

Translation symmetry restoration under random unitary dynamics

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The finite parts of a large, locally interacting many-body system prepared out of equilibrium eventually equilibrate. Characterizing the underlying mechanisms of this process and its timescales, however, is particularly hard as it requires one to decouple universal features from observable-specific ones. Recently, additional insight was gained by studying how certain symmetries of the dynamics that are broken by the initial state are restored at the level of the reduced state of a given subsystem. This provides a high-level, observable-independent probe. Until now, this idea has been applied to the restoration of internal symmetries, e.g., $U(1)$ symmetries related to charge conservation. Here we show that the same logic can be applied to the restoration of space-time symmetries, and hence can be used to characterize the relaxation of fully generic systems. We illustrate this idea by considering the paradigmatic example of “generic” many-body dynamics, i.e., a local random unitary circuit, where our method leads to exact results. We show that the restoration of translation symmetry in these systems only happens on timescales that are proportional to the subsystem’s volume. In fact, for large enough subsystems, the time of symmetry restoration becomes initial-state independent (as long as the latter breaks the symmetry at time zero) and coincides with the thermalization time. For intermediate subsystems, however, one can observe the “quantum Mpemba effect,” where the state of the system restores a symmetry faster if it is initially more asymmetric. We provide an exact characterization of this effect in a nonintegrable system.

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One of the most basic, yet lucrative tasks a scientist can perform when analyzing a physical system is to identify its symmetries and their breaking mechanisms. This can immediately return a wealth of information regarding the possible processes that can occur within the system, the nature of the excitations, and its potential phases. When applied to quantum many-body systems at equilibrium, this analysis has provided enormous insight into the allowed states of quantum matter and facilitated breakthroughs in all areas of theoretical physics, from the standard model to superconductivity [1,2]. More recently, this perspective has also been successfully applied in out-of-equilibrium settings [3–22].

In the latter context, a simple, yet instructive question concerns symmetry restoration from nonsymmetric initial states. Namely, one considers an initial state that breaks a symmetry of the evolution operator and tracks the symmetry restoration within a subsystem. Reference [16] showed that this process can be characterized in a direct, observable-independent way by measuring the distance between the evolution of the state and its symmetrized counterpart. This also reveals an unexpected phenomenon that has been dubbed the *quantum Mpemba effect*, which occurs when a symmetry is restored more rapidly when it is broken more by the initial state [16].

This genuine quantum effect owes its name to its similarity with the “classical” Mpemba effect, arising when hot water freezes faster than cold water [23], and has been observed in a range of different scenarios [16,17,24–39], including trapped ion quantum simulators [40].

Up to now, however, this observable-independent characterization of the symmetry restoration has only been performed for systems with certain special internal symmetries, mostly $U(1)$ or its discrete subgroups. Here we extend this approach to the case of arbitrary symmetries including *spatial* ones. As a relevant application, we then use it to characterize *exactly* the restoration of translation symmetry in random unitary circuits, where the evolution is implemented by local unitary matrices that are independently drawn from the Haar distribution at each space-time point [41]. These systems possess no internal symmetries, relax locally to the infinite-temperature state, and provide an efficient scrambling of quantum information, making them the paradigmatic example of a quantum chaotic system [18,42–47]. Their averaged dynamics, however, does exhibit space translation symmetry and one can ask how the latter is restored when broken by the initial state.

In this setting, the intuition based on the entanglement dynamics suggests almost immediate restoration of the translational symmetry. Indeed, when quenching from product states, Rényi entropies lose any initial-state dependence upon averaging [43], while for a more complicated initial state, the state dependence is subleading [48]. Given the fact that Rényi entropies characterize the full spectrum of the density matrix, these results seem to imply that the latter should lose its initial-state dependence very quickly. In contrast, however,

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we find that the symmetry is not fully restored for any finite time—signaling that even in these random systems, subsystems preserve memory of the initial state for long times. If we ask for ϵ -approximate restoration, i.e., for symmetry restoration up to a correction ϵ , we find that for large enough subsystems, the restoration time scales linearly with the subsystem size and is initial-state independent. Despite this, we observe occurrences of the quantum Mpemba effect for subsystems of intermediate sizes, and we provide an exact characterization in a nonintegrable system.

More specifically, we consider a quantum many-body system with Hilbert space $\mathcal{H}$ and where the time-evolution operator $\mathbb{U}$ has a symmetry group $\mathcal{G}$ admitting a unitary representation $\mathcal{G} \ni g \mapsto \Pi_g \in \text{End}(\mathcal{H})$ [49]. Then we focus on a quantum quench protocol preparing the system in a state $|\Psi\rangle$ that is neither an eigenstate of $\mathbb{U}$ nor of the symmetry group. To quantify the amount of symmetry breaking caused by the initial state, and to study its restoration in time, we measure the distance between the reduced density matrix of a subsystem A , $\rho_A(t) = \text{tr}_A[|\Psi(t)\rangle\langle\Psi(t)|]$, and the reduced *symmetrized* state,

$$\bar{\rho}_A(t) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \rho_A^{(g)}(t), \quad (1)$$

$$\rho_A^{(g)}(t) = \text{tr}_A[\mathbb{U}^t \Pi_g |\Psi(0)\rangle\langle\Psi(0)| \Pi_g^{-1} \mathbb{U}^{-t}], \quad (2)$$

where $|\mathcal{G}|$ is the order of the group [50]. Note that when the representation of the symmetry group factorizes between A and its complement, i.e., $\Pi_g = \Pi_{g,A} \otimes \Pi_{g,\bar{A}}$ for all g , the symmetrized state can be written as a completely positive trace-preserving map on $\rho_A(t)$, i.e.,

$$\bar{\rho}_A(t) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \Pi_{g,A} \rho_A(t) \Pi_{g,A}^{-1}. \quad (3)$$

A notable example where this factorized form holds is when $\mathcal{G}$ is generated by a charge with density supported on a single site, e.g., particle number. So far, all the studies of subsystem symmetry restoration after quenches have focused on this special case [16,17,24–40], while here we do not impose this restriction.

To quantify the distance between $\rho_A(t)$ and $\bar{\rho}_A(t)$, we compute the normalized Frobenius distance,

$$\Delta_2(t) = \frac{\|\rho_A(t) - \bar{\rho}_A(t)\|_2}{\|\rho_A(t)\|_2}, \quad (4)$$

where $\|O\|_2 = \sqrt{\text{tr}[OO^\dagger]}$ denotes the Frobenius norm. By definition, this quantity is always positive and equal to zero only if $\rho_A(t) = \bar{\rho}_A(t)$. We have used (4), instead of the quantum relative entropy as in Ref. [16] and subsequent works, because whenever the simplified form (3) does not apply, the latter becomes much harder to access analytically. Essentially, this is because to compute it one needs to introduce a replica trick [51] that breaks the non-negativity property.

Our goal is to use (4) to characterize translation symmetry restoration in random unitary circuits. In particular, we consider a one-dimensional chain of $2L$ qudits with local Hilbert space $\mathcal{H} = \mathbb{C}^q$, $2 \leq q \in \mathbb{Z}$. The coordinates of the qudits are indicated by $x \in \mathbb{Z}/2$ and we assume periodic boundary conditions, $x + L = x$. The dynamics of the system are generated

by a brickworklike, random quantum circuit such that the state of the system after $t + 1$ time steps is given by

$$|\Psi(t+1)\rangle = \mathbb{U}(t+1)|\Psi(t)\rangle, \quad (5)$$

and where the time-evolution operator is

$$\mathbb{U}(t+1) = \bigotimes_{x \in \mathbb{Z}_L + \frac{1}{2}} U_{x,x+\frac{1}{2}}(x,t) \bigotimes_{x' \in \mathbb{Z}_L} U_{x',x'+\frac{1}{2}}(x',t). \quad (6)$$

Here, $U_{x,x+1/2}(x,t)$ are operators acting on the whole chain, but acts nontrivially—as the matrix $U(x,t)$ —only at sites $x, x + 1/2$, and as the identity everywhere else. The $q^2 \times q^2$ matrices $U(x,t)$ are known as the *local gates*. In our system, we take them to be chosen independently from the Haar random ensemble.

For each different realization of the $2Lt$ local gates, the system exhibits no conserved quantities and has no symmetry. Upon averaging over different realizations, however, the dynamics acquires a $\mathbb{Z}_L$ symmetry under discrete translations. In particular,

$$\langle \Pi \mathbb{U}(t) \Pi^\dagger \rangle = \langle \mathbb{U}(t) \rangle, \quad (7)$$

where we have denoted the ensemble average by $\langle \cdot \rangle$ and introduced the two-site shift operator Π , which translates the entire system by two sites to the right ($\Pi^L = \mathbb{1}$). Namely, for an operator $\mathcal{O}_x$ acting on site x , we have $\Pi \mathcal{O}_x \Pi^\dagger = \mathcal{O}_{x+1}$. This emergent symmetry is manifested in the long-time limit of the state of the subsystem A , which we take to be a block of 2ℓ qubits. It is easy to see that after averaging over different realizations, the latter is given by the infinite-temperature state $\mathbb{1}/q^{2\ell}$. Therefore, the long-time steady state is invariant under the action of two-site shifts and $\lim_{t \rightarrow \infty} \langle \rho_A(t) \rangle = \mathbb{1}/q^{2\ell} = \lim_{t \rightarrow \infty} \langle \bar{\rho}_A(t) \rangle$.

We wish to study the emergence of this spatial symmetry for a system prepared in a state that explicitly breaks it. We focus on states which are formed from products over two sites of the form

$$|\Psi(0)\rangle = \bigotimes_{x=1}^L |m_x\rangle, \quad |m_x\rangle = \sum_{i,j=1}^q [m_x]_{ij} |i\rangle_x \otimes |j\rangle_{x+\frac{1}{2}}, \quad (8)$$

where $|i\rangle_x$, $i = 1, \dots, q$ span the set of states of the qudit at site x and $\{m_x\}$ is a set of $q \times q$ matrices characterizing the initial state which are taken to be properly normalized, $\text{tr}[(m_x)^\dagger m_x] = 1$. In addition, we impose that the states are translationally invariant only over shifts of 2ν sites with $\nu > 1$ [52]. Note that under translational invariant dynamics, the operator Π_g in Eq. (2) commutes with the time-evolution operator. Therefore, one can either apply it to the initial state or to the state at time t . This is not the case for random unitary circuits, as the translational symmetry emerges only after ensemble averaging.

Since by construction the Frobenius distance is non-negative and we have $\langle \Delta_2(0) \rangle > 0$ and $\langle \Delta_2(\infty) \rangle = 0$, we can use $\langle \Delta_2(t) \rangle$ to track the emergence of the spatial symmetry. In fact, to aid our analytic calculations, we consider

$$\langle \Delta_2(t) \rangle \approx \langle \Delta_2(t) \rangle_a \equiv \sqrt{\frac{\langle \|\rho_A(t) - \bar{\rho}_A(t)\|_2^2 \rangle}{\langle \|\rho_A(t)\|_2^2 \rangle}}, \quad (9)$$

where we have taken the average inside the square root and separately averaged the numerator and denominator. Previ-

ous work has established that these manipulations do not appreciably change the result due to small circuit-to-circuit fluctuations [34,39,43–45]. Note also that the quantity on the right-hand side of (9) is non-negative and only vanishes for $\rho_A(t) = \bar{\rho}_A(t)$; therefore, it provides a good measure of the restoration of spatial symmetry.

Let us now illustrate the main steps in the calculation of $\langle \Delta_2(t > 0) \rangle_a$ and discuss its central features. For pedagogical reasons, we restrict ourselves to case $\nu = 2$, with the general case discussed in [53]. As in previous studies of subsystem symmetry restoration [16,17,24–38,40], the central object to study is the averaged overlap between the states in the linear combination (1). Specifically, setting $\rho_A^{(0)}(t) = \rho_A(t)$ and $\rho_A^{(1)}(t) = \rho_A^{(\Pi)}(t)$, we express (9) as

$$\langle \Delta_2(t) \rangle_a^2 = \frac{1}{4} + \frac{\langle \text{tr}[\rho_A^{(1)}(t)^2] \rangle - 2\langle \text{tr}[\rho_A^{(0)}(t)\rho_A^{(1)}(t)] \rangle}{4\langle \text{tr}[\rho_A^{(0)}(t)^2] \rangle}. \quad (10)$$

Noting that $\text{tr}[\rho_A^{(0)}(t)\rho_A^{(1)}(t)] \geq 0$, because $\rho_A^{(0)}(t)$ and $\rho_A^{(1)}(t)$ are both positive semidefinite, we have that the Frobenius distance attains its maximal value (of $1/2$) when the two states are orthogonal. Conversely, it is zero only if $\rho_A^{(0)}(t) = \rho_A^{(1)}(t)$.

Using the special form of the Haar-averaged local gate, one can show that averaged overlaps fulfill a simple recurrence relation, which can be solved in closed form. To see this, it is convenient to represent the overlaps diagrammatically using the standard graphical notation of tensor networks [54]. Specifically, we represent the averaged, folded, local gate as

$$W = \langle (U(x, t) \otimes_r U^*(x, t))^{\otimes r^2} \rangle = \text{diag}, \quad (11)$$

where we have denoted by $\otimes_r$ the tensor product over different replicas. We also introduce the two special states on the replicated Hilbert space and their corresponding graphical depiction,

$$\begin{aligned} |\bigcirc\rangle &= \sum_{s_{1,2} \in \mathbb{Z}_q} |s_1, s_1, s_2, s_2\rangle = \text{diag}, \\ |\square\rangle &= \sum_{s_{1,2} \in \mathbb{Z}_q} |s_1, s_2, s_2, s_1\rangle = \text{diag}. \end{aligned} \quad (12)$$

Using the known expressions for the averages over the unitary group [55], one can show that the averaged local gate obeys

$$\begin{aligned} \text{diag} &= \text{diag} \otimes \text{diag}, & \text{diag} &= \text{diag} \otimes \text{diag}, \\ \text{diag} &= \alpha (\text{diag} \otimes \text{diag} + \text{diag} \otimes \text{diag}), & \text{diag} &= \alpha (\text{diag} \otimes \text{diag} + \text{diag} \otimes \text{diag}), \end{aligned} \quad (13)$$

where $\alpha = q/(q^2 + 1)$. Similarly, one can establish the validity of the time-reversed expressions in which the diagrams are flipped about the horizontal axis. These expressions are exact for all q .

In the folded representation, we introduce two types of local initial states,

$$(|m_x\rangle \otimes |m_x\rangle^*) \otimes_r (|m_{x+y}\rangle \otimes |m_{x+y}\rangle^*) = \text{diag}, \quad (14)$$

where x refers to the position on the lattice and $y \in \{0, 1\}$ characterizes the shift between the two copies. Setting $y = 0$

gives the actual local initial state on the replicated space, whereas $y = 1$ implies a two-site shift between replicas. Using these ingredients, we can write [56]

$$\langle \text{tr}[\rho_A^{(r)}(t)\rho_A^{(s)}(t)] \rangle \equiv D_{|r-s|}(-t-1, 2t+1, 2\ell), \quad (15)$$

where we introduced the diagram

$$D_y(x, p, q) = \text{diag}, \quad (16)$$

which we have depicted for $p = 6$ and $q = 2$. Employing the relations (13) to reduce the topmost line of gates in (16), we can relate $D_y(x, p, q)$ to a sum of four similar diagrams with $p-1$ rows, but with different values of x and q . Repeating this procedure, we find that $D_y(x, p, q)$ satisfies the recurrence relation

$$\begin{aligned} D_y(x, p, q) &= \alpha^2 [D_y(x, p-1, q+1) + D_y(x, p-1, q) \\ &\quad + D_y(x+1, p-1, q) \\ &\quad + D_y(x+1, p-1, q-1)], \end{aligned} \quad (17)$$

supplemented by the initial conditions

$$\begin{aligned} D_y(x, 1, q) &= \gamma_y(x)\gamma_y(x+q) \prod_{j=1}^{q-1} \beta_y(x+j), \\ D_y(x, p, 0) &= 1, \end{aligned} \quad (18)$$

where we introduced

$$\begin{aligned} \beta_y(x) &= \text{tr}[m_x m_{x+y}^\dagger] \text{tr}[m_{x+y} m_x^\dagger] \equiv \text{diag}, \\ \gamma_y(x) &= \text{tr}[m_x m_x^\dagger m_{x+y} m_{x+y}^\dagger] \equiv \text{diag} \otimes \text{diag}. \end{aligned} \quad (19)$$

Note that $\beta_y, \gamma_y(x) \in [0, 1]$ and $\beta_0 = 1$ [57]. The recursion relation (17) can be solved exactly for arbitrary parameters. The full solution is somewhat bulky and is presented [53]; however, it takes a particularly simple and instructive form in the “short-time” regime $p \leq q$. In particular, plugging back into (15), we find

$$\langle \text{tr}[\rho_A^{(r)} \rho_A^{(s)}] \rangle = c_{r-s} \beta_{r-s}^{\ell-2t} [(1 + \beta_{r-s})\alpha]^{4t}, \quad (20)$$

where the difference $r-s$ is computed mod 2 and c_y is an initial-state-dependent constant. We see that for $r=s$, this expression reproduces the known result for the averaged purity in random unitary circuits [43]. For $s \neq r$, the expression changes in two ways. First it acquires an exponential decay in the portion of A that has not yet received information about $\bar{A}$, i.e., where the state is locally still pure. Second, its decay in time is dressed—sped up—by the initial-state-dependent parameter β_1 .

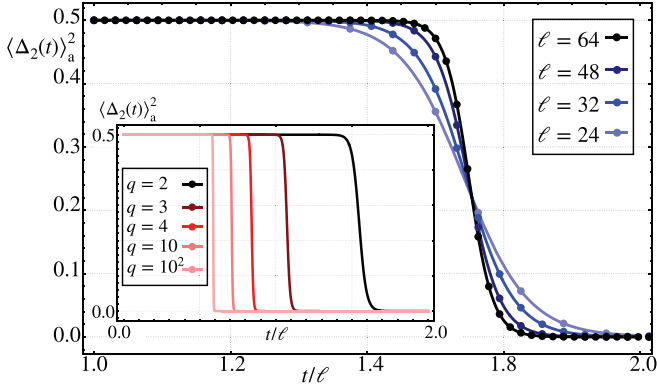


FIG. 1. Dynamical restoration of two-site translational symmetry in random unitary circuits. We plot the squared Frobenius distance, $\langle \Delta_2(t) \rangle_a^2$, between $\rho_A(t)$ and the symmetrized state $\bar{\rho}_A(t)$, as a function of t/ℓ for different subsystem sizes $|A| = 2\ell$. The initial state is parameterized by (24) with $c = 1$. Different lines correspond to increasing $\ell = 24, 32, 48, 64$, lighter to darker. The symbols are the exact result, while the solid lines are the result of (23). Inset: We plot the same for $\ell = 48$ and increasing local Hilbert space dimension $q = 2, 3, 4, 10, 100$, darker to lighter.

In fact, as illustrated in Fig. 1, for large ℓ , t , the overlap is well approximated by combining the short-time expression and eventual relaxed value in a continuous manner, i.e.,

$$\langle \text{tr}[\rho_A^{(r)} \rho_A^{(s)}] \rangle \simeq \begin{cases} c_{r-s} \beta_{r-s}^{\ell-2t} [(1 + \beta_{r-s}) \alpha]^{4t}, & t \leq \ell/v_{\beta-r-s} \\ q^{-2t}, & t \geq \ell/v_{\beta-r-s}, \end{cases} \quad (21)$$

where we introduced the β -dependent relaxation velocity,

$$v_\beta = -2 \frac{\ln[\alpha(\beta^{1/2} + \beta^{-1/2})]}{\ln(q\beta^{1/2})} \in [v_E, 2], \quad (22)$$

and the entanglement velocity $v_E = -2\ln(2\alpha)/\ln(q)$. Plugging (21) into (10), we find

$$\langle \Delta_2(t) \rangle_a^2 \simeq \begin{cases} 1 - \frac{c_1}{c_0} \beta_1^{\ell-2t} \left(\frac{1 + \beta_1}{2} \right)^{4t}, & t \leq \frac{\ell}{v_{\beta_1}} \\ 1 - \frac{1}{c_0} \frac{1}{q^{2t} (2\alpha)^{4t}}, & \frac{\ell}{v_{\beta_1}} \leq t \leq \frac{\ell}{v_E} \\ 0, & t \geq \frac{\ell}{v_E}. \end{cases} \quad (23)$$

We see that although Eq. (23) displays some explicit initial-state dependence, the latter is almost entirely washed away for large enough subsystems: for fixed $t/\ell < 1/v_E$ and large ℓ , Eq. (23) approaches $1/2$ up to exponentially small corrections. Nevertheless, we see that for $\beta_1 < 1$, translation symmetry is approximately restored only for times $t \simeq \ell/v_E$. More precisely, if, for a given $\epsilon > 0$, we ask what is the first time at which $\langle \Delta_2(t) \rangle_a \leq \epsilon$, we find $t \gtrsim \ell/v_E$. This is also the time at which the averaged Frobenius distance of $\rho_A(t)$ and $\mathbb{1}/q^{2\ell}$ approaches ϵ . This means that the reduced state restores translation symmetry only when it thermalizes and, furthermore, it does so abruptly rather than gradually. As discussed above, this appears to be in tension with the entanglement spectrum dynamics. To alleviate this, we note that $\Delta_2(t)$ encodes further

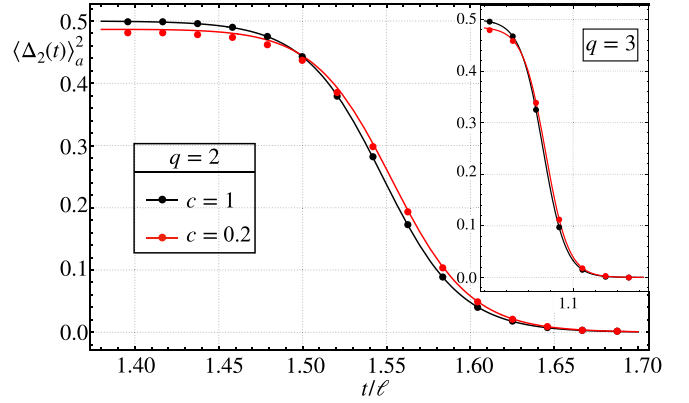


FIG. 2. The quantum Mpemba effect in random unitary circuits. In the main figure, we plot $\langle \Delta_2(t) \rangle_a^2$ for two different states specified by (24) using $c = 1$ (black) and $c = 0.2$ (red) with $q = 2$ and $\ell = 48$ as a function of t/ℓ . The symbols represent the exact solution of (17) and (18), while the solid lines use the analytic approximations (23) (black) and an improved approximation (red) discussed in [53]. We see that the lines cross, indicating a weak quantum Mpemba effect, whereby the initially less symmetric state (black) ϵ approximately restores the symmetry faster. For these parameters, we have $v_\beta \approx 1.1$ (black) and $v_\beta \approx 0.68$ (red), while $v_E \approx 0.64$. Inset: We plot the same quantity for $q = 3$, showing a far less pronounced crossing. Here, $v_\beta \approx 1.34$ (black) and $v_\beta \approx 0.96$ (red), while $v_E \approx 0.93$.

information about the full matrix $\rho_A(t)$ beyond merely its spectrum, meaning that some initial-state dependence remains encoded in its matrix structure. Finally, note that Eq. (23) vanishes identically for $\beta_1 = 1$, showing that the limit of the symmetric state commutes with that of large ℓ and t .

These features are seen in Fig. 1, where we plot $\langle \Delta_2(t) \rangle_a^2$ for the state parametrized as

$$m_{2x-1} = \frac{\mathbb{1} + \sigma^z}{2}, \quad m_{2x} = \frac{\mathbb{1} + \sigma^z + c\sigma^x}{\sqrt{4 + 2c^2}}, \quad (24)$$

where $\sigma^{x,z}$ are Pauli matrices acting on a two-dimensional subspace of $\mathbb{C}^q$ and $c \in \mathbb{R}$. We see that as ℓ increases, $\langle \Delta_2(t) \rangle_a$ behaves more like a step function and details of the state are forgotten. Likewise, for increasing q , the symmetry restoration is increasingly sharp. The case of $v > 2$ can be analyzed following the same logic and shows similar qualitative features [53].

For finite-size subsystems, memory of the initial state is retained until large times. This is exemplified in Fig. 2, where we again plot $\langle \Delta_2(t) \rangle_a^2$ but for two choices of the parameter c and Hilbert-space dimension $q = 2, 3$. We see that the initially more asymmetric state (black lines and symbols) exhibits ϵ -approximate symmetry restoration *before* the more symmetric state (red lines and symbols), signifying the occurrence of the quantum Mpemba effect. The effect is less prominent for larger q and vanishes for $q \rightarrow \infty$, which can be interpreted as the vanishing of the quantum Mpemba effect in the semiclassical limit.

To summarize, in this work, we have studied translation symmetry restoration in random unitary circuits. After introducing an observable-independent framework to conduct our analysis, we showed that in contrast to what is suggested by the entanglement dynamics, the restoration of translation

symmetry only happens on timescales proportional to the volume of the subsystem. Moreover, we have shown that for large enough subsystems, the symmetry restoration process only depends on the degree of symmetry breaking introduced by the initial state, i.e., the number of two-site shifts under which it is invariant; no other detail of the state is retained. At intermediate sizes, the initial-state dependence is still visible and one can observe instances of the quantum Mpemba effect which, nevertheless, vanishes in both the semiclassical and scaling limits.

Our results on the approximate symmetry restoration are in agreement with the fact that random unitary circuits form approximate unitary k -designs for times linear in the system size [58–66]. In essence, this means that when evolving up to those times, random unitary circuits become indistinguishable from all-to-all random unitary matrices acting on the full chain (and therefore all information about the initial state is lost). Our setting differs from that studied in this context in that we work in the thermodynamic limit. However, we find that a finite subsystem loses memory of the initial state only in a time that is linear in the size of the subsystem.

Our work raises many interesting questions. An immediate one is whether more realistic systems, e.g. Hamiltonian

systems without noise, also show simultaneous occurrence of spatial symmetry restoration and thermalization. Our results suggest this to be the case: realistic systems should restore more slowly than random unitary ones (they retain more information), but all the symmetries are restored upon thermalization. Another interesting question concerns the quantum Mpemba effect, which has previously been observed in nonintegrable settings only through numerical results on small systems [35,39]. By analytical means, we find that the effect invariably vanishes when system sizes are large. This is in contrast to integrable systems, raising the question of how the effect is stabilized therein and its origins in the present case.

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