text{Im} G_{ii} \rangle \approx \langle \text{Re} G_{ii} \rangle \langle \text{Im} G_{ii} \rangle + \frac{\beta_4}{p} \text{Re}(\langle G_{ii} \rangle^3) \text{Im}(\langle G_{ii} \rangle^3). \quad (18)$$

Again, these expressions can also be derived using the diagrammatic formalism [23]. We note that we need only to evaluate these moments to order $O(p^{-1})$ in order for the neglected contribution in the exponent of Eq. (16) to be of order $O(p^{-3})$. Having already found $\langle G \rangle$, we thus now have the requisite approximation for the second moments. Finally, we may also approximate

$$((k_X \text{Re}[G_{jj}] + k_Y \text{Im}[G_{jj}])^3) \approx (k_X \text{Re}\langle G \rangle + k_Y \text{Im}\langle G \rangle)^3 \quad (19)$$

in Eq. (16). We thus have all the information that we need to evaluate the distribution of the self-energy. Namely, we insert Eqs. (17), (18), and (19) into Eq. (16) to obtain the generating function accurate to order $O(p^{-2})$. We then use the Edgeworth approximation scheme to invert the Fourier transform [as discussed after Eq. (16)] and therefore obtain $P_\Sigma(X, Y)$.

$$P_{\text{LDOS}}(x) = \int_{\omega - \frac{1}{2\pi x}}^{\omega + \frac{1}{2\pi x}} dX \left[\frac{\pi[Y_+^2 + (X - \omega)^2]^2}{Y_+^2 - (X - \omega)^2} P_\Sigma(X, Y_+(x, X)) - \frac{\pi[Y_-^2 + (X - \omega)^2]^2}{Y_-^2 - (X - \omega)^2} P_\Sigma(X, Y_-(x, X)) \right], \quad (21)$$

where $Y_\pm(x, X) = \frac{1 \pm \sqrt{1 - 4\pi^2 x^2 (\omega - X)^2}}{2\pi x}$ and the integration range ensures that $Y_\pm(x, X)$ is real. The expression in Eq. (21) is the most general form for the LDOS distribution, and it can be evaluated numerically by first computing an analytical expression for the distribution $P_\Sigma(X, Y_-(x, X))$ using the Edgeworth approach, as discussed in the previous section. One thus obtains the solid black line in Fig. 2, which demonstrates good agreement with numerics, especially in the tail region where x is large.

B. Asymptotic behavior

We see from Fig. 2 that there appears to be a power-law behavior in the tail region of the curve $P_{\text{LDOS}}(x)$. We can confirm this and extract the power-law exponent by making a large- x approximation of the integral in Eq. (21) using Laplace's method. For large x , one finds that the value of X at which the integrand in Eq. (21) is maximum is $X^* = \omega + O(x^{-2})$. This is to be expected since, in order for the right-hand side of Eq. (20) to be large, we must have $X \approx \omega$ and small Y . One thus arrives at (see Appendix B for details of the calculation)

$$P_{\text{LDOS}}(x) \approx \frac{1}{(\sqrt{\pi x})^3} P_\Sigma\left(X = \omega, Y = \frac{1}{\pi x}\right). \quad (22)$$

We therefore see that the distribution of the LDOS has a power-law tail with exponent 3 as long as $P_\Sigma(X = \omega, Y = \frac{1}{\pi x})$ tends to a constant for large x . We expect this to be the case as long as the distribution of W_{ij}^2 (and consequently Y_i) has a nonzero and finite value of its pdf at small argument. We emphasize that the value of 3 here is not a result of our approximation scheme for $P_\Sigma(X, Y)$ and is instead a generic property of the weighted ER graph ensembles. This is evident

In summary, by truncating the approximation scheme of the moments at a certain order in $1/p$, we can close the set of equations, and we can express the higher moments of G_{ii} in terms of the mean, for which we obtain a closed-form expression. As a result, we can evaluate the probability distribution of the self-energy to a specified accuracy.

V. LDOS DISTRIBUTION AND POWER-LAW TAIL

A. General LDOS distribution expression

Now that we have approximated the distribution of the self-energy, we may evaluate the object that is most central to our considerations, the distribution of the LDOS. Since the LDOS can be written in terms of the self-energy using Eqs. (9) and (10),

$$\rho_i = \frac{1}{\pi} \frac{Y_i}{(X_i - \omega)^2 + Y_i^2}, \quad (20)$$

we can write the LDOS distribution $P_{\text{LDOS}}(\rho_i = x)$ in terms of the distribution of the self-energy as follows. Using a simple change of variables, we have

from the generality of the reasoning in Appendix B, where we do not appeal to a specific form for $P_\Sigma(X, Y)$.

A symmetry that applies to the LDOS distribution in many cases is [58]

$$P_{\text{LDOS}}(x) = (x/\rho)^{-3} P_{\text{LDOS}}(\rho^2 x^{-1}). \quad (23)$$

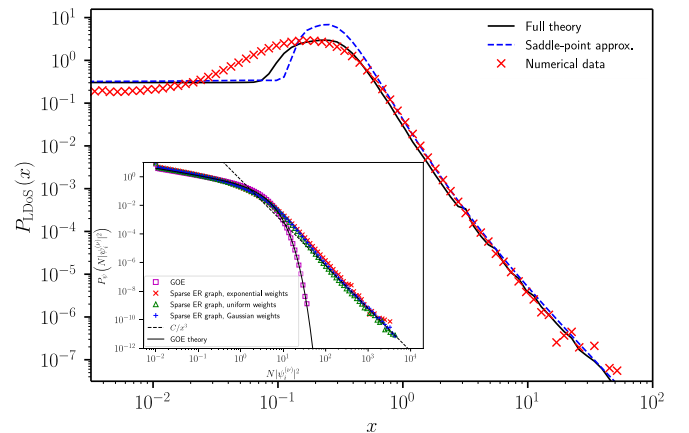


FIG. 2. Local density of states distribution for the ER graph adjacency matrix with uniformly distributed $u_{ij} = W_{ij}^2$ at $\omega = 0.4$. Numerics are for the giant component (we compare with results from the full graph in Appendix F). Comparison is made with Eq. (21) (solid black line) and the sum of Eqs. (22) and (B12) (dashed blue line). (Inset) The distribution $P_\psi(y)$ for the three edge weight distributions described in Sec. II. The theory line $P_\psi(y) = B/y^3$ is shown as a dashed line (with the constant B numerically fitted). Parameters: $p = 3.4$, $\omega = 0.4$, $N = 10^4$, and results are averaged over 1000 trials.

This symmetry arises from an analysis of the nonlinear sigma model [5,24], and has been verified to apply in a broad range of circumstances [2,31,43]. In the present case, the symmetry would imply that $P_{\text{LDOS}}(x) \sim x^0$ for small x . Indeed, we can show that this symmetry is satisfied asymptotically for small x by once again performing a saddle-point approximation. This procedure is also outlined in Appendix B. We discuss how the symmetry in Eq. (23) manifests in the eigenvector statistics in Sec. VI.

The sum of the two asymptotic predictions (small and large x) is compared against both the numerical integration of Eq. (21) and numerically generated random matrices in Fig. 2. We indeed find the same power-law exponent in the tail for a range of random matrix ensembles, as should be the case, given the generality of our reasoning. We note that the exponent of 3 is close to the values found numerically (ranging between 2.2 and 3.2) in Ref. [43] for weighted sparse ER graphs. However, we note that the weight distribution in this work did not satisfy $0 < \lim_{u \rightarrow 0} P(u) < \infty$ (where, again, $u_{ij} = W_{ij}^2$), which could account for the small discrepancy.

C. Onset of power-law behavior and crossover to ergodicity

We can also obtain a rule-of-thumb estimate for the value of x at which the onset of the power-law behavior occurs. Expanding Eq. (22) in x^{-1} , we obtain

$$P_{\text{LDOS}}(x) \approx \frac{P_{\Sigma}(\omega, 0)}{(\sqrt{\pi}x)^3} \left[1 + \frac{1}{\pi^2 x} \frac{\omega p^2}{4\beta_4^2 \rho \text{Re}\langle G \rangle |\langle G \rangle|^4} + \dots \right]. \quad (24)$$

We enter the power-law regime when the second term in the square brackets becomes negligible in comparison to the first. This crossover occurs when $x \approx p^2/[2\pi^2 \beta_4^2 \rho(\omega)]$, where we use that $\text{Re}\langle G \rangle \approx \omega/2$ and $|\langle G \rangle| \approx 1$ to leading order in $1/p$. That is, the power-law tail becomes suppressed as the average degree of the ER graph is increased, also in agreement with the numerical findings of Ref. [43] that found a similar gradual crossover from nonergodic to ergodic behavior. We verify this in Appendix F. However, it should be noted that nonergodic behavior was also found numerically in weighted ER graph ensembles for $p \sim N^{1-\gamma/2}$ for values of $0 < \gamma < 2$ more generally [32].

D. Breakdown of the x^{-3} behavior

While the x^{-3} behavior in the LDOS distribution is common across the weighted ER graph ensembles that we study in this work, it is not a completely general phenomenon. In particular, as can be seen from Eq. (22), the asymptotic behavior of $P_{\text{LDOS}}(x)$ for large x is related to the asymptotic behavior of $P_{\Sigma}(X, Y)$ for small Y . That is, we only obtain a $1/x^3$ tail of the LDOS distribution if the self-energy distribution tends to a constant at small values its argument. It is for this reason that we chose weights whose squares $u_{ij} = W_{ij}^2$ had pdf $P(u)$ with nonzero and finite value at zero argument.

Related to this, we noted earlier that it was necessary to go to second order in $1/p$ in our approximation scheme to understand the tail of the distribution $P_{\text{LDOS}}(x)$. The reason for this is that if we ignore the $O(1/p)$ terms in Eq. (18) and the other contribution to the $O(1/p^2)$ in Eq. (16), we find that $\mathcal{F}_{\Sigma}$

is simply the generating function of two linearly dependent Gaussian variables. That is, we could effectively write $\delta^{(X)} = z\text{Re}\langle G \rangle$ and $\delta^{(Y)} = z\text{Im}\langle G \rangle$, where z is a centered Gaussian random variable with variance β_4/p . The linear dependence of X and Y would mean that, according to Eq. (20), ρ_i would have a cutoff at $(\text{Re}\langle G \rangle + |\langle G \rangle|)/(2\omega\text{Im}\langle G \rangle)$, which we do not observe. Including the $1/p$ correction in Eq. (18), in particular, breaks the linear dependence of X and Y , and ultimately gives rise to the tail of $P_{\text{LDOS}}(x)$.

Another way in which the $1/x^3$ behavior can be suppressed is by moving ω to the edge of the continuous part of the eigenvalue spectrum, whereupon we observe a different behavior. As has been found previously in the context of Anderson localization [4,28,59–61], we find a power-law behavior of $P_{\text{LDOS}}(x) \sim x^{-3/2}$ close to the mobility edge. This is discussed further in Sec. VIII. This same power-law exponent is carried over to the localized phase, where the spectrum is no longer continuous but is instead pointlike, and a nontrivial dependence of the LDOS distribution on the regularizer ϵ emerges [27,38].

Finally, one also obtains different asymptotic behavior in the special case where $\omega = 0$ precisely. This is because the local resolvent in this case is purely imaginary. As such, we may write simply $\rho_i = (\pi Y_i)^{-1}$. If we define $P_Y(Y)$ to be the distribution of Y in this case, using that $X = 0$, one thus obtains

$$P_{\text{LDOS}}(x; \omega = 0) = \frac{1}{\pi x^2} P_Y\left(\frac{1}{\pi x}\right). \quad (25)$$

We discuss why this means that states with $\omega = 0$ are localized in Sec. VIII. Such localized states have been identified in complex network adjacency matrices previously in Ref. [36].

VI. SINGULARITY SPECTRUM

Given the relationship between the LDOS and the eigenvectors shown in Eq. (7), we surmise that the eigenvector distribution itself must also have a power-law tail with the same exponent [35,43]. Indeed, this is verified in the inset of Fig. 2. Our characterization of the $1/x^3$ tail therefore allows us to make some deductions about the singularity spectrum $f(\alpha)$.

The power-law tail, which occurs for large values of $y = N|\psi_i^{(v)}|^2 = N^{1-\alpha_i^{(v)}}$, corresponds to small values of α . Substituting $P_{\psi}(y) = By^{-3}$ into Eq. (4), one obtains

$$f(\alpha) = 2\alpha - 1, \quad (26)$$

noting that the contribution from the constant B vanishes in the limit $N \rightarrow \infty$. To understand the behavior of the eigenvectors on smaller scales, previous studies have exploited the symmetry $f(1 + \alpha) = f(1 - \alpha) + \alpha$ [31,43]. This symmetry derives from the previously mentioned symmetry $P_{\text{LDOS}}(x) = (x/\rho)^{-3} P_{\text{LDOS}}(\rho^2 x^{-1})$ [see the discussion around Eq. (22)]. If the symmetry were to hold for the eigenvectors, we would expect $f(\alpha) = 2 - \alpha$ for $\alpha \gtrsim 1$ and thus

$$f(\alpha) \approx \begin{cases} 2\alpha - 1 & \text{for } \alpha \ll 1, \\ 2 - \alpha & \text{for } \alpha \gg 1. \end{cases} \quad (27)$$

With that being said, what we notice in numerics is behavior more akin to GOE statistics for small $y = N|\psi_i^{(v)}|^2$ (see the

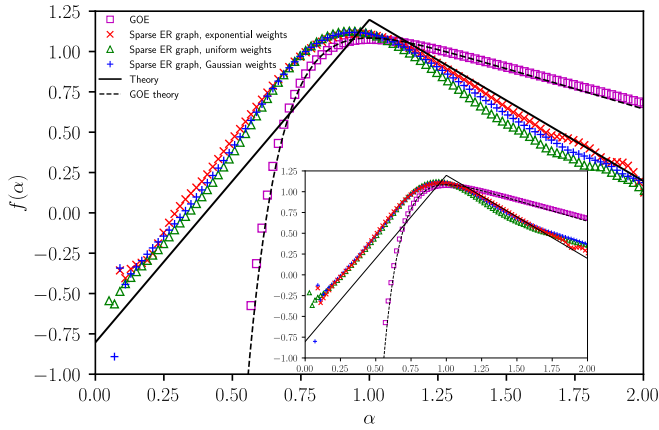


FIG. 3. Singularity spectrum for the three weight distributions mentioned in Sec. II at $\omega = 0.4$, with $p = 3.4$ and $N = 10^4$ averaged over 100 trials. GOE is also given for contrast. The solid black line is the result in Eq. (27). (Inset) The case of nonzero diagonal disorder with $W = 0.3$. The singularity spectrum remains unchanged when W (the amount of diagonal disorder) is sufficiently small.

inset of Fig. 2). However, as was pointed out in Ref. [31], this is because rapid GOE and Porter-Thomas fluctuations dominate when $y = N|\psi_i^{(v)}|^2$ is small. To obtain the slowly varying envelope that characterizes multifractal eigenstates, one must separate the fast fluctuation via a deconvolution procedure. That is, one assumes $\log y = \log y_{\text{GOE}} + \log y_{\text{env}}$, where y_{GOE} obeys Porter-Thomas statistics. Following the procedure detailed in Ref. [31], we may therefore extract the distribution $P_{\text{env}}(y_{\text{env}})$ numerically, which yields the singularity spectrum via Eq. (4). Indeed, after the deconvolution procedure, the singularity spectrum seems to obey the symmetry $f(1 + \alpha) = f(1 - \alpha) + \alpha$ and therefore Eq. (27), as is shown in Fig. 3.

VII. WEAK MULTIFRACTALITY AND NONERGODICITY

Given the behavior of the singularity spectrum, we may now extract the fractal dimensions using Eq. (6), and therefore gain insight into the ergodicity (or lack thereof) of the eigenvectors in the extended phase.

The precise behavior of $f(\alpha)$ for $\alpha > 1$ turns out to be unimportant for constructing $\tau(q)$ using the Legendre transform. All that matters is that the behavior $f(\alpha) = 2\alpha - 1$ stops at $\alpha \approx 1$, where the function $f(\alpha)$ is maximum, and beyond which $f(\alpha)$ decreases. We thus have

$$D(q) = \begin{cases} 1 & \text{for } 0 \leq q \leq 2, \\ 1/(q-1) & \text{for } 2 \leq q. \end{cases} \quad (28)$$

The degenerate nature of the fractal dimensions here is the signature of multifractality. The scaling dimensions $\tau(q) = (1-q)D(q)$ are plotted in Fig. 4. We see, however, that the multifractality here manifests only in the higher moments of the eigenvectors. This is then “weak” multifractality, in the sense used by Evers and Mirlin [2], in that the behavior does not deviate much from the typical “metallic” kind for a certain range of q . This is in contrast with stronger forms of multifractality, as seen in the modified Rosenzweig-Porter models [38,41,42], the Anderson model on the Bethe lattice [7,27], and at the Anderson critical point in finite dimensions

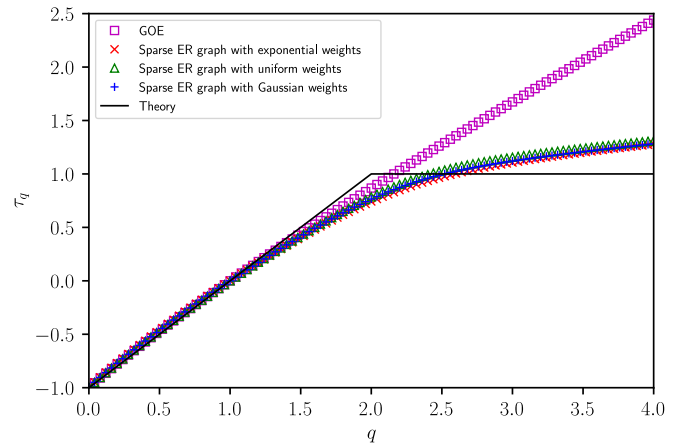


FIG. 4. The scaling exponent $\tau_q = D(q)(q-1)$ for the same ensembles and parameters as in Fig. 3. The solid line is given by the result in Eq. (28).

[2,3]. As was also noted in Ref. [43], this weaker kind of behavior is similar to that of generalized β ensembles [34,62], for which the fractal dimension differs from $D(q) = 1$ only for $q > 1$. Intriguingly, the behavior of $f(\alpha)$ that we obtain here is the same for $\alpha < 1$ as in Ref. [63], where a deterministic tridiagonal Hamiltonian is considered (obtained by taking $\beta \rightarrow \infty$), which exhibits power-law decaying eigenstates.

While sometimes nonergodicity and multifractality are considered as almost synonymous, nonergodicity can be considered somewhat distinct from the (multi)fractality of the eigenvectors and states. Nonergodicity has been defined in several ways in different contexts, leading to various (not always commensurate) criteria for its presence [10]. Criteria for nonergodicity include, for example, a deviation from Wigner-Dyson eigenvalue statistics [10], violation of the eigenstate thermalization hypothesis [64,65], a lack of convergence of $\sum_i |\psi_i^{(v)}|^{2q}$ to the ensemble average value [31], or indeed a deviation of the fractal dimensions from the values expected of fully extended states.

We have already demonstrated that the fractal dimensions deviate from $D_q = 1$ when $q > 2$. However, two of the scaling dimensions D_1 and D_2 have particular physical relevance for quantifying nonergodicity [10,27,66,67], as we discuss below. We here use D_1 and D_2 to understand in what sense the eigenvectors of the random matrix ensembles under present consideration are nonergodic.

A widely used measure of ergodicity and localization is the inverse participation ratio (IPR) I_2 . The reason for this is that the IPR is related to the infinite time average of the survival probability of a quantum state [10,66], and so $1/I_2 = N^{D_2}$ is an obvious quantifier of the number of sites occupied by that state. One also has that $I_2 \rightarrow 0$ for extended states, and $I_2 \sim N^0$ for localized states, and hence a nonergodic extended state would be one for which $I_2 \rightarrow 0$, but $1/(NI_2) \rightarrow 0$ as $N \rightarrow \infty$, i.e., one in which only a subextensive number of sites are accessed. Here, due to the power-law behavior of $P_\psi(y)$ at large y , we find that (see Appendix E for details)

$$I_2 = \int_0^N dy y^2 N^{-1} P_\psi(y) \sim \frac{\ln N}{N}, \quad (29)$$

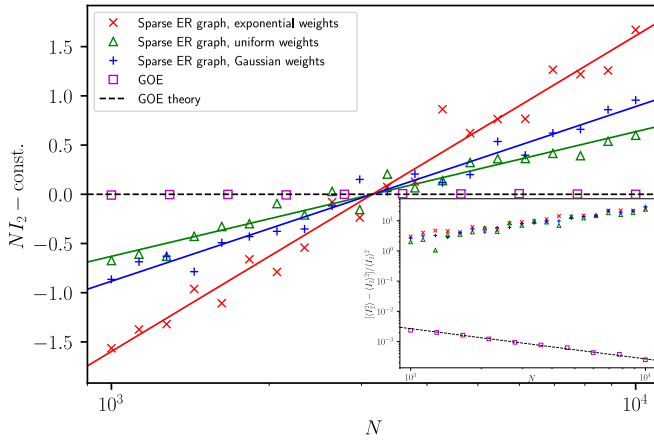


FIG. 5. Logarithmic scaling of the IPR with N for system parameters and ensembles given in Fig. 3. We have shifted the data so that each set of points passes through the origin. We show lines of best fit as guide to the eye. Since the constant B varies between ensembles [where $P_\psi(y) = B/y^3$ asymptotically], the gradients of the best fit lines in the above plot also vary. (Inset) The relative fluctuations of the IPR $[(I_2^2) - (I_2)^2]/(I_2)^2$ grow with N while they vanish for the GOE.

meaning that states indeed occupy a vanishing fraction of the sites [with $1/(NI_2) \rightarrow 0$]. This behavior is verified in Fig. 5. However, we also have $D_2 \approx 1 - \ln \ln N / \ln N \rightarrow 1$ in the thermodynamic limit. So, we have a somewhat special situation. One can say that, by the above criterion, we have nonergodicity, but only a weak kind in which the relevant scaling dimension tends to unity for $N \rightarrow \infty$ (albeit very slowly). We further verify in the inset of Fig. 5 that the relative fluctuations (another indicator of nonergodicity [31]) of the IPR are nonvanishing, and in fact grow with N , while in contrast the IPR fluctuations decrease with N for the fully extended states of the GOE.

Let us now turn to another common quantifier of nonergodicity, D_1 , which is related to the Shannon entropy via $S = D_1 \ln N$. Following Kravtsov *et al.* [28,68], the support set, of which D_1 is the scaling dimension, can be defined as the least number of components required to satisfy the normalization constraint up to some prescribed accuracy ϵ (see Appendix C for further details). One can show that the support set of the eigenvectors is $M_\epsilon = A_\epsilon N^{D_1}$. The criterion in Ref. [28] regards eigenvectors with $M_\epsilon \sim N^0$ as localized and those with $M_\epsilon \rightarrow \infty$ as extended (as $N \rightarrow \infty$), but if $M_\epsilon/N \rightarrow 0$ then the state is nonergodic. One immediately sees that if $0 < D_1 < 1$, then the state is nonergodic and extended by this definition.

Since $D_1 = 1$ in the present case [see Eq. (28)], the support set is extensive (this was also pointed out in the context of sparse random graphs in Ref. [43]). We can confirm this by computing the support set explicitly according to the prescription of Ref. [28], which we do in Appendix C. We find that the eigenvector support set is indeed extensive $M_\epsilon \approx A_\epsilon N$, where $A_\epsilon \approx (1 - \epsilon)^2/(2B)$, where B is same constant that appears in the power law $P_\psi(y) = B/y^3$. This is in contrast to the nonergodic extended states in models such as the Lévy-Rosenzweig-Porter ensemble [38] or the Anderson model

on the Bethe lattice [27,28], where $D_1 < 1$. We also note that the dimension D_1 is related to the peak of the LDOS distribution via $e^{(\ln \rho_i)} \sim N^{D_1-1}$ [27]. The peak of the LDOS distribution in our case is smaller than the mean, but not parametrically so, and this is expressed accordingly by the fractal dimension $D_1 = 1$.

In summary, although according to the criteria involving I_2 and its fluctuations the eigenvectors here are in some sense nonergodic, and although the scaling dimensions for $q > 2$ deviate from $D_q = 1$, this is a weak effect in comparison to some other random matrix ensembles.

VIII. SPECTRAL EDGE, LOCALIZED PHASE, AND VARYING DIAGONAL ELEMENTS

Having identified and understood the nature of the nonergodic extended phase in the weighted ER graph ensembles, we examine what happens at the edge of the continuous part of the eigenvalue spectrum, where this behavior terminates, and we see the onset of $I_2 \sim N^0$ (see Fig. 1). As has been shown in the context of sparse graphs [53,69,70], it is local network “defects” (i.e., well-connected hubs or heavily weighted edges) that give rise to these eigenvalues outside the continuous part of the eigenvalue spectrum. One can indeed show that the corresponding eigenvectors are exponentially localized about such defects.

As discussed in Appendix D, we demonstrate that at the spectral edge the LDOS distribution no longer obeys the power law in Eq. (22). Instead, a different power-law behavior emerges, which is characteristic of the transition to a localized phase, and has a power-law exponent of $\frac{3}{2}$, as has also been found previously in the context of Anderson localization [4,28,59–61]. This same power-law behavior is carried over into the localized phase, but a dependence on the regularizer emerges [27,38].

On the subject of localization, we may also ask what happens when we include nonzero diagonal elements J_{ii} , for instance drawn from a uniform distribution on the interval $[-W, W]$, as is common in the classic Anderson tight-binding model [2,7,71]. For values of ω and W such that $|\omega \pm W| < \omega_c$, where $\omega_c = 2 + O(p^{-1})$ is the spectral edge [23,52,55], we expect that the power-law tail of the LDOS should be preserved.

The reason for this can be seen by the following simple argument. Suppose we consider the subset of sites with $J_{ii} = E$. The LDOS statistics of these sites would simply be given by the LDOS statistics of the model without disorder at $\omega - E$. In the tail region, we would thus have

$$P_{\text{LDOS}}(x; \omega, E) = P_{\text{LDOS}}(x; \omega - E, 0) \approx \frac{A(\omega - E)}{x^3}, \quad (30)$$

with $A(\cdot)$ being a function that is independent of x . Therefore, integrating over the possible values of E , we obtain the same power-law behavior in the tail. If, however, W is large enough so that $\omega - E$ can be outside the continuous spectrum, this reasoning no longer applies since the LDOS distribution no longer maintains the same tail behavior. We instead expect that the LDOS distribution will pick up a dependence on the regularizer, the $\frac{3}{2}$ power-law behavior will dominate over the quicker power law in the bulk, and states will become

localized in the usual way. That the eigenvector components maintain the same multifractal statistics in the extended phase is verified in the inset of Fig. 3.

Finally, as we remarked earlier in Sec. V D, the states with precisely $\omega = 0$ are also localized. One can see this from the asymptotic behavior $P_{\text{LDOS}}(x) \sim x^{-2}$ for large x , which connotes $P_\psi(y) \sim y^{-2}$ for large y . We can therefore calculate the IPR using Eq. (29), and we obtain $I_2 \sim O(N^0)$. As was discussed in Ref. [43], these states are localized on leaves of the network, and must be orthogonal to the states with $\omega \neq 0$ due to the Hermiticity of the model. So, given the extensive number of leaves, this offers a mechanistic explanation of the extended nonergodic states via “orthogonality-induced multifractality”.

IX. SUMMARY AND DISCUSSION

In this work, we have studied the multifractal properties of the eigenvectors of certain weighted Erdős-Rényi graphs. By a simple cavity approach [44,45], we were able to use a systematic approximation scheme to quantify the distribution of the local density of states. Upon finding that the tail of this distribution exhibited a power-law behavior with exponent 3, we were able also to understand the statistics of the eigenvectors.

We found that, as a consequence, the eigenvectors were weakly multifractal. We also showed that the inverse participation ratio of the eigenvectors scaled as $\ln N/N$, meaning that the eigenvectors were extended and weakly nonergodic. In other words, the number of sites occupied by the eigenvectors (as judged by the IPR) diverged in the $N \rightarrow \infty$ limit, but the occupied fraction of the total number of sites was vanishing, and scaled as $1/\ln N$. This is in contrast to stronger kinds of nonergodicity, where the occupied number of sites goes as $\sim N^{D_2}$ with $0 < D_2 < 1$, connoting a nontrivial fractional scaling dimension. Only in the higher moments of the eigenvector components do we see a deviation from fully ergodic behavior that affects the fractal dimensions.

We note that the peculiar weak nonergodicity and multifractality displayed here has a fingerprint in the eigenvalue correlations also. Random matrix models exhibiting strong multifractality are associated with “minibands” of N^{D_1} hybridized eigenvalues in the nonergodic extended phase [38,41], and a deviation from Wigner-Dyson statistics of the microscopic eigenvalue correlations. In contrast, we have $D_1 = 1$ in our case, and as was shown in Ref. [23], the microscopic eigenvalue statistics remain of the Wigner-Dyson type. However, there are instead long-range eigenvalue correlations, which are not present for ensembles like the GOE. Such long-range correlations still give rise to a nonzero level compressibility $0 < \chi < 1$, marking a halfway point between the usual Wigner-Dyson result $\chi \rightarrow 0$ and the value for Poisson statistics $\chi \approx 1$. This somewhat mirrors the aforementioned unusual “in-between” nature of the IPR found here, in that there is a marked deviation from the usual fully extended eigenvector behavior, but not one that is detected in the fractal dimension D_1 .

Physically, the origin of the multifractal extended states is a consequence of the combination of two model aspects: the sparsity of the ER graph and the weights being allowed to come arbitrarily close to zero. As was also observed in Ref. [43], this makes it possible for hidden isolated clusters to form (i.e., distinct well-connected clusters with only weak connections between them), even when we consider only the giant component of the ER graph. What is only elucidated through analysis, however, is the power-law exponent of 3 that we have found here in the case where $0 < \lim_{u \rightarrow 0} P(u) < \infty$. We anticipate that a range of exponents (different from 3) could be found for other weight distributions, and this remains a subject of our investigation. Conversely, we have verified numerically (see Appendix F) that the same exponent of 3 is seen in configuration model networks with the same weight distributions that were studied here.

A natural extension to the analysis performed here would be to examine the LDOS distribution of other random matrix ensembles using the cavity approach, particularly those exhibiting strong multifractality. The use of a scheme whereby one determines the self-energy distribution self-consistently, in the context of localization phenomena, is not unprecedented [38,46,50], but certainly there is opportunity for a more thorough evaluation of the LDOS distribution in many important random matrix models. However, we note that one advantage that was available to us here was the fact that the generating function of the self-energy was independent of N , and was amenable to a perturbative treatment, which allowed closure of the self-consistent equations. In models such as the Lévy-Rosenzweig-Porter ensemble [38,40], the N dependence and statistics of the self-energy are somewhat more complicated, and may require nontrivial amendments to the analytical procedure, but this remains an intriguing avenue for future inquiry.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: EVALUATION OF $\langle G \rangle$

Let us begin with the expression in Eq. (10), recalling the definitions $\delta_i^{(X)} = X_i - \text{Re}\langle G \rangle$, $\delta_i^{(Y)} = Y_i - \text{Im}\langle G \rangle$, and $\delta \Sigma_i = \delta_i^{(X)} + i\delta_i^{(Y)}$. Given that from Eq. (16) we expect higher cumulants to give rise to higher-order terms in $1/p$, we may perform a systematic approximation of $\langle G \rangle$ to the desired order in $1/p$.

Proceeding as such, we have

$$\begin{aligned} \langle G \rangle &= \left\langle \frac{1}{\omega - \langle G \rangle - \delta_i^{(X)} - i\delta_i^{(Y)}} \right\rangle \\ &\approx \frac{1}{\omega - \langle G \rangle} + \frac{\langle (\delta \Sigma)^2 \rangle}{(\omega - \langle G \rangle)^3} + \frac{\langle (\delta \Sigma)^3 \rangle}{(\omega - \langle G \rangle)^4} + \frac{\langle (\delta \Sigma)^4 \rangle}{(\omega - \langle G \rangle)^5} \\ &\approx \frac{1}{\omega - \langle G \rangle} + \frac{\beta_4 \langle G^2 \rangle}{p(\omega - \langle G \rangle)^3} + \frac{\beta_6 \langle G^3 \rangle}{p^2(\omega - \langle G \rangle)^4} \\ &\quad + \frac{3\beta_4^2 \langle G^2 \rangle^2}{p^2(\omega - \langle G \rangle)^5}, \end{aligned} \quad (\text{A1})$$

where we have kept only the terms that contribute to order $O(p^{-2})$. Now, we wish to write the expression for $\langle G \rangle$ in terms of only itself. To do this, we exploit the fact that our approximation scheme only requires us to approximate $\langle G^2 \rangle$ to order $O(p^{-1})$ and $\langle G^3 \rangle$ to order $O(p^0)$ in order for the expression in Eq. (A1) to be valid to order $O(p^{-2})$. This permits us to close the set of equations.

Therefore, for the third moment, we may simply use $\langle G^3 \rangle \approx \langle G \rangle^3$. For the second moment, we have instead

$$\begin{aligned} \langle G^2 \rangle &= \left\langle \left[\frac{1}{\omega - \langle G \rangle - \delta_i^{(X)} - i\delta_i^{(Y)}} \right]^2 \right\rangle \\ &\approx \left[\frac{1}{\omega - \langle G \rangle} \right]^2 + \frac{2\beta_4 \langle G^2 \rangle}{p(\omega - \langle G \rangle)^3} + \frac{\beta_4 \langle G^2 \rangle}{p(\omega - \langle G \rangle)^4} \\ &\approx \langle G \rangle^2 + \frac{\beta_4 \langle G^2 \rangle}{p(\omega - \langle G \rangle)^4} \approx \langle G \rangle^2 + \frac{\beta_4 \langle G \rangle^2}{p(\omega - \langle G \rangle)^4}, \end{aligned} \quad (\text{A2})$$

where we have used the expression in Eq. (A1) [up to order $O(p^{-1})$], and in the last line, we reinserted the expression for $\langle G^2 \rangle$ back into the right-hand side, and ignored $O(p^{-2})$ terms. Inserting these into Eq. (A1), we obtain

$$\begin{aligned} \langle G \rangle &\approx \frac{1}{\omega - \langle G \rangle} + \frac{\beta_4 \langle G \rangle^2}{p(\omega - \langle G \rangle)^3} + \frac{\beta_4^2 \langle G \rangle^2}{p^2(\omega - \langle G \rangle)^7} \\ &\quad + \frac{\beta_6 \langle G \rangle^3}{p^2(\omega - \langle G \rangle)^4} + \frac{3\beta_4^2 \langle G \rangle^4}{p^2(\omega - \langle G \rangle)^5}. \end{aligned} \quad (\text{A3})$$

We have thus succeeded in finding an approximate expression for $\langle G \rangle$ in terms of only itself. We can simplify the expression somewhat by reinserting the right-hand side back into itself, ignoring terms that contribute to $O(p^{-3})$. To first order in p^{-1} , we obtain $\langle G \rangle \approx (\omega - \langle G \rangle)^{-1} + \beta_4 p^{-1} \langle G \rangle^2 (\omega - \langle G \rangle)^{-3} \approx (\omega - \langle G \rangle)^{-1} + \beta_4 p^{-1} \langle G \rangle^5$, which then gives us an approximation for $(\omega - \langle G \rangle)^{-1}$. Inserting this back into Eq. (A3), we finally arrive at Eq. (17).

APPENDIX B: LAPLACE'S METHOD APPLIED TO EQ. (21)

We wish now to approximate Eq. (21) for large and small x using Laplace's method. We begin with the large- x case. We define

$$P_\Sigma(X, Y) = \exp[F(X, Y)], \quad (\text{B1})$$

and we keep our reasoning as generally applicable to different forms of $F(X, Y)$ as possible. We focus on the two contributions to the integrand in Eq. (21) individually. For the

integrand involving Y_+ , the maximum occurs at $dL/dX = 0$ where

$$\begin{aligned} L(X, Y_+(X)) &= F(X, Y_+(X)) \\ &\quad + \log \left[\frac{[Y_+^2 + (X - \omega)^2]^2}{Y_+^2 - (X - \omega)^2} \right]. \end{aligned} \quad (\text{B2})$$

One obtains

$$\begin{aligned} \frac{dL}{dX} &= \frac{\partial F}{\partial X} + \frac{\partial F}{\partial Y_+} \frac{\partial Y_+}{\partial X} + \frac{2Y_+[Y_+^2 - 3(X - \omega)^2]}{Y_+^4 - (X - \omega)^4} \frac{dY_+}{dX} \\ &\quad + \frac{2(X - \omega)[3Y_+^2 - (X - \omega)^2]}{Y_+^4 - (X - \omega)^4}, \end{aligned} \quad (\text{B3})$$

where we also have

$$\begin{aligned} Y_+ &= \frac{1 + \sqrt{1 - 4\pi^2 x^2 (\omega - X)^2}}{2\pi x}, \\ \frac{dY_+}{dX} &= \frac{2\pi x(\omega - X)}{\sqrt{1 - 4\pi^2 x^2 (\omega - X)^2}}. \end{aligned} \quad (\text{B4})$$

We note that the integration range in Eq. (21) ensures the reality of $Y_+(X)$. So, necessarily, $(X - \omega)^2 < 1/(4x^2)$ and therefore $Y_+ \sim 1/x$ for large x . This means that we have

$$\begin{aligned} \frac{dL}{dX} &\approx \frac{\partial F}{\partial X} + a \frac{\partial F}{\partial Y_+} x(X - \omega) \\ &\quad + b(X - \omega)x^2 + cx^4(X - \omega)^3, \end{aligned} \quad (\text{B5})$$

where a , b , and c are constants of order $O(x^0)$ and $O[(X - \omega)^0]$. Let us now attempt to find the order of magnitude of $(X - \omega)$ in terms of x by using $dL/dX = 0$. Assuming that $\partial F/\partial X \sim \partial F/\partial Y_+ \sim x^0$, we see that we must have $(X - \omega) \sim 1/x^2$ in order for the solution of $dL/dX = 0$ to be consistent. This means that the maximum of the integrand for large x is given by

$$Y_+(X^*) = \frac{1}{\pi x} + O(x^{-3}), \quad X^* = \omega + O(x^{-2}). \quad (\text{B6})$$

One can verify the assumption that indeed $\partial F/\partial X \sim \partial F/\partial Y_+ \sim x^0$ at the maximum *a posteriori* for the approximation for $P_\Sigma(X, Y)$ discussed in Sec. IV. By similar reasoning, we find that the contribution to the integrand in Eq. (21) involving Y_- must be subleading for large x since one has instead $Y_- \sim x^{-3}$ with $(X - \omega) \sim 1/x^2$.

Now that we understand where the integrand of Eq. (21) is maximum, we can perform Laplace's approximation. That is, we expand the logarithm of the integrand in Eq. (21) around the maximum up to second order. This provides a Gaussian approximation for the integral, which we then evaluate. We have

$$\begin{aligned} \frac{d^2 F(X, Y_+(X))}{dX^2} &= \frac{\partial^2 F}{\partial X^2} + 2 \frac{\partial^2 F}{\partial Y \partial X} \frac{dY_+}{dX} + \frac{\partial^2 F}{\partial Y^2} \left(\frac{dY_+}{dX} \right)^2 \\ &\quad + \frac{\partial F}{\partial Y} \frac{d^2 Y_+}{dX^2}, \end{aligned} \quad (\text{B7})$$

where in addition to Eq. (B4) we have

$$\frac{d^2 Y_+}{dX^2} = \frac{2\pi x}{[1 - 4x^2\pi^2(\omega - X)^2]^{3/2}}. \quad (\text{B8})$$

Assuming that the power series for F has no terms proportional to Y^{-r} for $r > 0$, we see that the largest power of x that does not vanish when we set $X = \omega$ in Eq. (B7) comes from the term proportional to $\frac{d^2 Y_+}{dX^2}$. This is proportional to x .

However, if we examine the second derivative of the other term in Eq. (B2), we obtain at $X = \omega$

$$\frac{d^2}{dX^2} \log \left[\frac{[Y_+^2 + (X - \omega)^2]^2}{Y_+^2 - (X - \omega)^2} \right] \Big|_{X=\omega} = 2\pi^2 x^2. \quad (\text{B9})$$

This therefore provides the dominant term in L'' for large x , which is proportional to x^2 , and this ought to be the case generically for any $F(X, Y)$ that does not have poles at $Y = 0$. Simple application of Laplace's method then yields Eq. (22).

One notes that this reasoning breaks down if the maximum in Eq. (B6) is somehow inaccessible. This is the case, for example, when $\beta_4/p = O(1/N)$ or when X and Y are otherwise linearly dependent, as was discussed in Sec. VD.

It remains for us to understand the behavior for small x . Again, since $P_{\text{LDOS}}(x)$ is small for small x in comparison to the peak (i.e., small values of x are “rare events”), we expect the integrand to be dominated by contributions near the maximum value, and so Laplace's method again ought to yield the asymptotic behavior. We focus first on the second contribution to the integrand in Eq. (21), which involves Y_- .

From Eq. (B3), and using that for small x we have $Y_- \approx \pi x(\omega - X)^2$ and $dY_-/dX \approx 2\pi x(X - \omega)$ (assuming $\omega - X \sim x^0$), we have that

$$\frac{dL}{dX} \approx \frac{\partial F}{\partial X} + \frac{2}{X - \omega}. \quad (\text{B10})$$

Setting $\frac{dL}{dX} = 0$ in this expression yields the maximum X^* , at which indeed $\omega - X \sim x^0$. We may now evaluate the second derivative at this value, obtaining for small x

$$\frac{d^2 L}{dx^2} \approx \frac{\partial^2 F}{\partial X^2} + \frac{2}{(\omega - X)^2} \sim x^0. \quad (\text{B11})$$

If we instead consider the contribution involving Y_+ , one finds $Y_+ \sim 1/x$ but $dY_+/dX \sim x(\omega - X)$. We thus have $X - \omega \sim 1/x$. In this case, the contribution to $P_{\text{LDOS}}(x)$ is then dominated by the factor $P_\Sigma(X, Y_+)$, which is exponentially small. The dominant contribution thus comes from the term involving Y_- . We therefore finally obtain via Laplace's method for $x \rightarrow 0$

$$P_{\text{LDOS}}(x) \approx (\omega - X^*)^2 \sqrt{\frac{2\pi^3}{L''(X^*)}} P_\Sigma[X^*, \pi x(\omega - X^*)^2], \quad (\text{B12})$$

which is constant in x .

APPENDIX C: EIGENVECTOR SUPPORT SET

Consider the set of eigenvector components for a state v , $\psi_i^{(v)}$. We order them in terms of magnitude such that

$$|\psi_1^{(v)}|^2 \geq |\psi_2^{(v)}|^2 \geq |\psi_3^{(v)}|^2 \geq \dots \geq |\psi_N^{(v)}|^2. \quad (\text{C1})$$

The support set [28] is defined as the least number M_ϵ of these components that are necessary to satisfy the normalization

constraint up to some prescribed accuracy ϵ . That is,

$$\sum_{i=1}^{M_\epsilon} |\psi_i^{(v)}|^2 \leq 1 - \epsilon < \sum_{i=1}^{M_\epsilon+1} |\psi_i^{(v)}|^2. \quad (\text{C2})$$

Taking the limit $N \rightarrow \infty$ in Eq. (C2), we have

$$\begin{aligned} N \int_{y_\epsilon}^N dy P_\psi(y) &= M_\epsilon, \\ \int_{y_\epsilon}^N dy y P_\psi(y) &= 1 - \epsilon. \end{aligned} \quad (\text{C3})$$

Let us compute M_ϵ for the eigenvectors of the random matrix ensembles that we have studied. As long as y_ϵ is in the power-law tail of the eigenvector distribution (as it should be for small ϵ), we find

$$\frac{B}{2y_\epsilon^2} = \frac{M_\epsilon}{N}, \quad \frac{B}{y_\epsilon} = 1 - \epsilon. \quad (\text{C4})$$

Eliminating y_ϵ we therefore find for the support set

$$M_\epsilon = \frac{N(1 - \epsilon)^2}{2B}. \quad (\text{C5})$$

Crucially, we must understand the scaling of B with N to obtain the scaling of the support set with N . We can obtain an estimate the constant B as follows. We assume that the distribution $P_\psi(y)$ follows a scaled Porter-Thomas form up to a value $y = y_c$, beyond which we have the power-law behavior $P_\psi(y) = B/y^3$. By requiring continuity and normalization of both the distribution $P_\psi(y)$ and the eigenvectors, we arrive at the conclusion that $B \sim N^0$ (see Appendix E). Therefore, we find that the support set $M_\epsilon \sim N$ is extensive. We note that in reality the constant B varies between weight distributions and is dependent on other system parameters such as p and ω .

APPENDIX D: BEHAVIOR CLOSE TO THE SPECTRAL EDGE AND BEYOND

Following Ref. [28], let us suppose that the imaginary part of the self-energy is small in comparison to the real part. This is obviously the case at the edge of the spectrum where the eigenvalue density approaches zero. We therefore approximate [cf. Eq. (20)]

$$\text{Im} G_{ii} \approx \frac{Y_i}{(\omega - X_i)^2}. \quad (\text{D1})$$

Let us briefly examine the relationship between Y_i and X_i . From Eq. (16), we see that, to leading order in p^{-1} , X and Y are linearly dependent since the fluctuations of G_{ii} go as p^{-1} (this is also discussed in Sec. VD). For this reason, we may write $Y = z \langle \text{Im} G \rangle$ and $X = z \langle \text{Re} G \rangle$, where z is distributed according to

$$P_z(z) = \frac{1}{\sqrt{2\pi\beta_4/p}} e^{-\frac{(z-1)^2}{2\beta_4/p}}. \quad (\text{D2})$$

We note that this approximate linear dependence is not used in the previous calculation of $P_{\text{LDOS}}(x)$ in Secs. IV and V. In fact, the slight fluctuations about this solution are crucial to obtain large values of x when $\langle Y \rangle$ is not small, as was discussed in Sec. VD.

With this recasting of X and Y in terms of the single variable z , and writing $\langle Y \rangle = \pi\rho$, we obtain

$$\begin{aligned} P(\text{Im}G_{ii} = x) &\approx \int dz \delta\left(x - \frac{\pi\rho z}{(\omega - \langle X \rangle z)^2}\right) \frac{1}{\sqrt{2\pi\beta_4/p}} e^{-\frac{(z-1)^2}{2\beta_4/p}} \\ &= \left| \frac{\sqrt{\pi\rho(\pi\rho + 4\langle X \rangle \omega x)} - \pi\rho}{2\langle X \rangle^2 x^2} - \frac{\omega\sqrt{\pi\rho}}{\langle X \rangle x \sqrt{\pi\rho + 4\langle X \rangle \omega x}} \right| \frac{\exp\left[-\frac{(z(x)-1)^2}{2\beta_4/p}\right]}{\sqrt{2\pi\beta_4/p}}, \\ z(x) &= \frac{\pi\rho + 2\langle X \rangle \omega x - \sqrt{\pi\rho(\pi\rho + 4\langle X \rangle \omega x)}}{2\langle X \rangle^2 x}. \end{aligned} \quad (\text{D3})$$

In the limit $x \rightarrow \infty$, we have

$$P(\text{Im}G_{ii} = x) \approx \frac{\sqrt{\pi\rho\omega}}{2(\langle X \rangle x)^{3/2}} \frac{\exp\left[-\frac{(\omega/\langle X \rangle - 1)^2}{2\beta_4/p}\right]}{\sqrt{2\pi\beta_4/p}}. \quad (\text{D4})$$

In reality, however, this behavior will only be exhibited asymptotically when we are exactly at the spectral edge.

We now show that this same power-law behavior is preserved in the pointlike part of the spectrum (i.e., where the eigenvectors are localized). In this region of the spectrum, the resolvent has simple poles at various locations E_ν , corresponding to the outlier eigenvalues generated by the hubs of the network. One can indeed show that these poles correspond to localized states [53], and we can also characterize the LDOS distribution in this region of the spectrum following Ref. [27]. For small ϵ we can write the following using the polelike nature of the local resolvent above (when $\omega \approx E_\nu$):

$$\text{Im}G_{ii}(\omega) = \frac{|\psi_i^{(\nu)}|^2 \epsilon}{(\omega - E_\nu)^2 + \epsilon^2}. \quad (\text{D5})$$

By a simple change of variables, one thus finds

$$P_{\text{LDOS}}(x) \propto \frac{\sqrt{\epsilon}}{x^{3/2}} \quad (\text{D6})$$

until a cutoff of $x \sim \frac{1}{\epsilon}$.

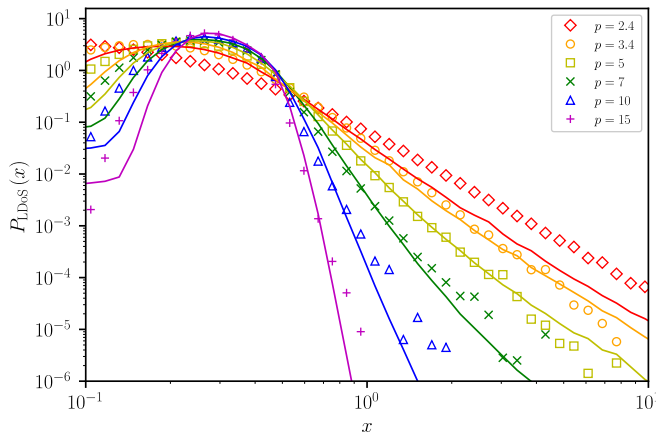


FIG. 6. LDOS distribution for various p . Here, $N = 10^4$ and simulations were averaged over 100 trials. A uniform distribution was used for the squared weights $u_{ij} = W_{ij}^2$.

APPENDIX E: EIGENVECTOR DISTRIBUTION

In order to understand clearly the N dependence of the IPR, we make an ansatz for the eigenvector distribution. Namely, given what we observe numerically in Fig. 2, we imagine that for small values of y , the eigenvector components exhibit Porter-Thomas statistics, whereas for larger values, we observe the power-law tail. We therefore employ a stepwise ansatz

$$P_\psi(y) = \frac{B_1}{\sqrt{2\pi}y} e^{-y/2} \Theta(y_c - y) + \frac{B_2}{y^3} \Theta(y - y_c). \quad (\text{E1})$$

We impose three conditions on this ansatz: (1) continuity, (2) distributional normalization, and (3) eigenvector normalization. More specifically, we require

$$\begin{aligned} \frac{B_1}{\sqrt{2\pi}y_c} e^{-y_c/2} &= \frac{B_2}{y_c^3}, \\ \int_0^N dy P_\psi(y) &= \int_0^{y_c} dy \frac{B_1}{\sqrt{2\pi}y} e^{-y/2} + \int_{y_c}^N dy \frac{B_2}{y^3} = 1, \\ \int_0^N dy y P_\psi(y) &= \int_0^{y_c} dy \frac{B_1 \sqrt{y}}{\sqrt{2\pi}} e^{-y/2} + \int_{y_c}^N dy \frac{B_2}{y^2} = 1, \end{aligned} \quad (\text{E2})$$

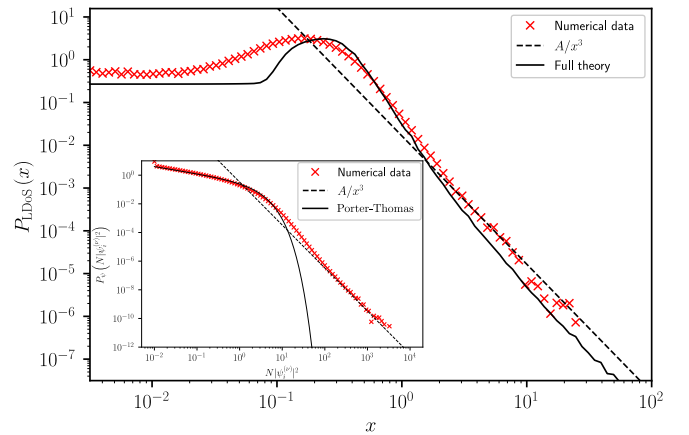


FIG. 7. LDOS distribution (main) and eigenvector distribution (inset) for a weighted configuration model network with Poissonian degree distribution and $p = 3.4$. Here, $N = 10^4$ and simulations were averaged over 100 trials. A uniform distribution was used for the squared weights $u_{ij} = W_{ij}^2$.

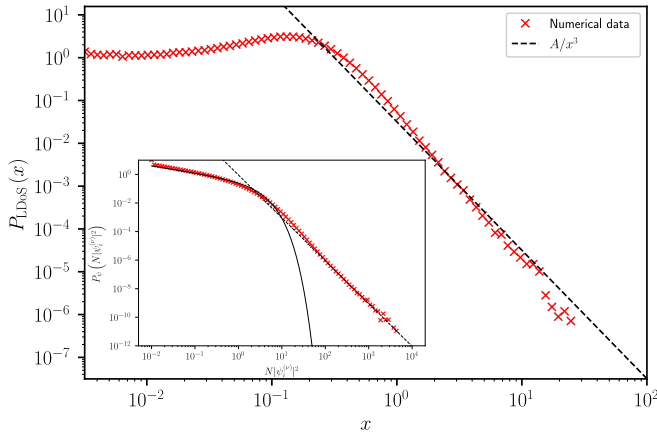


FIG. 8. LDOS distribution (main) and eigenvector distribution (inset) for a weighted configuration model network with degrees uniformly chosen from $\{1, 2, 3, 4, 5\}$. Here, $N = 10^4$ and simulations were averaged over 100 trials. A uniform distribution was used for the squared weights $u_{ij} = W_{ij}^2$.

where the last condition uses that $\sum_i |\psi_i^{(v)}|^2 = N^{-1} \sum_i y_i^{(v)} = 1$. Performing the integrations, we obtain

$$B_1 \operatorname{erf}\left(\sqrt{\frac{y_c}{2}}\right) + \frac{B_2}{2} \left[\frac{1}{y_c^2} - \frac{1}{N^2} \right] = 1, \\ B_1 \left[\operatorname{erf}\left(\sqrt{\frac{y_c}{2}}\right) - \sqrt{\frac{2y_c}{\pi}} e^{-y_c/2} \right] + B_2 \left[\frac{1}{y_c} - \frac{1}{N} \right] = 1. \quad (\text{E3})$$

After some elementary algebra, we find

$$\frac{\operatorname{erf}\left(\sqrt{\frac{y_c}{2}}\right) + \frac{y_c^{5/2}}{2\sqrt{2\pi}} e^{-y_c/2} \left[\frac{1}{y_c^2} - \frac{1}{N^2} \right]}{\operatorname{erf}\left(\sqrt{\frac{y_c}{2}}\right) - \sqrt{\frac{2y_c}{\pi}} e^{-y_c/2} + \frac{y_c^{5/2}}{\sqrt{2\pi}} e^{-y_c/2} \left[\frac{1}{y_c} - \frac{1}{N} \right]} = 1. \quad (\text{E4})$$

We see that the terms involving $1/N$ are negligible, and one thus has $y_c \approx N^0$, $B_1 \approx N^0$, and $B_2 \approx N^0$. Therefore, if we now compute the IPR based on this ansatz we have

$$I_2 = \int_0^N dy y^2 N^{-1} P_\psi(y) \\ \approx \frac{\int_0^{y_c} dy y^2 P_\psi(y) + \ln(N/y_c)}{N}, \quad (\text{E5})$$

where $\int_0^{y_c} dy y^2 P_\psi(y)$ is a constant independent of N . We therefore arrive at Eq. (29) as $N \rightarrow \infty$.

APPENDIX F: ADDITIONAL NUMERICAL TESTS

In this appendix, we provide some additional numerical tests of our theory. We also examine how robust our findings are under some simple natural changes of the underlying model.

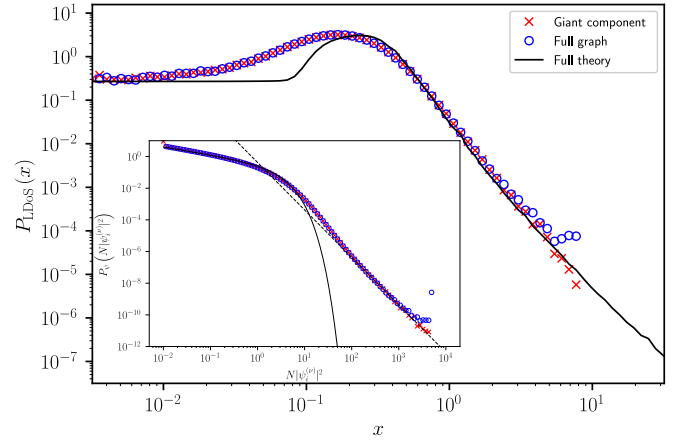


FIG. 9. LDOS distribution (main) and eigenvector distribution (inset) for the full ER graph and the giant component. Here, $N = 10^4$ and simulations were averaged over 100 trials. A uniform distribution was used for the squared weights $u_{ij} = W_{ij}^2$. We see a small discrepancy in the tails of both distributions corresponding to eigenvectors localized on disconnected clusters.

First, in Fig. 6, we explore the accuracy of the approximate theory in Eq. (21) as the average degree p is varied. We find that while the range of validity of theory is remarkable, some inaccuracy is found as the connectivity is lowered below $p \approx 3$. We also see how the power-law tail is suppressed as p is increased, as was stated in the discussion around Eq. (24).

Second, we examine how the LDOS behaves when we replace the ER graph (to which our theory applies most accurately) with a configuration model network with a Poissonian degree distribution (see Fig. 7). We find that while the power-law behavior is preserved, there is some disagreement with the theory, which is to be expected.

We also verify that the power-law behavior holds for sparse configuration model networks with uniform degree distributions (see Fig. 8). Indeed, we believe that the behavior observed here ought to carry over to many kinds of sparse networks.

Finally, we compare the eigenvector and LDOS distributions of the giant component and the full ER graph in Fig. 9. There is a small discrepancy, which manifests at large $y = N|\psi_i^{(v)}|^2 \approx N/2$. The reason for this is that pairs of isolated nodes can have localized eigenvectors, with an eigenvalue corresponding to the weight of the edge that connects them. There are further contributions from other isolated clusters of other sizes, but these are smaller. As N is increased, this discrepancy is pushed to larger values of y and x .

As discussed in Sec. IV, the $1/p$ expansion does not capture the contribution of the isolated clusters of the full ER graph. The theory is thus better suited to describing the LDOS and eigenvector distributions of the giant component, which is what we are interested in.

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