

Efficient quantum thermal state preparation via local driving: Lindbladian simulation with provable guarantees

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Preparing the thermal density matrix $\rho_\beta \propto e^{-\beta H}$ corresponding to a given Hamiltonian H is a task of central interest across quantum many-body physics, and is particularly salient when attempting to study it with quantum computers. Although solved in principle by recent constructions of efficiently simulable Lindblad master equations—that provably have ρ_β as a steady state [C.-F. Chen *et al.*, [Nature](#) **646**, 561 (2025)]—the implementation of these “exact Gibbs samplers” requires large-scale quantum computing resources and is hence challenging in practice on current or even near-term quantum devices. Here, we propose a scheme for approximately simulating an exact Gibbs sampler up to a rigorously bounded error that only requires the (repeated) implementation of three readily available ingredients: (a) analog simulation of H ; (b) strictly local but time-dependent couplings to ancilla qubits; and (c) reset of the ancillas. We give rigorous guarantees on the difference between the fixed point reached by our protocol and the exact thermal state, which only depend on parameters of the protocol and its *mixing time*. The procedure is efficiently implementable on near-term devices if H is local and the mixing time scales mildly with both system size and protocol parameters. While guaranteeing the latter for Hamiltonians of interest remains an important problem for future work, here we lay the groundwork for developing fully efficient thermal state preparation protocols on quantum simulators.

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I. INTRODUCTION

Preparing thermal states of many-body systems is a key goal for a wide range of quantum devices, since it enables the exploration of such systems through “quantum numerical experiments”. However, the ability of analog quantum simulators to access low-temperature regimes of target systems such as the Hubbard model is often limited by the absence of scalable, problem-agnostic techniques for removing entropy in a controlled manner so as to reach the target energy density. Similarly, while a proposed near-term use case for digital quantum computers is to attack problems in quantum chemistry, this goal is often obstructed by the absence of efficient ways to prepare a thermal density matrix. While various physically motivated approaches have been proposed to meet this challenge [1–12], these are typically of a heuristic and case-by-case nature. As such, their systematic error is poorly understood: In other words, it is often unclear how exactly one has to scale the available resources to obtain a close approximation to a given target state.

In counterpoint to this are recent rigorous results in the quantum computer science literature [13–23]. A subset of particular interest to this work concerns *quantum Gibbs sampling* [24–30]: The problem of engineering a dissipative quantum dynamics whose steady state corresponds to a specified Gibbs density matrix. Although a formal solution has long been available in terms of so-called Davies generators [31–33], the

corresponding dynamics involves a purely dissipative Lindblad evolution under highly nonlocal and hence unphysical “jump operators”, making it unfeasible in practice. The recent work upends this conventional wisdom by demonstrating that Gibbs sampling is possible with controllable accuracy (in the sense above), while inducing dissipation using only *quasilo-*cal jump operators, but at the cost of introducing a specially tailored coherent evolution. The steady state of the resulting Lindblad evolution is exactly given by ρ_β , which follows from the fact that the Lindbladian satisfies a certain (quantum) detailed balance property [34–37]. The sole remaining unknown is the mixing time τ_{mix} of the Lindbladian, which controls the approach to the steady state and can be long for physically meaningful reasons, much as in the classical case. Thus, although Gibbs samplers are unlikely to speed up problems that are classically hard because of glassy landscapes [38–44] (such as generic optimization problems), they are not limited by the sign problem or entanglement growth, which are the usual “intrinsically quantum” obstacles to simulating many-body physics.

Despite this remarkable breakthrough, convenient protocols to implement these exact Gibbs samplers (or their close approximations) in physically realistic settings relevant to current and near-term quantum hardware remain largely unexplored (but see, e.g., Refs. [45,46] for concurrent work). Proposed approaches rely on block-encoding Lindbladians [27,29]; doing so at appreciable scale is likely out of the question in the near term. It is this gap between formal theory and practical implementation that we bridge here, while bolstering a physically motivated picture with mathematical rigor.

Specifically, we devise a protocol that *approximately* prepares thermal states of an arbitrary Hamiltonian up to a finite

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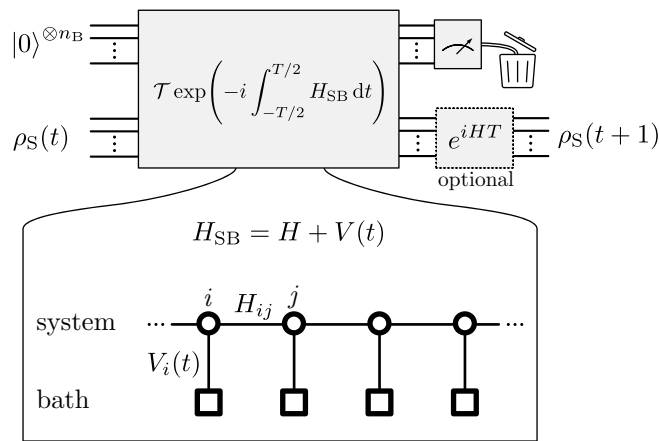


FIG. 1. Schematic of one time step of the local driving sampler. In each step, the “bath” qubits are initialized in the all-zero state, followed by an entangling step consisting of time evolution with the system Hamiltonian H together with a dynamically driven system-bath coupling $V(t) = \sum_i V_i(t)$ [cf. Eq. (16)]. Finally, the bath qubits are measured and the outcomes discarded. The final unitary “rewinding” e^{iHT} can be dropped with only a modest increase to the error bound, and can be ignored with no penalty if a quasiparticle picture applies.

nonzero but rigorously bounded error. Crucially, we only require three relatively standard features of the current generation of quantum devices: (a) “analog” simulation of the n_S -qubit Hamiltonian; (b) a dynamically tunable coupling to an (arbitrary) number of n_B ancilla qubits (usually, we will take $n_B = \mathcal{O}(n_S)$); and (c) the ability to reset the ancillas (see Fig. 1).

The proposed protocol implements a controlled discrete-time approximation of a certain Lindblad evolution. The corresponding Lindbladian has exactly the same dissipative part as the exact sampler, but the “wrong” coherent part: a Lamb shift correction rather than the coherent dynamics necessary for exact thermal state preparation. However, using the so-called secular approximation [5,47,48], we show that this difference in coherent evolution only incurs a controlled, arbitrary-small error per Lindbladian simulation time step. Assuming that the mixing time, and hence the simulation time necessary to reach the steady state, is short, this implies also that the fixed point of our protocol is close to the thermal state. In practice, the mixing time generally depends on parameters of the protocol that are also used to control the error per time step. This limits the minimum achievable fixed-point error. Developing a better understanding of, and eventually overcoming, these limitations—which also impede the practicality of the exact samplers—remains an important open question raised by our work.

Aficionados of near-term quantum algorithms will recognize a resemblance between our proposal and the “quasiparticle cooling” approach of Refs. [9,10]. Formally, the difference between the two protocols is that ours implements a certain additional unitary “rewinding” procedure. However, this step is unnecessary *if* one assumes the validity of a quasiparticle picture. We note that our motivation here is also distinct: We aim to engineer a specific time evolution of the system-bath coupling, such that the dissipative portion of our Lindbladian

coincides with that of an exact Gibbs sampler. This then admits an analysis that does not rely on a quasiparticle picture, allowing us to arrive at rigorous error bounds at arbitrary temperature and for arbitrary Hamiltonians; *inter alia*, our results also provide a rigorous (but less tight) error bound for the quasiparticle cooling protocol. Nevertheless, the link to Refs. [9,10] highlights the feasibility of our protocol. It also suggests one reason why their approach to thermal state preparation appears to work even when a quasiparticle description is absent (see Ref. [10] for examples), and it can be fashioned into a versatile tool with broader applicability.

The remainder of this paper is organized as follows. In Sec. II, we provide some relevant background on the history of quantum thermal state preparation, with an emphasis on recent work on exact Gibbs samplers. (The latter may also serve as a useful physicists’ guide to recent results in the quantum computer science literature.) We then introduce our local driving protocol in Sec. III, and both state and motivate bounds on its effectiveness and its efficiency, in terms of the scaling of resources required to achieve a specified proximity (in 1-norm distance) to the target density matrix. We then proceed to give a rigorous proof of these bounds in Sec. IV, which contains the bulk of the technical material. Readers primarily interested in physical intuition can skip directly to Sec. IV C, which gives the resource estimate implied by these bounds. Section V simulates the protocol numerically on small systems, demonstrating agreement with various aspects of the error bounds. We close with a discussion in Sec. VI, where we also outline promising directions for future study.

II. STATE PREPARATION, DETAILED BALANCE, AND EXACT GIBBS SAMPLING

In order to introduce concepts and place our work in context, we first give some relevant background on thermal state preparation. After a brief historical orientation, our main emphasis is on recent constructions of Lindblad operators whose steady state corresponds to a specified density matrix ρ , by virtue of their satisfying a so-called Kubo–Martin–Schwinger (KMS) detailed balance condition with respect to ρ .

A. Historical background

As noted in the Introduction, the use of quantum computers to study many-body systems in thermal equilibrium usually requires the preparation of a thermal (Gibbs) density matrix $\rho_\beta = Z^{-1}e^{-\beta H}$ (where $Z \equiv \text{tr } e^{-\beta H}$) of a given Hamiltonian H . A conceptually significant effort to meet this challenge was made by Terhal and DiVincenzo [2], who proposed adapting the physical picture of equilibration under system-bath dynamics to an algorithmic, computational setting. They argued that repeated cycles of initializing a bath of ancilla qubits, evolving its interaction with the system for finite time (e.g., using Hamiltonian simulation) and performing a reset of the ancillas, asymptotically prepare a thermal state of the system. However, this result is exact only in the limit of infinite bath size and vanishing system-bath coupling familiar in the derivation of master equations. Thus, their analysis does not address the feasibility of the protocol in more practical

settings, where making general statements in the absence of any foreknowledge of H is challenging.

Reference [15] instead proposed an algorithm of a very different spirit, by lifting the technique of importance-sampling configurations from the classical to the quantum setting. Such “quantum Metropolis sampling” involves a Metropolis-weighted random walk on eigenstates, using quantum phase estimation to determine energies.

It has recently been claimed that this algorithm is not provably efficient because of the finite energy resolution: the error in estimating the energy is argued to propagate to the error (e.g., in norm distance) between the actual and desired fixed points of the sampling algorithm, viewed as a quantum channel on the Hilbert space. (For a recent claimed resolution via weak measurements, see Ref. [23].)

A similar issue also applies to system-bath models. One way of ensuring that the Lindbladian governing the evolution of the system has ρ_β as a fixed point is to demand that it satisfy quantum detailed balance [49]; while we will make this notion precise below, a lucid discussion oriented towards physicists may be found in Ref. [33]. A formal solution satisfying detailed balance, the so-called “Davies’ Lindbladian”, requires nonlocal jump operators that distinguish individual eigenstates, so that its construction again requires exponentially fine energy resolution (or equivalently, exponentially long time evolution). Quantum Metropolis sampling relies on access to the eigenbasis to ensure detailed balance. Thus, both approaches fail because of the well-known [50] unphysicality of eigenstates in many-body systems.

Alternative approaches have employed classical detailed balance together with the eigenstate thermalization hypothesis to construct Gibbs samplers [7,46]. However, this assumption may limit their applicability in low-temperature regimes or in systems that fail to act as their own bath.

A recent burst of activity has provided an elegant *exact* solution to the longstanding challenge of preparing thermal states. First, Ref. [26] observed that for any geometrically local Hamiltonian H , the Davies’ jump operators can be replaced by “filtered” analogues that are quasilocal (in essence, generated by finite-time evolution of local operators) yet leading to a Lindbladian whose fixed point is only polynomially far from the thermal state ρ_β . Shortly thereafter, a subset of those authors showed [27] that adding a suitably tailored coherent evolution leads to a Lindbladian with ρ_β as an *exact* fixed point. The central insight in both works is that one should impose a suitable form of “Kubo–Martin–Schwinger” (KMS) detailed balance condition on the Lindbladian, and adjust the filter function and corresponding coherent evolution to ensure that this is satisfied. This construction was then generalized [28,29], leading to the identification of the most general family of Lindbladians satisfying the KMS condition. We turn next to a summary of these recent results, which will serve as essential background to our work.

B. Exact Gibbs samplers

Since KMS detailed balance and exact Gibbs sampling play a central role in our work, we give a brief overview of these topics to establish notation. Since we take a somewhat abstract perspective when introducing the necessary formal-

ism, readers may wish to initially skip or just skim this section, and return to it after the more concrete derivations of Sec. III, but before studying the derivation of error bounds in Sec. IV.

Much of what follows, including our notation, is adapted from Refs. [27,29], to which the reader is referred for details. While we attempt to give as much detail as feasible, more laborious computations are relegated to Appendix A.

1. Kubo–Martin–Schwinger detailed balance

The Lindbladian superoperator implements time evolution of states (density matrices) in the Schrödinger picture, according to [51]

$$\frac{d\rho}{dt} = \mathcal{L}[\rho], \quad (1)$$

so that formally we have $\rho(t) = e^{t\mathcal{L}}[\rho(0)]$. For $\mathcal{L}$ to represent a sensible time evolution, we require that $e^{t\mathcal{L}}$ is a completely positive trace-preserving (CPTP) map on density matrices. Trace preservation $\text{tr} e^{t\mathcal{L}}[\rho] = \text{tr} \rho$ in turn requires that $\text{tr} \mathcal{L}[\rho] = 0$.

The corresponding Heisenberg evolution of operators is implemented by the adjoint Lindbladian $\mathcal{L}^\dagger$, taken with respect to the Frobenius inner product $\langle A, B \rangle \equiv \text{tr}[A^\dagger B]$: for any state ρ and operator $\mathcal{O}$, we have $\text{tr}[\mathcal{L}[\rho]\mathcal{O}] = \text{tr}[\rho\mathcal{L}^\dagger[\mathcal{O}]]$. Requiring that the Schrödinger evolution preserves the trace is equivalent to requiring that the Heisenberg evolution be *unital*, i.e., preserves the identity, $e^{t\mathcal{L}^\dagger}[\mathbb{1}] = \mathbb{1}$, which in turn requires $\mathcal{L}^\dagger[\mathbb{1}] = 0$.

Given any full-rank density matrix ρ , we also define a self-adjoint “weighting” superoperator Γ_ρ ,

$$\Gamma_\rho[\cdot] := \rho^{1/2}(\cdot)\rho^{1/2} = \Gamma_\rho^\dagger[\cdot]. \quad (2)$$

A Lindbladian $\mathcal{L}$ satisfies KMS detailed balance with respect to a full-rank density matrix ρ if

$$\mathcal{L}^\dagger = \Gamma_\rho^{-1} \circ \mathcal{L} \circ \Gamma_\rho. \quad (3)$$

An immediate corollary of KMS detailed balance is that ρ is a fixed point of $\mathcal{L}$. To see this, observe that Eq. (3) is equivalent to the condition $\mathcal{L} = \Gamma_\rho \circ \mathcal{L}^\dagger \circ \Gamma_\rho^{-1}$, so that we have

$$\mathcal{L}[\rho] = \rho^{1/2}\mathcal{L}^\dagger[\rho^{-1/2}(\rho)\rho^{-1/2}]\rho^{1/2} = 0, \quad (4)$$

where the second equality follows from $\mathcal{L}^\dagger[\mathbb{1}] = 0$.

In its stated form Eq. (3), the KMS condition does not immediately resemble the classical notion of detailed balance, which is often formulated in terms of the “reversibility” of the transition probabilities. The connection can be sharpened by defining the *KMS inner product* with respect to ρ ,

$$\langle A, B \rangle_\rho \equiv \text{tr} \sqrt{\rho} A \sqrt{\rho} B. \quad (5)$$

The KMS detailed balance condition [Eq. (3)] is then the statement that $\mathcal{L}^\dagger$ is self-adjoint with respect to the inner product in Eq. (5). Equivalently, we may also say that $\mathcal{L} \circ \Gamma_\rho = \Gamma_\rho \circ \mathcal{L}^\dagger$ is self-adjoint with respect to the Frobenius inner product. Either perspective brings the KMS condition in consonance with classical detailed balance, which can be defined as the transition matrix being symmetric when weighted appropriately by the Boltzmann weights.

2. Structure of KMS-detailed balanced Lindbladians

To understand the structure imposed on $\mathcal{L}$ by KMS detailed balance, it is convenient to pass to a representation in terms of the Bohr frequencies (i.e., energy differences) of the Hamiltonian H that specifies the target thermal density matrix ρ_β ; we denote the set of such frequencies $\nu \in \mathfrak{B}$. We may then write $A(t) \equiv e^{iHt} A e^{-iHt} = \sum_{\nu \in \mathfrak{B}} A_\nu e^{i\nu t}$, where

$$A_\nu \equiv \sum_{\substack{\omega_1, \omega_2 \in \text{Spec}(H) \\ \omega_1 - \omega_2 = \nu}} |\omega_1\rangle \langle \omega_1| A |\omega_2\rangle \langle \omega_2|, \quad (6)$$

so that $(A_{-\nu})^\dagger = (A^\dagger)_\nu$.

We will first write $\mathcal{L}$ in terms of the Bohr frequency representation of some set of jump operators $A_a \in \mathcal{A}$. In this representation, a generic Lindbladian $\mathcal{L} = \mathcal{G} + \mathcal{T} + \mathcal{R}$ can be decomposed into “transition”, “decay”, and “coherent parts” given by

$$\mathcal{T}[\cdot] = \sum_{a \in \mathcal{A}} \sum_{\nu_1, \nu_2 \in \mathfrak{B}} \alpha_{\nu_1, \nu_2} A_{\nu_1}^a(\cdot) (A_{\nu_2}^a)^\dagger, \quad (7a)$$

$$\mathcal{R}[\cdot] = -\frac{1}{2} \sum_{a \in \mathcal{A}} \sum_{\nu_1, \nu_2} \alpha_{\nu_1, \nu_2} \{ (A_{\nu_2}^a)^\dagger A_{\nu_1}^a, \cdot \}, \quad (7b)$$

$$\mathcal{G}[\cdot] = -i \sum_{a \in \mathcal{A}} \sum_{\nu_1, \nu_2} g_{\nu_1, \nu_2} [(A_{\nu_2}^a)^\dagger A_{\nu_1}^a, \cdot], \quad (7c)$$

where $\sum_{a \in \mathcal{A}} \dots \equiv \sum_{a=1}^{|\mathcal{A}|}$, and we require that $\alpha_{\nu_1, \nu_2} = (\alpha_{\nu_2, \nu_1})^*$, $g_{\nu_1, \nu_2} = (g_{\nu_2, \nu_1})^*$ and that α_{ν_1, ν_2} is a positive semidefinite matrix for $\mathcal{L}$ to be CPTP. The potentially counterintuitive ordering of the frequency labels in $\mathcal{T}$ relative to $\mathcal{R}$ and $\mathcal{G}$ is required by the Lindbladian structure.

Note that while the kernel α_{ν_1, ν_2} of $\mathcal{T}$ is fixed to that of $\mathcal{R}$ by trace preservation, that of the coherent part g_{ν_1, ν_2} is arbitrary. However, as we will see, this is linked to α_{ν_1, ν_2} by detailed balance.

We now impose KMS detailed balance Eq. (3) with respect to a thermal density matrix $\rho_\beta = \frac{1}{Z} e^{-\beta H}$ and ask how this constrains $\mathcal{L}$.

First, we consider the transition part. From a straightforward calculation (Appendix A), we have

$$(\Gamma_\beta^{-1} \circ \mathcal{T} \circ \Gamma_\beta)[\cdot] = \sum_{a \in \mathcal{A}} \sum_{\nu_1, \nu_2 \in \mathfrak{B}} \alpha_{\nu_1, \nu_2} e^{\frac{\beta(\nu_1 + \nu_2)}{2}} \times A_{\nu_1}^a(\cdot) (A_{\nu_2}^a)^\dagger. \quad (8)$$

Meanwhile, under the assumption that the set of jump operators $\mathcal{A}$ is closed under Hermitian conjugation (i.e., $A_a \in \mathcal{A}$ iff $A_a^\dagger \in \mathcal{A}$), a slightly more involved computation (detailed in Appendix A) yields

$$\mathcal{T}^\dagger[\cdot] = \sum_{a \in \mathcal{A}} \sum_{\nu_1, \nu_2 \in \mathfrak{B}} \alpha_{-\nu_2, -\nu_1} A_{\nu_1}^a(\cdot) (A_{\nu_2}^a)^\dagger. \quad (9)$$

Comparing Eqs. (8) and (9), we see that $\mathcal{T}$ is KMS detailed balanced, i.e., $\mathcal{T}^\dagger = \Gamma_\beta^{-1} \circ \mathcal{T} \circ \Gamma_\beta$, if

$$\alpha_{-\nu_2, -\nu_1} = \alpha_{\nu_1, \nu_2} e^{\frac{\beta(\nu_1 + \nu_2)}{2}}. \quad (10)$$

While $\mathcal{T}$ satisfies detailed balance on its own, for generic choices of α_{ν_1, ν_2} satisfying Eq. (10), the decay and coherent parts $\mathcal{R}$ and $\mathcal{G}$ do not satisfy detailed balance individually, but

instead are intertwined by the action of Γ_β . However, as first noted by Ref. [27] (and here summarized in Appendix A), $\mathcal{R}$ and $\mathcal{G}$ together satisfy KMS detailed balance, i.e., $(\mathcal{G} + \mathcal{R})^\dagger = \Gamma_\beta^{-1} \circ (\mathcal{G} + \mathcal{R}) \circ \Gamma_\beta$, if we choose

$$g_{\nu_1, \nu_2} = -\frac{1}{2i} \tanh \frac{\beta(\nu_1 - \nu_2)}{4} \alpha_{\nu_1, \nu_2}. \quad (11)$$

Any set of coefficients α_{ν_1, ν_2} and g_{ν_1, ν_2} that satisfy the conditions in Eqs. (10) and (11) generates a Lindbladian via Eq. (7) that exactly satisfies KMS detailed balance, and hence has the thermal state ρ_β as a fixed point. Two choices are especially salient. The first is to take

$$\alpha_{\nu_1, \nu_2} = \delta_{\nu_1, \nu_2} \gamma(\nu_1) \quad \text{with} \quad \gamma(-\nu) = \gamma(\nu) e^{\beta\nu}, \quad (12)$$

in which case $G = 0$. This corresponds to the well-known Davies Lindbladian. The drawback, as noted previously, is that generating such Lindblad dynamics by evolving local jump operators under a local H requires a time exponentially long in system size (since, in order to resolve the smallest Bohr frequencies, we need to evolve until the Heisenberg time).

A different choice, and our main focus, is [29]

$$\alpha_{\nu_1, \nu_2} = \hat{f}(-\nu_1) [\hat{f}(-\nu_2)]^* \quad (13)$$

with

$$\hat{f}(\nu) = e^{+\beta\nu/4} q(\nu), \quad \text{and} \quad q(-\nu) = q^*(\nu). \quad (14a)$$

With the choice of $\hat{f}(\nu)$, the transition and decay parts in Eq. (7) can be brought into the form

$$\mathcal{L}_\beta[\cdot] = -i[G, \cdot] + \sum_{a \in \mathcal{A}} L_a(\cdot) L_a^\dagger - \frac{1}{2} \{L_a^\dagger L_a, \cdot\}, \quad (14b)$$

where

$$G = \frac{i}{2} \sum_{a \in \mathcal{A}} \sum_{\nu \in \mathfrak{B}} \tanh \left(\frac{\beta\nu}{4} \right) (L_a^\dagger L_a)_\nu \quad (14c)$$

and

$$L_a = \sum_{\nu \in \mathfrak{B}} \hat{f}(-\nu) (A_a)_\nu = \int_{-\infty}^{\infty} f(t) A_a(t) dt, \quad (14d)$$

with $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\nu) e^{i\nu t} d\nu$ being the inverse Fourier transform of $\hat{f}(\nu)$ [as can be seen using Eq. (6)]. We have added a subscript β to emphasize that Eq. (14b) is detailed balanced with respect to ρ_β . For a suitable choice of $q(\nu)$ (e.g., a Gaussian), the time-domain filter function decays quickly, so we can view Eq. (14d) as a “smoothing” of the time-evolved jump operator. As such, for local H , the L_a obtained in this way will be quasilocal because of Lieb–Robinson bounds.

The KMS condition [Eq. (3)] with respect to ρ_β is then conveniently stated as

$$\rho_\beta^{-1/2} L_a \rho_\beta^{1/2} = L_a^\dagger. \quad (15)$$

Reference [29] shows that Lindbladians that satisfy KMS detailed balance (i.e., exact Gibbs samplers) can always be written in the form of Eq. (14).

III. THE LOCAL DRIVING SAMPLER

The form of the exact Gibbs sampler, where quasilocall jump operators are generated by time evolution of strictly local operators, suggests that one might be able to generate such a Lindbladian by leveraging the operator spreading that naturally occurs under Hamiltonian dynamics. Combining this observation with known results on Lindbladian simulation via Hamiltonian simulation, we show below that we can implement, efficiently and with controlled error, a Lindbladian that reproduces an exact Gibbs sampler *up to the coherent part*. While the latter is important to *exactly* satisfy KMS detailed balance, this is inessential if we require only that the thermal state be an *approximate* steady state [26,48]. We show below in Sec. IV that the Lindbladian that we implement is indeed such an approximate Gibbs sampler with controlled fixed-point error, assuming that the mixing time does not scale adversely with either system size or protocol parameters. To derive such guarantees, the closeness to a Lindbladian that satisfies the KMS-detailed balance condition is essential. (To sequester the technical portion of the work to a single section, here we only discuss the high-level picture and relegate detailed derivations to the sections in which we prove error bounds in Sec. IV and Appendixes B and C.)

Our setup, sketched in Fig. 1, consists of a system with Hamiltonian H acting on n_S system qubits, coupled to a bath of n_B ancilla qubits, one per jump operator A_a , $a = 1, 2 \dots |A| = n_B$.

The Hamiltonian for the coupled system is

$$H_{SB} = I_B \otimes H + H_B \otimes I_S + \sum_a J \{ f(t) B_a^\dagger A_a + f^*(t) B_a A_a^\dagger \}, \quad (16)$$

where $f(t)$ is taken to coincide with the filter function appearing in the exact sampler [cf. Eqs. (14a) and (14d)] and $B = \frac{1}{2}(X_B - iY_B)$, $B^\dagger = \frac{1}{2}(X_B + iY_B)$ are lowering and raising operators for the ancilla, which satisfy $B^2 = (B^\dagger)^2 = 0$ and $B^\dagger|0\rangle = |1\rangle$, $B|0\rangle = 0$. We will often be interested in the case where the bath Hamiltonian is trivial, but for now allow there to be some nontrivial bath dynamics governed by H_B ; note, however, that we will *only* consider the case where the bath is noninteracting, i.e., the bath Hamiltonian H_B does not involve coupling between the ancillas. [The case where the bath interactions are purely diagonal can be shown to be equivalent to this, but may prove useful, e.g., in engineering appropriate $f(t)$'s.]

We now move to the interaction picture with respect to $H + H_B$: the corresponding interaction-picture Hamiltonian is then denoted by a tilde, and consists purely of a system-bath interaction

$$\tilde{H}_{SB}(t) = J \sum_a \{ f(t) B_a^\dagger(t) \otimes A_a(t) + f^*(t) B_a(t) \otimes A_a^\dagger(t) \} \quad (17)$$

with $\mathcal{O}_B(t) \equiv e^{iH_B t} \mathcal{O}_B e^{-iH_B t}$ for $\mathcal{O} = X, Y$, and $A(t) \equiv e^{iH t} A e^{-iH t}$ as above.

We assume that the $f(t)$ remains operational over some finite time interval $[-T/2, T/2]$, chosen to be symmetric for convenience. Returning to the Schrödinger picture, we see that

this implements the unitary $e^{-iHT} \tilde{V}$. We complete the cycle by “resetting” the bath to $\rho_B^0 = |0\rangle\langle 0|_B$, while implementing a “rewinding step” only on the system: a *backwards* time evolution with H for a period T . In other words, a single cycle of the time evolution implements the following channel on the *Schrödinger picture* density matrix ρ :

$$\mathcal{K}[\rho] = \text{tr}_B[\tilde{V}(\rho_B^0 \otimes \rho)\tilde{V}^\dagger]. \quad (18)$$

We emphasize that here,

$$\tilde{V} \equiv \mathcal{T}_t \exp \left(-i \int_{-T/2}^{T/2} dt \tilde{H}_{SB}(t) \right) \quad (19)$$

is the *interaction-picture* time-evolution operator for the composite system comprising the target system and the bath of ancillas, even though Eq. (18) describes the evolution of the *Schrödinger* density matrix. This explains the point of the “rewinding” step: to allow a repeated application of the channel corresponding to evolution under $\tilde{V}$ followed by a bath reset, which is *not* equivalent to simply evolving under $H + V(t)$ for time T followed by a reset.

The idea is that such a channel implements a Liouvillian time evolution over a controllably small interval $\Delta\tau = J^2$ up to an error $O(J^4)$. To see this, expand $\tilde{V}$ in powers of J up to third order via the Magnus expansion: we may write

$$\left\| \tilde{V} - \exp \left(-i \sum_{n=1}^3 \tilde{H}_M^{(n)} \right) \right\|_\infty = O \left(n_B \left(\frac{JT}{\sigma} \right)^4 \exp \left(\frac{\beta^2}{2\sigma^2} \right) \right), \quad (20)$$

as is derived in Appendix B1. Here, $\tilde{H}_M^{(n)}$ represents the n th-order term in the Magnus expansion in the interaction picture, and $\|A\|_\infty = \sup_{\|v\|=1} \|Av\|$ where $\|v\| = \sqrt{\sum_i |v_i|^2}$ denotes the spectral norm.

We now insert the final expression (20) for $\tilde{V}$ into Eq. (18) and expand the exponential. Owing to the assumption that the ancilla qubit is reset after each cycle, we can restrict our attention to terms in the expansion that involve $H_F^{(m)} \cdot H_F^{(n)}$ with $m+n$ being even, which enter at $O(J^{m+n})$. Accordingly, we have

$$\begin{aligned} \mathcal{K}[\rho] &= \rho + \text{tr}_B[\tilde{H}_M^{(1)}(\rho_B^0 \otimes \rho)\tilde{H}_M^{(1)}] \\ &\quad - \frac{1}{2} \text{tr}_B[\{(\tilde{H}_M^{(1)})^2, (\rho_B^0 \otimes \rho)\}] \\ &\quad - i \text{tr}_B[\tilde{H}_M^{(2)}, \rho_B^0 \otimes \rho] + O(n_B J^4). \end{aligned} \quad (21)$$

A straightforward but tedious calculation, presented in Appendix B, now yields the fact that

$$\mathcal{K}[\rho] = e^{J^2 \mathcal{L}_T}[\rho] + O(n_B J^4), \quad (22)$$

where the effective Lindbladian $\mathcal{L}_T$ can be written in a very appealing form: we have

$$\mathcal{L}_T[\cdot] = -i[H_{LS}, \cdot] + \mathcal{L}_{\text{diss},T}[\cdot], \quad (23a)$$

where we have separated the f -smoothed Lindblad dynamics into a purely dissipative evolution

$$\mathcal{L}_{\text{diss},T}[\cdot] = \sum_a L_{a,T} \cdot L_{a,T}^\dagger - \frac{1}{2} \{L_{a,T}^\dagger L_{a,T}, \cdot\} \quad (23b)$$

with the jump operator,

$$L_{a;T} \equiv \int_{-T/2}^{T/2} f(t) A_a(t) dt, \quad (23c)$$

and a coherent Lamb shift contribution governed by the Hamiltonian

$$H_{LS;T} = -\frac{1}{2i} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 f^*(t_2) f(t_1) \times \text{sgn}(t_1 - t_2) \sum_a A_a^\dagger(t_2) A_a(t_1) \quad (23d)$$

$$= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2} h_{v_1, v_2}^{(T)} (A_{v_2}^a)^\dagger A_{v_1}^a, \quad (23e)$$

where

$$h_{v_1, v_2}^{(T)} = -\frac{1}{2i} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 f(t_1) f^*(t_2) \times \text{sgn}(t_1 - t_2) e^{i v_1 t_1 - i v_2 t_2}. \quad (23f)$$

Note that the transition, decay, and coherent (Lamb shift) contributions each, respectively, arise from the three different traces over the bath in Eq. (21).

Observe that $\mathcal{L}_T$ has almost precisely the form of the exact Gibbs sampler in Eq. (14), up to a finite-time truncation of the f -filtered Heisenberg evolution, but with a distinct coherent term: instead of the form G required by the exact sampler [Eq. (14c)], the coherent evolution generated by the local drive is a “physical” Lamb shift [48].

This second distinction between the exact and local driving samplers is most easily seen by taking $T \rightarrow \infty$ in the latter; the resulting Lindbladian has transition and decay parts that take the form of Eqs. (7a) and (7b) with α_{v_1, v_2} given by Eq. (13), but with a different coherent part; instead of a kernel g_{v_1, v_2} in Eq. (7c) that satisfies the condition in Eq. (11) for exact detailed balance, we instead have

$$g_{v_1, v_2} \rightarrow h_{v_1, v_2} \equiv \lim_{T \rightarrow \infty} h_{v_1, v_2}^{(T)}. \quad (24)$$

We have so far made no specific choice of filter function $f(t)$. Henceforth, we make the convenient choice of a shifted Gaussian [29],

$$f(t) = \sqrt{\frac{2}{\pi \sigma^2}} e^{-\frac{2}{\sigma^2} (t - \frac{i\beta}{4})^2}, \quad (25)$$

corresponding to taking $q(v) = e^{-\frac{(\sigma v)^2}{8}}$ in Eq. (14a). This fixes both the exact KMS-detailed balanced and the local-driving Lindbladians that we consider in the remainder of this paper. In the next section, we bound the error in approximating the former by the latter.

IV. BOUNDS ON THE ACCURACY

In this section, we summarize the different error sources and their contributions to the final state when using the setup described in Sec. III to prepare an approximate thermal state of a Hamiltonian H . We use these results to derive an upper bound on the Hamiltonian simulation time necessary to prepare the thermal state ρ_β up to some error per qubit ϵ in trace norm. This will also clarify the conditions under which this

error can be made arbitrary small, and how to estimate the minimal error that can be achieved by our protocol.

We summarize the different approximations made in Fig. 2. The total error of implementing the local driving protocol as opposed to the exact sampler in Eq. (14) stems, roughly speaking, from two sources. First, the physical protocol implements a finite-time evolution with a Lindbladian $\mathcal{L}_T$, up to an error that is controlled by the truncation error of the third-order Magnus expansion in the interaction picture [52]. As explained below, this in turn is governed by the time step $\Delta\tau$ in the discretization of the target Lindblad evolution, which in effect scales as the second power of the system-bath coupling J . The second source of error is the fact that even if we could perfectly realize the Lindbladian $\mathcal{L}_T$, it has a steady state ρ_{SS} (i.e., $\mathcal{L}_T[\rho_{SS}] = 0$), which is only approximately equal to the thermal state, i.e., $\rho_{SS} \approx \rho_\beta$. This *fixed-point error* is ultimately controlled by the parameter T of $\mathcal{L}_T$, the width σ of the filter function, and the mixing time τ_{mix} . These quantities enter the error bound in two ways. Clearly, T affects the extent to which the finite- T smoothing of jump operators in Eq. (23c) approximates that of the exact sampler in Eq. (14d). Less obviously, σ , T , and τ_{mix} together govern the modification of the fixed point owing to the “wrong” coherent evolution, via their influence on the “secular approximation” discussed in Sec. IV B. Note that the quantities entering the fixed-point error are not independent, as the mixing time in general will depend on both σ and T . As discussed in Sec. IV C, if this dependence is too strong, it can limit the minimal accuracy achievable by the protocol.

We bound the fixed-point error in terms of the trace norm,

$$\|\rho - \sigma\|_1 = \text{tr} \sqrt{(\rho - \sigma)(\rho - \sigma)^\dagger}. \quad (26)$$

The trace norm gives an upper bound on the error in observables, as $\text{tr}(A(\rho - \sigma)) \leq \|A\|_\infty \|\rho - \sigma\|_1$. Moreover, in order to prove the bounds, we make use of the Schatten p -norms defined as

$$\|\rho\|_p = (\text{tr}(\sqrt{\rho \rho^\dagger})^p)^{1/p} \quad (27)$$

and the induced superoperator norms

$$\|K[\cdot]\|_{p \rightarrow p} = \sup_{\|X\|_p=1} \|K[X]\|_p. \quad (28)$$

A. Lindbladian simulation error

As we advertised, our protocol implements a Lindbladian $\mathcal{L}_T$ evolution with Liouvillian time step $\Delta\tau = J^2$ approximately by performing a time-dependent Hamiltonian simulation followed by a reset; the key is that for a suitable choice of $f(t)$, the drive approximates the jump operators that enter the exact sampler. This approximation stems from replacing the time-ordered integral $\tilde{V}$ in Eq. (19) by its third-order Magnus expansion (cf. Sec. III). Therefore, to bound this error we can leverage recent results on Hamiltonian simulation in the interaction picture [52], which show that

$$\left\| \tilde{V} - \exp\left[-i \left(\sum_{k=1}^3 \tilde{H}_{\text{SB}}^{(k)} \right)\right] \right\|_1 = O(n_B(dJT)^4), \quad (29)$$

where $\tilde{H}_{\text{SB}}^{(k)}$ is the k th-order term in the Magnus expansion of $\tilde{H}_{\text{SB}}$, and $d = \max_t |f(t)|$. We use this, in Appendix B, to

Fixed point	Lindbladian/ Channel	Parameter choices
$\rho_{\text{fix}} = \rho_\beta \propto e^{-\beta \hat{H}}$	$\mathcal{L}_\beta = -i[G, \bullet] + \mathcal{L}_{\text{diss}}$	$q(\nu) = e^{-\frac{(\sigma\nu)}{8}}$
$\ \rho_\infty - \rho_\beta\ _1 = \tilde{O}(n\epsilon)$	$\mathcal{L}_\infty = -i[H_{\text{LS}}, \bullet] + \mathcal{L}_{\text{diss}}$	$\sigma \gtrsim \beta \tau_{\text{mix}} \epsilon^{-1}$
$\ \rho_T - \rho_\beta\ _1 = \tilde{O}(n\epsilon)$	$\mathcal{L}_T = -i[H_{\text{LS},T}, \bullet] + \mathcal{L}_{\text{diss},T}$	$T \gtrsim \sigma \sqrt{\log(\tau_{\text{mix}} \epsilon^{-1})}$
$\ \rho_{\mathcal{K}} - \rho_\beta\ _1 = \tilde{O}(n\epsilon)$	$\mathcal{K}[\bullet] = e^{\Delta\tau \mathcal{L}_T} \bullet + O((\Delta\tau)^2)$	$\Delta\tau = J^2 = (\epsilon/\tau_{\text{mix}})^2$ $N_{\text{steps}} = \tau_{\text{mix}} (\Delta\tau)^{-1}$

FIG. 2. Summary of different approximations made during the derivation of the local driving sampler protocol. Overall, the setup in Fig. 1, with the parameter choices indicated in the rightmost column above, implements a channel $\mathcal{K}$ that approximates an exact Gibbs sampler and hence has an approximately thermal fixed point. As discussed in Sec. IV D, the parameter choices indicated in the last column may be nontrivial.

show that for all ρ ,

$$\begin{aligned} & \|e^{J^2 \mathcal{L}_T}[\rho] - \mathcal{K}[\rho]\|_1 \\ &= O\left(J^4 \left(n_B \left(\frac{T}{\sigma}\right)^4 + n_B^2\right) e^{\frac{\beta^2}{2\sigma^2}}\right), \end{aligned} \quad (30)$$

where $\mathcal{L}_T$ is given exactly by Eq. (23).

Effectively, our protocol implements a Lindbladian time step of size $\Delta\tau = J^2$ up to an error $O((\Delta\tau)^2)$, which allows one in principle to implement evolution with controlled error for arbitrary long times.

B. Fixed-point error

Accepting the fact that our protocol implements $\mathcal{L}_T$ up to controllable error, it remains to bound the “fixed-point error”, i.e., the difference between the steady state of the dynamics generated by $\mathcal{L}_T$ and the thermal state ρ_β . The fixed-point error again has two distinct sources, that is (i) the fact that we implement the dissipative part of the exact Gibbs sampler only up to some finite evolution time T , and (ii) the fact that $\mathcal{L}_T$ even for $T \rightarrow \infty$ implements the “wrong” coherent part of the evolution. It turns out that both of these lead to bounded errors per time step. The total error is then the error per time step multiplied by the mixing time, which governs the total Lindbladian evolution time necessary to reach the thermal state.

For a choice of filter function $f(t)$ that decays quickly on some finite time scale σ , restricting $\mathcal{L}_T$ to finite T incurs only a minor error in the Lindbladian evolution in the sense that $\mathcal{L}_\infty$ and $\mathcal{L}_T$ are close in superoperator norm for $T \gg \sigma$. This then bounds the difference between their respective fixed points,

$$\begin{aligned} \|\rho_T - \rho_\infty\|_1 &\leq 4 \tau_{\text{mix}}(\mathcal{L}_T) \|\mathcal{L}_T - \mathcal{L}_\infty\|_{1 \rightarrow 1} \\ &\leq \frac{24\sqrt{2}}{\sqrt{\pi}} n_B \left(\frac{\sigma}{T}\right) e^{\frac{\beta^2}{4\sigma^2}} e^{-\frac{T^2}{2\sigma^2}} \tau_{\text{mix}}(\mathcal{L}_T), \end{aligned} \quad (31)$$

where the first inequality is Lemma II.1 of Ref. [26], and the second, derived in Appendix C 1, follows for the specific choice of a Gaussian filter function with standard deviation σ . Here and below, we define the mixing time of a Lindbladian as the smallest time τ_{mix} for which, given any ρ_1, ρ_2 , we have

$$\|e^{\tau_{\text{mix}} \mathcal{L}}[\rho_1 - \rho_2]\|_1 \leq \frac{1}{2} \|\rho_1 - \rho_2\|_1. \quad (32)$$

Given the error bound in Eq. (31), it remains to bound the difference between the fixed point ρ_∞ of $\mathcal{L}_\infty$ and the thermal state. While this is technically the most involved step of the whole derivation, it follows from a clear intuition. Compared to the exact sampler in Eq. (14), the nontruncated local driving sampler $\mathcal{L}_\infty$ has an “added” coherent part $B = H_{\text{LS}} - G$. Now, writing B in terms of energy transfers corresponding to the Bohr frequencies of H ,

$$B = \sum_{v_1, v_2 \in \mathfrak{B}} b_{v_1, v_2} (A_{v_2})^\dagger A_{v_1}, \quad (33)$$

we can show that if the matrix b_{v_1, v_2} is fast decaying for $|v_1 - v_2| \gg \mu$ and some $\mu > 0$, then the operator B almost commutes with ρ_β . Therefore, adding it to $\mathcal{L}_\beta$ as a coherent evolution should leave the fixed point almost unchanged. The challenge is to quantify the “almost”, which technically requires employing the so-called “secular approximation” [5,26,47,48]. In our case, we show in Appendix C 2, that

$$\|\rho_\infty - \rho_\beta\|_1 = \tilde{O}\left(n_B \frac{\beta}{\sigma} \max(\tau_{\text{mix}}(\mathcal{L}_\infty), \tau_{\text{mix}}(\mathcal{L}_\beta))\right), \quad (34)$$

where $\tilde{O}$ denotes an upper bound up to (poly)logarithmic corrections. As long as the mixing time τ_{mix} does not grow superlinearly with the width of the filter function in the time domain σ , this part of the fixed-point error is hence controlled by σ compared to the inverse temperature β .

Combining the two sources of fixed-point error, we find

$$\|\rho_T - \rho_\beta\|_1 = \tilde{O}\left(n_B \left(\left(\frac{\sigma}{T}\right) e^{\frac{\beta^2}{4\sigma^2}} e^{-\frac{T^2}{2\sigma^2}} + \frac{\beta}{\sigma}\right) \tau_{\text{mix}}^*\right), \quad (35)$$

where $\tau_{\text{mix}}^* \equiv \max(\tau_{\text{mix}}(\mathcal{L}_T), \tau_{\text{mix}}(\mathcal{L}_\infty), \tau_{\text{mix}}(\mathcal{L}_\beta))$. This informs the choice of filter function and finite-time truncation: we want to choose $T \gg \sigma \gg \beta$ [cf. Eq. (36)].

C. Resource estimate

Using the error bounds derived above, we can estimate the total cost of preparing a thermal state ρ_β up to some precision ϵ . For that, we want both the fixed-point error in Eq. (35), as well as the total accumulated simulation error in Eq. (30) to be $\tilde{O}(\epsilon)$.

Assuming for a moment that all quantities appearing in the error bounds are independent, then for a given target total error ϵ , we choose

$$\sigma = \frac{\beta \tau_{\text{mix}}^*}{\epsilon}, \quad (36a)$$

$$T = \sigma \sqrt{\log \frac{\tau_{\text{mix}}^*}{\epsilon}} = \frac{\beta \tau_{\text{mix}}^*}{\epsilon} \sqrt{\log \frac{\tau_{\text{mix}}^*}{\epsilon}}, \quad (36b)$$

which, upon substitution into Eq. (35), yields

$$\|\rho_T - \rho_\beta\|_1 = \tilde{O}(\epsilon (1 + \sqrt{\log \epsilon^{-1}})) \quad (37)$$

$$= \tilde{O}(\epsilon). \quad (38)$$

Further, the total accumulated error during a Lindbladian time evolution of time $\propto \tau_{\text{mix}} \equiv \tau_{\text{mix}}(\mathcal{L}_T)$ is $O(n_B^2 \tau_{\text{mix}} J^2) = O(\epsilon)$ if $J^2 = \tilde{O}(\epsilon n_B^{-2} \tau_{\text{mix}}^{-1})$, and hence we need $N_{\text{step}} = O(\tau_{\text{mix}}^2 \epsilon^{-1} n_B^{-2})$ elementary steps of the protocol to reach the steady state. The total Hamiltonian evolution time necessary to prepare the thermal state ρ_β up to error $\tilde{O}(\epsilon)$ in trace norm then scales as

$$T_{\text{tot}} = O(N_{\text{step}} T) \quad (39a)$$

$$= \tilde{O}\left(\frac{n_B^2 \beta \tau_{\text{mix}}^2 \tau_{\text{mix}}^*}{\epsilon^2} \sqrt{\log \frac{\tau_{\text{mix}}^*}{\epsilon}}\right) \quad (39b)$$

$$= \tilde{O}\left(\frac{n_B^2 \beta \tau_{\text{mix}}^2 \tau_{\text{mix}}^*}{\epsilon^2}\right). \quad (39c)$$

This remains polynomial in system size even if we demand that the fixed-point be reached up to an $O(1)$ total error, assuming a polynomial-in-system size mixing time. In practice, the method is most effective if the mixing time has a mild (e.g., logarithmic) scaling with n_S , and ϵ can be chosen to be extensive, i.e., $\epsilon = n_S \epsilon_{\text{loc}}$ for some constant ϵ_{loc} . The latter we expect to be sufficient in many cases to obtain, e.g., local observables in the thermal state up to an $\tilde{O}(\epsilon)$ error.

The mixing time is in general controlled by spectral properties of the Lindbladian [53,54]. Although there exist Hamiltonians for which any dynamics with (quasi)local jump operators can be shown to have a mixing time that is exponentially slow in system size [41–44], fast mixing has been established for many models and temperature regimes of interest [55–62].

The discussion above is true even in the case where the mixing time τ_{mix}^* depends on σ and T , as long as we can make the choice of parameters in Eq. (36). However, if τ_{mix}^* depends on σ , this choice becomes nontrivial, or even impossible, for some ϵ . Because of this, in general there exists a minimal total error ϵ^* that can be achieved, which we discuss in the following.

D. Optimal accuracy

While the derived bounds on the fixed-point error apply to arbitrary choices of Hamiltonians, temperatures, and parameter σ in the filter function, it remains an open question to understand the extent to which the fixed point ρ_β can be approached by the channel $\mathcal{K}[\rho]$: In other words, to establish the values of ϵ for which the parameter choice in Eq. (36) is possible. The choice is nontrivial because the mixing time τ_{mix}^* in general depends on the Lindbladian, and hence the parameters σ and T .

The dependence on σ in particular is problematic since it can be very strong. This is because increasing the temporal filter width σ correspondingly decreases the width of the filter in frequency space, and transitions with energy ΔE are exponentially suppressed if $\Delta E \gg \sigma^{-1}$. In systems with a discrete spectrum (e.g., few-body or noninteracting systems), and without fine-tuning the temperature, one thus expects the mixing time to diverge *exponentially*, as soon as the inverse filter width σ^{-1} is smaller than the level spacing. The same problem also arises in the arguably more interesting setting of gapped systems at temperatures smaller than the gap (recall that we chose $\sigma \gg \beta$). In these cases, the fixed-point error cannot be arbitrarily reduced using the channel $\mathcal{K}[\rho]$, and there exists an optimal choice σ^* that minimizes it.

The dependence of the mixing time on σ becomes more involved in interacting systems at intermediate temperatures. In this case, the level spacing is expected to be exponentially small, and there are possible transitions for any σ . Still, even in this case numerical results for the mixed field Ising model (Appendix E) show an exponential increase of the mixing time with σ . This suggests that even in this case, there exists a finite σ^* , which minimizes the fixed-point error.

We note that the strong dependence of the mixing time τ_{mix}^* on the parameter σ limits not only our approximate protocol, but also poses problems for exact samplers, where $\sigma = \beta$ is a common choice of parameters [27,29]. Specifically, while such a choice does not limit the *accuracy* of exact samplers (obviously, given the name), it does limit their *efficiency* in preparing the thermal state from arbitrarily initial conditions. Developing a better understanding of the dependence of the mixing time on σ is therefore an important task for future work, both to find regimes of best applicability and how to avoid this dependence altogether. We note that since an early version of this work appeared in preprint form, there have been two notable proposals in this direction [63,64].

Despite these obvious drawbacks, we note that our protocol numerically yields good performance in certain regimes, as we explore for a simple example in Sec. V.

E. Reduction to quasiparticle cooling and the “rewinding” step

We now comment on the precise link between our work and the quasiparticle cooling protocol of Refs. [9,10]. Relative to the latter, our protocol (see Fig. 1) implements an additional backwards time evolution after each step. As noted above, this step is what allows us to build a channel that involves repeated application of the interaction-picture unitary $\tilde{V}$. However, evidently this step is superfluous if for each cycle $[\rho(t), H] = 0$

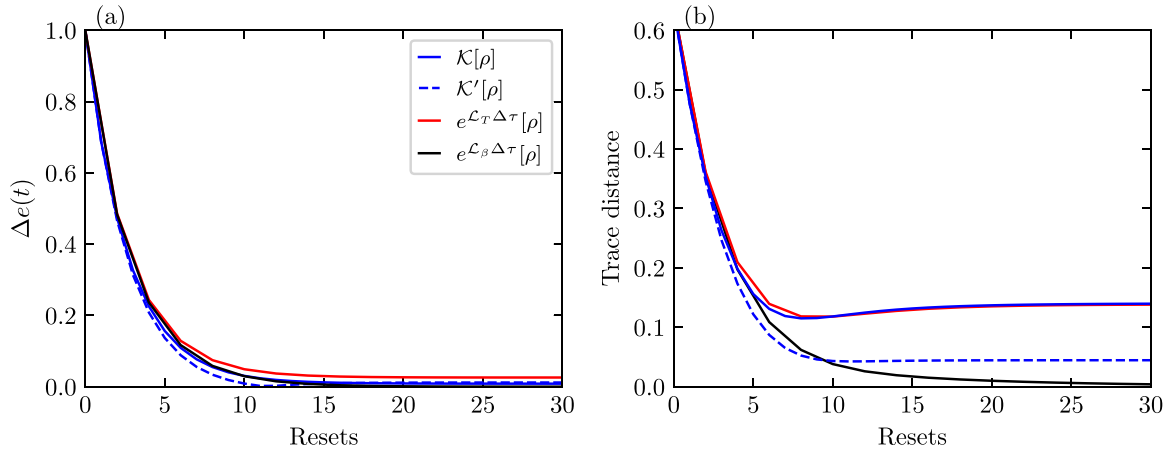


FIG. 3. Comparison of the time evolution of thermal state preparation for the mixed-field Ising model Eq. (41) under the exact Gibbs sampler $\mathcal{L}$ [cf. Eq. (14b)] (black), the Lindbladian $\mathcal{L}_T$ [cf. Eq. (23a)] (red), the local driving protocol $\mathcal{K}[\rho]$ [cf. Eq. (18)] (blue), and the protocol $\mathcal{K}'[\rho]$ without rewinding (dashed blue) starting from the maximally mixed state $\rho_0 = 2^{-n_S} \mathbb{1}$. Parameters are $\beta = 1.0$, $\sigma = 0.5$, $J = 0.5$, $n_S = 8$, $T = 6\sigma$. (a) Time evolution of the error in the average energy density $\Delta e(t) = \frac{(H - H_\beta)}{\langle H_\beta \rangle}$. The energy density approaches the exact result within less than two percent. (b) Evolution of the trace distance to the Gibbs state ρ_β . The trace distance converges to values below 0.2, indicating convergence toward a state close to the Gibbs state.

at a time t after the application of $\tilde{V}$. This in essence is the “quasiparticle assumption” at the heart of Refs. [9,10].

However, we note that, since it involves forward time evolution under $-H$, the rewinding step is potentially tricky to implement in practice for analog simulators, and may be costly in any case as it doubles the duration of the protocol. However, we show (in Appendix D) that even in the absence of any quasiparticle assumption, dropping the rewinding step only increases the fixed-point error by an additional factor of τ_{mix} , viz.,

$$\|\rho'_T - \rho_\beta\| \leq 4\delta\tau_{\text{mix}} \quad (40)$$

where δ is the error of the protocol *with* rewinding in Eq. (35). This leads to an additional factor of τ_{mix}^* in the cost estimate in Eq. (39). While this can be severe if the mixing time scaling is particularly adverse, the total Hamiltonian time evolution cost nevertheless remains polynomial in system size for any H for which the full protocol has $\text{poly}(n)$ scaling.

V. NUMERICAL CASE STUDY

To illustrate the feasibility of our protocol, we numerically simulate our protocol as a discrete-time quantum channel, and compare the resulting evolution to the dissipative evolution under the Lindbladians $\mathcal{L}_T$ [cf. Eq. (23a)] (to confirm that we approximately simulate it) and the exact Gibbs sampler $\mathcal{L}_\beta$ [Eq. (14b)] (whose simulation is our objective). We emphasize that the numerical simulations in this section are not meant to be numerical validations of our bound.

As a concrete example, we consider thermal state preparation for the mixed-field Ising model,

$$H = \sum_i Z_i Z_{i+1} + gX_i + hZ_i. \quad (41)$$

We set $g = 0.9045$ and $h = 0.809$, parameters for which the model is known to thermalize rapidly [65].

To simulate the Lindbladian dynamics, we numerically solve the Lindblad equation using the adaptive Runge–Kutta method of order 5(4) [66]. The initial density matrix ρ is taken to be the maximally mixed ensemble.

The results are shown in Fig. 3, where we present data for $\beta = 1.0$, $\sigma = 0.5$, $J = 0.5$, $T = 6\sigma$, and a system size of $n_S = 8$ qubits. The evolution under the exact Gibbs sampler is shown in black, while the results for the Lindbladian $\mathcal{L}_\infty$ [cf. Eq. (23)] are shown in red and under the local driving protocol $\mathcal{K}$ are shown in blue [cf. Eq. (18)]. We note that we are in a regime with $\beta \gtrsim \sigma$ where the error bounds in our protocol are not necessarily optimal. Results corresponding to evolving with the Lindbladian $\mathcal{L}$ or $\mathcal{L}_\infty$ only are shown as dashed lines. In the Schrödinger picture, this corresponds to the case where reverse-time evolution $U = e^{iHt}$ is applied after the time-dependent system-bath interaction.

In Fig. 3(a), we display the dynamics of the average energy, divided by the energy density of the thermal state

$$\Delta e(t) = \frac{\langle H - H_\beta \rangle}{H_\beta}, \quad (42)$$

with $H_\beta = \text{Tr}(H\rho_\beta)$ being the average energy of the Gibbs state.

In both cases, the energy rapidly converges to the thermal value within less than 2%.

Because of the deviations from an exact Gibbs sampler, the results converge to a state close to the Gibbs state. This is shown in Fig. 3(b), where the trace distance with respect to the thermal state ρ_β is displayed. The trace distance quickly decays, demonstrating rapid convergence to the steady state.

These results indicate that the local driving sampler yields a good approximation of the exact Gibbs sampler and provides evidence for the effectiveness of our protocol for thermal state preparation.

An open and interesting question is the effect of the rewinding step of the protocol $\mathcal{K}[\rho]$. In order to investigate its effect, we consider the channel $\mathcal{K}'[\rho]$, which is obtained by

removing the rewinding step

$$K'[\rho] = \text{tr}_B[e^{-iHT} \tilde{V}(\rho_B^0 \otimes \rho) \tilde{V}^\dagger e^{iHT}]. \quad (43)$$

The results for this protocol are indicated by a dashed blue line in Fig. 3(a). It turns out that the accuracy of these results improves in comparison to the protocol $K[\rho]$. Understanding the reason for this improvement remains an open question for future work.

VI. CONCLUDING REMARKS

In this work, we have demonstrated that it is in principle possible to prepare approximate thermal states with rigorously controlled error bounds using ingredients commonly available in the current generation of quantum devices: implementation of a local Hamiltonian, time-dependent local driving, and the ability to reset ancillas. The local driving protocol we introduce and study *exactly* simulates the dissipative portions of an exact Gibbs sampler (whose steady state is the thermal density matrix ρ_β), but rather than the specifically tailored coherent evolution of the latter, it involves a physical Lamb shift.

Our bounds explicitly depend on the mixing time of the approximated Lindbladians, which in turn is controlled by the properties of the filter functions. Understanding this dependence is therefore crucial to estimate the accuracy to which our approximate protocol can approximate a thermal state, and indeed is important more generally in determining the efficiency of exact Gibbs samplers, if not their accuracy. Since mixing times must often be determined on a case-by-case basis, this represents a challenging question for the future—although early results in this direction suggest that at least a measure of generality may be possible [63,64].

A notable feature of our proposal is that it does not require complicated “quantum programming” such as the block-encoding of operators. As such, it can be readily adapted to analog quantum simulators [67], such as those involving ultracold atoms, for which the most nontrivial obstacles will likely be a convenient implementation of the ancilla reset step (essential) and the backward time evolution (desirable).

The ready applicability of our protocol is underscored by the fact that for a suitable choice of jump operators, it resembles “quasiparticle cooling” in Refs. [9,10]; this also indicates that (modulo mixing-time considerations) the existence of an approximate quasiparticle description is not a necessary condition for such state-preparation methods.

The key distinction from the protocols in Refs. [9,10] is the requirement of backward time evolution, which is essential in order that the dissipation match that of the exact Gibbs sampler. From a practical standpoint, eliminating this rewinding step is highly desirable, as it doubles the simulation time and may be challenging to implement in experimental setups. Surprisingly, our numerics indicate that removing the rewinding step improves the performance of the protocol. Formally, this loosens the accuracy bound by a power of the mixing time, so future work [68,69] will explore whether this can be done without compromising the performance of the protocol in general.

Clearly, it would be highly desirable to develop similar readily implementable protocols that more closely approximate the exact coherent term of Refs. [27,29]. Possible routes

to this might involve a judicious choice of forward and backward time evolution to “echo out” spurious Lamb shift terms, and similar techniques of quantum control. A more careful study of the optimal parameter choice for efficient state preparation—especially one organized around the physical properties of the target state—as well as a more principled understanding of mixing times for local driving samplers, remain important directions for future work. Finally, it would be interesting to ask how these methods compare operationally with others for controlled approximation of thermal observables on quantum or even classical [70–73] computers.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: STRUCTURE OF THE EXACT GIBBS SAMPLER FROM KMS DETAILED BALANCE

In this Appendix, we provide a detailed derivation of the constraints imposed on the detailed balance Lindbladian by detailed balance, following Ref. [27]. Our starting point is the Bohr-frequency representation Eq. (7). We will assume (as in the main text) that $\mathcal{L}$ consists either of Hermitian jump operators, or else is “paired” such that A_a appears in the set of jump operators $\mathcal{A}$ if and only if $A_a^\dagger$ also appears in $\mathcal{A}$. The reason for this will become apparent below.

Before proceeding, recall that we have already introduced the “weighting” operator Γ_ρ involving conjugation with powers of any full-rank state ρ ,

$$\Gamma_\rho[\cdot] := \rho^{1/2}(\cdot)\rho^{1/2}, \quad (A1)$$

It is convenient to also define a second such operator,

$$\Lambda_\rho[\cdot] := \rho^{-1/2}(\cdot)\rho^{1/2}, \quad (A2)$$

such that for a Hermitian operator X we have $\Gamma_\rho[X]^\dagger = \Gamma_\rho[X]$ and $\Lambda_\rho[X]^\dagger = \Lambda_\rho^{-1}[X]$.

First, we consider the detailed balance condition on the transition part $\mathcal{T}$; a short calculation gives us that

$$\begin{aligned} (\Gamma_{\rho}^{-1} \circ \mathcal{T} \circ \Gamma_{\rho})[\cdot] &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{v_1, v_2} \rho_{\beta}^{-1/2} A_{v_1}^a \rho_{\beta}^{1/2}(\cdot) \rho_{\beta}^{1/2} (A_{v_2}^a)^{\dagger} \rho_{\beta}^{-1/2} \\ &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{v_1, v_2} e^{\frac{\beta(v_1 + v_2)}{2}} A_{v_1}^a(\cdot) (A_{v_2}^a)^{\dagger}, \end{aligned} \quad (\text{A3})$$

where we used the fact that $\rho^{-s} A_v \rho^s = e^{\beta s v} A_v$ and hence $\rho^s (A_v)^{\dagger} \rho^{-s} = e^{\beta s v} (A_v)^{\dagger}$, which follow from the Bohr frequency representation Eq. (6).

On the other hand, by using $\text{tr}\{\mathcal{OT}[\rho]\} = \text{tr}\{\rho \mathcal{T}^{\dagger}[\mathcal{O}]\}$ to determine $\mathcal{T}^{\dagger}$, we have

$$\begin{aligned} \mathcal{T}^{\dagger}[\cdot] &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{v_1, v_2} (A_{v_2}^a)^{\dagger}(\cdot) A_{v_1}^a \\ &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{v_2, v_1} (A_{v_1}^a)^{\dagger}(\cdot) A_{v_2}^a && \text{(Relabeling } v_1 \leftrightarrow v_2) \\ &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{-v_2, -v_1} (A_{-v_1}^a)^{\dagger}(\cdot) A_{-v_2}^a && \text{(Since if } v \in \mathfrak{B} \text{ then } -v \in \mathfrak{B}) \\ &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{-v_2, -v_1} ((A^{\dagger})_{-v_1})^{\dagger}(\cdot) (A^{\dagger})_{-v_2} && \text{(Since if } A^a \in \mathcal{A} \text{ then } A^{a\dagger} \in \mathcal{A}) \\ &= \sum_{a \in \mathcal{A}} \sum_{v_1, v_2 \in \mathfrak{B}} \alpha_{-v_2, -v_1} A_{v_1}^a(\cdot) (A_{v_2}^a)^{\dagger} && \text{(Since } (A_v)^{\dagger} = (A^{\dagger})_{-v}). \end{aligned} \quad (\text{A4})$$

Comparing Eqs. (A3) and (A4), we see that in order for $\mathcal{T}$ to satisfy the detailed balance condition $\mathcal{T}^{\dagger} = \Gamma_{\rho}^{-1} \circ \mathcal{T} \circ \Gamma_{\rho}$, we require that the kernel α_{v_1, v_2} satisfies the condition

$$\alpha_{-v_2, -v_1} = \alpha_{v_1, v_2} e^{\frac{\beta(v_1 + v_2)}{2}}. \quad (\text{A5})$$

We highlight the crucial role played by the requirement that every jump operator in $\mathcal{T}$ is “paired” with its Hermitian conjugate. Without this, since the left/right action of A and $A^{\dagger}$ are exchanged under the transformation from $\mathcal{T}$ to $\mathcal{T}^{\dagger}$, it would not be possible to relate $\mathcal{T}$ and $\mathcal{T}^{\dagger}$. Physically, the detailed balance parametrized by Eq. (3) [or Eq. (A5)] relates “downhill” transitions driven by A_v with “uphill” transitions driven by $(A^{\dagger})_v$. Note that this is related to the observation (Appendix B of Ref. [10]) that the rate equation for “quasiparticle cooling” is detailed-balanced only if the jump operators are chosen to be Hermitian.

The terms $\mathcal{G}$ and $\mathcal{R}$ are of the forms $\mathcal{G}[\cdot] = -i[G, \cdot]$ and $\mathcal{R}[\cdot] = -\frac{i}{2}[R, \cdot]$, respectively. We now define $K = G - i\frac{R}{2}$, so that $(\mathcal{G} + \mathcal{R})[\cdot] = -iK(\cdot) + i(\cdot)K^{\dagger}$, and $(\mathcal{G} + \mathcal{R})^{\dagger}[\cdot] = iK^{\dagger}(\cdot) - i(\cdot)K$. We can then write

$$\begin{aligned} (\mathcal{G} + \mathcal{R})^{\dagger}[\cdot] - [\Gamma_{\rho}^{-1} \circ (\mathcal{G} + \mathcal{R}) \circ \Gamma_{\rho}][\cdot] &= i(K^{\dagger}(\cdot) - (\cdot)K) + i(\Lambda_{\rho_{\beta}}[K](\cdot) - (\cdot)\Lambda_{\rho_{\beta}}^{-1}[K]) \\ &= i(K^{\dagger} + \Lambda_{\rho_{\beta}}[K])(\cdot) - i(\cdot)(K + \Lambda_{\rho_{\beta}}[K]^{\dagger}), \end{aligned} \quad (\text{A6})$$

where we have used the fact that if $\mathcal{Q}[\cdot] = \mathcal{Q}_1(\cdot) \pm (\cdot)\mathcal{Q}_2$, then

$$\Gamma_{\rho}^{-1} \circ \mathcal{Q} \circ \Gamma_{\rho}[\cdot] = \rho^{-1/2}(\mathcal{Q}_1(\rho^{1/2}(\cdot)\rho^{1/2}) \pm (\rho^{1/2}(\cdot)\rho^{1/2})\mathcal{Q}_2)\rho^{-1/2} = \Lambda_{\rho}[\mathcal{Q}_1](\cdot) \pm (\cdot)\Lambda_{\rho}^{-1}[\mathcal{Q}_2], \quad (\text{A7})$$

and that $\Lambda_{\rho}^{-1}[K^{\dagger}] = \Lambda_{\rho}^{-1}[G] + \frac{i}{2}\Lambda_{\rho}^{-1}[R] = \Lambda_{\rho}[G]^{\dagger} + \frac{i}{2}\Lambda_{\rho}[R]^{\dagger} = (\Lambda_{\rho}[G] - \frac{i}{2}\Lambda_{\rho}[R])^{\dagger} = \Lambda_{\rho}[K]^{\dagger}$.

Since $\mathcal{G} + \mathcal{R}$ must satisfy detailed balance Eq. (3) independently of $\mathcal{T}$, we require that the RHS of Eq. (A6) vanishes. This in turn requires that

$$K^{\dagger} + \Lambda_{\rho_{\beta}}[K] = 0 + \lambda I, \quad (\text{A8})$$

where $\lambda \in \mathbb{R}$. For simplicity, we will set $\lambda = 0$. Now, since G, R are Hermitian, we can write

$$K^{\dagger} + \Lambda_{\rho_{\beta}}[K] = G + \frac{i}{2}R + \Lambda_{\rho_{\beta}}[G] - \frac{i}{2}\Lambda_{\rho_{\beta}}[R]. \quad (\text{A9})$$

Using the explicit forms of G and R in Eq. (7) and

$$\Lambda_{\rho_{\beta}}[(A_{v_2}^a)^{\dagger} A_{v_1}^a] = \rho_{\beta}^{-1/2} (A_{v_2}^a)^{\dagger} A_{v_1}^a \rho_{\beta}^{1/2} = e^{\frac{\beta(v_1 - v_2)}{2}} (A_{v_2}^a)^{\dagger} A_{v_1}^a, \quad (\text{A10})$$

we find that

$$K^{\dagger} + \Lambda_{\rho_{\beta}}[K] = \sum_{a \in \mathcal{A}} \sum_{v_1, v_2} \left[\left(1 + e^{\frac{\beta(v_1 - v_2)}{2}}\right) g_{v_1, v_2} + \frac{i}{2} \left(1 - e^{\frac{\beta(v_1 - v_2)}{2}}\right) \alpha_{v_1, v_2} \right] (A_{v_2}^a)^{\dagger} A_{v_1}^a \quad (\text{A11})$$

Setting this to zero gives us a condition on kernels of the decay and coherent parts such that they together satisfy detailed balance,

$$g_{v_1, v_2} = \frac{i}{2} \frac{1 - e^{\frac{\beta(v_1 - v_2)}{2}}}{1 + e^{\frac{\beta(v_1 - v_2)}{2}}} \alpha_{v_1, v_2} = -\frac{1}{2i} \tanh \frac{\beta(v_1 - v_2)}{4} \alpha_{v_1, v_2}. \quad (\text{A12})$$

APPENDIX B: DETAILED DERIVATION OF ERROR BOUNDS I: LINDBLADIAN EVOLUTION AND MAGNUS EXPANSION

In this Appendix, we provide a detailed derivation of the first of the two error bounds used in the main text to derive the upper bound on the total Hamiltonian evolution time needed to prepare a thermal state up to some error $n_B \epsilon$ in trace distance.

Here, we bound the errors incurred by the Lindbladian evolution and the Magnus expansion: we show that given $\mathcal{K}$ as in Eq. (18), and with $\mathcal{L}_T$ given in Eq. (23a), that

$$\|e^{J^2 \mathcal{L}_T}[\rho] - \mathcal{K}[\rho]\|_1 = O\left(J^4 \left(n_B \left(\frac{T}{\sigma}\right)^4 + n_B^2\right) e^{\frac{\beta^2}{2\sigma^2}}\right). \quad (\text{B1})$$

1. Bounding the Magnus approximation error

Recall that in the interaction picture, we implement the Hamiltonian

$$\tilde{H}_{\text{SB}}(t) = \sum_a J \{f(t) B_a^\dagger(t) \otimes A_a(t) + f^*(t) B_a(t) \otimes A_a^\dagger(t)\}. \quad (\text{B2})$$

Here, $B = \frac{1}{2}(X_B - iY_B)$, and $B^\dagger = \frac{1}{2}(X_B + iY_B)$ are lowering and raising operators for the ancilla, and $f(t)$ is given by Eq. (25).

Using the Magnus expansion, we may express the exact time-evolution operator $\tilde{V}$ [see Eq. (19), reproduced here for convenience]

$$\tilde{V} = \mathcal{T} \exp \left(-iJ \int_{-T/2}^{T/2} dt \tilde{H}_{\text{SB}}(t) \right) \quad (\text{B3})$$

up to third order as

$$\tilde{V} \approx e^{-i \sum_{n=1}^3 \tilde{H}_M^{(n)}} \quad (\text{B4})$$

with

$$\tilde{H}_M^{(1)} = \int_{-T/2}^{T/2} dt_1 \tilde{H}_{\text{SB}}(t_1), \quad (\text{B5})$$

$$\tilde{H}_M^{(2)} = \frac{1}{2i} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 [\tilde{H}_{\text{SB}}(t_1), \tilde{H}_{\text{SB}}(t_2)]$$

$$\begin{aligned} \tilde{H}_M^{(3)} = & -\frac{1}{6} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{t_1} dt_2 \int_{-T/2}^{t_2} dt_3 \\ & \times ([\tilde{H}_{\text{SB}}(t_1), [\tilde{H}_{\text{SB}}(t_2), \tilde{H}_{\text{SB}}(t_3)]] \\ & + [\tilde{H}_{\text{SB}}(t_3), [\tilde{H}_{\text{SB}}(t_2), \tilde{H}_{\text{SB}}(t_1)]]) \end{aligned} \quad (\text{B6})$$

Note that the higher-order terms include commutators between different jump operators (and hence, given our construction, different ancillas a, a').

The error incurred can be bounded by using (a mild generalization of) Theorem 1 of Ref. [52]. Adapting their notation to the present setting, Ref. [52] considers the question of

approximating the time-evolution operator corresponding to evolution under a (potentially time-dependent) k -local Hamiltonian H_{SB} , each term of which is supported on a subset X of at most k qubits, in the interaction picture of a geometrically local Hamiltonian H . We then have

$$\left\| \tilde{V} - \exp \left(-i \sum_{n=1}^q \tilde{H}_M^{(n)} \right) \right\|_\infty = O(n_B (JdT)^{q+1}), \quad (\text{B7})$$

where $d = \max_i \sum_{X: i \in X} \|H_{\text{SB}}\|_\infty$ is the *interaction degree* (where the maximum is taken over all sites in the system and bath) of B_X , and $\tilde{H}_{\text{SB}}(t) = e^{iHt} H_{\text{SB}} e^{-iHt}$ is the time-evolved interaction-picture operator. In our case, X consists of a single site and its ancilla, and $H_{\text{SB}}(t) = \sum_a f(t) A_a^\dagger(t) \otimes B_a(t) + \text{H.c.}$ is explicitly time dependent. Note that Ref. [52] explicitly considers time-independent B_X , but observe that the generalization to time-dependent B_X is immediate given the nature of their proof as long as d can be bounded at all times. A second comment is in order regarding the scaling with n_B in (B7); Ref. [52] does not have a separation of system and bath and so the operator that plays the role $\tilde{H}_{\text{SB}}$ simply scales with the system size. Analyzing the derivation of their bound (or equivalently, by applying it to each term in the sum within $\tilde{H}_{\text{SB}}$ separately) it is clear that the appropriate scaling in our case is with n_B rather than the total size of system and bath together, $n_S + n_B$. In practice, of course for effective cooling, $n_B \propto n_S$ so the distinction is not fundamental, but may be relevant for practical considerations where prefactors matter.

We see that given the choice of filter function, since $\|A_a^\dagger(t) \otimes B_a(t)\|_\infty = 1$, we have

$$d = \max_t |f(t)| = \sqrt{\frac{2}{\pi \sigma^2}} e^{\frac{\beta^2}{8\sigma^2}}. \quad (\text{B8})$$

Combining this with Eq. (B7), we arrive at Eq. (20).

2. Bounding the error of the reset protocol

It remains to bound the error of the reset protocol, i.e., the fact that we implement a Lindbladian by effectively “Trotterizing”: the drive followed by reset results in a quantum channel that approximates evolution with the Lindbladian for a short time.

Recall that each time step of our evolution implements the quantum channel

$$\mathcal{K}[\rho] = \text{tr}_B[\tilde{V}(\rho_B^0 \otimes \rho)\tilde{V}^\dagger], \quad (\text{B9})$$

with $\rho_B^0 = \bigotimes_{a=1}^{n_B} |0_a\rangle\langle 0_a|_B$. Consider $\mathcal{L}_T$ as defined in Eq. (23). We argue that if we take $J = \sqrt{\Delta\tau}$, then

$$\|e^{\Delta\tau \mathcal{L}_T}[\rho] - \mathcal{K}[\rho]\|_1 = O\left(J^4 \left(n_B \left(\frac{T}{\sigma}\right)^4 + n_B^2\right) e^{\frac{\beta^2}{2\sigma^2}}\right). \quad (\text{B10})$$

Note that the scaling is one order in J better than naively expected; this is because of the trace over the bath degrees of freedom. We now derive this in detail.

We begin by defining the Magnus-approximated channel

$$\mathcal{K}_M[\rho] = \text{tr}_B \left[\exp \left(-i \sum_{n=1}^q \tilde{H}_M^{(n)} \right) (\rho_B^0 \otimes \rho) \exp \left(i \sum_{n=1}^q \tilde{H}_M^{(n)} \right) \right]. \quad (\text{B11})$$

Using the triangle inequality,

$$\|e^{\Delta\tau\mathcal{L}_T}[\rho] - \mathcal{K}[\rho]\|_1 \leq \|e^{\Delta\tau\mathcal{L}_T}\rho - \mathcal{K}_M[\rho]\|_1 + \|\mathcal{K}_M[\rho] - \mathcal{K}[\rho]\|_1. \quad (\text{B12})$$

The second term has the scaling, using a triangle and Hoelder inequality

$$\begin{aligned} & \|\mathcal{K}_M[\rho] - \mathcal{K}[\rho]\|_1 \\ &= \left\| \text{tr}_B \left(\tilde{V} - e^{-i \sum_{n=1}^3 \tilde{H}_M^{(n)}} \right) \rho \left(\tilde{V}^\dagger - e^{i \sum_{n=1}^3 \tilde{H}_M^{(n)}} \right) \right\|_1 \\ &\leq \left(\|\tilde{V}^\dagger\|_\infty + \left\| e^{i \sum_{n=1}^3 \tilde{H}_M^{(n)}} \right\|_\infty \right) \|\rho\|_1 \left\| \tilde{V} - e^{-i \sum_{n=1}^3 \tilde{H}_M^{(n)}} \right\|_\infty \\ &= \|\rho\|_1 O \left(n_B J^4 \left(\frac{T}{\sigma} \right)^4 e^{\frac{\beta^2}{2\sigma^2}} \right), \end{aligned} \quad (\text{B13})$$

because of the Magnus expansion error bound Eq. (B4) combined with the fact that the partial trace (over the ancillas) is a contractive operation.

It remains to bound the first term. To do so, we show that $e^{\Delta\tau\mathcal{L}_T}\rho$ and $\mathcal{K}_M[\rho]$ agree up to $O(n_B J^4)$. We first bound the induced trace norm of $\mathcal{L}_T$: We obtain for the transition term, using Hoelder's inequality

$$\|L_{a;T}\rho L_{a;T}^\dagger\|_1 \leq \|L_{a;T}\|_\infty^2 \|\rho\|_1 \quad (\text{B14})$$

and

$$\|L_{a;T}\|_\infty \leq \int_{-T/2}^{T/2} |f(t)| \|A_a(t)\|_\infty dt \leq \int_{-\infty}^{\infty} |f(t)| dt = e^{\frac{\beta^2}{8\sigma^2}}. \quad (\text{B15})$$

Similarly, the dissipative and coherent parts can be bound, which gives

$$\|\mathcal{L}_T\|_{1 \rightarrow 1} \leq 3n_B e^{\frac{\beta^2}{4\sigma^2}}. \quad (\text{B16})$$

With the identification $J^2 = \Delta\tau$ and a Taylor expansion, this gives

$$\begin{aligned} \|e^{\Delta\tau\mathcal{L}_T}[\rho] - (\rho + \Delta\tau\mathcal{L}_T[\rho])\|_1 &= O(J^4 \|\mathcal{L}_T\|_{1 \rightarrow 1}^2) \\ &= O(J^4 n_B^2 e^{\frac{\beta^2}{2\sigma^2}}). \end{aligned} \quad (\text{B17})$$

On the other hand, we consider the expansion of $\mathcal{K}_M[\rho]$. Because of the trace over ancilla qubits at the end of the protocol and the structure of the system-bath interactions, all odd orders in J cancel. It follows that

$$\begin{aligned} \mathcal{K}_M[\rho] &= \rho + \text{tr}_B [\tilde{H}_M^{(1)} (\rho_B^0 \otimes \rho) \tilde{H}_M^{(1)}] \\ &\quad - \frac{1}{2} \text{tr}_B [(\tilde{H}_M^{(1)})^2 (\rho_B^0 \otimes \rho) + (\rho_B^0 \otimes \rho) (\tilde{H}_M^{(1)})^2] \\ &\quad - i \text{tr}_B [\tilde{H}_M^{(2)} (\rho_B^0 \otimes \rho) - (\rho_B^0 \otimes \rho) \tilde{H}_M^{(2)}] + O(n_B^2 J^4 e^{\frac{\beta^2}{2\sigma^2}}). \end{aligned} \quad (\text{B18})$$

Because of the tracing out of bath degrees of freedom, all jump operators have to appear in pairs and we obtain a more favorable scaling with n_B^2 instead of n_B^4 .

We consider the first term in detail; the treatment of the other terms proceeds analogously. We have

$$\begin{aligned} & \text{tr}_B [\tilde{H}_M^{(1)} (\rho_B^0 \otimes \rho) \tilde{H}_M^{(1)}] \\ &= J^2 \sum_{a,a'} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 \text{tr}_B [(f(t_1) B_a^\dagger(t_1) \otimes A_a(t_1) \\ &\quad + f^*(t_1) B_a(t_1) \otimes A_a^\dagger(t_1)) (\rho_B^0 \otimes \rho) \\ &\quad \times (f(t_2) B_{a'}^\dagger(t_2) \otimes A_{a'}(t_2) + f^*(t_2) B_{a'}(t_2) \otimes A_{a'}^\dagger(t_2))]. \end{aligned} \quad (\text{B19})$$

Tracing out the bath degrees of freedom forces $a = a'$. Meanwhile, placing the bath degree of freedom in the $|0\rangle$ state (the reset step) forces the application of $B_a^\dagger(t_1)$ on the left and $B_a(t_2)$ on the right. Together, these give

$$\begin{aligned} & \text{tr}_B [\tilde{H}_M^{(1)} (\rho_B^0 \otimes \rho) \tilde{H}_M^{(1)}] \\ &= J^2 \sum_a \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 f(t_1) f^*(t_2) A_a(t_1) \rho A_a^\dagger(t_2). \end{aligned} \quad (\text{B20})$$

Identifying $\Delta\tau = J^2$, this gives the transition part in Eq. (23). Similarly, the remaining terms in the expansion Eq. (B18) of $\mathcal{K}_M[\rho]$ can be, respectively, identified with the decay and coherent parts of $\mathcal{L}_T$, completing the proof.

APPENDIX C: DETAILED DERIVATION OF ERROR BOUNDS II: FIXED-POINT ERROR

In this Appendix, we consider the second of the two error bounds used in the main text. Namely, we obtain an upper bound on the overall fixed-point error, i.e., the difference between the fixed point of the local driving sampler $\mathcal{L}_T$ in Eq. (23a), (that we denote ρ_T), and the thermal state ρ_β .

We split the total error into two parts using the triangle inequality, viz.,

$$\|\rho_T - \rho_\beta\|_1 \leq \|\rho_T - \rho_\infty\|_1 + \|\rho_\infty - \rho_\beta\|_1, \quad (\text{C1})$$

where ρ_∞ is the fixed point of $\mathcal{L} \equiv \lim_{T \rightarrow \infty} \mathcal{L}_T$, the infinite time-evolution limit of the local driving sampler, i.e., $\mathcal{L}[\rho_\infty] = 0$. The two terms on the right-hand side (RHS) of the above equation capture different pieces of physics. The first is the error incurred by the fact that the exact sampler involves an infinite-time smoothing, while the local driving sampler is truncated at time T . The second essentially comes from the fact that even as $T \rightarrow \infty$, the local driving sampler has the “wrong” coherent part, given by the Lamb shift, which we tackle via the so-called “secular approximation”. We address these in turn.

Note that throughout this appendix, to streamline notation, we consider a single jump operator A ; the generalization to multiple jump operators A_a is straightforward and simply involves a sum over a , and all the errors simply pick up an overall factor of $n_B = |\mathcal{A}|$.

1. Error from finite-time evolution

Here, we prove that

$$\begin{aligned} \|\rho_T - \rho_\infty\|_1 &\leq 4 \tau_{\text{mix}} \|\mathcal{L}_T - \mathcal{L}_\infty\|_{1-1} \\ &\leq \frac{24\sqrt{2}}{\sqrt{\pi}} \left(\frac{\sigma}{T}\right) e^{\frac{\beta^2}{4\sigma^2}} e^{-\frac{T^2}{2\sigma^2}} t_{\text{mix}}(\mathcal{L}_T). \end{aligned} \quad (\text{C2})$$

We can use Lemma II.1 of [26] to write

$$\|\rho_T - \rho\|_1 \leq 4 t_{\text{mix}}(\mathcal{L}_T) \|\mathcal{L}_T - \mathcal{L}_\infty\|_{1-1}. \quad (\text{C3})$$

Recall that

$$\begin{aligned} \mathcal{L}_T[\cdot] &= \sum_a \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 f(t_1) f^*(t_2) \\ &\quad \times \left(-i \left[-\frac{\text{sgn}(t_1 - t_2)}{2i} A^{a\dagger}(t_2) A^a(t_1), \cdot \right] \right. \\ &\quad \left. + A^a(t_1)(\cdot) A^{a\dagger}(t_2) - \frac{1}{2} [A^\dagger(t_2) A^a(t_1), \cdot] \right). \end{aligned} \quad (\text{C4})$$

Recall that $\mathcal{L}_\infty = \lim_{T \rightarrow \infty} \mathcal{L}_T$, and consider the transition parts of the Lindbladians: We obtain, using Hölder's and the triangle inequality and $\|A(t)\|_\infty \leq 1$,

$$\begin{aligned} \|L_{a;T} \rho L_{a;T}^\dagger - L_{a;\infty} \rho L_{a;\infty}^\dagger\|_1 &= \|(L_{a;T} - L_{a;\infty}) \rho L_{a;T}^\dagger + L_{a;\infty} \rho (L_{a;T}^\dagger - L_{a;\infty}^\dagger)\|_1 \\ &\leq \|\rho\|_1 \|L_{a;T} - L_{a;\infty}\|_\infty (\|L_{a;T}\|_\infty + \|L_{a;\infty}\|_\infty) \\ &\leq 2\|\rho\|_1 \left(\int_{-\infty}^{\infty} dt |f(t)| \right) \left(\int_{\mathbb{R}_{[T]}} dt |f(t)| \right), \end{aligned} \quad (\text{C5})$$

where we have defined $\mathbb{R}_{[T]} = \mathbb{R} \setminus [-T/2, T/2]$. Similarly, we can bound the dissipative part and the coherent part, using $|\frac{\text{sgn}(t_1 - t_2)}{2}| < \frac{1}{2}$. Since this applies for all ρ , we obtain for the induced norm

$$\|\mathcal{L}_T - \mathcal{L}_\infty\|_{1-1} \leq 6n_B \left(\int_{-\infty}^{\infty} dt |f(t)| \right) \left(\int_{\mathbb{R}_{[T]}} dt |f(t)| \right). \quad (\text{C6})$$

For our specific choice of filter function

$$f(t) = \sqrt{\frac{2}{\pi\sigma^2}} e^{-\frac{2}{\sigma^2}(t - \frac{i\beta}{4})^2}, \quad (\text{C7})$$

we obtain

$$\int_{\mathbb{R}_{[T]}} dt |f(t_1)| = e^{\frac{\beta^2}{8\sigma^2}} \text{erfc}\left(\frac{T}{\sigma\sqrt{2}}\right) \leq \frac{\sqrt{2}\sigma}{\sqrt{\pi}T} e^{\frac{\beta^2}{8\sigma^2}} e^{-\frac{T^2}{2\sigma^2}}, \quad (\text{C8})$$

where the last bound holds for $\frac{T^2}{2\sigma^2} > 1$, which is guaranteed by the choice in Eq. (36) as $\epsilon \rightarrow 0$. This leads to the final bound

$$\|\rho_T - \rho_\infty\|_1 \leq \frac{24\sqrt{2}}{\sqrt{\pi}} n_B \left(\frac{\sigma}{T}\right) e^{\frac{\beta^2}{4\sigma^2}} e^{-\frac{T^2}{2\sigma^2}} t_{\text{mix}}(\mathcal{L}_T). \quad (\text{C9})$$

2. Secular approximation for the Lamb shift

We next derive the inequality

$$\|\rho_\infty - \rho_\beta\|_1 = \tilde{O}\left(n_B \frac{\beta}{\sigma} \max(\tau_{\text{mix}}(\mathcal{L}_\infty), \tau_{\text{mix}}(\mathcal{L}_{\text{diss}}))\right). \quad (\text{C10})$$

The derivation ultimately proceeds by means of implementing the so-called ‘‘secular approximation’’, but first we need to perform some preliminary manipulations involving eigenvector perturbation theory. Much of our treatment is agnostic to specific choices of the filter function, except the very last portions of the proof, which impose some requirements of the time-domain asymptotics of $f(t)$. Accordingly, we will only specify our choice of $f(t)$ where necessary.

a. Preliminaries

We first establish some notation. We denote by $\mathcal{L}_\infty$, $\mathcal{L}_{\text{sec}}$, and $\mathcal{L}_\beta$ the Lindbladians that, respectively, correspond to the $T \rightarrow \infty$ limit of the local-driving sampler $\mathcal{L}_T$, its *secular approximation* (to be explained below), and the exact sampler. These all have identical dissipative parts, but differ in their coherent parts,

$$\begin{aligned} \mathcal{L}_\beta[\cdot] &= -i[G, \cdot] + \mathcal{L}_{\text{diss}}[\cdot] \\ \mathcal{L}_\infty[\cdot] &= -i[H_{\text{LS}}, \cdot] + \mathcal{L}_{\text{diss}}[\cdot] = -i[B, \cdot] + \mathcal{L}_\beta[\cdot] \\ \mathcal{L}_{\text{sec}}[\cdot] &= -i[B_{\text{sec}}, \cdot] + \mathcal{L}_\beta[\cdot] = -i[B_{\text{sec}} + G, \cdot] + \mathcal{L}_{\text{diss}}[\cdot] \end{aligned} \quad (\text{C11})$$

where we have defined $B = H_{\text{LS}} - G$, the difference between the coherent parts of the local-driving and exact samplers, and its secular approximation B_{sec} (to be specified below). Note that in contrast to Ref. [26] (Appendix D), we *do not* make a secular approximation for the dissipative part of the sampler. This is because we are bounding the difference between the local-driving sampler and the *exact* Gibbs sampler, whereas Ref. [26] were bounding the difference between a physical system-bath problem and a suitably smoothed approximation of a Davies Lindbladian *without* the coherent part necessary to make the latter an exact Gibbs sampler. This difference considerably streamlines our proof relative to that of Ref. [26].

The Lindbladians introduced above satisfy the following fixed-point relations:

$$\mathcal{L}_\beta[\rho_\beta] = 0; \quad \mathcal{L}_\infty[\rho_\infty] = 0; \quad \mathcal{L}_{\text{sec}}[\rho_{\text{sec}}] = 0, \quad (\text{C12})$$

where the final equation can be viewed as a *definition* of ρ_{sec} .

We now have, by the triangle inequality, that

$$\|\rho_\infty - \rho_\beta\|_1 \leq \|\rho_\infty - \rho_{\text{sec}}\|_1 + \|\rho_{\text{sec}} - \rho_\beta\|_1. \quad (\text{C13})$$

Of the remaining two expressions in Eq. (C13), $\|\rho_\infty - \rho_{\text{sec}}\|_1$ can be bounded using Lemma II.1 of Ref. [26], yielding

$$\|\rho_\infty - \rho_{\text{sec}}\|_1 \leq 4\|\mathcal{L}_\infty - \mathcal{L}_{\text{sec}}\|_{1-1} \cdot t_{\text{mix}}(\mathcal{L}_\infty). \quad (\text{C14})$$

As we will frequently do, we can use Hölder's inequality to relate $p - p$ norms of superoperators $\mathcal{Q}[\cdot] = -i[\mathcal{Q}, \cdot]_\pm$ whose sole action is commutation (−) or anticommutation (+) with some operator $\mathcal{Q}$ to the operator norm of $\mathcal{Q}$, yielding [74]

$$\|\rho_\infty - \rho_{\text{sec}}\|_1 \leq 8\|B - B_{\text{sec}}\|_\infty \cdot t_{\text{mix}}(\mathcal{L}_\infty). \quad (\text{C15})$$

The second piece is trickier to bound, and requires the use of eigenvector perturbation theory. Specifically, we use the following identity from Ref. [26], Appendix E: given two linear (super)operators [75] $\mathcal{M}, \mathcal{M}'$ (such that $\mathcal{M}'$ is a “small” perturbation of $\mathcal{M}$), with eigenvectors ρ, ρ' corresponding to eigenvalues λ, λ' , we have

$$\|\rho' - \rho\|_2 \leq \frac{2\sqrt{2}\|\mathcal{M}' - \mathcal{M}\|_{2-2} + |\lambda' - \lambda|}{\zeta_{-2}(\mathcal{M} - \lambda I)}, \quad (\text{C16})$$

where $\zeta_{-2}(\mathcal{Q})$ is the second smallest singular value of $\mathcal{Q}$.

Although, in general, eigenvector perturbation theory is poorly behaved for generic non-Hermitian operators (such as $\mathcal{M}, \mathcal{M}'$), it is possible to choose both to share a common eigenvalue $\lambda = \lambda' = 0$: simply choose superoperators $\mathcal{M}, \mathcal{M}'$, which have $\rho_\beta^{-1/4} \rho_{\text{sec}} \rho_\beta^{-1/4}$ and $\rho_\beta^{1/2}$ as fixed points (eigenstates with eigenvalue 0). It follows then

$$\begin{aligned} \|\rho_\beta - \rho_{\text{sec}}\|_1 &= \|\rho_\beta^{1/4}(\rho_\beta^{1/2} - \rho_\beta^{-1/4} \rho_{\text{sec}} \rho_\beta^{-1/4})\rho_\beta^{1/4}\|_1 \\ &\leq \|\rho_\beta^{1/4}\|_4^2 \|\rho_\beta^{1/2} - \rho_\beta^{-1/4} \rho_{\text{sec}} \rho_\beta^{-1/4}\|_2 \\ &= \|\rho_\beta^{1/2} - \rho_\beta^{-1/4} \rho_{\text{sec}} \rho_\beta^{-1/4}\|_2, \end{aligned} \quad (\text{C17})$$

where we used Hoelder’s inequality in the second line and $\|\rho_\beta^{1/4}\|_4 = 1$ in the last step. The remaining term can then be bounded using Eq. (C16).

We now choose $\mathcal{M}$ as

$$\mathcal{M}[\cdot] = \rho_\beta^{-1/4} \mathcal{L}_{\text{sec}}[\rho_\beta^{1/4} \cdot \rho_\beta^{1/4}] \rho_\beta^{-1/4}. \quad (\text{C18})$$

However, there is a wide range of possible superoperators that can be chosen to have $\rho_\beta^{1/2}$ as a fixed point, because of the latter’s central role in defining detailed balance. The nontrivial idea introduced in Ref. [26] is that this flexibility can be leveraged to obtain a tight bound. An elegant choice is

$$\mathcal{M}'[\cdot] = \rho_\beta^{1/4} \mathcal{L}_2^\dagger[\rho_\beta^{-1/4} \cdot \rho_\beta^{-1/4}] \rho_\beta^{1/4}, \quad (\text{C19})$$

with $\mathcal{L}_2^\dagger[\cdot] = -i[B_{\text{sec}}, \cdot] + \mathcal{L}_\beta^\dagger[\cdot]$. Observe that $\mathcal{M}'[\rho_\beta^{1/2}] = \rho_\beta^{1/4} \mathcal{L}_2^\dagger[\mathbb{1}] \rho_\beta^{1/4} = 0$, by using the fact that $\mathcal{L}_2^\dagger[\mathbb{1}] = 0$ as long as $\mathcal{L}_2$ is a valid Lindbladian.

We then have, using the definition Eq. (C11) of $\mathcal{L}_{\text{sec}}$ and by rewriting the detailed balance condition as

$$\rho_\beta^{-1/4} \mathcal{L}_\beta[\rho_\beta^{1/4} \cdot \rho_\beta^{1/4}] \rho_\beta^{-1/4} - \rho_\beta^{1/4} \mathcal{L}_\beta^\dagger[\rho_\beta^{-1/4} (\cdot) \rho_\beta^{-1/4}] \rho_\beta^{1/4} = 0$$

that

$$\begin{aligned} (\mathcal{M}' - \mathcal{M})[\cdot] &= \rho_\beta^{1/4} \mathcal{L}_2^\dagger[\rho_\beta^{-1/4} (\cdot) \rho_\beta^{-1/4}] \rho_\beta^{1/4} - \rho_\beta^{-1/4} \mathcal{L}_{\text{sec}}[\rho_\beta^{1/4} (\cdot) \rho_\beta^{1/4}] \rho_\beta^{-1/4} \\ &= -i\rho_\beta^{1/4} [B_{\text{sec}}, \rho_\beta^{-1/4} (\cdot) \rho_\beta^{-1/4}] \rho_\beta^{1/4} + \rho_\beta^{1/4} \mathcal{L}_\beta^\dagger[\rho_\beta^{-1/4} (\cdot) \rho_\beta^{-1/4}] \rho_\beta^{1/4} \\ &\quad + i\rho_\beta^{-1/4} [B_{\text{sec}}, \rho_\beta^{1/4} (\cdot) \rho_\beta^{1/4}] \rho_\beta^{-1/4} - \rho_\beta^{-1/4} \mathcal{L}_\beta[\rho_\beta^{1/4} (\cdot) \rho_\beta^{1/4}] \rho_\beta^{-1/4} \\ &= i[(\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4})(\cdot) - (\cdot)(\rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4} - \rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4})]. \end{aligned} \quad (\text{C20})$$

Using the triangle and Hölder inequalities again, we then have

$$\begin{aligned} \|\mathcal{M}' - \mathcal{M}\|_{2-2} &\leq \|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty \\ &\quad + \|\rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4} - \rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4}\|_\infty \\ &= 2\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty, \end{aligned} \quad (\text{C21})$$

where in the second line we have used $\|Q\|_* = \|Q^\dagger\|_*$, with $Q = \rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}$.

Using the Fan–Hoffmann inequality [76], Proposition III.5.1] on the second-smallest singular value [77] of $\mathcal{M}$ (not necessarily Hermitian), we have

$$\begin{aligned} \zeta_{-2}(\mathcal{M} - \lambda I) &\geq -\lambda_2\left(\frac{\mathcal{M} + \mathcal{M}^\dagger}{2} - \text{Re}(\lambda)I\right) \\ &= -\lambda_2(\mathcal{L}_\beta - \text{Re}(\lambda)I), \end{aligned} \quad (\text{C22})$$

where $\lambda_2(\mathcal{Q})$ denotes the second largest eigenvalue of $\mathcal{Q}$. Note that in the final step we have simply used Eq. (C11) with $\frac{\mathcal{M} + \mathcal{M}^\dagger}{2} = \rho_\beta^{-1/4} \mathcal{L}_\beta[\rho_\beta^{1/4} \cdot \rho_\beta^{1/4}] \rho_\beta^{-1/4}$, which has the same spectrum as $\mathcal{L}_\beta$. The spectral gap can be lower bounded by

the mixing time via

$$-\lambda_2(\mathcal{L}_{\text{beta}}) \geq \frac{\ln(2)}{t_{\text{mix}}(\mathcal{L}_\beta)}. \quad (\text{C23})$$

We can now assemble a bound on the second term in Eq. (C13), by using Eqs. (C16) and (C17) with $\lambda = \lambda' = 0$ and the results in Eqs. (C21)–(C23),

$$\begin{aligned} \|\rho_{\text{sec}} - \rho_\beta\|_1 &\leq \frac{1}{\ln 2} \|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty \\ &\quad \cdot t_{\text{mix}}(\mathcal{L}_\beta). \end{aligned} \quad (\text{C24})$$

Putting together Eqs. (C15) and (C24), we finally arrive at a bound on the error of the fixed point of local-driving sampler in terms of secular-approximable quantities and mixing times

$$\begin{aligned} \|\rho_\infty - \rho_\beta\|_1 &\leq 8\|B - B_{\text{sec}}\|_\infty \cdot t_{\text{mix}}(\mathcal{L}_\infty) \\ &\quad + \frac{1}{\ln 2} \|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty \cdot t_{\text{mix}}(\mathcal{L}_\beta) \\ &\leq 8(\|B - B_{\text{sec}}\|_\infty + \|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty) \\ &\quad \cdot \max(t_{\text{mix}}(\mathcal{L}_\infty), t_{\text{mix}}(\mathcal{L}_\beta)). \end{aligned} \quad (\text{C25})$$

We now turn to computing these norms using the secular approximation, which will allow us to bound the terms in the RHS of Eq. (C25).

b. Implementing the secular approximation

The secular approximation is most transparently phrased in frequency space (although bounding various contributions is often easier in the time domain). Consider a generic operator of the form $B = \sum_{v_1, v_2 \in \mathfrak{B}} b_{v_1, v_2} (A_{v_2})^\dagger A_{v_1}$. The secular approximation of B is obtained by restricting the sum to the “almost diagonal” pieces, such that $|v_1 - v_2| \lesssim \mu$, where μ is some suitably chosen cutoff. Formally, we do this by multiplying the kernel b_{v_1, v_2} by a suitably chosen “bump function”,

$$\begin{aligned} B &= \sum_{v_1, v_2 \in \mathfrak{B}} b_{v_1, v_2} (A_{v_2})^\dagger A_{v_1} \Rightarrow B_{\text{sec}} \\ &= \sum_{v_1, v_2 \in \mathfrak{B}} b_{v_1, v_2} w\left(\frac{v_-}{\mu}\right) (A_{v_2})^\dagger A_{v_1}, \end{aligned} \quad (\text{C26})$$

where $v_- \equiv v_1 - v_2$ and $w(x)$ is a smooth (i.e., infinitely differentiable) function with

$$w(x) = \begin{cases} 1, & x = 0 \\ 0, & |x| > 1 \\ < 1, & \text{else} \end{cases}. \quad (\text{C27})$$

For our purposes, it is convenient to choose a bump function such that it remains close to 1 except in some finite interval near the boundaries at ± 1 . While a specific choice will not be essential to our argument (modulo this requirement) a concrete choice is

$$w(x) = w_1\left(\frac{x+1}{1-\lambda}\right) w_1\left(\frac{1-x}{1-\lambda}\right), \quad (\text{C28})$$

where we define $w_1(x)$ in terms of another auxiliary function

$$w_1(x) = \frac{w_2(x)}{w_2(x) + w_2(1-x)}, \quad w_2(x) = \begin{cases} \exp(-\frac{1}{x}), & x > 0 \\ 0, & \text{else.} \end{cases} \quad (\text{C29})$$

Note that w is zero outside of $[-1, 1]$ and one in the interval $[-\lambda, \lambda]$. By choosing λ close to one, we can make the function arbitrarily sharp.

The reason to implement the secular approximation is that in the limit where we take $\mu \rightarrow 0$, the secular piece is purely diagonal in Bohr frequencies and hence commutes with ρ_β . An important point is that, depending on the specific function we wish to approximate, a sharp cutoff in frequencies may prove difficult to bound; implementing the secular approximation then requires a judicious use of bump functions (smooth functions that are strictly vanishing outside a compact domain) in order to remove high-frequency terms. This will be the case for our coherent parts.

Evidently, in order to use the secular approximation we must bound the two separate terms in Eq. (C25): the “almost-commuting” piece $\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}$ and the nonsecular piece $B - B_{\text{sec}}$. The strategy of proving these bounds is distinct, so we tackle them in turn.

c. Bounding the almost-commuting part

First, we consider the “almost-commuting” piece $\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty$. Note that we can always choose to write any operator in terms of its Bohr frequency expansion as $B = \sum_{v \in \mathfrak{B}} B_v$. Using the secular approximation, it then follows that

$$\begin{aligned} &\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty \\ &= \left\| \sum_{v \in \mathfrak{B}} w\left(\frac{v}{\mu}\right) (e^{\frac{\beta v}{4}} - e^{-\frac{\beta v}{4}}) B_v \right\|_\infty. \end{aligned} \quad (\text{C30})$$

We now exploit the fact that $w(\frac{v}{\mu})(e^{\frac{\beta v}{4}} - e^{-\frac{\beta v}{4}}) < \beta \mu$ for $\beta \mu \leq 1$, which places a restriction on the choice of μ . From this, it follows that

$$\begin{aligned} &\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty \\ &= O\left(\left\| \sum_{v \in \mathfrak{B}} B_v \right\|_\infty \beta \mu\right) = O(\|B\|_\infty \beta \mu), \end{aligned} \quad (\text{C31})$$

where we used $\sum_{v \in \mathfrak{B}} B_v = B$. Since $B = H_{\text{LS}} - G$, the final step in bounding the almost-commuting piece is therefore to bound the norms of the operators that appear in the coherent evolution of the exact sampler and the local driving sampler (the latter being the Lamb shift), which we may reduce to bounding the norm of each operator individually. For both cases, we can show that this is $O(1)$, by writing the relevant operator as a time-domain integral of an expression quadratic in the jump operators multiplied by some weight function that depends quadratically on the smoothing functions. Since the jump operators can always be fixed to have unit norm by a suitable choice of normalization, we can use triangle inequalities to simplify the bound to one in terms of time-domain integrals over the weight functions. For the Lamb shift the time-domain representation is shown in Eq. (23d); the weight functions are just a product of filter functions, and so the resulting bound on the norm is clearly $O(1)$ the result of the normalization of the filter functions. For the exact coherent part, the introduction of the $\tanh \frac{\beta v_-}{4}$ complicates the weight functions. It is straightforward but tedious to show (cf. Ref. [27], Appendix A) that

$$G = \int_{-\infty}^{\infty} \tilde{g}_-(t_-) e^{-iHt_-} \left(\int_{-\infty}^{\infty} \tilde{g}_+(t_+) A(t_+) A(-t_+) dt_+ \right) e^{iHt_-} dt_- \quad (\text{C32})$$

with

$$\begin{aligned} \tilde{g}_-(t) &= \frac{1}{2i} \int_{-\infty}^{\infty} dv_- e^{iv_- t} g_-(v_-) \\ &= \left[\frac{2\sqrt{2}\pi}{\beta \cosh\left(\frac{2\pi t}{\beta}\right)} \right] *_{t_-} \left[\frac{\sqrt{2}}{\sigma} e^{\frac{\beta^2 - 16t^2}{4\sigma^2}} \sin\left(\frac{2\beta t_-}{\sigma^2}\right) \right] \end{aligned} \quad (\text{C33})$$

and

$$\tilde{g}_+(t) = \int_{-\infty}^{\infty} dv_+ e^{iv_+ t} g_+(v_+) = \frac{4\sqrt{\pi} e^{-\frac{2}{\sigma^2}(t_+ - \frac{i\beta}{4})^2}}{\sigma}. \quad (\text{C34})$$

With this in hand, we can use triangle inequalities to write

$$\|G\|_\infty \leq \left(\int_{-\infty}^{\infty} dt_+ |\tilde{g}_+(t_+)| \right) \left(\int_{-\infty}^{\infty} dt_- |\tilde{g}_-(t_-)| \right). \quad (\text{C35})$$

Since $g_+(t_+)$ is a shifted Gaussian, its integral is bounded. Now, $g_-(t_-)$ is the convolution of two bounded functions, and we can use Young's convolution inequality for the L_1 norm to bound the norm of this convolution in terms of the product of the norms. In each case, because of the choice of overall normalization, the bound is $O(1)$. Combining the result above, we find that for a single jump operator,

$$\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty = O(\beta\mu). \quad (\text{C36})$$

Finally, for multiple jump operators, $n_B = |\mathcal{A}|$, the errors add independently, so that we finally have

$$\|\rho_\beta^{-1/4} B_{\text{sec}} \rho_\beta^{1/4} - \rho_\beta^{1/4} B_{\text{sec}} \rho_\beta^{-1/4}\|_\infty = O(n_B \beta\mu). \quad (\text{C37})$$

d. Bounding the nonsecular part

The operator B that appears in the error bounds involves the difference between the coherent parts of the local driving sampler (the Lamb shift) and the exact sampler. A sufficient bound obtains by simply using the triangle inequality to bound these pieces separately. Therefore, we will proceed initially being agnostic to the form of B , making general arguments as to the form of the kernel such that the nonsecular part $B - B_{\text{sec}}$ is bounded, before we specialize to the form dictated by our choice of filter functions.

Consider the nonsecular piece

$$B - B_{\text{sec}} = \sum_{v_1, v_2 \in \mathfrak{B}} b_{v_1, v_2} \left(1 - w\left(\frac{v_-}{\mu}\right) \right) (A_{v_2})^\dagger A_{v_1}. \quad (\text{C38})$$

Now, let us *define* a time-domain kernel $\mathcal{W}(t_1, t_2)$ implicitly via

$$b_{v_1, v_2} \left(1 - w\left(\frac{v_-}{\mu}\right) \right) \equiv \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(t_1, t_2) e^{iv_1 t_1 - iv_2 t_2}. \quad (\text{C39})$$

Using Eq. (C39) in our expression for the nonsecular part, we have

$$\begin{aligned} B - B_{\text{sec}} &= \sum_{v_1, v_2 \in \mathfrak{B}} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(t_1, t_2) e^{iv_1 t_1 - iv_2 t_2} (A_{v_2})^\dagger A_{v_1} \\ &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(t_1, t_2) A^\dagger(t_2) A(t_1), \end{aligned} \quad (\text{C40})$$

where we have simply used the representation of time evolution in terms of Bohr frequencies. The reason for this step is that the $A(t)$ are obtained by time-evolving bounded local operators, and therefore their norm has some $O(1)$ bound (which can be set to 1 by a suitable choice of normalization, which we henceforth assume). We then find, using Eq. (C40) and the triangle inequality, that the nonsecular part can be bounded by a certain two-dimensional time-domain integral,

$$\|B - B_{\text{sec}}\|_\infty \leq \left| \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(t_1, t_2) \right|. \quad (\text{C41})$$

Now, we can invert Eq. (C39) by recognizing that it is just a Fourier transformation, so that

$$\mathcal{W}(t_1, t_2) = \int_{-\infty}^{\infty} \frac{dv_1}{2\pi} \int_{-\infty}^{\infty} \frac{dv_2}{2\pi} b_{v_1, v_2} \left(1 - w\left(\frac{v_-}{\mu}\right) \right) e^{-iv_1 t_1 + iv_2 t_2}. \quad (\text{C42})$$

To proceed, we need to consider the form of b_{v_1, v_2} . For now, we will simply assume that this has the form $b_{v_1, v_2} = b_+(v_1 + v_2)b_-(v_1 - v_2)$; this will be the form of the kernel for both $H_{\text{LS}, f}$ and G . We then see, by combining Eq. (C41) (with a judicious sign change) with Eq. (C42), and defining $t_\pm = \frac{t_1 \pm t_2}{2}$, $v_\pm = v_1 \pm v_2$, that

$$\begin{aligned} \|B - B_{\text{sec}}\|_\infty &\leq \left| \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(t_1, t_2) \right| \\ &= \left| \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \mathcal{W}(-t_1, t_2) \right| \\ &= \left| \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} \frac{dv_1}{2\pi} \int_{-\infty}^{\infty} \frac{dv_2}{2\pi} b_+(v_+) b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right) \right) e^{iv_1 t_1 + iv_2 t_2} \right| \\ &\leq \int_{-\infty}^{\infty} dt_+ \left| \int_{-\infty}^{\infty} \frac{dv_+}{2\pi} b_+(v_+) e^{iv_+ t_+} \right| \times \int_{-\infty}^{\infty} dt_- \left| \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right) \right) e^{iv_- t_-} \right| \\ &\leq \int_{-\infty}^{\infty} dt_+ |\widehat{b}_+(t_+)| \times \int_{-\infty}^{\infty} dt_- \left| \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right) \right) e^{iv_- t_-} \right|, \end{aligned} \quad (\text{C43})$$

where $\widehat{b}_+(t_+)$ is the inverse Fourier transform of $b_+(v_+)$.

We find that in the two cases of interest to us, $|\widehat{b}_+(t_+)|$ takes the simple form

$$|\widehat{b}_+(t_+)| = |\widehat{h}_+(t_+)| = |\widehat{g}_+(t_+)| = \frac{1}{\sqrt{\pi}\sigma} e^{\beta^2/4\sigma^2} e^{-4t_+^2/\sigma^2}. \quad (\text{C44})$$

Therefore, we can express the first term in the product on the RHS of the final inequality Eq. (C43) above as

$$\int_{-\infty}^{\infty} dt_+ |\widehat{b}_+(t_+)| = \frac{e^{\beta^2/4\sigma^2}}{2} \quad (\text{C45})$$

in both the coherent and exact cases, which will only give an overall constant prefactor.

Perhaps unsurprisingly, the nontrivial bound involves the integration over t_- in Eq. (C43). It is useful to split this into a short-time contribution $\mathcal{I}_1$ with $t \in [-t_0, t_0]$ and a long-time contribution $\mathcal{I}_2$ with $|t| > t_0$, with t_0 to be specified below. The short-time piece is given by

$$\begin{aligned} \mathcal{I}_1 &\equiv \int_{-t_0}^{t_0} dt_- \left| \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) e^{iv_-t_-} \right| \\ &\leq 2t_0 \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} |b_-(v_-)| \left|1 - w\left(\frac{v_-}{\mu}\right)\right|. \end{aligned} \quad (\text{C46})$$

Given our choice of bump function, $1 - w(\frac{v_-}{\mu})$ is only nonzero for $v_- \gtrsim \mu$, where it is close to 1, so that the integral is dominated by the tails, and we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} |b_-(v_-)| \left|1 - w\left(\frac{v_-}{\mu}\right)\right| &= O\left(2 \int_{\mu}^{\infty} \frac{dv_-}{2\pi} |b_-(v_-)|\right) \\ &= O\left(\frac{e^{-\frac{\sigma^2 \mu^2}{16}}}{\sigma^2 \mu}\right), \end{aligned} \quad (\text{C47})$$

where in the second step we have used the fact that both $|b_-| \leq e^{-(\sigma v_-)^2/16}$ for either choice $b_- = g_-, h_-$. Combining Eqs. (C46) and (C47), we have

$$\mathcal{I}_1 = O\left(\frac{t_0 e^{-\frac{\sigma^2 \mu^2}{16}}}{\sigma^2 \mu}\right). \quad (\text{C48})$$

We turn next to the long-time contributions. By integrating by parts twice, we observe that

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) e^{iv_-t_-} \\ = -\frac{1}{t_-^2} \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} \frac{d^2}{dv_-^2} \left[b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) \right] e^{iv_-t_-}. \end{aligned} \quad (\text{C49})$$

We can then use this result to obtain a useful bound on the long-time contributions,

$$\begin{aligned} \mathcal{I}_2 &\equiv \int_{\mathbb{R} \setminus [-t_0, t_0]} dt_- \left| \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) e^{iv_-t_-} \right| \\ &\leq \frac{2}{t_0} \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} \left| \frac{d^2}{dv_-^2} \left[b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) \right] \right|. \end{aligned} \quad (\text{C50})$$

Using the product rule, we have that

$$\begin{aligned} \frac{d^2}{dv_-^2} \left[b_-(v_-) \left(1 - w\left(\frac{v_-}{\mu}\right)\right) \right] &= \left(1 - w\left(\frac{v_-}{\mu}\right)\right) b_-''(v_-) \\ &+ \frac{1}{\mu} b_-'(v_-) w'\left(\frac{v_-}{\mu}\right) + \frac{1}{\mu^2} b_-(v_-) w''\left(\frac{v_-}{\mu}\right), \end{aligned} \quad (\text{C51})$$

so that on inserting Eq. (C51) into Eq. (C50) and using the triangle inequality that

$$\begin{aligned} \mathcal{I}_2 &\leq \frac{2}{t_0} \int_{-\infty}^{\infty} \frac{dv_-}{2\pi} \left(\left| \left(1 - w\left(\frac{v_-}{\mu}\right)\right) b_-''(v_-) \right| \right. \\ &\quad \left. + \left| \frac{1}{\mu} b_-'(v_-) w'\left(\frac{v_-}{\mu}\right) \right| + \left| \frac{1}{\mu^2} b_-(v_-) w''\left(\frac{v_-}{\mu}\right) \right| \right). \end{aligned} \quad (\text{C52})$$

Let us now consider the two choices $b_- = h_-, g_-$ corresponding to the local driving and exact samplers. We have that $h_-(v_-) \propto e^{-(\sigma v_-)^2/16}$, whereas $h_-(v_-) \propto \tanh(\frac{\beta v_-}{4}) e^{-(\sigma v_-)^2/16}$. Crucially, the tanh that appears in the latter case has a bounded derivative everywhere, and so is relatively innocuous when inserted into Eq. (C52). Therefore, after a scaling analysis (that we do not reproduce here as it is straightforward but tedious) we can show that each term in Eq. (C52) takes the form $\text{Lpoly}(\beta v, \sigma v) \times e^{-(\sigma v_-)^2/16}$, where Lpoly represents some Laurent polynomial of bounded negative and positive degrees. Meanwhile, for our choice of bump function, $w'(v_-/\mu)$ and $w''(v_-/\mu)$ are both $O(1)$ around $v_-/\mu \sim 1$ or vanish otherwise, whereas as stated before $1 - w(v_-/\mu)$ is only nonzero outside the bounded domain $[-\mu, \mu]$. Combining these results, it is evident that we can bound the various terms in Eq. (C52) as

$$\mathcal{I}_2 = O\left(\frac{\sigma}{t_0} \text{Lpoly}(\beta \mu, \sigma \mu) e^{-(\sigma \mu)^2/16}\right), \quad (\text{C53})$$

which results from considering that the integrals only receive contributions either near $\pm\mu$ or $(\pm\mu, \infty)$, rescaling the integrand, and estimating the weight because of the Gaussian factors, and the β dependence will only appear for the exact sampler. Combining the bound in Eq. (C45) on the v_+ integral with the bounds in Eqs. (C48) and (C53) respectively and setting $\sigma = t_0$,

$$\|B - B_{\text{sec}}\|_{\infty} = O(\text{Lpoly}(\beta \mu, \sigma \mu) e^{-(\sigma \mu)^2/16}). \quad (\text{C54})$$

Again, for the case of multiple jump operators $n_B = |\mathcal{A}|$, the errors add independently, so that we have

$$\|B - B_{\text{sec}}\|_{\infty} = O(n_B \text{Lpoly}(\beta \mu, \sigma \mu) e^{-(\sigma \mu)^2/16}). \quad (\text{C55})$$

Now, substituting Eqs. (C55) and (C37) into Eq. (C25), we obtain

$$\begin{aligned} \|\rho_{\infty} - \rho_{\beta}\|_1 &= O(n_B [\beta \mu + \text{Lpoly}(\beta \mu, \sigma \mu) e^{-(\sigma \mu)^2/16}] \\ &\quad \times \max(t_{\text{mix}}(\mathcal{L}_{\infty}), t_{\text{mix}}(\mathcal{L}_{\text{diss}})), \end{aligned} \quad (\text{C56})$$

which, for some choice of $\mu = \tilde{O}(\sigma^{-1})$ (the $\tilde{O}$ now hiding extra-log factors necessary to suppress the second term above) yields the desired result in Eq. (C10).

e. Choice of filter functions and kernels

To complete the proof, we must obtain the relevant functions b_+, b_- for the two samplers in question, for a specific filter function, and confirm that they satisfy the properties assumed in the proof above. While this can be done for arbitrary sufficiently quickly decaying $f(t)$, for concreteness, here we present results for the specific choice in Eq. (25) in the main text.

With this choice, we see that the Lamb shift kernel in Eq. (23f) is given by (taking $T \rightarrow \infty$ as we have throughout the secular approximation, and dropping the superscript T)

$$\begin{aligned} -2ih_{v_1, v_2} &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \operatorname{sgn}(t_1 - t_2) f(t_1) f^*(t_2) e^{iv_1 t_1 - iv_2 t_2} \\ &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \operatorname{sgn}(t_1 + t_2) f(t_1) f^*(-t_2) e^{iv_1 t_1} e^{iv_2 t_2} \\ &= 2 \int_{-\infty}^{\infty} dt_+ \int_{-\infty}^{\infty} dt_- \operatorname{sgn}(t_+) f(t_+ + t_-) f^*(-t_+ + t_-) e^{iv_+ t_+} e^{iv_- t_-}. \end{aligned} \quad (\text{C57})$$

We can further simplify the integrand as follows:

$$\begin{aligned} f(t_+ + t_-) f^*(-t_+ + t_-) &= \frac{2}{\pi \sigma^2} \exp \left\{ -\frac{2}{\sigma^2} \left[\left(\left(t_+ - \frac{i\beta}{4} \right) + t_- \right)^2 + \left(\left(t_+ - \frac{i\beta}{4} \right) - t_- \right)^2 \right] \right\} \\ &= \frac{2}{\pi \sigma^2} \exp \left[-\frac{4}{\sigma^2} \left(t_+ - \frac{i\beta}{4} \right)^2 \right] \exp \left[-\frac{4}{\sigma^2} t_-^2 \right]. \end{aligned} \quad (\text{C58})$$

From this, we obtain

$$h_{v_1, v_2} = h_+(v_+) h_-(v_-). \quad (\text{C59})$$

Here,

$$\begin{aligned} h_+(v_+) &= -\frac{1}{2i} \int_{-\infty}^{\infty} dt_+ \\ &\quad \frac{2}{\sqrt{\pi} \sigma} \operatorname{sgn}(t_+) \exp \left[-\frac{4}{\sigma^2} \left(t_+ - \frac{i\beta}{4} \right)^2 \right] e^{-iv_+ t_+}, \end{aligned} \quad (\text{C60})$$

which gives us that

$$\begin{aligned} |\widehat{b}_+(t_+)| &= \left| \frac{i}{\sqrt{\pi} \sigma} \operatorname{sgn}(t_+) \exp \left[-\frac{4}{\sigma^2} \left(t_+ - \frac{i\beta}{4} \right)^2 \right] \right| \\ &= \frac{1}{\sqrt{\pi} \sigma} e^{\beta^2/4\sigma^2} e^{-4t_+^2/\sigma^2} \end{aligned} \quad (\text{C61})$$

and hence satisfies the assumption of Gaussian decay at large t_+ . Meanwhile, we have

$$h_-(v_-) = \int_{-\infty}^{\infty} \frac{dt_-}{\sqrt{\pi} \sigma} e^{-\frac{4}{\sigma^2} t_-^2} e^{-iv_- t_-} = \frac{1}{2} e^{-\frac{1}{16} \sigma^2 v_-^2}, \quad (\text{C62})$$

which has the form of a bounded function (in this case a constant) multiplied by a Gaussian.

For the exact sampler, instead, we have that $\alpha_{v_1, v_2} = \exp[-(\sigma v_1)^2/8 - (\sigma v_2)^2/8 - \beta v_1/4 - \beta v_2/4]$, which yields

$$g_{v_1, v_2} = -\frac{1}{2i} \tanh \frac{\beta(v_1 - v_2)}{4} \alpha_{v_1, v_2} = g_+(v_+) g_-(v_-) \quad (\text{C63})$$

with

$$\begin{aligned} g_+(v_+) &= \frac{i}{2} e^{-\beta v_+/4} e^{-\frac{(\sigma v_+)^2}{16}} \quad \text{and} \\ g_-(v_-) &= \tanh \frac{\beta v_-}{4} e^{-\frac{(\sigma v_-)^2}{16}}. \end{aligned} \quad (\text{C64})$$

Evidently, g_- once again has the form of a bounded function multiplying a Gaussian. Turning finally to the Fourier transform of g_+ , we have

$$|\widehat{g}_+(t_+)| = \left| \frac{i}{2} \int_{-\infty}^{\infty} \frac{dv_+}{2\pi} e^{-\beta v_+/4} e^{-\frac{(\sigma v_+)^2}{16}} e^{iv_+ t_+} \right|$$

$$\begin{aligned} &= \left| \frac{i}{\sqrt{\pi} \sigma} \exp \left[-\frac{4}{\sigma^2} \left(t_+ - \frac{i\beta}{4} \right)^2 \right] \right| \\ &= \frac{1}{\sqrt{\pi} \sigma} e^{\beta^2/4\sigma^2} e^{-4t_+^2/\sigma^2}, \end{aligned} \quad (\text{C65})$$

which is identical to $|\widehat{h}_+(t_+)|$ and in particular of exactly the form assumed in Eq. (C44).

APPENDIX D: BOUNDING THE ERROR IN DROPPING THE “REWINDING”

In this Appendix, we show that dropping the rewinding procedure in our protocol adds to the total error bound in a controllable way, namely that the error bound for the original protocol increases by a factor of $2\tau_{\text{mix}}$ [cf. Eq. (40) in the main text].

Concretely, we consider a channel $\mathcal{K}'$ defined as $\mathcal{K}'[\sigma] \equiv V\mathcal{K}[\sigma]V^\dagger$ for some unitary V , and a state ρ_β that is invariant under conjugation with V : $V\rho_\beta V^\dagger = \rho_\beta$. We assume that the fixed point of $\mathcal{K}$, ρ , ($\mathcal{K}[\rho] = \rho$) is close to ρ_β : $\|\rho - \rho_\beta\|_1 \leq \delta$ for some $\delta > 0$. We then show that

$$\|\rho' - \rho_\beta\|_1 \leq 4\delta\tau_{\text{mix}}, \quad (\text{D1})$$

where ρ' is the fixed-point of $\mathcal{K}'$ and τ_{mix} is the mixing time of $\mathcal{K}$, respectively.

As preliminaries, observe that since the channel $\mathcal{K}$ is contracting with respect to the Schatten 1-norm,

$$\|\mathcal{K}[\rho_\beta] - \rho\|_1 = \|\mathcal{K}[\rho_\beta] - \mathcal{K}[\rho]\|_1 \leq \|\rho_\beta - \rho\|_1 \leq \delta. \quad (\text{D2})$$

Furthermore, because the Schatten 1-norm is preserved under unitary conjugation,

$$\begin{aligned} \|\mathcal{K}'[\rho_\beta] - \rho_\beta\|_1 &= \|V\mathcal{K}[\rho_\beta]V^\dagger - \rho_\beta\|_1 \leq \|V(\mathcal{K}[\rho_\beta] - \rho)V^\dagger\|_1 \\ &\quad + \|V(\rho - \rho_\beta)V^\dagger\|_1 \leq 2\delta. \end{aligned} \quad (\text{D3})$$

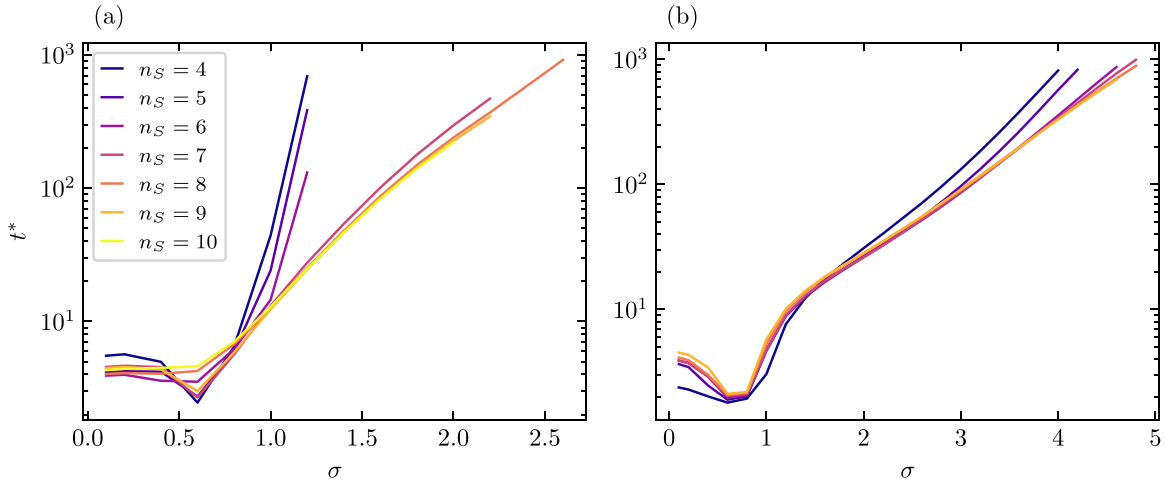


FIG. 4. The mixing time of the maximally mixed state t^* as a function of the parameter σ in the filter function, $\beta = 1.0$. (a) For the mixed-field Ising model defined in Eq. (41), (b) the transverse field Ising model defined in Eq. (E2). For large σ , t^* scales exponentially with σ .

Now, for the fixed point ρ' of $\mathcal{K}'$, we have

$$\begin{aligned} \|\rho' - \rho_\beta\|_1 &= \|\mathcal{K}'[\rho'] - \rho_\beta\|_1 \leq \|\mathcal{K}'[\rho'] - \mathcal{K}'[\rho_\beta]\|_1 \\ &\quad + \|\mathcal{K}'[\rho_\beta] - \rho_\beta\|_1 \leq \eta \|\rho' - \rho_\beta\|_1 + 2\delta, \end{aligned} \quad (\text{D4})$$

where $\eta \in [0, 1)$ is the contraction factor

$$\eta' \equiv \sup_{\omega \neq \zeta} \frac{\|\mathcal{K}'[\omega] - \mathcal{K}'[\zeta]\|_1}{\|\omega - \zeta\|_1} = \sup_{\omega \neq \zeta} \frac{\|\mathcal{K}[\omega] - \mathcal{K}[\zeta]\|_1}{\|\omega - \zeta\|_1} = \eta, \quad (\text{D5})$$

where the second inequality follows from the definition of $\mathcal{K}$. Note that $\eta < 1$ under the assumption that $\mathcal{K}$ is primitive (that is irreducible and aperiodic), i.e., that any initial state converges to a unique fixed point. Rearranging Eq. (D4) then gives

$$\|\rho' - \rho_\beta\|_1 \leq \frac{2\delta}{1 - \eta}. \quad (\text{D6})$$

What is left to show to obtain Eq. (D1) is then to relate the contraction factor η to the mixing time τ_{mix} . To this end, note that the contraction factor lower bounds the second-largest absolute value in the spectrum of $\mathcal{K}$, which in turn lower bounds the mixing time (see e.g., Lemma 30 of Ref. [53]),

$$\frac{1}{1 - \eta} \leq \frac{1}{1 - |\lambda_1|} \leq 2\tau_{\text{mix}}. \quad (\text{D7})$$

Substituting this into Eq. (D6) then yields the final result

$$\|\rho' - \rho_\beta\|_1 \leq \frac{2\delta}{1 - \eta} \leq 4\delta\tau_{\text{mix}}. \quad (\text{D8})$$

APPENDIX E: MIXING TIME ANALYSIS

In this Appendix, we present numerical results for the mixing time of the exact Gibbs sampler $\mathcal{L}_\beta$ as a function of the parameter σ in the filter function and the system size n_S . The analysis is important, since a different choice of σ in our

protocol leads to different jump operators in the approximated Lindbladian and as such quantities appearing in the accuracy bounds as the mixing time are not necessarily independent of the parameter sigma mixing time.

Since the mixing time itself is difficult to probe numerically, we use as a proxy the time to convergence, starting from the maximally mixed state ρ_0 ,

$$t^* = \min_t \|\rho_\beta - e^{\mathcal{L}_\beta t} \rho_0\|_1 < 0.1, \quad (\text{E1})$$

with ρ_0 being the maximally mixed state. t^* serves thus as a lower bound for the true mixing time. We compare t^* for the mixed-field Ising models defined in Eq. (41) and the transverse field Ising model,

$$H = \sum_i Z_i Z_{i+1} + gX_i \quad (\text{E2})$$

with $g = 0.9045$. While the mixed field Ising model is known to be thermalizing, the transverse field Ising model can be mapped to a model of free fermions, and the energy spectrum is described by discrete excitations with a finite spectral gap for $g \neq 1$. If σ is smaller than the spectral gap, we thus expect that transitions of these excitations are suppressed and the mixing time exponentially increases. The same argument essentially also holds for the mixed-field Ising model, at small temperatures below the gap. At high or intermediate temperatures, however, the effect of changing the filter is less obvious, since the level spacing at finite energy density is expected to be exponentially small in system size.

We choose Y_i on each site, as the set of jump operators, and $\beta = 1.0$. The results are shown in Fig. 4. For the transverse-field Ising model, t^* increases exponentially with σ with almost no dependence on system size. For the mixed-field Ising model, initially t^* decreases with system size, but for $n_S \geq 7$, this decrease saturates and we again observe an exponential increase of t^* with σ with little dependence on system size. Understanding the dependence of the mixing time on parameters, such as the parameter σ in the filter function, remains to be investigated in future work.

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- [77] Note that for any $n \times n$ matrix A , $\zeta_j(A) = \zeta_j(-A) \geq \lambda_j(-\frac{1}{2}(A + A^\dagger)) = -\lambda_{n-j+1}(\frac{1}{2}(A + A^\dagger))$, which yields the identity in Eq. (C22) if indexing is understood to be modulo n .