

Integrable and critical Haagerup spin chains

Luke Corcoran^{✉*} and Marius de Leeuw^{✉†}

School of Mathematics & Hamilton Mathematics Institute, Trinity College Dublin, Ireland



(Received 14 November 2024; revised 21 February 2025; accepted 7 April 2025; published 16 April 2025)

We construct the first integrable models based on the Haagerup fusion category H_3 . We introduce a Haagerup version of the anyonic spin chain and use the boost operator formalism to identify two integrable Hamiltonians of PXP type on this chain. The first of these is an analog of the golden chain; it has a topological symmetry based on H_3 and satisfies the Temperley-Lieb algebra with parameter $\delta = (3 + \sqrt{13})/2$. We prove its integrability using a Lax formalism, and construct the corresponding solution to the Yang-Baxter equation. We present numerical evidence that this model is gapless with a dynamical critical exponent $z \neq 1$. The second integrable model we find breaks the topological symmetry. We present numerical evidence that this model reduces to a CFT in the large volume limit with central charge $c \sim 3/2$.

DOI: [10.1103/PhysRevB.111.L140408](https://doi.org/10.1103/PhysRevB.111.L140408)

Introduction. Noninvertible symmetries have played a prominent role in recent years, with applications ranging widely from condensed matter systems and critical phenomena to topological lines in quantum field theory [1–5]. They appear notably in the context of the two-dimensional (2D) Ising model via the Kramers-Wannier duality [6], a generalization of which is given by the statistical AFM model [7,8]. By taking the anisotropic limit of this model, one can obtain one-dimensional quantum anyonic chains [9]. These models also enjoy noninvertible symmetries and are based on the data of fusion categories [10].

Fusion categories are noninvertible extensions of groups, and consist of a set of simple objects $a, b, c, \dots$, which can also be thought of as anyon species. These simple objects can be combined using the fusion rule

$$a \times b = \bigoplus_c N_{ab}^c c, \quad (1)$$

where N_{ab}^c are non-negative integers. The associativity of the usual group law is relaxed using the data of F -symbols:

$$\begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ p \quad \quad q \\ \diagup \quad \diagdown \\ u \end{array} = \sum_q (F_u^{abc})_{qp} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ \quad \quad q \\ \diagup \quad \diagdown \\ u \end{array}, \quad (2)$$

where $(F_u^{abc})_{qp} \in \mathbb{C}$. These F -symbols need to satisfy a set of consistency conditions which arise from four anyon fusion known as the pentagon equations. They are the key input for the construction of fusion category symmetric anyonic chains.

The simplest example is given by that of the golden chain [11], which is related to the Fibonacci fusion category [12]. This model has many interesting properties; it is a critical lattice model which corresponds to a conformal field theory (CFT) at both ends of its spectrum in the continuum limit. It is also an integrable model, since it is a special point of an integrable spin chain related to Baxter's hard square model [13,14]. This integrable model is related to the restricted solid-on-solid (RSOS) models of Andrews, Baxter, and Forrester [15,16], see also, Ref. [17]. The spin-chain Hilbert space for the golden chain is *constrained*; in particular it is a spin-1/2 chain where neighboring down spins are disallowed. This constraint is known as the Rydberg blockade [18]. The Rydberg-constrained Hilbert space admits several interesting models which do not commute with the fusion category symmetry, for example the PXP model [19–21], the Lesanovsky model [22–24], and the constrained XXZ model [25,26].

It is a natural question whether there are interesting anyonic chains based on fusion categories beyond the Fibonacci case. The simplest example of a fusion category which is not simply related to (affine) Lie algebras or quantum groups is the so-called Haagerup fusion category [27,28]. The F -symbols were calculated recently [29,30], which allowed for the investigation of Haagerup symmetric models at the level of an anyonic chain [31,32] and a 2D lattice model [33]. Both [32,33] identify critical models with $c = 2$, although the relation between these models is still unclear.

In this letter we study Haagerup anyonic spin chains from the point of view of integrability, which provides a new way to generate potentially interesting models. In recent years, the boost operator formalism has proven an effective method for generating integrable models on various Hilbert spaces [34–39]. This method has recently been adapted to higher range models [40] and constrained Hilbert spaces [41]. We apply this framework to partially classify the set of range-3 integrable models on the Haagerup-constrained Hilbert space. We identify one of the models obtained with the projector onto the identity fusion channel, P_1 , so that this model can be seen

*Contact author: lcorcoran@maths.tcd.ie

†Contact author: mdeleeuw@maths.tcd.ie

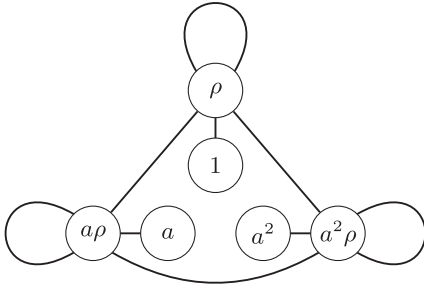


FIG. 1. Adjacency rules for the Haagerup Hilbert space. Nodes represent simple objects in H_3 . If there is no edge between a pair of objects then this is a disallowed pair in V^L .

as one generalization of the golden chain to the Haagerup Hilbert space. The alternative generalization, P_ρ , is critical but notably not integrable. The second model we find partially breaks the Haagerup topological symmetry, but appears to be a critical spin chain in the large volume limit with central charge $c \sim 3/2$.

Haagerup Hilbert space. The Haagerup fusion category H_3 consists of six simple objects $\text{Obj}(H_3) = \{1, a, a^2, \rho, a\rho, a^2\rho\}$. The nontrivial fusion rules are

$$\rho \times a^2 = a\rho, \quad \rho \times \rho = 1 + \rho + a\rho + a^2\rho, \quad (3)$$

and $a^3 = 1$. An anyonic chain based on this fusion category can be constructed, which describes the interaction of L ρ anyons. We denote the Hilbert space by V^L . The basis states are given via the fusion diagram [11,32]

$$|x_1 x_2 \cdots x_L\rangle = \left\| \begin{array}{c} \rho \quad \rho \quad \rho \quad \rho \\ | \\ x_1 \quad x_2 \quad \cdots \quad x_L \quad x_1 \end{array} \right\|, \quad (4)$$

where $x_i \in \text{Obj}(H_3)$ and slashed lines represent periodic identification. In order for Eq. (4) to represent an allowed state in the Hilbert space, it must correspond to a valid fusion diagram in H_3 . In other words, the fusion $x_i \times \rho$ must contain x_{i+1} , for each $i = 1, 2, \dots, L$.

V^L can be obtained from $(\mathbb{C}^6)^{\otimes L}$ by projecting away disallowed neighboring pairs. Of the 36 potential neighboring pairs in $\mathbb{C}^6 \otimes \mathbb{C}^6$, only 15 are allowed by the fusion diagram (4). The precise adjacency rules are summarized in Fig. 1. We introduce an explicit projector

$$\Pi^L = \prod_{i=1}^L \Pi_{i,i+1}, \quad \Pi^L : (\mathbb{C}^6)^{\otimes L} \rightarrow V^L, \quad (5)$$

where $\Pi_{i,i+1} \in \text{End}(\mathbb{C}^6 \otimes \mathbb{C}^6)$ projects away disallowed neighboring states. Due to this constraint the Hilbert space V^L does not have a natural tensor product form. Its dimension grows as $d_L \sim \psi^L$, where $\psi = (3 + \sqrt{13})/2$ is the quantum dimension of the object ρ .

Projectors and Haagerup symmetry. The simplest operators one can consider on V^L are those which project the fusion of neighboring ρ anyons onto a particular fusion channel $z \in \{1, \rho, a\rho, a^2\rho\}$. The definition is analogous to the XXX Heisenberg spin chain, which can be viewed as a projection of the fusion of neighboring spin- $\frac{1}{2}$ particles onto the

spin-0 singlet. The construction of such projectors for a general fusion category is described in many places, see for example [42].

To define these operators, we use the F -symbol identity

$$\begin{array}{c} \rho \quad \rho \\ | \quad | \\ x_{i-1} \quad x_i \quad x_{i+1} \end{array} = \sum_{\tilde{x}_i} (F_{x_{i-1}}^{\rho\rho x_{i+1}})_{\tilde{x}_i x_i} \begin{array}{c} \rho \quad \rho \\ \diagdown \quad \diagup \\ x_{i-1} \quad x_{i+1} \end{array} \tilde{x}_i, \quad (6)$$

where $(F_u^{abc})_{qp}$ are the F -symbols of H_3 [29]. We define a projector $P_z^{(i)}$ which acts across three sites $|x_{i-1}x_i x_{i+1}\rangle$ and selects the $\tilde{x}_i = z$ fusion channel. We use a further F -move to write the result in the original basis, such that the operator takes the form [43]

$$P_z^{(i)} : |x_{i-1}x_i x_{i+1}\rangle = \sum_{\tilde{x}_i} (F_{x_{i+1}}^{x_{i-1}\rho\rho})_{z\tilde{x}_i} (F_{x_{i-1}}^{x_{i+1}\rho\rho})_{\tilde{x}_i z} |x_{i-1}\tilde{x}_i x_{i+1}\rangle. \quad (7)$$

Since one fusion outcome is guaranteed, these operators satisfy $P_1^{(i)} + P_\rho^{(i)} + P_{a\rho}^{(i)} + P_{a^2\rho}^{(i)} = 1$. On the full Hilbert space we define the projectors as $P_z = -\sum_{i=1}^L P_z^{(i)}$. The critical model studied in [32] is P_ρ .

The projectors P_z are distinguished by the fact that they are symmetric under the action of the Haagerup fusion category. An operator $\mathcal{O} : V^L \rightarrow V^L$ has Haagerup symmetry if it commutes with a set of nonlocal topological operators Y_z , $z \in \text{Obj}(H_3)$. The action of Y_z is defined by fusing z into a state $|x_1 x_2 \cdots x_L\rangle$ below the chain [11,44,45]. Its matrix elements are given by

$$\langle y_1 y_2 \cdots y_L | Y_z | x_1 x_2 \cdots x_L \rangle = \prod_{i=1}^L (F_{y_{i+1}}^{z x_i \rho})_{x_{i+1} y_i}. \quad (8)$$

The operators Y_z furnish a representation of the fusion algebra, and therefore $\mathcal{O}$ only needs to commute with Y_a and Y_ρ to have Haagerup symmetry. Such operators decompose into various topological sectors which are labeled by the eigenvalues of Y_z . We note that while Y_ρ is invertible as a matrix, it is not invertible as a topological symmetry. This is because Y_ρ^{-1} does not belong to the fusion algebra [45].

We choose to use an F -symbol gauge where Y_a coincides with the operator fusing all $x \in \text{Obj}(H_3)$ with a . In other words, Y_a simply rotates the labels $1 \rightarrow a \rightarrow a^2 \rightarrow 1$ and $\rho \rightarrow a\rho \rightarrow a^2\rho \rightarrow \rho$ [46].

In the following we consider operators on V^L which do not necessarily commute with the full set of topological operators, i.e. operators which are not necessarily a linear combination of projectors P_z . We do this in analogy to the Rydberg atom Hilbert space, which admits interesting operators beyond those constructed directly from Fibonacci fusion category data.

Integrability. Here we are searching for *integrable* Hamiltonians $Q_2^{(\Pi)} : V^L \rightarrow V^L$ on the Haagerup Hilbert space, using the constrained boost operator formalism of [41]. We call a range-3 Hamiltonian $Q_2 : (\mathbb{C}^6)^{\otimes L} \rightarrow (\mathbb{C}^6)^{\otimes L}$ integrable on V^L if it is a member of an infinite tower of constraint compatible operators $Q_i : (\mathbb{C}^6)^{\otimes L} \rightarrow (\mathbb{C}^6)^{\otimes L}$ which mutually commute on the constrained subspace:

$$\Pi^L [Q_i, Q_j] \Pi^L = 0, \quad [Q_i, \Pi^L] = 0. \quad (9)$$

For translationally invariant spin chains the first charge $\mathbb{Q}_1$ can always be taken as the momentum operator, which generates translations along the chain. For Yang-Baxter integrable spin chains, $\mathbb{Q}_3$ can be obtained from the Hamiltonian using a higher-range analog of the so-called *boost operator* [40,47,48]:

$$\mathbb{Q}_3 = \sum_{i=1}^L [\mathcal{H}_{i,i+1,i+2}, \mathcal{H}_{i+1,i+2,i+3} + \mathcal{H}_{i+2,i+3,i+4}], \quad (10)$$

where $\mathbb{Q}_2 = -\sum_{i=1}^L \mathcal{H}_{i,i+1,i+2}$. Therefore, given a Hamiltonian density $\mathcal{H}$, a necessary condition for integrability is

$$\Pi^L[\mathbb{Q}_2, \mathbb{Q}_3]\Pi^L = 0, \quad (11)$$

where $\mathbb{Q}_3$ is constructed from $\mathbb{Q}_2$ using (10). This condition is believed to be sufficient; we are not aware of any models satisfying (11) which are not integrable. To prove integrability it is necessary to construct a Lax operator which generates the charges $\mathbb{Q}_i$ and a corresponding R -matrix.

Hamiltonian ansatz. We search for solutions to the integrability condition (11), and so require an appropriate ansatz for the Hamiltonian density $\mathcal{H}$. We require that $\mathbb{Q}_2$ is symmetric and invariant under spatial reflection. We further require that $\mathcal{H}_{i,i+1,i+2}$ takes a generalized PXP form

$$\mathcal{H}_{i,i+1,i+2} = D_i \mathcal{O}_{i+1} D_{i+2}, \quad (12)$$

where $D : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ is diagonal. Finally, we require partial Haagerup symmetry, and impose that $\mathbb{Q}_2^{(\Pi)} := \Pi^L \mathbb{Q}_2$ commutes with Y_a . We trivialize Y_a symmetry by decomposing the Hamiltonian density

$$\mathcal{H}_{i,i+1,i+2} = \tilde{\mathcal{H}}_{i,i+1,i+2} + \tilde{\mathcal{H}}_{i,i+1,i+2}^a + \tilde{\mathcal{H}}_{i,i+1,i+2}^{a^2}, \quad (13)$$

where $\tilde{\mathcal{H}}_{i,i+1,i+2}^a = Y_a \tilde{\mathcal{H}}_{i,i+1,i+2} Y_a^{-1}$. After imposing these requirements, there are 18 operators contributing to the reduced Hamiltonian $\tilde{\mathcal{H}}_{i,i+1,i+2}$. These operators can be written in terms of the following local operators.

Diagonal operators. The diagonal operators are spanned by the projectors P_x , for $x \in \text{Obj}(H_3)$, which act via $P_x|y\rangle = \delta_{xy}|y\rangle$.

Flips. We will denote by X, Y, Z the operators which flip between $|1\rangle$ and $|\rho\rangle$, $|\rho\rangle$, and $|a^2\rho\rangle$, respectively:

$$X : |1\rangle \leftrightarrow |\rho\rangle, \quad Y : |1\rangle \leftrightarrow |\rho\rangle, \quad Z : |1\rangle \leftrightarrow |a^2\rho\rangle. \quad (14)$$

Transpositions. There is one more independent operator which transposes noninvertible objects:

$$T : |\rho\rangle \leftrightarrow |a\rho\rangle. \quad (15)$$

Integrable solutions. To search for integrable Hamiltonians on V^L , we use a Hamiltonian ansatz which is a linear combination of the 18 operators contributing to $\tilde{\mathcal{H}}_{i,i+1,i+2}$. We use this to form the higher charge $\mathbb{Q}_3$ using (10), and impose the integrability condition (11). This leads to a set of highly coupled cubic equations in the coefficients of $\tilde{\mathcal{H}}$.

Model 1. We are able to fully classify the solutions of these equations which further commute with the topological operator Y_ρ . We find a single solution, where the reduced

Hamiltonian takes the form

$$\begin{aligned} \tilde{\mathcal{H}}_{i,i+1,i+2} = & \psi^{-1/2} P_\rho (X + Y + Z) P_\rho + P_\rho T P_\rho + P_{a\rho} T P_{a\rho} \\ & + P_{a^2\rho} T P_{a^2\rho} + \psi P_1 P_\rho P_1 + \psi^{-1} P_\rho P_1 P_\rho \\ & + P_\rho P_\rho P_\rho + P_\rho P_{a\rho} P_\rho + P_{a\rho} P_\rho P_{a\rho}, \end{aligned} \quad (16)$$

where for brevity we omitted indices on the right hand side. For example, the first term should be $\psi^{-1/2} P_{\rho,i} X_{i+1} P_{\rho,i+2}$. This Hamiltonian coincides with $\mathbb{P}_1$, as given by (7).

Model 2. We partially classified the solutions of (11) which do not commute with Y_ρ . One solution is

$$\begin{aligned} \tilde{\mathcal{H}}_{i,i+1,i+2} = & \gamma^{-1/2} P_\rho (X + Y + Z) P_\rho \\ & + P_\rho T P_\rho + P_{a\rho} T P_{a\rho} + P_{a^2\rho} T P_{a^2\rho} \\ & + (1 + 2\gamma) P_1 P_\rho P_1 + (1 + 2\gamma^{-1}) P_\rho P_1 P_\rho \\ & + 2\gamma (P_\rho P_\rho P_\rho + P_\rho P_{a\rho} P_\rho + P_{a\rho} P_\rho P_{a\rho}), \end{aligned} \quad (17)$$

where $\gamma = (1 + \sqrt{3})/2$.

Lax operator and Yang-Baxter. The Hamiltonians (16) and (17) obey the necessary condition (11) for integrability. In order to prove integrability, we need to exhibit both as a member of a larger family of commuting charges $\mathbb{Q}_i$. We do this by the construction of a *Lax operator* $\mathcal{L}_{A,j}(u) \in \text{End}(V_A \otimes V_j)$. $V_A \simeq \mathbb{C}^6 \otimes \mathbb{C}^6$ is the *auxiliary space* [49], $V_j \simeq \mathbb{C}^6$ is a copy of the local physical space, and u is a spectral parameter. The Lax operator is used to form a transfer matrix, which is a generating function for the set of commuting charges.

Model 1. Writing $\mathbb{Q}_2 = -\sum_{i=1}^L e_i$ we find that the generators e_i satisfy the Temperley-Lieb algebra

$$\begin{aligned} e_i^2 &= \psi e_i, & e_i e_{i\pm 1} e_i &= e_i, \\ [e_i, e_j] &= 0, & |i - j| > 1. \end{aligned} \quad (18)$$

In such cases the Lax operator can typically be obtained as a linear combination of the identity matrix and $\mathcal{H}$. Indeed, we find that

$$\begin{aligned} \mathcal{L}_{123}(u) &= \mathcal{P}_{13} \mathcal{P}_{23} \left(1 - \frac{1}{\frac{\psi}{2} - \alpha \coth \alpha u} \mathcal{H}_{123} \right) \\ &:= \mathcal{P}_{13} \mathcal{P}_{23} \check{\mathcal{L}}_{123}(u) \end{aligned} \quad (19)$$

is appropriate, where $\mathcal{P}_{ij}$ is the permutation operator on $V_i \otimes V_j$ and $\alpha := \sqrt{\frac{3}{4}}(\psi - 1)$. The transfer matrix is defined by

$$t(u) := \text{tr}_A[\mathcal{L}_{AL}(u) \mathcal{L}_{A,L-1}(u) \cdots \mathcal{L}_{A1}(u)], \quad (20)$$

and the corresponding charges $\mathbb{Q}_i$ are extracted from $t(u)$ via $i - 1$ logarithmic derivatives at $u = 0$.

In order to show the integrability condition (9), we look for a solution to the constrained $R\mathcal{L}\mathcal{L}$ equation [41]. We find the following R -matrix $R_{AB}(u, v)$:

$$\check{R}_{1234}(u, v) = \check{\mathcal{L}}_{123}(-v) \check{\mathcal{L}}_{234}(u - v) \check{\mathcal{L}}_{123}(u), \quad (21)$$

which satisfies appropriate projected $R\mathcal{L}\mathcal{L}$ and Yang-Baxter equations [40,41]. This R -matrix can be used to prove $\Pi^L[t(u), t(v)]\Pi^L = 0$ [50], from which the integrability condition (9) follows.

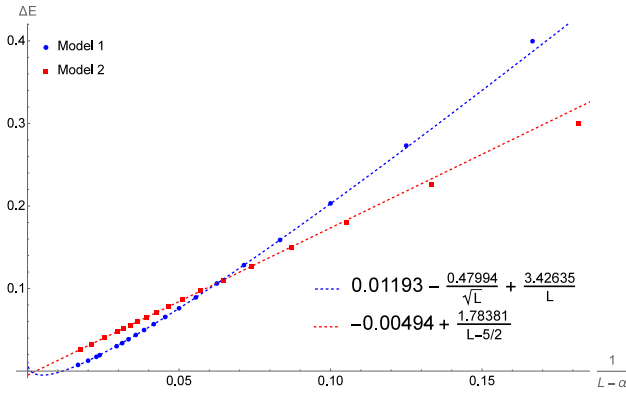


FIG. 2. Energy gap up to $L = 60$ on the periodic chain vs the inverse shifted length $(L - \alpha)^{-1}$. For model 1 we take $\alpha = 0$ and for model 2 we take $\alpha = 5/2$. For the model 2 fit we used the last 10 points.

Model 2. The Lax operator for model 2 is given by

$$\check{L}_{123}(u) = 1 + u\mathcal{H}_{123} + f(u)\mathcal{H}_{123}^2, \quad (22)$$

where $(u + 2\gamma f(u))^2 = f(u)(f(u) + 2)$.

Numerics. We investigate the low-lying spectrum of the obtained integrable models. For a gapless model the energy gap vanishes at large L :

$$\Delta E := E_1 - E_0 \propto \frac{1}{L^z}. \quad (23)$$

For a CFT we expect a dynamical critical exponent $z = 1$ and the half-chain entanglement entropy of the ground state on the periodic chain to scale as [51,52]

$$S \sim \frac{c}{3} \log L. \quad (24)$$

We computed both of these quantities numerically for the Hamiltonians (16) and (17). We studied the periodic chain because integrability is preserved and the ground state is nondegenerate for both models. We used methods of density matrix renormalization group with the iTensor package [53,54]. We verified these results up to $L = 14$ via exact diagonalisation.

For model 1 we find evidence of gaplessness. While we were unable to determine the analytic behavior of the gap, any Laurent series fit in L or $L^{1/2}$ leads to a small or negative gap in the large- L limit. Since the gap needs to be positive, this points towards a vanishing gap. In Fig. 2 we present a fit of the gap to $a + bL^{-1/2} + cL^{-1}$, which was one of the best fits we obtained. More data is needed to understand the precise analytic structure. We expect that the model is not a CFT since we find a dynamical critical exponent $z \neq 1$. This is supported by our results of the half-chain entanglement entropy, which does not scale as $\log L$, see Fig. 3.

For model 2 we find evidence that this model is critical in the large volume limit. We find in Fig. 2 that the gap scales as $(L - 5/2)^{-1}$ as $L \rightarrow \infty$. Furthermore, we find that the

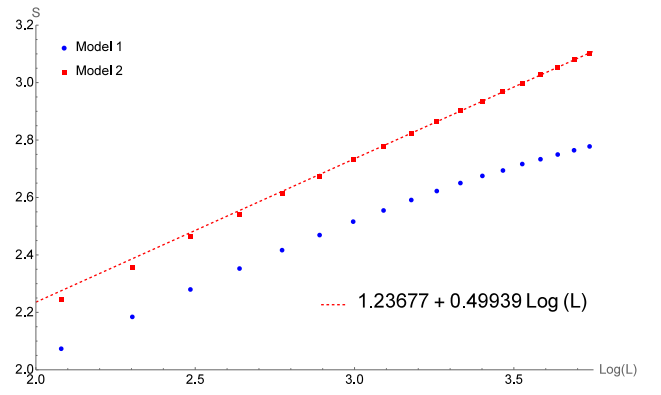


FIG. 3. Half-chain entanglement entropy up to $L = 42$ on the periodic chain. For the fit we used the last 10 points.

half-chain entanglement entropy scaling in Fig. 3 is consistent with Eq. (24) for a central charge $c \sim 3/2$. We see that for $L > 20$ both plots become nicely linear. We, therefore, propose model 2 as the first example of a critical Haagerup spin chain which is integrable.

Conclusions and outlook. In this letter we studied Haagerup spin chains from the perspective of integrability. We applied the boost operator formalism and presented two integrable spin chains on the Haagerup Hilbert space. We find that only one integrable Hamiltonian, P_1 , is compatible with the full Haagerup symmetry. The other model we present breaks this topological symmetry, but seems to correspond to a CFT in the continuum limit.

Due to integrability, the Temperley-Lieb algebra (18), and the fact that the model is one generalization of the golden chain, it was reasonable to expect that (16) also corresponds to a CFT in the continuum limit. We do not find evidence of this at our current level of numerics. This is potentially surprising, given that many TL models correspond to CFTs. A potential reason in this case is that the Temperley-Lieb parameter $\delta = \psi = q + q^{-1}$ does not correspond to q which is a root of unity, which differs from the golden chain case. Since the model still appears to be gapless, it is interesting to consider whether this TL algebra corresponds at least to some deformation of conformal symmetry. It would be interesting to consider whether there are noninvertible dualities between models based on this representation of the TL algebra, similarly to Ref. [55].

The golden chain corresponds to the projector P_1 constructed from the Fibonacci fusion category F -symbols, or equivalently to $P_\tau = 1 - P_1$. Therefore both P_1 and P_ρ can be considered as Haagerup extensions of the golden chain. We see that while P_ρ inherits criticality from the golden chain, it is P_1 which inherits the integrability properties. We conjecture that this pattern persists for Haagerup-Izumi fusion categories of higher rank [30].

We presented one more integrable model which appears to be critical in the large volume limit, with central charge $c \sim 3/2$. Since we find a shifted scaling behavior $\Delta E \sim 1/(L - \alpha)$, it is important to develop more specific numerical and analytic tools for anyonic spin chains, to push the computation of the gap and other to higher length.

This is also necessary to study the ferromagnetic versions of our models, which have highly degenerate ground states.

There are hints that both models we present are different points of a more general family of integrable Hamiltonians on the Haagerup Hilbert space. It would be interesting to identify this parent integrable model, which potentially further breaks Y_a symmetry, and study the corresponding integrable statistical mechanical model. In this case the situation would resemble the link between the golden chain and the one-parameter extension [13], which corresponds to Baxter's hard square model.

Finally, there are several possibilities for analytic computations at the interface between integrability and fusion categories. For example, it may be possible to prove when a

fusion category projector Hamiltonian is boost integrable by using the pentagon identity. While we have proven integrability of some models, their exact solvability is an open question. Therefore, It would be important to formulate a Bethe ansatz solution for these models. Finally, it would be interesting to make contact with recent the noninvertible S -matrix bootstrap programme [56,57].

Acknowledgments. We thank S. Komatsu, T. McLoughlin, and Y. Tachikawa for useful discussions. We thank B. Pozsgay for discussions and collaboration on a related project. M.d.L. was supported in part by SFI and the Royal Society for funding under Grants No. UF160578, No. RGF\R1\181011, No. RGF\8EA\180167, and No. RF\ERE\210373. M.d.L. is also supported by ERC-2022-CoG - FAIM 101088193. L.C. was supported by RF\ERE\210373.

- [1] J. McGreevy, Generalized symmetries in condensed matter, *Annu. Rev. Condens. Matter Phys.* **14**, 57 (2023).
- [2] C. Cordova, T. T. Dumitrescu, K. Intriligator, and S.-H. Shao, Snowmass white paper: Generalized symmetries in quantum field theory and Beyond, in *Snowmass 2021* (2022).
- [3] L. Bhardwaj, L. E. Bottini, L. Fraser-Taliente, L. Gladden, D. S. W. Gould, A. Platschorre, and H. Tillim, Lectures on generalized symmetries, *Phys. Rep.* **1051**, 1 (2024).
- [4] S. Schafer-Nameki, ICTP lectures on (non-)invertible generalized symmetries, *Phys. Rep.* **1063**, 1 (2024).
- [5] S.-H. Shao, What's done cannot be undone: TASI lectures on non-invertible symmetries, [arXiv:2308.00747](https://arxiv.org/abs/2308.00747).
- [6] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part II, *Phys. Rev.* **60**, 263 (1941).
- [7] D. Aasen, R. S. K. Mong, and P. Fendley, Topological defects on the lattice I: The Ising model, *J. Phys. A: Math. Theor.* **49**, 354001 (2016).
- [8] D. Aasen, P. Fendley, and R. S. K. Mong, Topological defects on the lattice: Dualities and degeneracies, [arXiv:2008.08598](https://arxiv.org/abs/2008.08598).
- [9] K. Inamura and K. Ohmori, Fusion surface models: 2+1d lattice models from fusion 2-categories, *SciPost Phys.* **16**, 143 (2024).
- [10] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor Categories* (American Mathematical Society, Providence, RI, 2015).
- [11] A. Feiguin, S. Trebst, A. W. W. Ludwig, M. Troyer, A. Kitaev, Z. Wang, and M. H. Freedman, Interacting anyons in topological quantum liquids: The golden chain, *Phys. Rev. Lett.* **98**, 160409 (2007).
- [12] S. Trebst, M. Troyer, Z. Wang, and A. W. W. Ludwig, A short introduction to fibonacci anyon models, *Prog. Theor. Phys. Suppl.* **176**, 384 (2008).
- [13] P. Fendley, K. Sengupta, and S. Sachdev, Competing density-wave orders in a one-dimensional hard-boson model, *Phys. Rev. B* **69**, 075106 (2004).
- [14] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, New York, 1982).
- [15] G. E. Andrews, R. J. Baxter, and P. J. Forrester, Eight-vertex sos model and generalized Rogers-Ramanujan-type identities, *J. Stat. Phys.* **35**, 193 (1984).
- [16] R. J. Baxter and P. J. Forrester, Further exact solutions of the eight-vertex SOS model and generalizations of the Rogers-Ramanujan identities, *J. Stat. Phys.* **38**, 435 (1985).
- [17] D. Bianchini, E. Ercolessi, P. A. Pearce, and F. Ravanini, RSOS quantum chains associated with off-critical minimal models and $\mathbb{Z}_n$ parafermions, *J. Stat. Mech.* (2015) P03010.
- [18] E. Urban, T. A. Johnson, T. Henage, L. Isenhower, D. D. Yavuz, T. G. Walker, and M. Saffman, Observation of Rydberg blockade between two atoms, *Nat. Phys.* **5**, 110 (2009).
- [19] I. Lesanovsky and H. Katsura, Interacting fibonacci anyons in a Rydberg gas, *Phys. Rev. A* **86**, 041601(R) (2012).
- [20] M. Serbyn, D. A. Abanin, and Z. Papić, Quantum many-body scars and weak breaking of ergodicity, *Nat. Phys.* **17**, 675 (2021).
- [21] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papić, Quantum scarred eigenstates in a Rydberg atom chain: Entanglement, breakdown of thermalization, and stability to perturbations, *Phys. Rev. B* **98**, 155134 (2018).
- [22] I. Lesanovsky, Many-body spin interactions and the ground state of a dense Rydberg lattice gas, *Phys. Rev. Lett.* **106**, 025301 (2011).
- [23] I. Lesanovsky, Liquid ground state, gap, and excited states of a strongly correlated spin chain, *Phys. Rev. Lett.* **108**, 105301 (2012).
- [24] D. K. Mark, C.-J. Lin, and O. I. Motrunich, Exact eigenstates in the Lesanovsky model, proximity to integrability and the PXP model, and approximate scar states, *Phys. Rev. B* **101**, 094308 (2020).
- [25] F. C. Alcaraz and R. Z. Bariev, An exactly solvable constrained XXZ chain, [arXiv:cond-mat/9904042](https://arxiv.org/abs/cond-mat/9904042).
- [26] F. C. Alcaraz and R. Z. Bariev, Integrable models of strongly correlated particles with correlated hopping, *Phys. Rev. B* **59**, 3373 (1999).
- [27] U. Haagerup, Principal graphs of subfactors in the index range $4 < [M : N] < 3 + \sqrt{2}$, in *Subfactors: Proceedings of the Taniguchi Symposium, Katata*, (World Scientific, Singapore, 1993), pp. 1–38.
- [28] P. Grossman and N. Snyder, Quantum subgroups of the Haagerup fusion categories, *Commun. Math. Phys.* **311**, 617 (2012).
- [29] T. J. Osborne, D. E. Stiegemann, and R. Wolf, The F -symbols for the H3 fusion category, [arXiv:1906.01322](https://arxiv.org/abs/1906.01322).
- [30] T.-C. Huang and Y.-H. Lin, The F -symbols for transparent Haagerup-Izumi categories with $G = \mathbb{Z}_{2n+1}$, [arXiv:2007.00670](https://arxiv.org/abs/2007.00670).

- [31] R. Wolf, Microscopic models for fusion categories, Ph.D. thesis, Leibniz University, Hannover, 2020.
- [32] T.-C. Huang, Y.-H. Lin, K. Ohmori, Y. Tachikawa, and M. Tezuka, Numerical evidence for a Haagerup conformal field theory, *Phys. Rev. Lett.* **128**, 231603 (2022).
- [33] R. Vanhove, L. Lootens, M. Van Damme, R. Wolf, T. J. Osborne, J. Haegeman, and F. Verstraete, Critical lattice model for a Haagerup conformal field theory, *Phys. Rev. Lett.* **128**, 231602 (2022).
- [34] M. de Leeuw, A. Pribytok, and P. Ryan, Classifying two-dimensional integrable spin chains, *J. Phys. A* **52**, 505201 (2019).
- [35] M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore, and P. Ryan, Classifying nearest-neighbor interactions and deformations of ads, *Phys. Rev. Lett.* **125**, 031604 (2020).
- [36] M. de Leeuw, A. Pribytok, A. L. Retore, and P. Ryan, New integrable 1D models of superconductivity, *J. Phys. A: Math. Theor.* **53**, 385201 (2020).
- [37] M. de Leeuw, C. Paletta, A. L. Pribytok, A. Retore, and P. Ryan, Yang-Baxter and the boost: Splitting the difference, *SciPost Phys.* **11**, 069 (2021).
- [38] M. de Leeuw, C. Paletta, A. Pribytok, A. L. Retore, and A. Torrielli, Free Fermions, vertex Hamiltonians, and lower-dimensional *AdS/CFT*, *J. High Energy Phys.* **02** (2021) 191.
- [39] L. Corcoran and M. de Leeuw, All regular 4×4 solutions of the Yang-Baxter equation, *SciPost Phys. Core* **7**, 045 (2024).
- [40] T. Gombor and B. Pozsgay, Integrable spin chains and cellular automata with medium-range interaction, *Phys. Rev. E* **104**, 054123 (2021).
- [41] L. Corcoran, M. de Leeuw, and B. Pozsgay, Integrable models on Rydberg atom chains, [arXiv:2405.15848](https://arxiv.org/abs/2405.15848).
- [42] L. Eck and P. Fendley, Generalizations of Kitaev's honeycomb model from braided fusion categories, [arXiv:2408.04006](https://arxiv.org/abs/2408.04006).
- [43] We take a set of F-symbols which are unitary and real.
- [44] C. Gils, E. Ardonne, S. Trebst, D. A. Huse, A. W. W. Ludwig, M. Troyer, and Z. Wang, Anyonic quantum spin chains: Spin-1 generalizations and topological stability, *Phys. Rev. B* **87**, 235120 (2013).
- [45] M. Buican and A. Gromov, Anyonic chains, topological defects, and conformal field theory, *Commun. Math. Phys.* **356**, 1017 (2017).
- [46] In practice we achieve this by using the F symbols of Ref. [29] with a mild gauge transformation.
- [47] M. G. Tetel'man, Lorentz group for two-dimensional integrable lattice systems, *Sov. J. Exper. Theoret. Phys.* **55**, 306 (1982).
- [48] J. Links, H.-Q. Zhou, R. H McKenzie, and M. D Gould, Ladder operator for the one-dimensional Hubbard model, *Phys. Rev. Lett.* **86**, 5096 (2001).
- [49] For models of range r , the auxiliary space contains $r - 1$ copies of the physical space.
- [50] The proof is closely related to the standard train-track argument of integrability, see, e.g., Ref. [58], which constructs an RTT relation from the RLL relation and proves $[t(u), t(v)] = 0$ by taking a trace. This has been adapted in Ref. [41] to integrable models on constrained Hilbert spaces by inserting appropriate projectors.
- [51] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, *Nucl. Phys. B* **424**, 443 (1994).
- [52] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, *J. Stat. Mech.* (2004) P06002.
- [53] We used a maximum truncation error of 10^{-9} and a maximum bond dimension of 1700.
- [54] M. Fishman, S. White, and E. Stoudenmire, The ITensor software library for tensor network calculations, *SciPost Physics Codebases*, 4 (2022).
- [55] L. Eck and P. Fendley, From the XXZ chain to the integrable Rydberg-blockade ladder via non-invertible duality defects, *SciPost Phys.* **16**, 127 (2024).
- [56] C. Copetti, L. Cordova, and S. Komatsu, Non-invertible symmetries, anomalies and scattering amplitudes, [arXiv:2403.04835](https://arxiv.org/abs/2403.04835).
- [57] C. Copetti, L. Cordova, and S. Komatsu, S-matrix bootstrap and non-invertible symmetries, [arXiv:2408.13132](https://arxiv.org/abs/2408.13132).
- [58] L. Faddeev, How algebraic Bethe ansatz works for integrable model, [arXiv:hep-th/9605187](https://arxiv.org/abs/hep-th/9605187).