

Optical manifestations and bounds of topological Euler class

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We analyze quantum-geometric bounds on optical weights in topological phases with pairs of bands hosting nontrivial Euler class, a multigap invariant characterizing non-Abelian band topology. We show how the bounds constrain the combined optical weights of the Euler bands at different dopings and further restrict the size of the adjacent band gaps. In this process, we also consider the associated interband contributions to dc conductivities in the flat-band limit. We physically validate these results by recasting the bound in terms of transition rates associated with the optical absorption of light, and demonstrate how the Euler connections and curvatures can be determined through the use of momentum and frequency-resolved optical measurements, allowing for a direct measurement of this multiband invariant. Additionally, we prove that the bound holds beyond the degenerate limit of Euler bands, resulting in nodal topology captured by the patch Euler class. In this context, we deduce optical manifestations of Euler topology within $\mathbf{k} \cdot \mathbf{p}$ models, which include quantized optical conductivity, and third-order jerk photoconductivities. We showcase our findings with numerical validation in lattice-regularized models that benchmark effective theories for real materials and are realizable in metamaterials and optical lattices.

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Introduction. Topological insulators and semimetals constitute an active field [1–3]. Topological insulators give rise to a condensed-matter realization of the θ vacuum, exhibiting a parity anomaly [4,5], resulting in a variety of physical phenomena [6–15]. Motivated by the recent progress in the understanding of such relationships between topology and optical responses, and further by the experimental observation of effects that result from this connection, a natural question is to what extent these insights translate to the context of recently discovered multigap topologies with non-Abelian properties.

While conventional “single-gap” topologies [16–29] are well understood, and can be classified by comparing general momentum-space constraints [22] and real-space conditions [27,28], a variety of problems concerning multigap topologies [30,31] remain open. A prominent example in this regard is the Euler class, which characterizes systems with real Hamiltonians (the reality condition is usually guaranteed by the presence of either C_2T or PT symmetry). In such systems, band degeneracies residing between adjacent pairs of bands can carry non-Abelian frame charges [32–34], akin to π -disclination defects in biaxial nematics [35–38], which can be changed upon braiding these nodes in momentum space. In

particular, two-band subspaces may feature a finite Euler class that physically manifests itself by having pairs of nodes with similarly valued frame charges that cannot annihilate each other. The Euler class of a given two-band subspace can be evaluated on any patch $\mathcal{D}$ in the Brillouin zone (BZ) as [34,39]

$$\chi = \frac{1}{2\pi} \int_{\mathcal{D} \subseteq \text{BZ}} d^2\mathbf{k} \text{Eu}(\mathbf{k}) - \frac{1}{2\pi} \oint_{\partial\mathcal{D}} d\mathbf{k} \cdot \mathbf{a}(\mathbf{k}). \quad (1)$$

Here, $\mathbf{a}$ denotes the Euler connection, defined as the Pfaffian of the non-Abelian Berry connection $\mathbf{A}_{n,n+1}(\mathbf{k}) \equiv \langle u_n(\mathbf{k}) | \nabla_{\mathbf{k}} u_{n+1}(\mathbf{k}) \rangle$ of the neighboring Bloch bands $|u_n(\mathbf{k})\rangle$ and $|u_{n+1}(\mathbf{k})\rangle$, while $\text{Eu}(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{a}(\mathbf{k})$ is the non-Abelian Euler curvature, equal to the exterior derivative of the Euler connection with respect to the quasimomentum $\mathbf{k}$ [34]. In particular, when the patch covers the entire Brillouin zone, the boundary $\partial\mathcal{D}$ vanishes and Eq. (1) acts as a real analog of the Chern number

$$C = \frac{1}{2\pi} \int_{\text{BZ}} d^2\mathbf{k} \mathcal{F}(\mathbf{k}), \quad (2)$$

which is, instead, given by the Abelian Berry curvature $\mathcal{F}(\mathbf{k}) = i \sum_n^{\text{occ}} \nabla_{\mathbf{k}} \times \mathbf{A}_{n,n}$, with $\mathbf{A}_{n,n}$ the single-band connection. We note that manifestations of multigap topologies, the braiding of non-Abelian charges, and the Euler class have been found in a variety of physical settings and phenomena, including quench dynamics [40,41], periodically driven Floquet systems [42], and contexts that range from phononic [43–46] and electronic systems [34,47], to twisted bilayer and magnetic systems [33,48], or acoustic and photonic metamaterials [49,50].

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Motivated by a range of works reporting an interplay between Chern numbers and matter-light responses [7,13,51], we here address the question of whether optical observables can capture the topological Euler class. Intuitively, these relationships arise from quantum-geometric constraints imposed by the invariant; quantum geometry is directly related to optical properties [52,53]. Using a recent formulation [53] of the many-band quantum metric using Plücker maps, we derive three analytical bounds on: (i) optical weights, detailing relations of interband contributions to dc conductivities induced by the quantum metric, (ii) combined optical transition rates under circularly polarized light, and (iii) gaps both above and below the Euler bands, providing a multigap extension of the single-gap bound on Chern insulators recently derived in Ref. [54]. We show that our results concerning optical weights hold in degenerate flat Euler bands, e.g., conjectured in the context of twisted bilayer graphene (TBG) [33], and beyond, i.e., when nodal Euler topology arises in the presence of dispersion [34,49]. Accordingly, we discover optical manifestations of such nodes in terms of (i) quantized optical conductivities, (ii) optical weights, and (iii) third-order jerk photoconductivities [55,56], within effective $\mathbf{k} \cdot \mathbf{p}$ models. We further demonstrate how the associated Euler invariants can be reconstructed in terms of optical transition rates. Finally, we numerically showcase the obtained identities in minimal lattice-regularized models, setting the stage for effective theories in real materials, whilst also offering direct implementations in ultracold atomic simulations and metamaterials.

Optical manifestations of Euler bands. We first consider the case of isolated Euler bands and discuss bounds on the optical observables captured by quantum-geometric quantities, which can be indeed used to infer the presence of a nontrivial Euler class. We label the magnitude of the (direct) gaps below and above the Euler bands $\{|u_n\rangle, |u_{n+1}\rangle\}$ of energies $E_n(\mathbf{k}), E_{n+1}(\mathbf{k})$ as $\Delta_{g-} \equiv \min_{\mathbf{k}}[E_n(\mathbf{k}) - E_{n-1}(\mathbf{k})]$ and $\Delta_{g+} \equiv \min_{\mathbf{k}}[E_{n+2}(\mathbf{k}) - E_{n+1}(\mathbf{k})]$, respectively [see Fig. 1(a)]. Before deriving the bounds, we recall the inequality imposed on quantum geometry due to the nontriviality of the Euler class [52,53], as it forms the basis of further calculations. The quantum metric is defined as $g_{ab}^x(\mathbf{k}) = \Re \sum_{n=1}^2 \sum_{m \neq 1,2} [r_{nm}^a r_{mn}^b]$, where $a, b = x, y$, and the $r_{nm}^a = i(1 - \delta_{n,m})A_{n,m}^a$ are transition dipole matrix elements captured by the components of the non-Abelian Berry connection $A_{n,m}^a$. The integral of the trace of g_{ab}^x over the BZ can be shown to be bounded from below by the Euler class of the two-band subspace,

$$\frac{1}{4\pi} \int d^2\mathbf{k} \text{Tr} \mathbf{g}^x \geq |\chi|, \quad (3)$$

where the trace is over $\mathbf{k}$ -space coordinates at each $\mathbf{k}$ point individually (see Refs. [53,57,58] and Supplemental Material (SM) [59]). On the other hand, in three-band models, the Euler class is captured by the winding of the third gapped vector $\mathbf{n}_3 \equiv |u_3\rangle$, and may be expressed as $\chi = \frac{1}{4\pi} \int d^2\mathbf{k} \mathbf{n}_3 \cdot (\partial_{k_x} \mathbf{n}_3 \times \partial_{k_y} \mathbf{n}_3)$ [40]. In this case, the left-hand side of Eq. (3) reduces to $\frac{1}{4\pi} \int d^2\mathbf{k} \text{Tr} \mathbf{g}^x = \frac{1}{4\pi} \int d^2\mathbf{k} (|\partial_{k_x} \mathbf{n}_3|^2 + |\partial_{k_y} \mathbf{n}_3|^2)$. In fact, this is the result of a stronger inequality that holds at every point in the BZ (i.e., between the integrands). Namely, $|\partial_{k_x} \mathbf{n}_3|^2 + |\partial_{k_y} \mathbf{n}_3|^2 \geq |\mathbf{n}_3 \cdot (\partial_{k_x} \mathbf{n}_3 \times \partial_{k_y} \mathbf{n}_3)|$, which is analogous to a similar

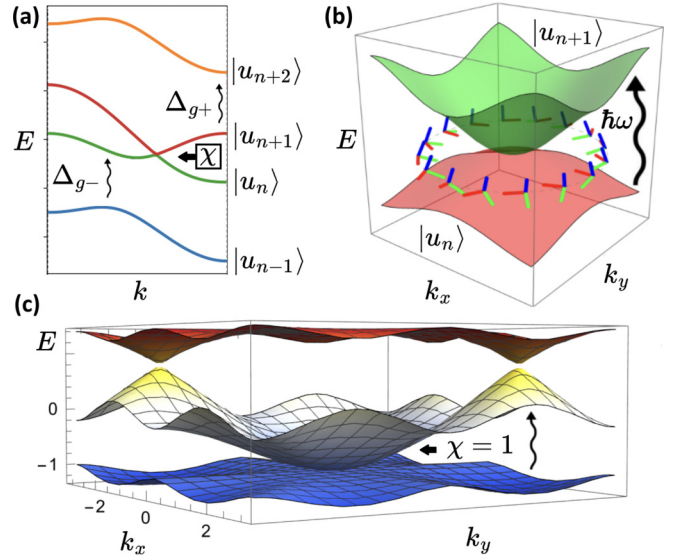


FIG. 1. (a) Optical transitions to/from Euler bands with lower/upper gaps having energies Δ_{g-}/Δ_{g+} . (b) Quadratic Euler node with nontrivial patch Euler class $\chi = 1$ in a $\mathbf{k} \cdot \mathbf{p}$ model. The associated eigenstate frame winding (colored vectors) corresponds to the winding of the non-Abelian connection $\mathbf{A}_{1,2}$, which is singular at the node. An optical transition within the Euler band subspace (for semimetallic doping) is schematically shown. (c) Lattice-regularized realization of quadratic Euler node in a kagome model. The optical conductivities, optical weights, as well as jerk conductivities, can be used to deduce the patch Euler class χ .

geometric bound in Chern insulators, which arises from the skyrmion formula [54,60]. This inequality suggests an interplay between the Euler class and optical phenomena.

In that context, the quantum metric is related to the intrinsic quadrupole moment, the quantity upon which electric quadrupole (E2) transitions depend [52,61]. In Euler phases, since the diagonal elements of the Berry connection vanish due to the reality condition, this culminates in the single-particle quadrupole moment given by $q_{ab}^n(\mathbf{k}) \equiv -\frac{e}{2} (\langle \partial_{k_a} u_n | \partial_{k_b} u_n \rangle + \langle \partial_{k_b} u_n | \partial_{k_a} u_n \rangle)$, where e is the elementary charge. Consequently, by employing $q_{ab}^1 + q_{ab}^2 = -eg_{ab}^x$ for a real Hamiltonian, we find that Eq. (3) leads to a bound on the integrated total quadrupole moment of the topological bands due to their finite Euler class

$$\frac{1}{4\pi} \int d^2\mathbf{k} |\text{Tr}(q_{ab}^1 + q_{ab}^2)| \geq e|\chi|. \quad (4)$$

Here, we consider coupling to an electric field $\mathcal{E} = (\mathcal{E}_x \cos \omega t, \mathcal{E}_y \cos(\omega t + \phi))$, where the phase shift ϕ controls the polarization, with $\phi = \pm\pi/2$ corresponding to left- and right-circularly polarized light (LCP/RCP). In this context, the quantum metric can be accessed by inspection of the appropriate optical weight $W_{ab}^1(\omega_{\max}) \equiv \int_0^{\omega_{\max}} d\omega \frac{\Re \epsilon \sigma_{ab}(\omega)}{\omega}$, for the optical conductivities $\sigma_{ab}(\omega)$ under the electric field up to some large frequency $\omega_{\max}$ [8,11,15,59,62].

We now focus on the optical weights associated with transitions to and from the Euler subspace, which apply to arbitrarily dispersive Euler bands such as those arising in lattice

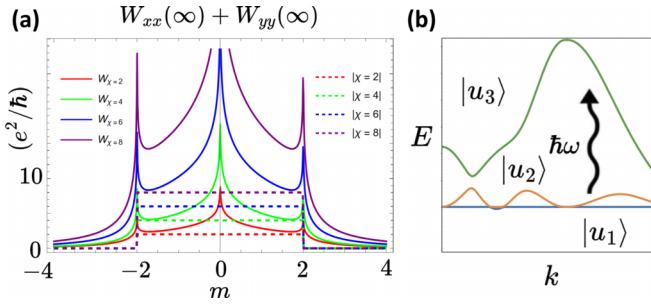


FIG. 2. (a) Optical weights in a three-band Euler Hamiltonian with the occupied Euler bands [lowest two bands as depicted in (b)] can be related to the interband contributions to linear dc conductivities. The combined optical weight is bounded by the Euler invariant, see Eq. (6). Singularities occur at topological phase transitions ($|m| = 2$), associated with the closing of the gap as the topological mass term m is changed. The limit of large $|m|$ corresponds to the trivial phase with $\chi = 0$. (b) A schematic picture of the corresponding transitions.

models (for examples see SM [59], Refs. [42,49,63]). As our first central result, we derive an optical bound constraining the optical weights W_{ab}^1 involving the Euler bands, across the gaps harbouring Δ_{g-} and Δ_{g+} , (see Fig. 1) [54,64]. We label optical transitions into the isolated Euler subspace from bands below, and out of this subspace into bands above by unoccupied (unocc) and occupied (occ), respectively. We find that these transitions involving the Euler bands are bounded as

$$W_{xx,(\text{occ})}^1(\infty) + W_{yy,(\text{occ})}^1(\infty) + W_{xx,(\text{unocc})}^1(\infty) + W_{yy,(\text{unocc})}^1(\infty) \geq \frac{e^2}{\hbar} |\chi|, \quad (5)$$

which holds in the high-frequency limit $\omega_{\max} \rightarrow \infty$ [59]. If there are no bands below the Euler subspace, $W_{aa,(\text{unocc})}^1(\infty)$ vanishes and this condition reduces to

$$W_{xx,(\text{occ})}^1(\infty) + W_{yy,(\text{occ})}^1(\infty) \geq \frac{e^2}{\hbar} |\chi|, \quad (6)$$

which is observable for topological phases with $\chi \neq 0$. We numerically corroborate these results in models with different Euler class (see Fig. 2). Finally, we point out that we can generalize these results to quantum-metric signatures in flat-band systems with dc linear conductivity due to the interband contributions [65], as detailed in the SM [59].

We remark that an analogous conclusion, viewed as a bound on the optical weight, can be also reached from Fermi's golden rule under circular polarization [8]. Namely, we find that the combined absorption rate $\tilde{\Gamma}^{\text{tot}}$ of LCP and RCP light [7,13], combining all possible transitions when Euler bands are fully occupied and are completely unoccupied, is bounded from below by the Euler class

$$\tilde{\Gamma}^{\text{tot}} = \frac{2\pi e^2 \mathcal{E}^2}{\hbar^2} \int d^2\mathbf{k} \text{Tr} \mathbf{g}^x \geq \frac{e^2 \mathcal{E}^2}{2\hbar^2} |\chi|. \quad (7)$$

This result is consistent with Eq. (5), on recognizing that combinations of LCP and RCP can be viewed as linear polarizations inducing corresponding excitations that appear in the optical weights [59].

We conclude by showing that the bound on the optical weight is additionally related to the size of the gaps. To prove this, we start by considering a three-band spectrum with the Euler bands at the bottom, and apply the f -sum rule [54], to recognize that $\int_0^\infty d\omega \frac{\Re \epsilon_{aa}}{\omega} \leq \int_0^\infty d\omega \Re \epsilon_{aa} \frac{\hbar}{\Delta_{g+}} = \frac{n_{e,\text{occ}} \hbar e^2}{2m\Delta_{g+}}$. We obtain that

$$\frac{n_{e,\text{occ}} \pi \hbar^2}{m\Delta_{g+}} \geq |\chi| \Rightarrow \frac{n_{e,\text{occ}} \pi \hbar^2}{m|\chi|} \geq \Delta_{g+}, \quad (8)$$

where $n_{e,\text{occ}}$ is the charge density in the occupied bands, and m is the bare electron mass. A tighter bound can be obtained upon replacing m (from the minimal-coupling Hamiltonian) with the effective mass m^* in the f -sum rule (within the $\mathbf{k} \cdot \mathbf{p}$ approach) [54]. In the flat-band limit, a nonzero Euler class $|\chi| \neq 0$ implies that the gap Δ_{g+} vanishes in the limit of large effective mass $m^* \rightarrow \infty$. Hence, to keep the gap finite, an Euler insulator needs dispersion in either the Euler bands or in the unoccupied band above the gap. In the many-band case, we analogously employ the sum rule to obtain

$$\frac{\pi \hbar^2}{m} \left(\frac{n_{e,\text{occ}}}{\Delta_{g-}} + \frac{n_{e,\text{unocc}}}{\Delta_{g+}} \right) \geq |\chi|, \quad (9)$$

where $n_{e,\text{occ}}, n_{e,\text{unocc}} \sim 1/a^2$, with lattice constant a , are the total charge densities for the cases where the Euler bands are occupied (as before) and unoccupied, respectively. By demanding that $n_{e,\text{occ}}$ is the carrier density in certain $\mathbf{k} \cdot \mathbf{p}$ low-energy models, e.g., for twisted bilayer graphene [66–68], a further bound can be obtained. Here, the Euler class constrains the size of *both* gaps as

$$\frac{n_{e,\text{unocc}} \pi \hbar^2}{m|\chi|} \geq \frac{n_{e,\text{unocc}}}{\frac{n_{e,\text{occ}}}{\Delta_{g-}} + \frac{n_{e,\text{unocc}}}{\Delta_{g+}}} > \frac{1}{2} \text{HM}(\Delta_{g-}, \Delta_{g+}), \quad (10)$$

where $\text{HM}(\Delta_{g-}, \Delta_{g+}) \equiv [(\Delta_{g+}^{-1} + \Delta_{g-}^{-1})/2]^{-1}$ is the harmonic mean of the band gaps, and we have used $n_{e,\text{occ}} > n_{e,\text{unocc}}$. This is the central result of this work, elucidating a quantum-geometric constraint on the multigap Euler topology in electronic materials, as manifested by a fundamental optical bound on the band gaps. Contrary to the bounds related to dc conductivity within noninteracting particle picture (see SM [59]), this kind of fundamental bound could even hold in interacting, strongly correlated systems [54].

Optical manifestations of patch Euler class. We now demonstrate how our findings generalize for a patch Euler invariant, which extend beyond subspaces separated by gaps $\Delta_{g\pm}$ (see Fig. 1). A patch of two bands with $\chi \neq 0$ necessarily hosts band nodes with same non-Abelian frame charges that are obstructed to annihilate [30], as long as the patch excludes band crossings of other bands [34]. However, accessing this invariant can be complicated due to the multiband character of the Euler class [30]. We here show that it can be probed with optical conductivities, associated optical weights, and third-order jerk photoconductivities [55,56].

A general $\mathbf{k} \cdot \mathbf{p}$ Hamiltonian for a bulk Euler node hosting a patch Euler class can be written as

$$H_\chi(\mathbf{k}) = \alpha(k_+^{2|\chi|} \sigma_- + k_-^{2|\chi|} \sigma_+), \quad (11)$$

where α is a dispersion constant, $k_\pm = k_x \pm ik_y$, and $\sigma_\pm = (\sigma_x \pm i\sigma_z)/2$ [69]. Note that the Hamiltonian is real by

construction. The elements of the non-Abelian Berry connection, the quantum metric, and associated optical responses may be computed from the eigenstates of $H_\chi(\mathbf{k})$. We find that in the proximity of the node, the optical conductivity is quantized by the patch Euler invariant χ as

$$\sigma_{aa}(\omega) = \frac{e^2}{8\hbar} |\chi|. \quad (12)$$

Furthermore, on doping with chemical potential μ away from the node (see SM [59]), we show that the optical weight is

$$W_{aa}^1(\omega_{\max}) = \frac{\pi e^2}{4\hbar} |\chi| \ln\left(\frac{\omega_{\max}}{2\mu}\right). \quad (13)$$

Since $\omega_{\max}$ and μ are controllable parameters, we predict that the patch Euler class can be not only deduced with optical conductivities, but also traced on doping.

As a next step, we demonstrate that independently of the dispersion proportionality constant α , the jerk conductivities [55] at any nonvanishing light frequency ω offer universal Euler class-dependent ratios [59]. Most interestingly, we obtain that

$$\frac{\sigma_{\text{jerk}}^{xxxx}(\omega)}{\sigma_{\text{jerk}}^{xyxy}(\omega)} = \frac{3|\chi| - 1}{|\chi| + 1}, \quad (14)$$

which is always nonvanishing and bounded: $1 < \sigma_{\text{jerk}}^{xxxx}/\sigma_{\text{jerk}}^{xyxy} < 3$. This holds provided that, at least to first order, the node (i) is rotationally symmetric, and (ii) hosts patch Euler class χ . We also note that the nature of the low-frequency divergence of jerk photoconductivities $\omega \rightarrow 0$ depends on χ , becoming weaker as χ increases (see SM [59]). All second-order photoconductivities vanish within the model, as the bulk node naturally enjoys the inversion symmetry [59]. When full rotational symmetry is not preserved, as in lattice models, we find that Eqs. (12)–(14) still hold closely to the nodes.

Reconstructing the Euler invariant. Finally, we demonstrate that the Euler invariant (1) can be probed by monitoring the transition rates in a frequency- and momentum-resolved way. Motivated by the reconstruction of the quantum-geometric tensor (QGT) [8,15], we devise a protocol upon considering *only* the bottom band of the Euler subspace to be occupied (not both), see Fig. 1(b). We recognize that both terms of Eq. (1) can be extracted from the elements of the non-Abelian Berry connection, which we identify as real part of the QGT. Specifically, the vector components can be deduced from the absorption rate of the linearly polarized light in the x and y direction, probing $g_{xx}^{12} = |A_{12}^x|^2$, and $g_{yy}^{12} = |A_{12}^y|^2$, see SM [59]. The non-Abelian connection vector can be thus obtained modulo the sign of each component. Imposing a smooth real gauge by hand to fix the sign as a gauge choice, we then compute the Euler curvature and the Euler invariant.

Numerical verification in representative models. Given the general formulation of Euler models using Plücker embeddings [30,63], we can readily corroborate our results with tight-binding models. We consider illustrative three-band and four-band models on square and kagome lattices, which can be synthesized experimentally in trapped-ion simulators and metamaterials [40,49]. We present our numerical findings for optical weights in Fig. 2, demonstrating how the optical

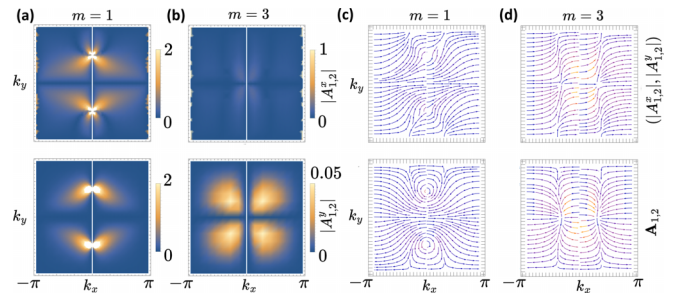


FIG. 3. Mapping out the Euler invariant. The absolute values of the components of the non-Abelian connection are reconstructed from the associated quantum metric probed by optical transition rates under linearly polarized light. (a),(b) $|A_{12}^x|$ (upper panel) and $|A_{12}^y|$ (lower panel) are mapped out for the Euler Hamiltonian with $\chi = 2$ on a square lattice, in the topological ($m = 1$) (a) and trivial phase ($m = 3$) (b). The non-Abelian connection vectors can be reconstructed by fixing the signs of these components in a maximally smooth gauge for topological (c) and trivial (d) phases.

bounds from Eq. (5) are satisfied in the topological phases [59]. Furthermore, in Fig. 3 we show how the Euler invariant can be reconstructed from the optical transitions, as proposed. Finally, we corroborate results from the $\mathbf{k} \cdot \mathbf{p}$ model of an Euler node in the lattice-regularized realizations (see SM [59]).

Discussion. We further comment on the scope and applicability of our results. As derived analytically and verified numerically, our bounds on optical responses offer a route to probe many-band topology in Euler Hamiltonians within the independent-particle picture. We note that these models may be implemented in trapped-ion simulators and metamaterials. Additionally, the bounds on optical weights can be understood microscopically in terms of the application of Fermi's golden rule to interband transitions. In the case of nodal Euler topology, the relation between the optical weight and doping of an Euler semimetal naturally arises, which is sensitive to the value of the patch Euler class. Here, the jerk conductivities, and other third-order effects, provide a valuable insight, especially by means of low-frequency divergences [70]. Even when the nodes are not rotationally symmetric, the ratios of photoconductivities obtained within the effective model may be modified. Additionally, on breaking the inversion symmetry assumed in the continuum model, linear injection, as well as circular shift photoconductivities [59], might arise, offering further insights into the physics induced by the non-Abelian connection constrained by the Euler invariant.

Furthermore, the Euler topology can be removed by breaking $\mathcal{C}_2\mathcal{T}/\mathcal{PT}$ symmetry. Upon breaking also $\mathcal{T}$ symmetry individually, a pair of bands with opposite Chern numbers may appear [63]. In this case, magnetic circular dichroism effects must emerge on doping or frequency tuning [7,59], whereas in Euler phases, such a dichroism vanishes by the reality condition.

Finally, the consequences of the derived bound on the gaps below and above the Euler bands can be contrasted with effective models for twisted layered graphene systems [66,67] assuming flat bands with $\chi = 1$, as well as with the single-gap Chern bounds in moiré MoTe_2 and WTe_2 [54].

Contrary to the single-gap condition for Chern insulators [54], we show that the Euler topology *necessarily* constrains two neighboring gaps through a multigap quantum-geometric bound, Eq. (10). This distinction can be traced back to the fact that the Euler invariants of occupied two-band subspaces are not additive, unlike the Chern numbers of the individual occupied bands, allowing the corresponding bound to be reduced to a single-gap condition. Consistently, our general optical bounds, Eqs. (5),(7), combine two distinct, gapped band occupation configurations (unocc/occ). All bounds thus reflect a multigap nature.

Conclusions. We demonstrated how the topological Euler class can be optically probed in electronic systems. In particular, it can be exactly reconstructed from the optical absorption/excitation experiments. Using quantum geometry, we showed that the presence of the two-band subspace hosting an Euler invariant enforces a lower bound on the optical weights and interband contributions to dc conductivities. Furthermore, we derived a fundamental bound constraining a harmonic mean of both gaps neighboring the Euler bands. Finally, we showed that the Euler class of a non-Abelian semimetal can be reconstructed from (quantized) optical conductivities, optical weights, and higher-order

photoconductivities, providing feasible avenues for experimental implementations.

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