

Localization transitions in a half-filled helical Aubry-André model

Taylan Yildiz  and B. Tanatar *

Department of Physics, *Bilkent University*, 06800 Ankara, Türkiye

Balázs Hetényi

Department of Theoretical Physics, *Budapest University of Technology and Economics*, H-1111 Budapest, Hungary;

MTA-BME Lendület “Momentum” Open Quantum Systems Research Group, Institute of Physics, Budapest University of Technology and Economics, Műegyetem rkp. 3, H-1111 Budapest, Hungary;

and Institute for Solid State Physics and Optics, HUN-REN Wigner Research Centre for Physics, H-1525 Budapest, P. O. Box 49, Hungary



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We investigate localization in a one-dimensional quasiperiodic lattice obtained by extending the Aubry-André model with a structured N th-neighbor hopping term of strength J_N . This additional channel connects successive windings of an effective helical chain, introducing both a hopping scale and a geometric length scale into the localization problem. Under periodic boundary conditions, we identify the localization transition using a geometric Binder cumulant U_4 constructed from the polarization amplitudes of the occupied many-body Slater determinant. The finite-size critical potential is extracted from the zero crossing of U_4 . At fixed helical range N , increasing J_N shifts the transition toward larger quasiperiodic potential strengths. In contrast, varying N produces a strongly nonmonotonic phase boundary with pronounced spikelike enhancements of the critical potential. Using Fibonacci system sizes together with a Zeckendorf-shift construction for the particle number, we show that the dominant spike structure survives the thermodynamic-limit extrapolation. The largest enhancements are associated with helical ranges for which the phase shift $2\pi\beta N$ approaches an in-phase or antiphase resonance condition. We supplement the polarization analysis with occupied-state IPR and NPR, energy-resolved IPR maps, and the single-particle Fermi gap. The Fermi gap begins to open at the same crossover onset at which U_4 departs from its conducting-regime plateau, providing a complementary spectral signature of the same insulating transition. The participation ratios show that, in the half-filled sectors studied here, this transition is also accompanied by a gradual and energy-dependent localization of the single-particle eigenstates.

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I. INTRODUCTION

The delocalization-localization transition (DLT) in condensed matter physics has been a topic of intense interest for many decades. Anderson localization (AL) is a localization phenomenon that occurs in the presence of random on-site disorder [1]. In such systems, a DLT is only found in three dimensions; in one or two dimensions, extended states are absent [2,3]. Quasiperiodic systems, however, can exhibit extended states even in one dimension. The best known example of a quasiperiodic lattice is the Aubry-André (AA) model, where an incommensurate on-site modulation drives a self-dual DLT at a critical potential strength [4–7]. Quasiperiodic models, especially the AA model, offer a controlled setting for studying the DLT, mobility edges, and finite-size effects [8–16]. Extensions of the AA model are also

extensively studied in condensed matter physics, including those with longer-range tunneling [17–20], many-body extensions [21–24], multiple transitions [25–32], and non-Hermitian generalizations [33–36]. Because the AA model and its extensions are made with incommensurate lattice potentials, they are experimentally feasible and directly relevant to engineered lattices in cold atoms, optical lattices, and electrical circuits [37–43].

The present work combines three essential conceptual ingredients: localization, quasiperiodicity, and helical configuration. The question of particle localization is closely related to the distinction between conducting and insulating behavior. Following Kohn’s criterion [44], the insulating character of a system is determined by the response of the many-particle ground state rather than solely by the spatial extent of an individual single-particle orbital. This distinction makes polarization-based diagnostics particularly natural for the present problem. Quasiperiodicity provides the second ingredient: The Aubry-André model is a minimal setting for deterministic, nonrandom localization, and its extensions are known to produce richer behavior such as intermediate regimes, mobility edges, and multifractal states. The third ingredient, helical configuration, introduces a structured long-range hopping channel rather than a generic increase in

*Contact author: tanatar@fen.bilkent.edu.tr

connectivity. Helical symmetry appears in several condensed-matter and molecular systems, including carbon nanotubes [45–47] and DNA-like molecular chains [48–52], both of which are often modeled using tight-binding descriptions.

Motivated by these considerations, we study localization in a one-dimensional quasiperiodic lattice obtained by extending the Aubry-André hopping process. Specifically, we introduce an additional N th-neighbor hopping term with strength J_N that couples sites separated by a fixed distance N . This hopping channel admits an effective helical interpretation: If the one-dimensional lattice is wrapped such that each winding contains N sites, the usual nearest-neighbor hopping propagates along a winding, while the N th-neighbor hopping connects neighboring windings [53–56]. In this way, the parameter N is not only a hopping range, but also a geometric length scale that controls how the long-range tunneling samples the quasiperiodic potential.

The resulting helical AA model introduces a second tuning parameter J_N beyond the quasiperiodic potential strength, enabling a systematic exploration of how long-range tunneling reshapes the localization transition. A key motivation is that adding a structured long-range coupling can qualitatively modify transport and spectral properties. Moreover, because the N th-neighbor hopping connects sites whose on-site potentials differ by the phase shift $2\pi\beta N$, particular helical ranges can satisfy approximate in-phase or antiphase resonance conditions and produce pronounced enhancements of the critical potential.

A practical challenge is that many localization diagnostics are formulated in real space and implicitly assume open boundaries, while quasiperiodic models are often studied with periodic boundary conditions to minimize edge effects and to employ rational approximants of the incommensurate modulation. Under periodic boundary conditions, the position operator is not a well-defined observable on a ring [44], so directly using moments of position as localization indicators is subtle. We overcome this difficulty by employing the modern theory of polarization [57–63], where the relevant object is a twist, or polarization, operator whose expectation value remains well defined on a ring and distinguishes the conducting and insulating responses of the many-body ground state. For a noninteracting fermionic ground state, the polarization amplitudes Z_q can be evaluated efficiently as determinants of overlap matrices constructed from the occupied single-particle orbitals. From $|Z_1|$ and $|Z_2|$ one obtains approximate second and fourth moments of the centered polarization distribution, which in turn define the geometric Binder cumulant [64,65] U_4 .

The geometric Binder cumulant [66–68] provides a convenient criterion for the insulating transition. As the quasiperiodic potential Δ is increased, $U_4(\Delta)$ departs from its extended-regime value and subsequently changes sign. We extract the finite-size critical potential Δ_c from the zero crossing of $U_4(\Delta)$ and map its dependence on the long-range hopping amplitude J_N and the helical range N . Increasing J_N generally shifts Δ_c toward larger potential strengths, whereas varying N produces a strongly nonmonotonic phase boundary with pronounced resonancelike spikes.

To clarify the physical character of this transition, we supplement the polarization analysis with the single-particle

Fermi gap and with occupied-state inverse and normalized participation ratios. The Fermi gap begins to open at the same potential strength at which U_4 departs from its extended-regime plateau, providing a spectral signature of the same insulating transition. The participation ratios probe a different property: They characterize the spatial localization of the individual single-particle orbitals forming the occupied Slater determinant. In the half-filled sectors studied here, their evolution shows that the insulating transition is also accompanied by a gradual and energy-dependent localization of the single-particle states.

Finally, to take a controlled thermodynamic limit in a quasiperiodic setting, we work along Fibonacci system sizes $L = F_n$, which are natural rational approximants for incommensurate modulations under periodic boundary conditions. However, specifying only a target filling ratio does not uniquely determine how the many-body sector is approached as $n \rightarrow \infty$. We therefore employ a Zeckendorf-shift construction [69]: Starting from a reference particle number with a fixed Zeckendorf decomposition [70], we generate particle numbers for larger Fibonacci sizes by shifting the Fibonacci indices. This defines a consistent sequence of many-body sectors with a well-defined thermodynamic-limit filling, enabling controlled finite-size scaling of U_4 , the critical potential Δ_c , and the resonancelike structure of the phase boundary.

The paper is organized as follows. In Sec. II, we introduce the helical Aubry-André model and the additional N th-neighbor hopping term that produces the effective helical geometry. In Sec. III, we present the geometric Binder cumulant and explain how the finite-size critical potential is extracted from $U_4(\Delta)$. In Sec. IV, we define the occupied-state IPR and NPR and the single-particle Fermi gap, and clarify the distinct information provided by the polarization, spectral, and eigenstate-based diagnostics. In Sec. V, we present the results, including the dependence of the transition on J_N and N , the comparison of U_4 with the participation ratios and Fermi gap, the energy-resolved localization behavior, and the thermodynamic limit analysis along Zeckendorf-shifted Fibonacci sequences. We also discuss the phase-resonance origin of the spike structure and possible physical realizations of the model. Section VI summarizes our main conclusions. Technical details of the discrete geometric moments are provided in Appendix.

II. HELICAL AUBRY-ANDRÉ MODEL

We consider the Aubry-André Hamiltonian given by

$$\hat{H}_{\text{AA}} = -J \sum_{j=1}^L (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) + \Delta \sum_{j=1}^L v_j n_j, \quad (1)$$

where $v_j = \cos(2\pi\beta j + \phi)$ is the quasiperiodic part of the on-site potential, β is an irrational number which governs the quasiperiodicity, and ϕ is a phase offset. Since we will study the model under periodic boundary conditions, where the phase offset is not relevant, we will set $\phi = 0$. Δ and J correspond to potential and hopping strength, respectively. Operators $c_j^\dagger$ (c_j) are the creation (annihilation) operators for fermions.

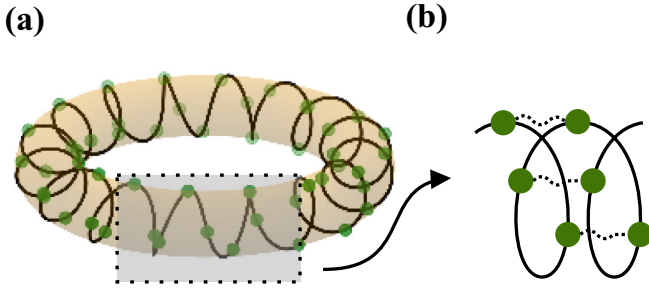


FIG. 1. (a) Schematic illustration of a one-dimensional lattice embedded on a toroidal geometry. The sites follow a helical path wrapped around the torus, such that periodic boundary conditions are naturally realized by the closed topology. The shaded region highlights a local segment of the lattice. (b) Effective representation of the helical structure, where sites on neighboring turns become spatially close. Solid lines denote nearest-neighbor hopping J , while dotted links indicate longer-range hopping J_N emerging from the geometric embedding.

Now we extend the AA model to a longer-range hopping denoted by the N th-hopping,

$$H_N = H_{AA} - J_N \sum_{j=1}^L (c_{j+N}^\dagger c_j + c_j^\dagger c_{j+N}), \quad (2)$$

where J_N is the strength of N th-neighbor hopping. The N th-hopping gives rise to a helical geometry, as illustrated in Fig. 1. In a helical chain, this long-range hopping is the tunneling between successive helical windings, and with this setting, each winding has N lattice points. The helical range N represents a microscopic geometric length scale. For a helix with radius R , pitch p , and nearest-neighbor spacing a measured along the helical trajectory, the number of sites in one winding is approximately

$$N \simeq \frac{\sqrt{(2\pi R)^2 + p^2}}{a}.$$

The microscopic quantities R , p , and a , and hence N , are kept fixed when the thermodynamic limit is taken. Increasing L therefore corresponds to adding more windings, so that $N/L \rightarrow 0$ as $L \rightarrow \infty$. Under periodic boundary conditions, N is defined modulo L , and N and $L - N$ generate the same hopping bonds; independent ranges may therefore be restricted to $N \leq L/2$. A limit in which N/L remains finite would correspond to a system-size-dependent helical geometry and is not considered here.

III. GEOMETRIC BINDER CUMULANT

For periodic boundary conditions (PBC), the usual position operator is not a well-defined observable on a ring, hence real-space localization diagnostics (based directly on x) are not immediate. This difficulty is overcome by the modern theory of polarization [59,60] (MTP), which replaces the expectation value of x by a geometric phase, usually known as the Zak phase, which arises upon integrating across the Brillouin zone [71]. The many-body generalization of this theory was developed some years later by Resta and Sorella [61,62]. This

methodology was first used to study the Aubry-André model by Varma and Pilati [72].

The geometric Binder cumulant [67,68] (GBC) is a combination of the modern theory of polarization with the techniques used in statistical mechanics for finite size scaling developed by Binder [64,65]. The central idea is to note that expectation value, first used by Resta [61,62] in evaluating the polarization and its variance in many-body crystalline systems, can be interpreted as a cumulant generating function, whose first moment or cumulant is the polarization (or the Zak phase in crystalline systems), and higher moments or cumulants can be generated by the well-known formulas from probability theory. One can then use their ratios as was done by Binder to obtain quantities which are sensitive to localization. The GBC formalism was already applied [66] to the Aubry-André model. Our derivation below follows Refs. [72] and [66].

We consider a noninteracting fermionic system of size L described by the helical Aubry-André Hamiltonian, introduced in Sec. II and we implement PBC, i.e., $c_{L+1} \equiv c_1$ and $c_{i+N} \equiv c_{\text{mod}(i+N-1,L)+1}$ (and similarly for negative indices). At a given filling $\nu = N_p/L$, the many-body ground state is a Slater determinant of the N_p lowest-energy single-particle eigenstates,

$$|\Psi_0\rangle = \prod_{\lambda=1}^{N_p} d_\lambda^\dagger |0\rangle, \quad d_\lambda^\dagger = \sum_{j=1}^L \phi_\lambda(j) c_j^\dagger, \quad (3)$$

where $\phi_\lambda(j)$ are normalized single-particle eigenvectors of the PBC Hamiltonian.

The central object of MTP is the polarization amplitude [61,62] (twist expectation value)

$$Z_q \equiv \langle \Psi_0 | \hat{U}^{(q)} | \Psi_0 \rangle, \quad (4)$$

where we have

$$\hat{U}^{(q)} = \exp\left(\frac{2\pi i q}{L} \hat{X}\right), \quad q \in \mathbb{Z} \quad (5)$$

and $\hat{X}$ is the many-body position operator. For a Slater determinant, Z_q can be evaluated as a determinant of an $N_p \times N_p$ matrix,

$$Z_q = \det[U_{\lambda\lambda'}^{(q)}], \quad U_{\lambda\lambda'}^{(q)} = \sum_{j=1}^L \phi_\lambda^*(j) e^{i\frac{2\pi q}{L} j} \phi_{\lambda'}(j). \quad (6)$$

The set $\{Z_q\}$ is the discrete cumulant-generating function of the total position X of the charge carriers on the ring. The corresponding distribution $P(X)$ may be obtained with an inverse discrete Fourier transform,

$$P(X) = \frac{1}{L} \sum_{q=0}^{L-1} Z_q \exp\left(i\frac{2\pi q}{L} X\right). \quad (7)$$

To construct a Binder-type indicator, we first center the distribution by replacing $Z_q \mapsto |Z_q|$, which removes an overall phase and shifts $P(X)$ to be centered within a unit cell. The use of Z_q is analogous to using the characteristic function of a probability distribution. For an ordinary centered distribution, the small wave-vector expansion of its characteristic function

contains the spatial moments,

$$Z(k) \simeq 1 - \frac{k^2}{2} M_2 + \frac{k^4}{24} M_4 + \dots$$

On a finite ring, the allowed wave vectors are discrete, $k_q = 2\pi q/L$. Therefore the lowest harmonics, $q = 1$ and $q = 2$, can be used to construct finite-difference estimates of the second and fourth geometric moments. This gives

$$M_2 = \frac{L^2}{2\pi^2} (1 - |Z_1|), \quad (8)$$

$$M_4 = \frac{L^4}{8\pi^4} (|Z_2| - 4|Z_1| + 3). \quad (9)$$

Finally, the geometric Binder cumulant is defined as

$$U_4 \equiv 1 - \frac{1}{3} \frac{M_4}{M_2^2}. \quad (10)$$

The quantities M_2 and M_4 , obtained in this way from the polarization amplitudes Z_1 and Z_2 , can be interpreted as geometric moments of the many-body polarization distribution on the ring. The second moment M_2 measures the spread of this distribution, while M_4 measures its higher-order shape. The Binder combination U_4 compares the fourth moment with the square of the second moment, and is therefore sensitive to changes in the shape of the polarization distribution rather than only to its overall width.

Physically, a conducting many-body ground state can respond to a twist around the ring and produce a characteristic broad polarization distribution, giving a positive value of U_4 . As the quasiperiodic potential is increased and localization develops, the polarization distribution changes shape and U_4 decreases. In the convention used here, a nondegenerate delocalized ground state gives $U_4 = 1/2$, while degeneracies can give $U_4 = 1/3$. We use the subsequent sign change of $U_4(\Delta)$ as a finite-size criterion for identifying the delocalization-localization transition. In practice, the critical point Δ_c is extracted by locating the size-independent crossing, or zero crossing, of $U_4(\Delta)$ computed for several system sizes L at fixed filling ν and fixed helical parameters (N, J_N) .

For a given parameter set (L, ν, Δ, J_N, N) the algorithm is: (i) Diagonalize the single-particle PBC Hamiltonian and collect eigenvectors $\{\phi_\lambda(j)\}$; (ii) Occupy the lowest $N_p = \nu L$ states to form $U^{(q)}$ for $q = 1, 2$; (iii) Compute Z_1, Z_2 , then M_2, M_4 , and finally U_4 ; and (iv) Repeat for multiple L and locate Δ_c from the finite-size behavior of U_4 .

A technical issue concerns the shape of $P(X)$: For certain parity combinations of particle number N_p and system size L , the distribution can become bimodal near criticality, in which case the above GBC construction is not directly applicable. In computations, we therefore focus on opposite-parity cases (odd L with even N_p , or vice versa), for which $P(X)$ remains unimodal, and the GBC analysis is robust.

IV. SINGLE-PARTICLE AND SPECTRAL DIAGNOSTICS

To complement the polarization-based geometric Binder cumulant, we also employ conventional single-particle localization and spectral diagnostics. For a normalized single-particle eigenstate $\psi_n(j)$, the inverse participation ratio (IPR) and normalized participation ratio (NPR) are defined as

[73,74]

$$\text{IPR}_n = \sum_{j=1}^L |\psi_n(j)|^4, \quad \text{NPR}_n = \frac{1}{L \sum_{j=1}^L |\psi_n(j)|^4}. \quad (11)$$

For an extended eigenstate, $\text{IPR}_n \sim L^{-1}$ and $\text{NPR}_n \sim \mathcal{O}(1)$, whereas for a localized eigenstate, $\text{IPR}_n \sim \mathcal{O}(1)$ and $\text{NPR}_n \sim L^{-1}$.

Since the geometric Binder cumulant is constructed from the occupied many-body Slater determinant, we compare it with participation ratios averaged over the occupied single-particle states [75],

$$\langle \text{IPR} \rangle_{\text{occ}} = \frac{1}{N_p} \sum_{n=1}^{N_p} \text{IPR}_n, \quad \langle \text{NPR} \rangle_{\text{occ}} = \frac{1}{N_p} \sum_{n=1}^{N_p} \text{NPR}_n. \quad (12)$$

It is important that these diagnostics are not generally equivalent. The IPR and NPR characterize the spatial extent of individual single-particle orbitals, whereas U_4 probes the polarization fluctuations and insulating response of the occupied many-body Slater determinant. Similarly, the Fermi gap measures the spectral separation between occupied and unoccupied levels rather than the spatial localization of their eigenstates. A gapped insulating Slater determinant may, in general, be constructed from spatially extended orbitals. We therefore use the IPR, NPR, and Fermi gap as complementary diagnostics, rather than as alternative definitions of the critical potential extracted from U_4 .

V. RESULTS AND DISCUSSION

We scale energies by J throughout the results, and use the nearest opposite-parity sector to half filling whenever the filling is not otherwise specified; for $L = 987$, this corresponds to $N_p = 494$.

Figure 2 shows representative traces of the geometric Binder cumulant $U_4(\Delta)$ at fixed helical range $N = 40$ and system size $L = 987$ for several values of the N th-neighbor hopping amplitude J_N . For weak quasiperiodic modulation, U_4 stays in the extended phase, and we see that $U_4 = 1/2$ or $U_4 = 1/3$. If $U_4 = 1/2$, the ground state is nondegenerate, the distribution of $P(X)$ is flat and if $U_4 = 1/3$ the ground state is degenerate, the distribution is a raised cosine. As Δ is increased, $U_4(\Delta)$ decreases and ultimately changes sign, signaling a qualitative change in the polarization fluctuations consistent with the onset of localization. Following the procedure described in Sec. III we define the finite-size critical potential $\Delta_c(L)$ as the zero crossing of $U_4(\Delta)$; the corresponding values are marked by the vertical dotted lines and listed above each panel.

In the Aubry-André limit $J_N = 0$, the sign change occurs at $\Delta_c \simeq 1.98$, consistent with the well-known critical value $\Delta_c = 2$ in the units used here. Turning on a small helical hopping shifts the crossing only mildly ($\Delta_c \simeq 2.02$ for $J_N = 0.01$), whereas increasing J_N leads to a pronounced stabilization of the extended phase: The critical potential rises to $\Delta_c \simeq 2.45$ for $J_N = 0.1$, to $\Delta_c \simeq 4.67$ for $J_N = 0.5$, and reaches $\Delta_c \simeq 9.3$ for $J_N = 1.0$. Physically, the additional long-range tunneling increases the effective kinetic

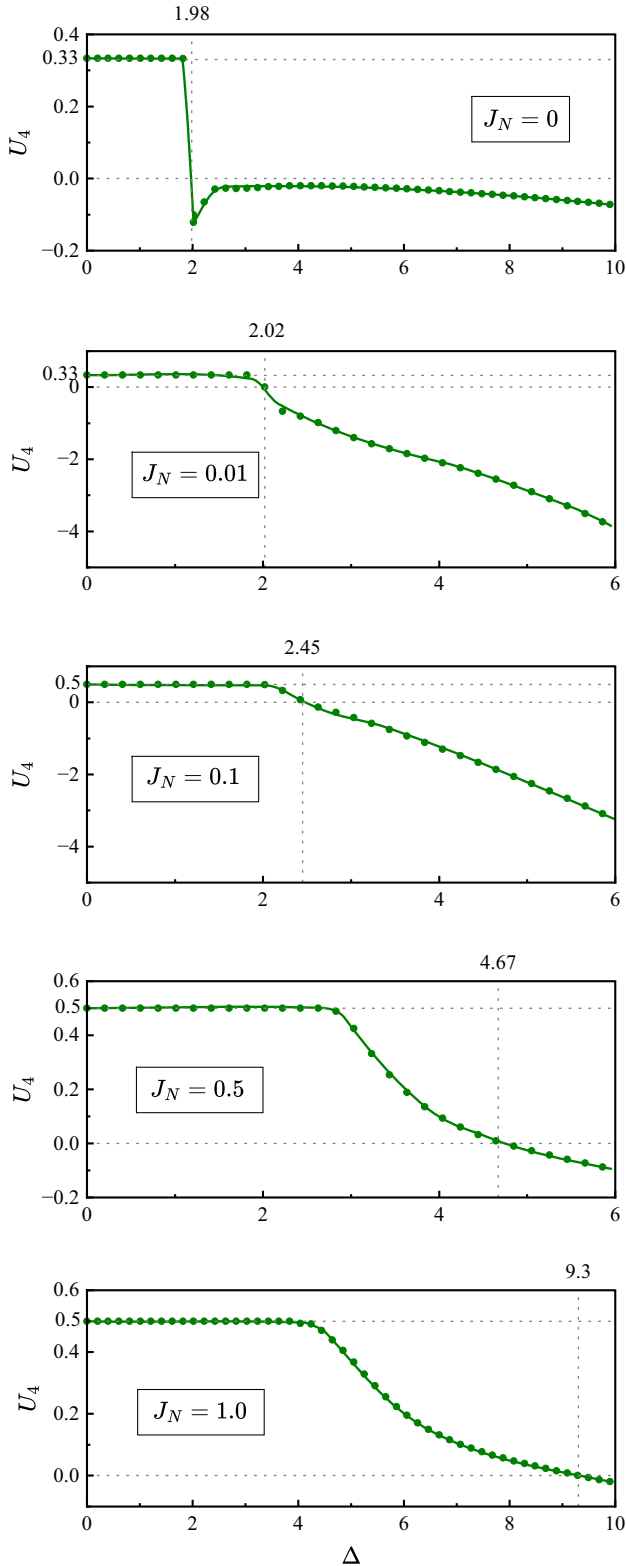


FIG. 2. Geometric Binder cumulant U_4 as a function of quasiperiodic potential strength Δ for $L = 987$ and $N = 40$, shown for several N th-neighbor hopping amplitudes J_N (panel labels). Horizontal dotted lines indicate $U_4 = 1/3$, $1/2$, and $U_4 = 0$. The vertical dotted line marks the critical potential Δ_c defined by the zero crossing of $U_4(\Delta)$.

connectivity of the chain (or, in the helical picture, enhances interwinding transport), so a stronger quasiperiodic modulation is required to suppress coherent motion and drive localization. The evolution of $U_4(\Delta)$ also becomes smoother when the helical hopping J_N is introduced. This should not be interpreted solely from a single finite-size curve as the disappearance of the localization transition. The sharp behavior in the $J_N = 0$ Aubry-André limit is tied to the self-dual structure of the nearest-neighbor model, whereas the additional N th-neighbor hopping breaks this simple self-duality and introduces an additional length scale associated with the helical range N . Consequently, finite-size curves of $U_4(\Delta)$ can display a broadened crossoverlike region before entering the localized regime. Beyond the crossing, U_4 becomes negative and decreases further with Δ , indicating increasingly strong localization-induced polarization fluctuations.

These panel-by-panel $U_4(\Delta)$ curves serve as the basic input for constructing the phase boundary. In the remainder of this section, we repeat the same extraction of Δ_c over a dense grid of (J_N, N) and for multiple Fibonacci sizes, and then perform finite-size scaling to obtain thermodynamic-limit estimates of $\Delta_c(J_N, N)$ and to identify commensurability-induced anomalies in the N dependence. Figure 3 demonstrates that this stabilizing effect of helical tunneling persists upon changing the helical range. Here we keep the system size fixed at $L = 987$ but increase the winding to $N = 100$. The qualitative behavior of $U_4(\Delta)$ remains the same across all J_N : an extended-phase plateau at small Δ , followed by a monotonic decrease and a sign change at a well-defined Δ_c . However, the location of the crossing is markedly shifted compared to Fig. 2 while the AA limit again yields $\Delta_c \simeq 2$ (here $\Delta_c \simeq 1.98$ for $J_N = 0$), the critical potential rises much more rapidly with J_N at this larger winding: $\Delta_c \simeq 2.1$ for $J_N = 0.01$, $\Delta_c \simeq 3.15$ for $J_N = 0.1$, $\Delta_c \simeq 8.4$ for $J_N = 0.5$, and $\Delta_c \simeq 16.4$ for $J_N = 1.0$. Thus, for the representative parameters considered here, the critical potential at fixed J_N is larger for $N = 100$ than for $N = 40$. This comparison demonstrates that the transition is highly sensitive to the helical range. It should not, however, be interpreted as a monotonic enhancement of interwinding transport with increasing N , since the number and strength of the additional hopping bonds remain unchanged. Instead, varying N changes the relative phase $2\pi\beta N$ sampled by the helical hopping, leading to the strongly nonmonotonic behavior examined later.

A. Phase boundary versus helical hopping strength

Having established how Δ_c is extracted from the zero crossing of $U_4(\Delta)$, we now summarize the resulting phase boundary in terms of the two helical control parameters. Figure 4 shows the critical quasiperiodic potential Δ_c as a function of the long-range hopping amplitude J_N for a fixed chain length $L = 987$ at half filling, for two representative windings: (a) $N = 100$ and (b) $N = 40$. In both cases, $\Delta_c(J_N)$ increases monotonically with J_N , demonstrating that helical tunneling stabilizes the extended phase: A larger quasiperiodic modulation is required to drive the ground state into the localized regime when the additional N th-neighbor channel

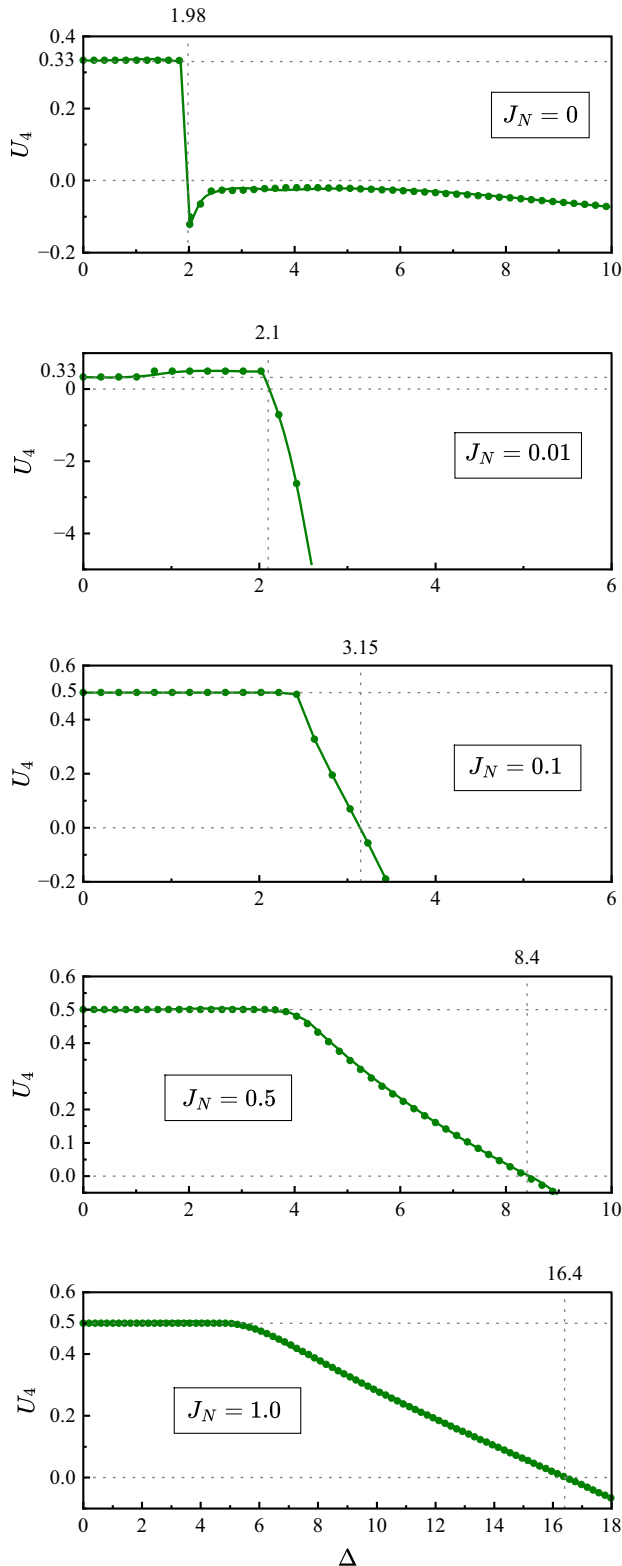


FIG. 3. Geometric Binder cumulant U_4 as a function of quasiperiodic potential strength Δ for $L = 987$ and $N = 100$, shown for several N th-neighbor hopping amplitudes J_N (panel labels). Horizontal dotted lines indicate $U_4 = 1/3$, $1/2$, and $U_4 = 0$. The vertical dotted line marks the critical potential Δ_c defined by the zero crossing of $U_4(\Delta)$.

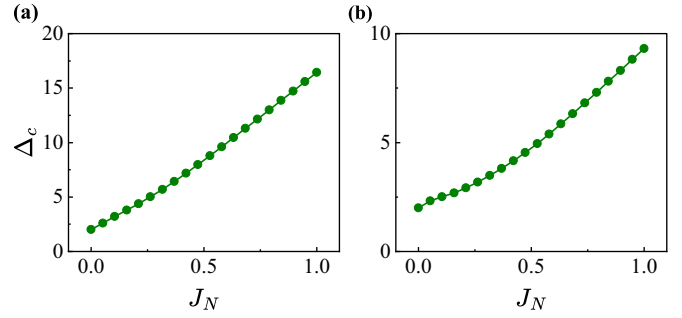


FIG. 4. Critical quasiperiodic potential strength Δ_c as a function of the long-range (helical) hopping amplitude J_N for a chain of length $L = 987$ at half filling. The critical value Δ_c is extracted from the zero crossing of the geometric Binder cumulant $U_4(\Delta)$ upon increasing Δ , with the helical hopping range fixed to (a) $N = 100$ (b) $N = 40$ and J_N varied linearly from 0 to 1.

is present. The growth is substantially stronger for $N = 100$ than for $N = 40$, consistent with the comparison between Figs. 2 and 3 and highlighting that the influence of J_N depends on the helical geometry set by N . The dependence on J_N establishes how the strength of the additional helical hopping shifts the transition. We now turn to the role of the hopping range N , which introduces a geometric length scale and controls how the long-range hopping samples the quasiperiodic modulation.

B. Dependence on helical range and phase-resonance effects

We next fix the helical hopping amplitude and scan the winding N . Since the N th-neighbor hopping connects sites j and $j + N$, changing N changes the phase shift with which the hopping probes the quasiperiodic potential. Therefore, the N

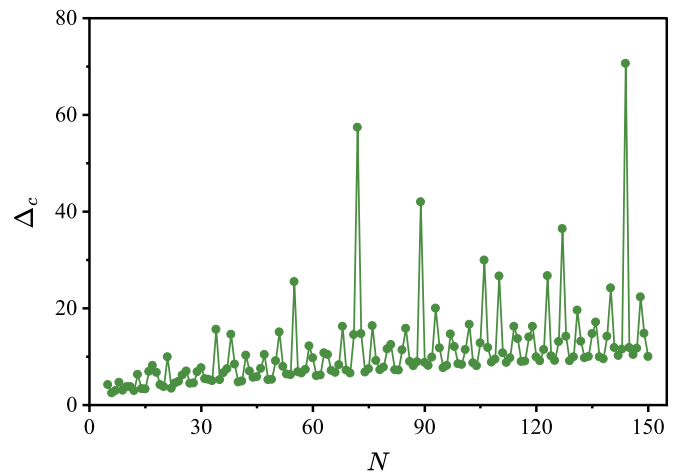


FIG. 5. Dependence of the critical quasiperiodic potential strength Δ_c on the helical winding (long-range hopping range) N for a chain of length $L = 987$ and $J_N = 0.5$ at half filling, where Δ_c is extracted from the zero crossing of the geometric Binder cumulant.

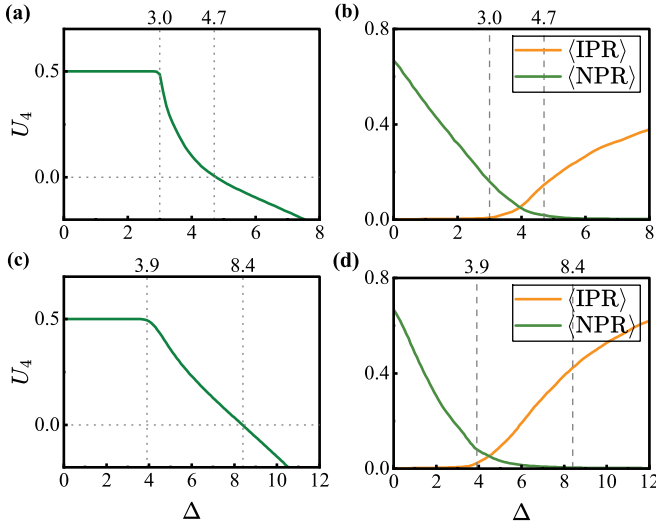


FIG. 6. Comparison between the geometric Binder cumulant and occupied-state participation ratios for $L = 987$ and $J_N = 0.5$. (a) and (b) show the results for $N = 40$, while (c) and (d) show the results for $N = 100$. (a) and (c) display $U_4(\Delta)$, and (b) and (d) display $\langle \text{IPR} \rangle_{\text{occ}}$ and $\langle \text{NPR} \rangle_{\text{occ}}$. The vertical dashed lines denote two reference scales obtained from U_4 : the departure from its extended-regime plateau and its zero crossing. These lines are included to compare the corresponding crossover intervals and are not independently extracted critical points of the participation ratios.

dependence is sensitive not only to the strength of the additional hopping channel, but also to the phase relation between the helical hopping range and the Aubry-André modulation.

Figure 5 shows Δ_c as a function of N for a broad low-resolution scan at half filling and $L = 987$, with fixed helical hopping amplitude $J_N = 0.5$. The scan shows that the dependence of Δ_c on N is strongly nonsmooth: Pronounced spikes appear at specific windings over a wide interval. We select the interval $N = 40$ – 60 for the detailed Fibonacci finite-size scaling analysis in Sec. V. We emphasize that this choice does not imply that resonance effects occur only in this interval; rather, it provides a representative window in which the finite-size stability of the spike structure can be tested.

Taken together, Figs. 4 and 5 show that the phase boundary depends on the helical hopping in two distinct ways. The hopping strength J_N sets the overall scale of the transition, while the hopping range N introduces a geometric length scale that produces strong phase-resonance-induced anomalies in $\Delta_c(N)$. In the following, we therefore repeat the extraction of Δ_c across multiple Fibonacci system sizes and perform finite-size scaling to determine which features persist in the thermodynamic limit.

C. Single-particle localization in the occupied sector

We first examine whether the polarization crossover identified by U_4 is accompanied by spatial localization of the occupied single-particle orbitals. Figure 6 compares $U_4(\Delta)$ with $\langle \text{IPR} \rangle_{\text{occ}}$ and $\langle \text{NPR} \rangle_{\text{occ}}$ for $J_N = 0.5$ and the representative helical ranges $N = 40$ and $N = 100$.

For $N = 40$, the departure of U_4 from its extended-regime plateau is accompanied by an increase of $\langle \text{IPR} \rangle_{\text{occ}}$ and a

corresponding decrease of $\langle \text{NPR} \rangle_{\text{occ}}$. The participation ratios evolve over a finite interval of Δ , rather than changing sharply at a single finite-size point. Nevertheless, their evolution occurs over the same broad crossover region in which the polarization fluctuations are strongly modified.

For $N = 100$, both the decrease of U_4 and the evolution of the participation ratios extend over a broader interval. The occupied-state IPR increases gradually, while the NPR decreases, consistent with a progressive localization of the occupied orbitals. This broader single-particle crossover is compatible with the smoother thermodynamic-limit behavior of U_4 found for the longer helical range.

The vertical lines in Fig. 6 indicate reference scales extracted from U_4 : the departure from the extended-regime plateau and the zero crossing. They are not independently determined IPR/NPR critical points. Exact coincidence between these scales is not expected because the geometric Binder cumulant and participation ratios characterize different properties of the system.

The comparison nevertheless establishes an important result for the half-filled sectors studied here: The polarization crossover is accompanied by an increasing spatial localization of the occupied single-particle states. Thus, the participation ratios provide an independent eigenstate-based characterization of the transition, without being interpreted as quantities equivalent to U_4 .

D. Energy-resolved inverse participation ratio

The occupied-state averages characterize the overall localization tendency of the filled sector but do not show how the evolution varies across the single-particle spectrum. We therefore examine the eigenstate-resolved IPR as a function of the quasiperiodic potential strength. For every eigenstate $\psi_n(j)$, the IPR is evaluated using Eq. (11), and the corresponding eigenenergy is represented by the normalized value

$$\tilde{\epsilon}_n = \frac{\epsilon_n - \epsilon_{\min}}{\epsilon_{\max} - \epsilon_{\min}}. \quad (13)$$

Figure 7 presents finite-size IPR maps for $N = 40$ and $N = 100$ at $L = 987$ and $J_N = 0.5$. At weak quasiperiodic potential, most eigenstates have small IPR, indicating predominantly extended behavior. As Δ is increased, enhanced-IPR regions first appear in selected energy intervals, while other portions of the spectrum retain substantially smaller IPR values. The simultaneous presence of lower- and higher-IPR states at the same value of Δ indicates that the localization crossover is energy dependent.

These maps should be interpreted as finite-size evidence for mobility-edge-like localization patterns rather than as a complete determination of thermodynamic mobility edges. Establishing energy-dependent mobility-edge trajectories would require tracking individual spectral regions over multiple system sizes and performing state-resolved finite-size scaling. The present results nevertheless show that helical hopping does not merely shift a uniform single-particle localization threshold, but produces strongly energy-dependent localization tendencies across the spectrum.

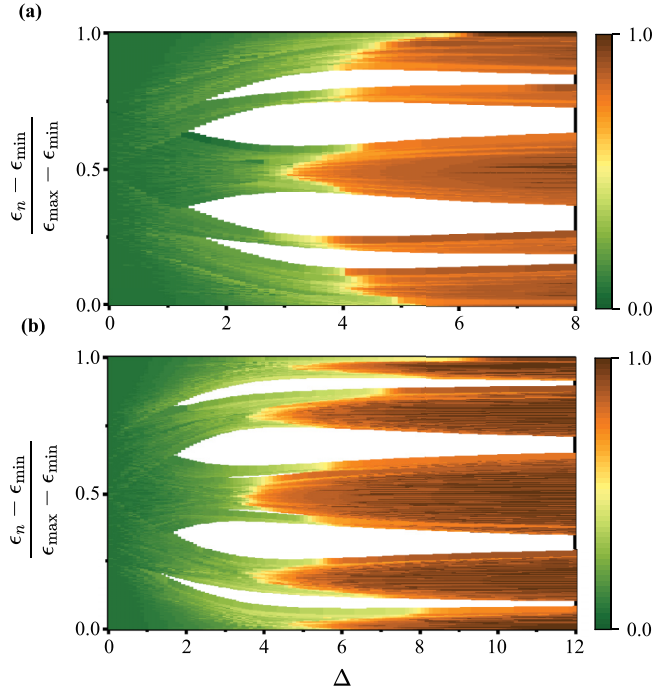


FIG. 7. Energy-resolved inverse participation ratio for $L = 987$ and $J_N = 0.5$. (a) and (b) correspond to $N = 40$ and $N = 100$, respectively. The vertical axis is the normalized single-particle energy $\tilde{\epsilon}_n = (\epsilon_n - \epsilon_{\min}) / (\epsilon_{\max} - \epsilon_{\min})$, and the color scale represents IPR_n . The coexistence of lower- and higher-IPR spectral regions indicates an energy-dependent localization crossover.

E. Fermi gap and the insulating transition

We next examine the evolution of the single-particle Fermi gap in order to provide a spectral characterization of the transition identified by the geometric Binder cumulant. For N_p occupied levels, the normalized Fermi gap is defined as

$$\tilde{\Delta}_F = \frac{\epsilon_{N_p+1} - \epsilon_{N_p}}{\epsilon_{\max} - \epsilon_{\min}}, \quad (14)$$

where the single-particle eigenvalues are ordered as $\epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_L$. Thus, ϵ_{N_p} and ϵ_{N_p+1} are the highest occupied and lowest unoccupied levels, respectively.

Figure 8 compares the geometric Binder cumulant U_4 with the normalized Fermi gap for representative values of the helical hopping parameters. In the weak-potential regime, the Fermi gap remains close to zero and U_4 stays near its extended-state plateau. Upon increasing the quasiperiodic potential, the Fermi gap begins to open at the same potential strength at which U_4 departs from this plateau. Therefore, within the numerical resolution, the opening of the Fermi gap coincides with the onset of the transition identified from U_4 .

To place the Fermi-gap behavior in a broader context, we also examine how the entire single-particle spectrum changes with Δ . Figure 9 plots the normalized energies $\tilde{\epsilon}_n = (\epsilon_n - \epsilon_{\min}) / (\epsilon_{\max} - \epsilon_{\min}) \in [0, 1]$ as a function of Δ (logarithmic horizontal axis) for $L = 4181$. White regions correspond to parameter intervals with no eigenvalues at the plotted normalized energy, i.e., spectral gaps in the rescaled spectrum. Figure 9 highlights the dependence on the helical hopping amplitude J_N at fixed winding (left set $N = 40$, right set

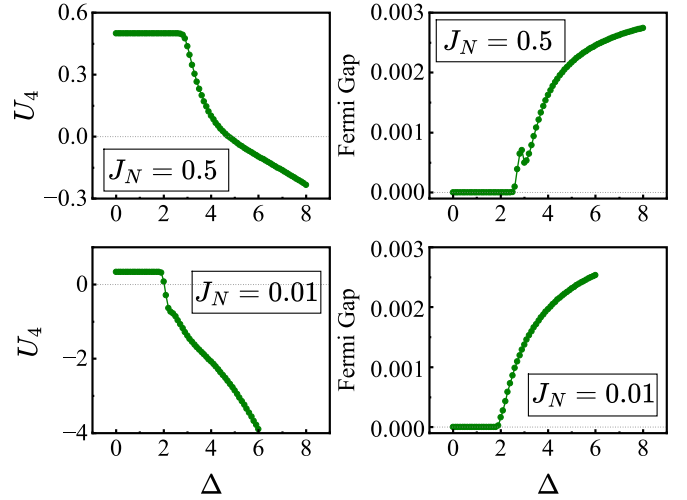


FIG. 8. Comparison of the geometric Binder cumulant $U_4(\Delta)$ and the single-particle normalized Fermi gap $\tilde{\Delta}_F = (\epsilon_{N_p+1} - \epsilon_{N_p}) / (\epsilon_{\max} - \epsilon_{\min})$, as functions of the quasiperiodic potential strength Δ for $L = 987$. Top row: strong helical hopping $J_N = 0.5$; Bottom row: weak helical hopping $J_N = 0.01$. In each row, U_4 is shown on the left and the Fermi gap on the right. The gap remains close to zero at small Δ and begins to open in the parameter region where U_4 departs from its extended-phase plateau.

$N = 100$): Increasing J_N substantially reshapes the quasiperiodic band structure, shifting and splitting prominent bands and modifying the gap pattern across a wide range of Δ .

The geometric Binder cumulant and the Fermi gap consequently provide two complementary signatures of the same transition. The opening of the Fermi gap suppresses low-energy particle-hole excitations across the occupied boundary and produces a gapped insulating Slater-determinant ground state. The corresponding change in the many-body polarization distribution is detected by U_4 . Their agreement, therefore, gives a direct spectral interpretation of the transition obtained from the polarization analysis.

This interpretation should be distinguished from that of the single-particle IPR and NPR. The Fermi gap and U_4 characterize the insulating nature of the occupied many-body ground state, whereas the IPR and NPR characterize the spatial extent of the individual single-particle eigenstates. Consequently, a gap-driven insulating state can be detected by the Fermi gap and U_4 even when the individual orbitals remain extended.

F. Zeckendorf decomposition and thermodynamic limit

In quasiperiodic lattice models it is natural to study finite systems whose length is taken from the Fibonacci sequence, $L = F_n$, because rational approximants of the incommensurate modulation are then compatible with periodic boundary conditions. An important point is that, when the thermodynamic limit is taken along the Fibonacci subsequence, specifying the filling only through a rational value N_p/L (e.g., $1/2$) does not uniquely determine how the many-body sector is approached as $n \rightarrow \infty$. Different sequences of particle numbers $N(F_n)$ that all satisfy $N_p(F_n)/F_n \rightarrow \nu$ can correspond to different sectors and may lead to distinct finite-size trends.

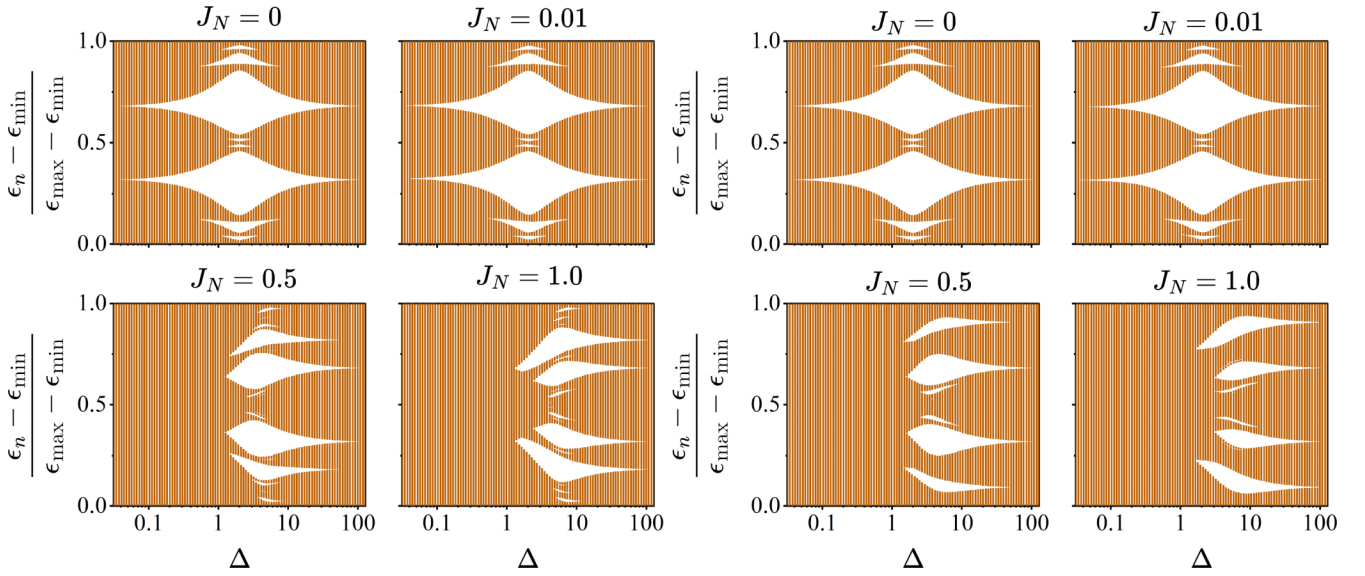


FIG. 9. Normalized single-particle energy spectrum of the helical Aubry-André model as a function of the quasiperiodic potential strength Δ (logarithmic scale on the horizontal axis) for $L = 4181$. Each panel shows the rescaled eigenenergies $\tilde{\epsilon}_n = (\epsilon_n - \epsilon_{\min})/(\epsilon_{\max} - \epsilon_{\min}) \in [0, 1]$ versus Δ for different helical hopping strengths J_N (indicated above each, subplot). Left set: helical range $N = 40$. Right set: helical range $N = 100$. White regions correspond to parameter intervals with no eigenvalues (spectral gaps), illustrating how increasing J_N reshape the gap structure.

To define a controlled thermodynamic limit we follow a Zeckendorf construction. Every positive integer N_p admits a unique Zeckendorf decomposition,

$$N_p = \sum_{\alpha} F_{\alpha}, \quad (15)$$

meaning that N_p is written as a sum of nonconsecutive Fibonacci numbers. We choose a reference system size $L_0 = F_{n_0}$ and a reference particle number N_0 , compute its Zeckendorf indices $\{\alpha\}$, and then generate particle numbers for larger Fibonacci sizes by an index shift:

$$L_m = F_{n_0+m}, \quad N_{p,m} = \sum_{\alpha} F_{\alpha+m}. \quad (16)$$

This procedure keeps the digit pattern in the Zeckendorf representation fixed while increasing the system size, and therefore defines a unique sequence of many-body sectors along the Fibonacci subsequence.

The filling factor $v_m = N_{p,m}/L_m$ converges as $m \rightarrow \infty$ because $F_{n-k}/F_n \rightarrow \varphi^{-k}$, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. More explicitly, writing $d_{\alpha} = n_0 - \alpha$,

$$v_{\infty} = \lim_{m \rightarrow \infty} \frac{N_{p,m}}{L_m} = \sum_{\alpha} \varphi^{-d_{\alpha}}, \quad (17)$$

which is generally irrational. Thus, the Zeckendorf representation provides a well-defined thermodynamic limit $L_m = F_{n_0+m} \rightarrow \infty$ at fixed Zeckendorf filling ratio.

In the calculations below we fix the initial pair to $(L_0, N_0) = (987, 494)$. The corresponding Zeckendorf decomposition is

$$494 = 377 + 89 + 21 + 5 + 2 = F_{14} + F_{11} + F_8 + F_5 + F_3, \quad (18)$$

and the shifted sequence yields the particle numbers

$$(L, N_p) = (987, 494), (1597, 799), (2584, 1293), \\ (4181, 2092), (6765, 3385) \quad (19)$$

which approach an asymptotic filling close to one half. Thermodynamic-limit estimates are then obtained by comparing results across the Fibonacci sizes and by extrapolating observables in powers of $1/L$.

To illustrate how the Zeckendorf-shift sequence stabilizes the finite-size scaling, in Fig. 10 we plot $U_4(\Delta)$ for the Fibonacci sizes $L = 987, 1597, 2584, 4181$, and 6765 using the shifted particle numbers in Eq. (16), and construct an $L \rightarrow \infty$ estimate by extrapolating U_4 at fixed Δ with a linear fit in $1/L$ (dark green curve). In both panels the finite-size curves approach the extrapolated thermodynamic-limit profile smoothly, showing that the Zeckendorf prescription yields a well-behaved sequence of many-body sectors for comparing different L . The horizontal dashed line marks $U_4 = 0$, separating the delocalized regime ($U_4 > 0$) from the localized regime ($U_4 < 0$), while the vertical dashed line indicates the corresponding thermodynamic-limit critical point extracted from the sign change of the extrapolated curve.

For the shorter helical range, $N = 40$ [Fig. 10(a)], the thermodynamic-limit extrapolation gives $\Delta_c \simeq 3.72$. Although the finite-size curves show a broadened intermediate region, this structure is progressively reduced with increasing Fibonacci system size. In the extrapolated $L \rightarrow \infty$ curve, $U_4(\Delta)$ undergoes a comparatively sharp sign change, indicating that the finite-size broadening does not survive as a separate crossover window in this case.

By contrast, for the longer helical range, $N = 100$ [Fig. 10(b)], the transition is shifted to a substantially larger value, $\Delta_c \simeq 6.4$, and the extrapolated curve remains

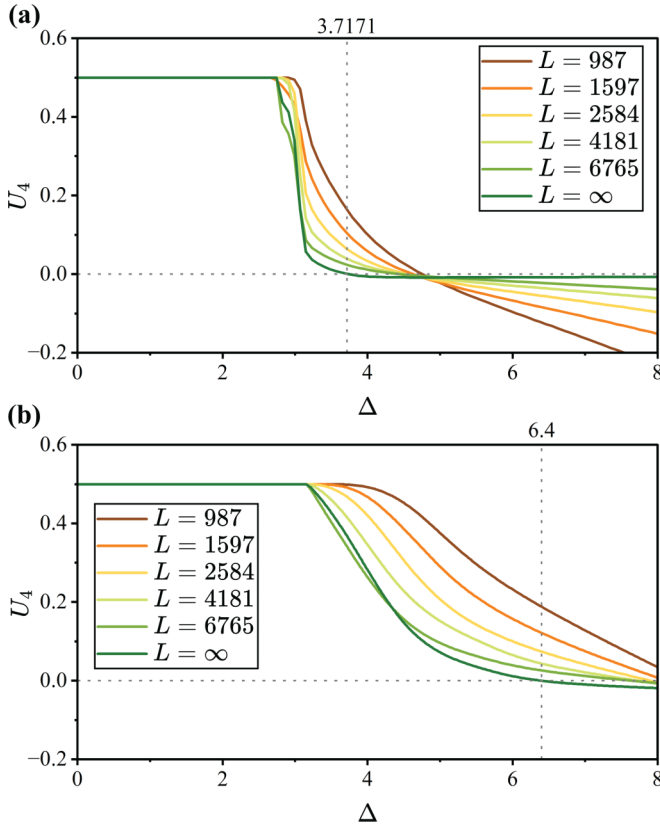


FIG. 10. Geometric Binder cumulant U_4 as a function of quasiperiodic potential strength Δ for Fibonacci system sizes $L = 987, 1597, 2584, 4181, 6765$ at the fixed-filling sequence defined by the Zeckendorf-shift construction where (a) $N = 40, J_N = 0.5$ (b) $N = 100, J_N = 0.5$. The dark green curve shows the thermodynamic-limit estimate $L \rightarrow \infty$ obtained by extrapolating at fixed Δ (linear fit in $1/L$). The horizontal dashed line marks $U_4 = 0$, separating the delocalized regime ($U_4 > 0$) from the localized regime ($U_4 < 0$), and the vertical dashed line indicates the critical points.

noticeably smoother. This indicates that the longer-range helical hopping sustains a broader crossoverlike regime before full localization sets in. Therefore, the smoothening observed for $J_N > 0$ is not universal, but depends on the helical range N : For smaller N it is largely a finite-size effect, whereas for larger N the smoother evolution persists more strongly in the thermodynamic-limit extrapolation. Thus, in finite systems, the evolution should be viewed as crossoverlike, while the zero crossing of the extrapolated U_4 curve provides an operational estimate of the localization transition in the thermodynamic-limit sense.

To further clarify the role of the helical winding, we now examine the thermodynamic-limit behavior of Δ_c as a function of N over the range $N = 40$ – 60 at fixed $J_N = 0.5$. Figure 11 shows the finite-size critical potentials extracted for the Fibonacci sizes $L = 987, 1597, 2584, 4181$, and 6765 , together with the $L \rightarrow \infty$ estimate obtained from a linear extrapolation in $1/L$. In contrast to the stronger finite-size sensitivity suggested by the single-size scan, the five finite-size curves now lie very close to one another throughout the entire interval. This near-collapse indicates that once the many-body sectors are chosen consistently, the N dependence

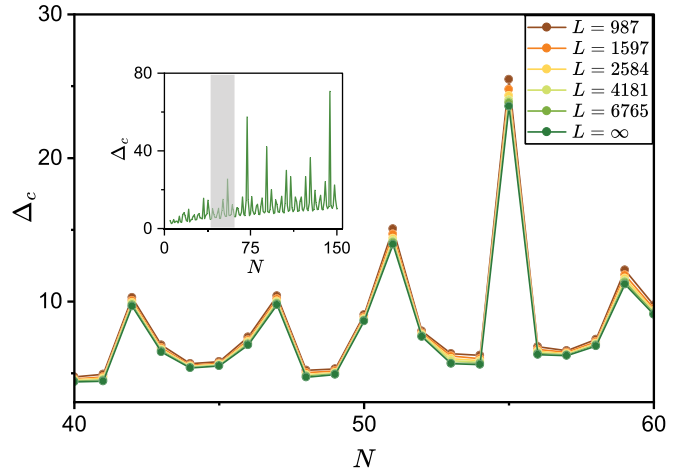


FIG. 11. Critical quasiperiodic potential strength Δ_c as a function of the helical winding (long-range hopping range) N for $N = 40$ – 60 at fixed helical hopping amplitude $J_N = 0.5$. The finite-size values are shown for Fibonacci system sizes $L = 987, 1597, 2584, 4181$, and 6765 , where Δ_c is extracted from the zero crossing of the geometric Binder cumulant $U_4(\Delta)$. The dark green curve denotes the thermodynamic-limit estimate $L \rightarrow \infty$, obtained by extrapolating Δ_c linearly in $1/L$. The inset shows a broader low-resolution exploratory scan of $\Delta_c(N)$ for $L = 987$ and $J_N = 0.5$. The shaded region marks the interval $N = 40$ – 60 selected for the detailed Fibonacci finite-size scaling analysis.

of Δ_c is already well converged at the accessible Fibonacci sizes. Most importantly, the nonmonotonic structure survives the extrapolation: The pronounced maximum at $N = 55$ remains the dominant feature of the curve, and the smaller enhancements near $N = 51$ and $N = 59$ are also retained in the thermodynamic-limit estimate. We therefore conclude that the main spike structure is not an artifact of a particular finite size, but a robust property of the helical Aubry-André model.

The physical origin of the spike structure can be understood from the way the helical hopping modifies the quasiperiodic potential. Since the long-range term connects sites j and $j + N$, the on-site modulation at the shifted site is

$$v_{j+N} = \cos(2\pi\beta j + 2\pi\beta N). \quad (20)$$

Thus, varying the winding N changes the phase shift

$$\theta_N = 2\pi\beta N$$

with which the helical hop probes the Aubry-André modulation. Special values of N occur when this phase shift is close to either 0 or π modulo 2π . In the first case, $v_{j+N} \simeq v_j$, so the helical hopping connects sites with nearly equal on-site energies. In the second case, $v_{j+N} \simeq -v_j$, so the hopping connects sites with nearly opposite on-site energies. Both cases correspond to strongly constrained phase relations along the helical hopping direction and can enhance the reorganization of the occupied states, producing anomalously large critical potentials.

To make this resonance picture quantitative, we define the detuning from the nearest 0 or π phase condition as

$$\eta_N = |2\beta N - \text{round}(2\beta N)|. \quad (21)$$

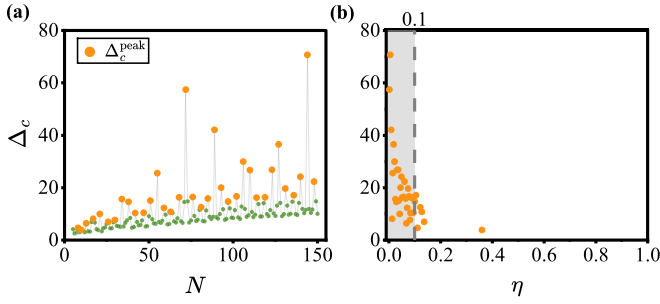


FIG. 12. Quantitative comparison between the spike structure in $\Delta_c(N)$ and the helical phase-detuning condition. (a) shows the broad $L = 987$ scan of the critical quasiperiodic potential Δ_c as a function of the helical range N , with the local maxima marked by orange symbols. (b) shows the values of these local maxima, Δ_c^{peak} , plotted against the phase detuning η_N .

This follows from the condition $\theta_N = 2\pi\beta N \simeq m\pi$, which is equivalent to $2\beta N \simeq m$, with $m \in \mathbb{Z}$. Therefore, small η_N means that the helical phase shift is close to either the in-phase condition $\theta_N \simeq 0$ or the antiphase condition $\theta_N \simeq \pi$ modulo 2π .

Figure 12 tests this interpretation using the broad $L = 987$ scan of $\Delta_c(N)$. We first identify the local maxima of $\Delta_c(N)$ from the discrete N -scan. A point N_i is classified as a local maximum if its critical potential is larger than those of the two neighboring points in the scan,

$$\Delta_c(N_i) > \Delta_c(N_i - 1), \quad \Delta_c(N_i) > \Delta_c(N_i + 1).$$

The full $\Delta_c(N)$ curve and the selected local maxima are shown in Fig. 12(a). In Fig. 12(b), the heights of these local maxima are plotted against the detuning η_N . The dominant peaks are concentrated at small η_N , showing that the largest critical potentials occur when the helical phase shift is close to one of the proposed resonance conditions. The scatter in the data also shows that η_N does not determine the peak height by itself; rather, it identifies favorable values of N where enhanced peaks can occur.

G. Physical motivation and possible realizations

The helical hopping term can be viewed as a structured long-range tunneling process, and it is therefore most naturally realized in engineered platforms where the connectivity graph can be designed directly. Topoelectrical circuits [76,77] are a particularly suitable setting, since the tight-binding connectivity is encoded in the circuit Laplacian [78]. Nearest-neighbor hopping can be implemented through links between adjacent circuit nodes, while the N th-neighbor helical hopping corresponds to additional capacitive or inductive links between nodes j and $j + N$. A quasiperiodic on-site modulation can be introduced through node-dependent grounding elements. In this sense, the helical Aubry-André model is directly compatible with platforms where synthetic lattice connectivity can be engineered at the level of individual links.

Photonic waveguide arrays provide another possible route. The spatial layout of the waveguides can be designed so that sites separated by N steps in the effective one-dimensional indexing are brought close in physical space, thereby generating

an additional controlled coupling channel. Cold-atom realizations of quasiperiodic lattices already provide a natural way to implement the Aubry-André on-site modulation through bichromatic optical lattices [79]. Engineering a single selected N th-neighbor hopping is more demanding in this setting, but Raman-assisted [80] or Floquet-induced [81] tunneling can in principle be used to generate longer-range hopping processes.

The model is also motivated by physical systems with intrinsic helical structure. Helical symmetry arises in several condensed-matter and molecular contexts, such as carbon nanotubes and DNA-like molecular chains, where tight-binding descriptions are commonly used. In the context of DNA, disorder and sequence correlations are important [82]: The relevant disorder is not expected to be purely uncorrelated Anderson-type disorder, but may instead contain correlated or deterministic structure [83].

A related example is the de Moura-Lyra model, where long-range correlations in the on-site disorder were shown to modify one-dimensional localization and allow extended-state behavior even in a disordered chain [84]. This idea has also been discussed in connection with charge transport in DNA-like systems, although its applicability to realistic molecular sequences remains subtle and has been debated [85,86]. In this respect, quasiperiodicity offers a controlled deterministic modulation that is simpler than realistic sequence disorder, while still going beyond uncorrelated randomness.

The present model provides a simple setting in which quasiperiodicity and helical geometry can be studied together.

VI. CONCLUSION

We have investigated the localization properties of a helical extension of the Aubry-André model containing an additional N th-neighbor hopping term of strength J_N . The additional hopping acts as a structured long-range tunneling channel connecting neighboring windings of an effective helical lattice. We identified the insulating transition under periodic boundary conditions using the zero crossing of the geometric Binder cumulant U_4 , constructed from the polarization amplitudes of the occupied many-body Slater determinant.

At fixed helical range, increasing J_N shifts the transition toward larger quasiperiodic potential strengths, demonstrating that the additional tunneling channel stabilizes the extended regime. The dependence on the hopping range N is substantially more complex. Rather than producing a smooth variation of the critical potential, changing N generates a strongly nonmonotonic phase boundary containing pronounced spikelike enhancements. Thus, the transition is controlled not only by the magnitude of the additional hopping, but also by the geometric phase relation between the hopping range and the quasiperiodic modulation.

To distinguish genuine thermodynamic behavior from finite-size commensurability effects, we performed finite-size scaling along Fibonacci system sizes using a Zeckendorf-shift construction for the particle number. This prescription fixes the occupied many-body sector consistently as the system size is increased. The dominant spike structure, including the large enhancement near $N = 55$ in the range studied in detail, survives the thermodynamic-limit extrapolation. The principal

anomalies are therefore not artifacts of an isolated Fibonacci size.

The spike structure can be understood through the phase shift experienced along the helical hopping direction. Since the additional term connects j with $j + N$, it samples the on-site modulation with a relative phase $2\pi\beta N$. The largest critical potentials are concentrated near values of N for which this phase approaches either zero or π modulo 2π . These conditions correspond to helical links joining sites with approximately equal or opposite on-site potentials. The phase-detuning analysis supports this resonance interpretation, although the residual scatter shows that the detuning alone does not completely determine the height of every peak.

The thermodynamic evolution of U_4 also depends on the helical range. For the shorter representative range $N = 40$, the broadened finite-size crossover is progressively reduced and the extrapolated curve displays a comparatively sharp sign change. For the longer range $N = 100$, the transition occurs at a substantially larger potential and retains a smoother crossoverlike profile in the thermodynamic extrapolation. The smoothing introduced by the helical term is therefore not purely a universal finite-size effect, but depends on the additional length scale N .

The supplementary diagnostics clarify the physical character of the transition. The single-particle Fermi gap begins to open at the same potential at which U_4 departs from its extended-regime plateau. The Fermi gap and geometric Binder cumulant therefore provide complementary spectral and polarization signatures of the same insulating transition. The occupied-state IPR and NPR address a different question: They measure the spatial localization of the individual orbitals forming the Slater determinant. Their evolution shows that, in the half-filled helical sectors studied here, the insulating transition is also accompanied by increasing single-particle localization. The energy-resolved IPR further indicates that this localization develops nonuniformly across the spectrum, producing finite-size mobility-edge-like coexistence regions.

Overall, the helical Aubry-André model provides a minimal setting in which quasiperiodicity, structured long-range tunneling, spectral gapping, and geometric resonance effects can be studied together. The model is particularly suitable for engineered platforms such as topoelectrical circuits and photonic lattices, where the connectivity between selected pairs of sites can be designed directly. More generally, the results demonstrate that the range of a structured long-range hopping process can reorganize a quasiperiodic localization phase boundary in a highly nonmonotonic manner, even when the hopping strength itself is held fixed.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: DERIVATIONS

For a discrete set of points we let $Z(k) = \langle e^{ikx} \rangle$ with $k_q = 2\pi q/L$ and with $Z(0) = 1$. Now we have

$$M_2 = -\frac{d^2 Z(k)}{dk^2} \approx -\frac{Z(k_1) + Z(-k_1) - 2Z(0)}{k_1^2} \approx \frac{-(Z_1 + Z_{-1} - 2Z_0)}{k_1^2}, \quad (Z_1 + Z_{-1} = 2|Z_1|). \quad (A1)$$

Hence, we get

$$M_2 = \frac{-(2|Z_1| - 2)}{k_1^2} = \frac{-L^2}{4\pi^2} (2|z_1| - 2) = \frac{L^2}{2\pi^2} (1 - |Z_1|). \quad (A2)$$

Now for M_4 we have

$$\frac{d^4 Z(k)}{dk^4} = \frac{Z(-2k_1) - 4Z(-k_1) + 6Z(0) - 4Z(k_1) + Z(2k_1)}{k_1^4}. \quad (A3)$$

Then,

$$M_4 = \frac{Z_{-2} - 4Z_{-1} + 6Z_0 - 4Z_1 + Z_2}{k_1^4} = \frac{(Z_2 + Z_2^*) - 4(Z_1 + Z_1^*) + 6}{k_1^4}. \quad (A4)$$

Therefore, we get

$$M_4 = \frac{2|Z_2| - 8|Z_1| + 6}{16\pi^4/L^4} = \frac{L^4}{8\pi^4} (|Z_2| - 4|Z_1| + 3). \quad (A5)$$

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