

Adiabatic transformations in dissipative and non-Hermitian phase transitionsPavel Orlov^{1,2,3}, Georgy V. Shlyapnikov^{2,4,5,6} and Denis V. Kurlov^{1,2,7}¹*Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana 1000, Slovenia*²*Russian Quantum Center, Skolkovo, Moscow 143025, Russia*³*Nanocenter CENN, Jamova 39, SI-1000 Ljubljana, Slovenia*⁴*Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region 141701, Russia*⁵*Université Paris-Saclay, CNRS, LPTMS, F-91405 Orsay, France*⁶*Van der Waals–Zeeman Institute, Institute of Physics, University of Amsterdam, Science Park 904, NL-1098 XH Amsterdam, The Netherlands*⁷*National University of Science and Technology “MISIS”, Moscow 119049, Russia*

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The quantum geometric tensor has established itself as a general framework for the analysis and detection of equilibrium phase transitions in isolated quantum systems. We propose a generalization of the quantum geometric tensor, which offers a universal approach to studying phase transitions in non-Hermitian quantum systems. Our generalization is based on the concept of the generator of adiabatic transformations and can be applied to systems described by either a Liouvillian superoperator or by an effective non-Hermitian Hamiltonian. We illustrate the proposed method by analyzing the non-Hermitian Su-Schrieffer-Heeger model and a generic quasifree dissipative fermionic system with a quadratic Liouvillian. Our findings reveal that this method effectively identifies phase transitions across all examined models, providing a universal tool for investigating general non-Hermitian systems.

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Introduction. Equilibrium quantum phase transitions (QPTs) have been the subject of extensive research over recent decades [1,2]. One of the approaches to equilibrium QPTs in isolated quantum systems is based on information geometry and relies on the notion of the quantum geometric tensor (QGT), which was introduced in Ref. [3]. The real part of the QGT coincides with the Fubini-Study metric tensor (also known as a quantum metric), while the imaginary part is nothing but the Berry curvature. The QGT-based approach is particularly powerful because it does not require knowledge of an order parameter and can be applied universally to all equilibrium QPTs, including the topological ones. Indeed, a QPT is associated with the nonanalytic change in the ground state of a system. Hence, phase transitions can be detected by the nonanalytic behavior of the metric tensor [4–6] or the Berry curvature [7–10], i.e., through the singular behavior of the QGT.

In recent years, there has been significant interest in dissipative phase transitions, which occur in open quantum systems [11–17]. This interest is partly driven by remarkable experimental advances that enable the engineering and control of dissipation across various quantum systems, including ultracold neutral atoms [18], trapped ions [19], quantum optical systems [20], and superconducting circuits [21]. These developments open up avenues for exploring

the complex physics of dissipative and non-Hermitian (NH) quantum systems, alongside potential applications in quantum technologies [22,23].

For Markovian open systems described by a Liouvillian superoperator, phase transitions manifest themselves as a non-analytic behavior of the system's nonequilibrium steady state (NESS), which is generically a mixed state. Remarkably, the information-geometric approach to QPT can also be extended to dissipative phase transitions and mixed states. Specifically, the Bures metric [24,25], the Uhlmann curvature [26], and the quantum Fisher tensor [27,28] generalize the Fubini-Study metric tensor, the Berry curvature, and the QGT, correspondingly. Both the Bures metric and Uhlmann curvature have been successfully exploited to describe QPTs in open quantum systems [29–35]. However, these quantum geometric measures for mixed states do not allow one to study phase transitions in quantum systems with non-Hermitian Hamiltonians. The latter appear as an effective phenomenological description of an open system and can be useful for the investigation of a wide range of physical phenomena, such as NH topological [36] and parity-time ($\mathcal{PT}$) symmetric phases [37]. Various generalizations of the QGT have been proposed to deal with NH Hamiltonians [38–41]. However, all proposed measures are either not gauge invariant, or vanish identically for kinematic reasons when applied to Liouvillian superoperators. Thus, at present there are two distinct sets of measures for open and NH quantum systems, since no existing quantity applies to the study of phase transitions in both Liouvillians and NH Hamiltonians.

In this Letter we introduce a non-Hermitian generalization of the quantum geometric tensor (NH-QGT). It is gauge

invariant and can be used for generic non-Hermitian systems, including those described by Liouvillians and NH Hamiltonians. Thus, the proposed generalization provides a universal framework for investigating phase transitions in such systems. Our generalization is based on the generator of adiabatic transformations, known as the adiabatic gauge potential (AGP), which has proven to be extremely useful for isolated systems [42,43]. We test our proposal on a few exactly solvable models with a rich phase diagram structure, where the NH-QGT can be calculated explicitly. The proposed NH-QGT correctly captures critical points in all analyzed cases.

Adiabatic gauge potential and NH-QGT. For an isolated system described by a parameter-dependent Hermitian Hamiltonian $\hat{H}(\lambda)$, the QGT for an eigenstate $|n(\lambda)\rangle$ is defined as [3]

$$\chi_{\mu\nu}^{(n)} = \langle \partial_\mu n | \partial_\nu n \rangle - \langle \partial_\mu n | n \rangle \langle n | \partial_\nu n \rangle. \quad (1)$$

There are several noteworthy properties associated with this quantity. To begin with, QGT is both Hermitian and positive semidefinite. Its real part, denoted as $g_{\mu\nu}^{(n)} = \text{Re}[\chi_{\mu\nu}^{(n)}]$, is a symmetric positive-semidefinite tensor known as the Fubini-Study metric, while its imaginary part, $F_{\mu\nu}^{(n)} = -\frac{1}{2}\text{Im}[\chi_{\mu\nu}^{(n)}]$, is an antisymmetric tensor known as the Berry curvature [42]. Additionally, QGT exhibits gauge invariance under the transformation $|n\rangle \rightarrow e^{i\gamma(\lambda)}|n\rangle$, related to the phase ambiguity of eigenstates. The choice of the second term in Eq. (1) is specifically made to ensure the gauge invariance.

Let us now consider a system described by a non-Hermitian matrix K , which can be either a non-Hermitian effective Hamiltonian H or a Liouvillian superoperator $\mathcal{L}$. In the latter case, we treat the matrix space as a vector space with the inner product defined as $\langle \rho_1 | \rho_2 \rangle = \text{Tr}(\rho_1^\dagger \rho_2)$, consequently considering the superoperator as an ordinary (albeit non-Hermitian) operator within this Hilbert space.

We now briefly summarize some key properties of non-Hermitian operators that are important for our purposes (see Ref. [44] for a detailed overview). For non-Hermitian operators the right and left eigenvectors do not coincide and one has

$$K|n_R\rangle = \Lambda_n|n_R\rangle, \quad K^\dagger|n_L\rangle = \Lambda_n^*|n_L\rangle. \quad (2)$$

Moreover, the left and right eigenvectors are not orthogonal individually, e.g., $\langle m_R | n_R \rangle = C_{mn} \neq \delta_{mn}$, but together they form a biorthogonal system, satisfying the relation $\langle m_L | n_R \rangle = \delta_{mn}$. The Gram matrices for the left and right eigenvectors are inverses of each other, i.e., $\langle m_L | n_L \rangle = (C^{-1})_{mn}$.

For non-Hermitian systems, eigenstates are defined not merely up to a phase but up to an arbitrary nonzero constant. This implies that any generalization of QGT must remain invariant under gauge transformations

$$|n_R\rangle \rightarrow e^{r_n(\lambda)}|n_R\rangle, \quad |n_L\rangle \rightarrow e^{-r_n^*(\lambda)}|n_L\rangle, \quad (3)$$

where $r_n(\lambda)$ is an arbitrary complex-valued function.

To construct a non-Hermitian generalization of the quantum geometric tensor, we introduce the generator of adiabatic transformations $\mathcal{A}_\lambda|n_R\rangle = |\partial_\lambda n_R\rangle$, known as the adiabatic gauge potential (AGP) [42,45]. From the biorthogonality condition one has $\mathcal{A}_\lambda^\dagger|n_L\rangle = -|\partial_\lambda n_L\rangle$. For Hermitian systems, the left and right eigenvectors coincide, so that $\mathcal{A}_\lambda$ is anti-Hermitian. As a result, the Hermitian QGT in Eq. (1) can be

expressed as a connected part of the expectation value in two equivalent ways:

$$\chi_{\mu\nu}^{(n)} = -\langle \mathcal{A}_\mu \mathcal{A}_\nu \rangle_c = \langle \mathcal{A}_\mu^\dagger \mathcal{A}_\nu \rangle_c. \quad (4)$$

However, for non-Hermitian systems, the AGP is no longer anti-Hermitian. Thus, Eq. (4) suggests that a non-Hermitian generalization of the QGT $\chi_{\mu\nu}^{(n)}$ can be constructed in two ways.

One approach uses $-\langle n_L | \mathcal{A}_\mu \mathcal{A}_\nu | n_R \rangle$ as a basis and then includes necessary counterterms to ensure gauge invariance. In this way, the first generalization is given by

$$\eta_{\mu\nu}^{(n)} = \langle \partial_\mu n_L | \partial_\nu n_R \rangle - \langle \partial_\mu n_L | n_R \rangle \langle n_L | \partial_\nu n_R \rangle. \quad (5)$$

This tensor has been already studied in a series of works [38–41] in the context of criticality in non-Hermitian Hamiltonians. Despite its successful application in describing phase transitions, this quantity cannot be utilized as a probe of criticality in NESS-QPTs. This is attributed to the fact that the left eigenvector, which corresponds to NESS, is the identity operator [46]. Consequently, for NESS $\eta_{\mu\nu}$ is trivially equal to zero.

This circumstance motivates us to define the second generalization of QGT. The same methodology can be applied if we use $\langle n_L | \mathcal{A}_\mu^\dagger \mathcal{A}_\nu | n_R \rangle$ as the starting point. The relation $\langle n_L | \mathcal{A}_\mu^\dagger = \sum_m (C^{-1})_{nm} \langle \partial_\mu m_R |$ allows us to readily introduce the required counterterms. Ultimately, we define our NH-QGT as

$$\zeta_{\mu\nu}^{(n)} = \sum_m (C^{-1})_{nm} \langle \partial_\mu m_R | \hat{1} - P_m^\dagger - P_n + P_m^\dagger P_n | \partial_\nu n_R \rangle, \quad (6)$$

where $P_m = |m_R\rangle\langle m_L|$ represents a set of nonorthogonal projectors on the right eigenstates.

The definition of the NH-QGT can be expressed in a simpler form. Gauge transformations of eigenstates allow us to nullify the diagonal matrix elements of AGP [see the Supplemental Material (SM) [47] for a more detailed discussion of the AGP properties]. In this gauge, the NH-QGT can be written as

$$\zeta_{\mu\nu}^{(n)} = \langle n_L | \mathcal{A}_\mu^\dagger \mathcal{A}_\nu | n_R \rangle, \quad (7)$$

with $\mathcal{A}_\lambda$ such that $\forall m \langle m_L | \mathcal{A}_\lambda | m_R \rangle = 0$. We will use this expression below to calculate the geometric tensor in dissipative models. Also, note that this definition can be straightforwardly generalized to the case of nondiagonalizable K as well [48].

In general, the NH-QGT is neither Hermitian nor positive semidefinite. However, for NESS its complex structure simplifies. In this case the NH-QGT is purely real since for Liouvillian systems AGP operators preserve Hermiticity and trace. Another important point is that the NH-QGT is not related to the geometry of a particular state, as it includes derivatives and the Gram matrix with the other eigenstates.

To motivate the introduction of $\zeta_{\mu\nu}^{(n)}$ even more, we define the covariant derivative of eigenstates as the gauge covariant part of the derivative operator

$$\begin{aligned} |D_\mu n_R\rangle &\equiv |\partial_\mu n_R\rangle - A_\mu^{(n)}|n_R\rangle, \\ \langle D_\mu n_L| &\equiv \langle \partial_\mu n_L| + A_\mu^{(n)}\langle n_L|, \end{aligned} \quad (8)$$

where $A_\mu^{(n)} = \langle n_L | \partial_\mu n_R \rangle = -\langle \partial_\mu n_L | n_R \rangle$ is a non-Hermitian analog of the Berry connection. Under the gauge

transformation Eq. (3) the covariant derivatives are transformed simply as

$$\begin{aligned} |D_\mu n_R\rangle &\rightarrow e^{r_n(\lambda)} |D_\mu n_R\rangle, \\ \langle D_\mu n_L| &\rightarrow e^{-r_n(\lambda)} \langle D_\mu n_L|. \end{aligned} \quad (9)$$

Using these derivatives, Eq. (6) can be rewritten in a form with a more transparent gauge invariance:

$$\zeta_{\mu\nu}^{(n)} = \sum_m \langle n_L | m_L \rangle \langle D_\mu m_R | D_\nu n_R \rangle. \quad (10)$$

Particularly, it can be seen that the sum in the relation for the NH-QGT is weighted by the overlap between left eigenvectors, giving a greater contribution to states that are closer to the state of interest.

Note that every term in the sum of Eq. (10) is itself gauge invariant. This means that instead of summing over the entire spectrum we can limit ourselves to the term with $m = n$, defining a limited version of the NH-QGT,

$$\tilde{\zeta}_{\mu\nu}^{(n)} = \langle n_L | n_L \rangle \langle D_\mu n_R | D_\nu n_R \rangle. \quad (11)$$

In principle, this quantity itself can be seen as a generalization of the QGT. It now depends only on one particular state and not the entire spectrum. Moreover, $\tilde{\zeta}_{\mu\nu}^{(n)}$ is Hermitian and positive semidefinite in contrast to the full NH-QGT. Our calculations indicate that both $\zeta_{\mu\nu}$ and $\tilde{\zeta}_{\mu\nu}$ can be used as a criticality measure. However, analytical results for $\zeta_{\mu\nu}$ are more feasible and transparent than those for $\tilde{\zeta}_{\mu\nu}$, as will be seen below. Therefore, in the main text, we focus on the results for $\zeta_{\mu\nu}$, while a detailed discussion of $\tilde{\zeta}_{\mu\nu}$ can be found in the SM [47].

In what follows we will illustrate that our NH-QGT introduced in this section is linked to the critical behavior of non-Hermitian systems.

Non-Hermitian SSH model. To demonstrate the capability of the NH-QGT in accurately detecting topological effects in non-Hermitian Hamiltonians, we apply it to the non-Hermitian analog of the Su-Schrieffer-Heeger (SSH) model [49,50],

$$\begin{aligned} H = & (t + \delta) \sum_n a_n^\dagger b_n + (t - \delta) \sum_n b_n^\dagger a_n \\ & + t' \sum_n (b_n^\dagger a_{n+1} + a_{n+1}^\dagger b_n), \end{aligned} \quad (12)$$

where t' represents the intercell hopping parameter, and the breaking of Hermiticity arises from distinct intracell hopping parameters $t + \delta$ and $t - \delta$. The corresponding Bloch Hamiltonian for periodic boundary conditions is given by

$$h(k) = \begin{pmatrix} 0 & t - \delta + t' e^{-ik} \\ t + \delta + t' e^{ik} & 0 \end{pmatrix}, \quad (13)$$

and the energy levels are $\pm \sqrt{\varepsilon(k)}$, where $\varepsilon(k) = 1 + t^2 - \delta^2 + 2t \cos(k) - 2i\delta \sin(k)$ (we put $t' = 1$). Different phases are separated by the band crossing $\varepsilon(k) = 0$. This occurs at $|t \pm \delta| = 1$. Thus, the parameter space is divided into four phases where $|t - \delta|$ and $|t + \delta|$ are either greater or less than one. In what follows we will denote these regions as (s_1, s_2) with $s_i = \pm$. Here, $s_1 = +$ or $s_1 = -$ means that $|t - \delta| > 1$ or $|t - \delta| < 1$ correspondingly (similarly for s_2 and $|t + \delta|$).

These phases are of a topological nature as they can be characterized by two winding numbers of the Hermitian part of the Hamiltonian around the exceptional points [50].

Explicit expressions for $\eta_{\mu\nu}$ [see Eq. (5)] in this model were obtained in Ref. [41]. To calculate $\zeta_{\mu\nu}$ we can apply Eq. (10) to all k sectors of the Hamiltonian independently. This leads to the NH-QGT of the following form,

$$\begin{aligned} \zeta_{tt} &= \sum_k \frac{\delta^2 + \sin^2(k)}{4|\varepsilon|^2}, \quad \zeta_{\delta\delta} = \sum_k \frac{[t + \cos(k)]^2}{4|\varepsilon|^2}, \\ \zeta_{t\delta} &= \zeta_{\delta t} = - \sum_k \frac{[t + \cos(k)]\delta}{4|\varepsilon|^2}, \end{aligned} \quad (14)$$

where sums go over the Brillouin zone. It is already seen that singularities of this tensor are exactly at points where $\varepsilon(k) = 0$, i.e., $|t \pm \delta| = 1$. It is even more apparent in the thermodynamic limit $L \rightarrow \infty$ where the NH-QGT becomes

$$\begin{aligned} \zeta_{tt(\delta\delta)} &= \frac{L}{16} \left(\frac{1}{|(\delta+t)^2-1|} + \frac{1}{|(\delta-t)^2-1|} \pm f(\delta, t) \right), \\ \zeta_{t\delta} &= \zeta_{\delta t} = \frac{L}{16} \left(\frac{1}{|(\delta+t)^2-1|} - \frac{1}{|(\delta-t)^2-1|} \right). \end{aligned} \quad (15)$$

Here, the top (bottom) sign corresponds to the index tt ($\delta\delta$) and the function $f(\delta, t)$ depends on the phase under consideration:

$$f(\delta, t) = \begin{cases} \frac{2}{(\delta-t)(\delta+t)}, & (+, +), \\ \frac{1}{\delta(\delta+t)}, & (-, +), \\ \frac{1}{\delta(\delta-t)}, & (+, -), \\ 0, & (-, -). \end{cases} \quad (16)$$

Each topological phase is described by its own tensor and the singularities of the NH-QGT are exactly at topological phase transition points. Note that $\zeta_{\mu\nu}$ has the Euclidean metric signature, in contrast to the Minkowski one for $\eta_{\mu\nu}$ in Ref. [41].

Dissipative quadratic models. As our next testing ground, we use fermionic dissipative Markovian models whose time evolution is given by quadratic Liouvillians $\partial_t \rho = \mathcal{L}(\rho)$ with

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right), \quad (17)$$

where Hamiltonian $H = \mathbf{w}^T \mathbf{H} \mathbf{w}$ is quadratic and jump operators $L_k = \mathbf{l}_k^T \mathbf{w}$ are linear in Majorana operators $\mathbf{w} = (w_1, \dots, w_{2n})^T$ that are defined as $w_{2j-1} = c_j + c_j^\dagger$, $w_{2j} = i(c_j - c_j^\dagger)$. Here, $c_j^\dagger$ and c_j are the creation and annihilation operators of n fermionic modes, $\mathbf{H}$ is a $2n \times 2n$ antisymmetric Hermitian matrix and $\mathbf{l}_k$ are arbitrary complex vectors of dimension $2n$.

Liouvillians of this nature can be exactly diagonalized. One of the most elegant methods to achieve this is through the use of the so-called third quantization technique, which was introduced in Ref. [11]. At the core of this method is a set of $2n$ superoperators that satisfy canonical anticommutation relations

$$\hat{a}_j(\rho) = -\frac{i}{2} W[w_j, \rho], \quad \hat{a}_j^\dagger(\rho) = -\frac{i}{2} W\{w_j, \rho\}, \quad (18)$$

where $W = i^n \prod_{l=1}^{2n} w_l$ and $\{\hat{a}_i^\dagger, \hat{a}_k\} = \delta_{ik}$. Direct calculations show that the Liouvillian in Eq. (17) can be rewritten using these superoperators as

$$\mathcal{L} = - \sum_{ij} \left(X_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} Y_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger \right), \quad (19)$$

where $X = 4iH + 2\text{Re}\{M\} = X^*$, $Y = -4i\text{Im}\{M\} = -Y^T = Y^\dagger$, and $M = \sum_k I_k I_k^\dagger = M^\dagger \geq 0$ is the so-called bath matrix.

For simplicity we assume that the matrix X is diagonalizable $X = U D_X U^{-1}$ with $D_X = \text{diag}(\{x_j\}_{j=1}^{2n})$. In this case, the Liouvillian can be diagonalized by a nonunitary Bogoliubov transformation [51] $\mathbf{b} = U^{-1}(\mathbf{a} + \Gamma \mathbf{a}^\dagger)$, $\mathbf{b}^\times = U^T \mathbf{a}^\dagger$, and its diagonal form is given by $\mathcal{L} = - \sum_{j=1}^{2n} x_j b_j^\times b_j$. Here, the matrix Γ is obtained from the solution of the Sylvester equation

$$X\Gamma + \Gamma X^T = Y. \quad (20)$$

The uniqueness of the solution [which is equivalent to $\text{Min}_j \text{Re}(x_j) > 0$] denotes the condition for the steady state to be unique. Furthermore, the NESS is a Gaussian state in terms of operators w_j and the two-point correlation function of this state coincides with the matrix Γ : $\langle [w_i, w_j] \rangle_{\text{NESS}} = 2\Gamma_{ij}$.

The third quantization allows us to determine the AGP operator. The AGP with the required gauge [Eq. (7)] for the λ -dependent quadratic Liouvillian is also quadratic in fermions, and its structure resembles Eq. (19),

$$\mathcal{A}_\lambda = \sum_{ij} \left(\mathcal{X}_{ij}^\lambda \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} \mathcal{Y}_{ij}^\lambda \hat{a}_i^\dagger \hat{a}_j^\dagger \right), \quad (21)$$

where

$$\mathcal{X}^\lambda = U A_X^{(\lambda)} U^{-1}, \quad \mathcal{Y}^\lambda = \partial_\lambda \Gamma + \mathcal{X}^\lambda \Gamma + \Gamma (\mathcal{X}^\lambda)^T, \quad (22)$$

and $A_X^{(\lambda)}$ is the off-diagonal part of $\partial_\lambda U^{-1} U$. The detailed derivation of Eqs. (21) and (22) is given in the SM [47]. Note that $\mathcal{X}^\lambda$ is a real matrix, while $\mathcal{Y}^\lambda$ is an imaginary antisymmetric matrix. This structure of the AGP is imposed by its trace and Hermiticity (but not positivity) preservation.

Using the explicit form of the AGP in Eq. (21) the NH-QGT in a steady state can be calculated more explicitly:

$$\zeta_{\mu\nu}^{\text{NESS}} = \frac{1}{2} \text{Tr}(\partial_\mu \Gamma \partial_\nu \Gamma) + \text{Tr}(\mathcal{X}^\mu \Gamma \partial_\nu \Gamma). \quad (23)$$

The first term in the last equation encodes how dramatically the two-point correlation function changes in a steady state. Given that Gaussian states are completely defined by their two-point correlation function, any nonanalytical traits within the NESS essentially correspond to nonanalyticity within the first term. The subsequent term contains both Γ and U matrices, illustrating the involvement of derivatives of other eigenstates in the generalized QGT [see Eq. (10)]. Here, we note that in order to exclude the contribution of other eigenstates, one can consider the limited version $\tilde{\zeta}_{\mu\nu}$ of the NH-QGT [Eq. (11)]. However, in this case manifestation of the correlation matrix Γ is more involved (see the SM [47]).

To illustrate this general framework with a specific example, we examine the Kitaev chain with local dissipation. The Hamiltonian for this model, with periodic boundary conditions imposed, is given by $\hat{H} = - \sum_{j=1}^L (c_j^\dagger c_{j+1} + \gamma c_j^\dagger c_{j+1}^\dagger +$

$\frac{h}{2} c_j^\dagger c_j + \text{H.c.})$. Additionally, two jump operators act on each site $L_j^- = g\mu_- c_j$, $L_j^+ = g\mu_+ c_j^\dagger$. This particular model was initially introduced in Ref. [52]. In the weak-coupling limit $g \rightarrow 0$, the system becomes critical in the same regions as the Kitaev chain Hamiltonian: For $\gamma \neq 0$ there is a critical field $h = 1$ and for $\gamma = 0$ the whole line $|h| < 1$ is critical [34,35,52].

To compute the steady-state geometric tensor in this theory, we can utilize Eq. (23). However, since the model exhibits translational invariance, it is beneficial to first apply the Fourier transform. Ultimately, the desired tensor is expressed as the sum over the Brillouin zone (detailed calculations can be found in the SM [47]),

$$\zeta_{\mu\nu}^{\text{NESS}} = \Lambda^2 \sum_k \sin^2(\varphi_k) \partial_\mu \varphi_k \partial_\nu \varphi_k, \quad (24)$$

where $\Lambda = (\mu_+^2 - \mu_-^2)/(\mu_+^2 + \mu_-^2)$, $\varphi_k = \arctan(\frac{\gamma \sin(k)}{h - \cos(k)})$, $\{\lambda_1, \lambda_2\} = \{h, \gamma\}$. The structure of the NH-QGT resembles the ground-state QGT for the Kitaev Hamiltonian [6], with the only difference being the additional term of $\sin^2(\varphi_k)$ under the summation. In the thermodynamic limit $L \rightarrow \infty$, one can replace the sums with integrals to derive analytical formulas. For instance, for $|h| < 1$ the geometric tensor can be expressed as

$$\zeta_{\mu\nu} = \frac{\Lambda^2 L}{8|\gamma|} \text{diag} \left(\frac{3}{(1-h^2)}, \frac{1+3|\gamma|}{(1+|\gamma|)^3} \right). \quad (25)$$

Expressions for $|h| > 1$ are given in the SM [47]. This tensor exhibits singular behavior precisely at the critical regions.

Conclusion. Using the generator of adiabatic transformations we introduced a non-Hermitian generalization of the QGT. We proposed that our generalization can serve as a universal tool for studying both dissipative and non-Hermitian phase transitions. We examined our proposal using several exactly solvable models, including a non-Hermitian version of the Su-Schrieffer-Heeger (SSH) model and generic quasifree fermionic Liouvillians. Furthermore, in the latter case we obtained an explicit expression for the AGP (super)operator.

Thus, our findings emphasize the significance of the adiabatic transformations in understanding phase transitions in non-Hermitian systems. We expect our results to be valuable in the field of quantum control, particularly for investigating regimes of counterdiabatic drive in open quantum systems, similar to those in isolated systems [42]. Another promising research direction is to explore potential connections between adiabaticity and dissipative quantum chaos, similar to those observed in Hermitian quantum systems [43,53].

Finally, it should be emphasized that the Hermitian QGT is an experimentally measurable quantity. A possible method of measuring it in polaritonic systems was proposed in Ref. [54]. This approach has recently been adapted for non-Hermitian polaritonic systems, thus providing a methodology for measuring $\eta_{\mu\nu}$ [55]. We note that this method can also be straightforwardly utilized to measure the NH-QGT introduced in our work, enabling an experimental observation of this quantity.

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