

Thermal transport properties of magnons on the α - T_3 lattice

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(Received 19 September 2024; revised 7 December 2024; accepted 7 January 2025; published 21 January 2025)

We theoretically investigate magnons on the α - T_3 lattice. Atomistic spin dynamics simulations show that next-nearest neighbor hopping and easy-axis anisotropy stabilize ferromagnetic order in the presence of Dzyaloshinskii-Moriya interaction. We identify one topologically trivial magnon insulator phase and three magnon Chern insulator phases. The topologically trivial magnon insulator phase exhibits a small but nonzero magnon thermal Hall conductivity, while in the magnon Chern insulator phases the Chern number of the lowest magnon band dominates the magnon thermal Hall conductivity. The sign of the magnon thermal Hall conductivity does not change at the topological phase boundaries, but distinct changes are observed in the magnitude.

DOI: [10.1103/PhysRevB.111.024416](https://doi.org/10.1103/PhysRevB.111.024416)

I. INTRODUCTION

Integrating topology with Bloch band theory [1] has enabled the discovery of new classes of matter [2]. Topological insulators [3–5], characterized by their robust metallic edge states with insulating bulk, exemplify how topology can lead to exotic electronic properties, such as dissipationless transport [6,7]. Additionally, topology has facilitated the exploration of various quasiparticles, including spinons [8,9], Majorana fermions [10,11], axions [12,13], magnetic monopoles [14,15], and fractionally charged vortices [16], which hold potential for application in high-performance electronics [17], spintronics [18,19], and topological quantum computation [20–22].

Among the quasiparticles intensively studied by topological band theory are also magnons [18], the quanta of spin waves in crystalline magnetic materials. These bosons without electrical charge are an alternative to electrons for information transport and processing. In particular, high-energy magnons have been intensively studied because they avoid Joule heating [18,23]. Magnons exhibit exotic phenomena such as the thermal Hall effect [24–29], spin Nernst effect [30–37], spin Seebeck effect [38–41], magnon-mediated Edelstein effect [42], and magnon orbital Nernst effect [43].

Intriguing magnon physics emerges for special crystal lattices. For example, antiferromagnets with Kagome lattice

exhibit the spin Seebeck and spin Nernst effects even in the absence of Dzyaloshinskii-Moriya (DM) interaction [44,45] due to noncollinear alignment of the magnetic moments [36]. Both ferromagnetic and antiferromagnetic magnons have been studied for the honeycomb [46–53], kagome [24,26–29,54], Lieb [55–57], and triangular [58] lattices. Theoretical proposals additionally suggest the existence of magnons in the newly discovered class of altermagnets [59–62].

The α - T_3 lattice [63] is interesting due to the similarity of its band structure to that of graphene except for an extra flat band at the Dirac point [64–66]. It is a honeycomb lattice with an additional atom at the center of each hexagon coupled only to one of the other two sublattices with a strength $\alpha > 0$. It becomes a honeycomb lattice for $\alpha = 0$ and a dice (or T_3) lattice for $\alpha = 1$. Numerous studies have explored the equilibrium and nonequilibrium properties of the α - T_3 lattice. The α -dependent band structure and Berry curvature [67] enable it to host properties such as super Klein tunneling [68–73], Anderson localization [74,75], valley-dependent transport [76–79], zitterbewegung [80], Ruderman-Kittel-Kasuya-Yosida interaction [81,82], linear and nonlinear magneto-optical effects [83–85], Majorana corner states [86], and Seebeck and Nernst effects [87]. The topological properties of electrons on the α - T_3 lattice have been explored in Ref. [88]. The α - T_3 lattice appears in $\text{Hg}_{1-x}\text{Cd}_x\text{Te}$ with an intermediate $\alpha = 3^{-1/2}$ [89,90] and also has been realized by fabricating a $\text{SrTiO}_3/\text{SrIrO}_3/\text{SrTiO}_3$ structure [91].

Despite significant research on the α - T_3 lattice, the properties of magnons remain unexplored. In this work, we study the α - T_3 lattice via a Heisenberg spin Hamiltonian with nearest-neighbor coupling, next-nearest neighbor coupling, DM interaction, and easy-axis anisotropy. The magnon band structure of the honeycomb lattice is recovered for $\alpha = 0$. We address the magnetic order by atomistic spin dynamics simulations and then extract the topological phase diagram. We calculate the magnon thermal Hall conductivity by determining the Berry curvature. The paper is organized as

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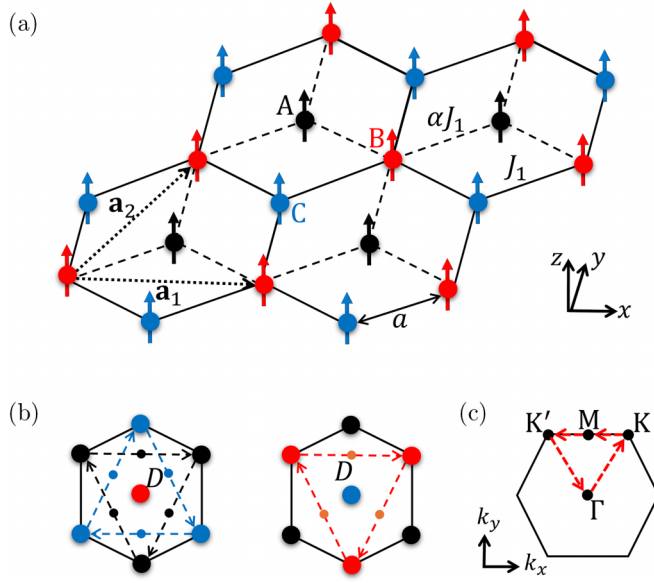


FIG. 1. (a) α - T_3 lattice with the spins aligned along the z axis. $\mathbf{a}_1$ and $\mathbf{a}_2$ are the lattice vectors. (b) DM interaction between next nearest neighbors. (c) High symmetry points in the first Brillouin zone and path chosen for plotting the magnon band structure.

follows: The spin Hamiltonian and its conversion into the magnon Hamiltonian are presented in Sec. II followed by the atomistic spin dynamics and thermal transport approaches. The magnetic order along with the magnon band structure, topological phase diagram, and magnon thermal transport properties are discussed in Sec. III. Section IV gives a summary.

II. MODEL AND METHOD

A. Spin Hamiltonian and magnon Hamiltonian

We consider the α - T_3 lattice shown in Figs. 1(a) and 1(b). The spin Hamiltonian is given by

$$H = - \sum_{\langle i,j \rangle} J_1^{ij} \mathbf{S}_i \cdot \mathbf{S}_j - J_2 \sum_{\langle\langle i,j \rangle\rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_{\langle\langle i,j \rangle\rangle} D v_{ij} \hat{z} \cdot (\mathbf{S}_i \times \mathbf{S}_j) - J_K \sum_i (S_i^z)^2, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ represents nearest neighbor sites, separated by a distance a , $\langle\langle \cdot, \cdot \rangle\rangle$ represents next-nearest neighbor sites, $\mathbf{S}_i$ represents the spin on site i , and $J_1^{ij} > 0$ and $J_2 > 0$ represent the nearest and next-nearest neighbor coupling strengths, respectively. We set $J_1^{ij} = \alpha J_1$ for the coupling between sublattices A and B, and $J_1^{ij} = J_1$ for the coupling between sublattices B and C. The third term in Eq. (1) represents the out-of-plane DM interaction with $D \geq 0$ analogous to the honeycomb lattice [92], $v_{ij} = +1$ for clockwise and -1 for counterclockwise hopping, and $\hat{z}$ denoting a vector of unit length. $J_K > 0$ represents the anisotropy strength defining the easy axis.

We utilize the linear spin-wave formalism [93,94] to derive the tight-binding magnon Hamiltonian in momentum space. To this aim, we represent H in terms of magnon

operators by applying the lowest order Holstein-Primakoff transformation [95]

$$\begin{aligned} S_i^+ &= S_i^x + iS_i^y \approx \sqrt{2S}b_i, \\ S_i^- &= S_i^x - iS_i^y \approx \sqrt{2S}b_i^\dagger, \\ S_i^z &= (S - b_i^\dagger b_i), \end{aligned} \quad (2)$$

where b_i and $b_i^\dagger$ are the magnon annihilation and creation operators, respectively, and we assume $|\mathbf{S}_i| = S$ for all i . Keeping only the quadratic terms in the magnon operators we then perform the Fourier transformation

$$b_i^{(\dagger)} = \frac{1}{\sqrt{L}} \sum_{\mathbf{k}} e^{(-)i\mathbf{t}_i \cdot \mathbf{k}} b_{\mathbf{k}}^{(\dagger)}, \quad (3)$$

where $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ are the magnon annihilation and creation operators in Fourier space, respectively, L is the number of unit cells, and $\mathbf{t}_i$ is the position vector of site i . We obtain

$$H = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger h(\mathbf{k}) \Psi_{\mathbf{k}}, \quad (4)$$

where $\Psi_{\mathbf{k}}$ is the magnon basis and $h(\mathbf{k}) = h_1(\mathbf{k}) + h_2(\mathbf{k}) + h_D(\mathbf{k}) + h_K$ with

$$\begin{aligned} h_1(\mathbf{k}) &= J_1 S \begin{bmatrix} 3\alpha & -\alpha\Gamma_{1,\mathbf{k}} & 0 \\ -\alpha\Gamma_{1,\mathbf{k}}^* & 3(1+\alpha) & -\Gamma_{1,\mathbf{k}} \\ 0 & -\Gamma_{1,\mathbf{k}}^* & 3 \end{bmatrix}, \\ h_2(\mathbf{k}) &= J_2 S (6 - \Gamma_{2,\mathbf{k}}) I_{3 \times 3}, \\ h_D(\mathbf{k}) &= D S \begin{bmatrix} \Gamma_{D,\mathbf{k}} & 0 & 0 \\ 0 & \Gamma_{D,\mathbf{k}} & 0 \\ 0 & 0 & -\Gamma_{D,\mathbf{k}} \end{bmatrix}, \\ h_K &= 2J_K S I_{3 \times 3}, \end{aligned} \quad (5)$$

and the abbreviations

$$\begin{aligned} \Gamma_{1,\mathbf{k}} &= \exp(ia k_y) + 2 \cos\left(\frac{\sqrt{3}a}{2} k_x\right) \exp\left(-\frac{ia}{2} k_y\right), \\ \Gamma_{2,\mathbf{k}} &= 2 \left(\cos(\sqrt{3}a k_x) + 2 \cos\left(\frac{\sqrt{3}a}{2} k_x\right) \cos\left(\frac{3a}{2} k_y\right) \right), \\ \Gamma_{D,\mathbf{k}} &= -2 \left(\sin(\sqrt{3}a k_x) - 2 \sin\left(\frac{\sqrt{3}a}{2} k_x\right) \cos\left(\frac{3a}{2} k_y\right) \right). \end{aligned} \quad (6)$$

$I_{3 \times 3}$ is the three-dimensional identity matrix. We set $a = 1$ nm in the following.

Note that H in Eq. (4) is restricted to quadratic order in the magnon operators, which is valid at low temperature. Higher-order corrections can be incorporated by including quadratic and higher terms in the magnon operators into the Holstein-Primakoff transformation and utilizing the methods proposed in Ref. [51]. These corrections encompass thermally activated processes and spontaneous decay [96]. While the former are frozen out at low temperature, the latter broaden the magnon spectrum [42]. Inclusion of higher-order corrections, such as magnon-magnon interactions, is out of the scope of this work. However, we anticipate that they do not alter the qualitative conclusions drawn from our analysis.

B. Atomistic spin dynamics

Spin dynamics simulations are conducted using the atomistic Landau-Lifshitz-Gilbert equation [97,98]

$$\frac{\partial \mathbf{S}_i}{\partial t} = -\frac{\gamma}{(1 + \Delta^2)\mu_i} \mathbf{S}_i \times \mathbf{B}_i^{\text{eff}} - \frac{\gamma \Delta}{(1 + \Delta^2)\mu_i} \mathbf{S}_i \times (\mathbf{S}_i \times \mathbf{B}_i^{\text{eff}}), \quad (7)$$

where γ is the gyromagnetic ratio, Δ is the Gilbert damping, μ_i is the spin moment, and $\mathbf{B}_i^{\text{eff}}$ is the effective magnetic field (which depends on H and at $T = 0$ is given by $\mathbf{B}_i^{\text{eff}} = \partial H / \partial \mathbf{S}_i$). The first term in Eq. (7) represents the quantum mechanical precession of spins, while the second term represents the damping that aligns the spin direction with an applied magnetic field. To numerically evolve a spin according to Eq. (7), we use the SPIRIT code that adopts a fourth-order Runge-Kutta solver [99].

C. Chern number and magnon thermal conductivity

Topological phases of matter are well investigated for electrons [3,100]. One example is the Chern insulator, which differs from a trivial insulator in that the Bloch states acquire along a closed path a nonzero Berry phase [101] (which defines a nonzero Berry curvature). A nontrivial topology of Bloch states thus is characterized by a nonzero Chern number. This concept of topology is not exclusive to electrons but can also be applied to magnon Bloch states [102–105]. In our case, the topology is driven by the DM interaction, as magnons acquire a phase from one site to another when $D \neq 0$, giving rise to the Berry curvature

$$\Omega_{n,\mathbf{k}}^z = -2\hbar^2 \text{Im} \sum_{\substack{m=1, \\ m \neq n}}^3 \frac{v_x^{nm}(\mathbf{k}) v_y^{mn}(\mathbf{k})}{(\epsilon_{n,\mathbf{k}} - \epsilon_{m,\mathbf{k}})^2} \quad (8)$$

of the n th magnon Bloch state. Here, $\hbar$ is Planck's constant, $v_i^{nm}(\mathbf{k}) = \langle n, \mathbf{k} | \partial h(\mathbf{k}) / \partial k_i | m, \mathbf{k} \rangle$ are the matrix elements of the velocity operator in direction i , and $|n, \mathbf{k}\rangle$ and $\epsilon_{n,\mathbf{k}}$ are the n th eigenstate and eigenvalue of $h(\mathbf{k})$, respectively. The Chern number of the n th magnon band is given by integration of $\Omega_{n,\mathbf{k}}^z$ over the first Brillouin zone,

$$C_n = \frac{1}{2\pi} \int d^2\mathbf{k} \Omega_{n,\mathbf{k}}^z. \quad (9)$$

$C_n \neq 0$ indicates that the magnon thermal Hall conductivity of the n th magnon band [51]

$$\kappa_{xy}^n = -\frac{k_B^2 T}{\hbar} \int \frac{d^2\mathbf{k}}{4\pi^2} \Omega_{n,\mathbf{k}}^z c_2(\epsilon_{n,\mathbf{k}}) \quad (10)$$

is nonzero, where the temperature gradient is applied in the y direction and the thermal current is measured in the x direction. We denote the Boltzmann constant as k_B and

$$c_2(\epsilon_{n,\mathbf{k}}) = -\int_{\epsilon_{n,\mathbf{k}}}^{\infty} d\epsilon \left(\frac{\epsilon}{k_B T} \right)^2 f_B'(\epsilon), \quad (11)$$

where $f_B'(\epsilon)$ is the derivative of the Bose distribution function. The magnon thermal Hall conductivity then is given by $\kappa_{xy} = \sum_{n=1}^3 \kappa_{xy}^n$. The longitudinal thermal conductivity

analogously is given by $\kappa_{xx} = \sum_{n=1}^3 \kappa_{xx}^n$ with

$$\kappa_{xx}^n = -\frac{\hbar}{T\eta} \sum_{m=1}^3 \int \frac{d^2\mathbf{k}}{4\pi^2} |v_x^{nm}(\mathbf{k})|^2 \epsilon_{n,\mathbf{k}}^2 f_B'(\epsilon_{n,\mathbf{k}}), \quad (12)$$

where η is the magnon broadening. The magnon density of states is calculated through the retarded Green's function as

$$\text{DOS}(\epsilon_{n,\mathbf{k}}) = -\frac{1}{\pi} \text{Im} \int \frac{d^2\mathbf{k}}{4\pi^2} \text{Tr} \left[\frac{1}{\epsilon_{n,\mathbf{k}} - h(\mathbf{k}) + i\eta} \right]. \quad (13)$$

III. RESULTS AND DISCUSSION

A. Magnetic order

We conduct atomistic spin dynamics simulations using the SPIRIT code [99] to determine the values of the parameters α , J_1 , J_2 , D , and J_K that support ferromagnetic order. They are performed for $10 \times 10 \times 1$ unit cells with periodic boundary conditions in the x and y directions. We solve Eq. (7) using the DEPOND solver, initiating the simulations with a random spin configuration at $t = 0$. The spins are allowed to evolve under a Gilbert damping of 0.05 with a time step of 1 fs. First, we examine the case $J_2/J_1 = D/J_1 = J_K/J_1 = 0$ to establish the effect of varying α from 0 to 1.5, finding for all values ferromagnetic order after a few ns. Figures 2(a)–2(e) illustrate the stability of the ferromagnetic order for $\alpha = 1.25$, as the energy density (energy per number of lattice sites) converges, the densities of the x and y spin components (spin per number of lattice sites) approach zero, and the density of the z spin component approaches unity. Second, for $D/J_1 = J_K/J_1 = 0$, we vary both α and J_2/J_1 from 0 to 1.5, finding always ferromagnetic order. Finally, by fixing J_2/J_1 and varying D/J_1 from 0 to 1 we determine the critical value of J_K/J_1 required to realize ferromagnetic order for different α . We obtain for $J_K/J_1 = 0$ spiral order as long as $D/J_1 > J_2/J_1$, as shown in Fig. 2(f). Figure 2(g) shows the critical value of J_K/J_1 at $J_2/J_1 = 0.3$ for different values of D/J_1 and α . As it does not significantly impact our qualitative analysis, we fix $J_2/J_1 = 0.3$ unless stated otherwise. Moreover, based on the above results, we constrain $0 \leq D/J_1 \leq 0.3$ and set $J_K/J_1 = 0$ to ensure ferromagnetic order.

B. Band structure and topological phase diagram

For simplicity, we set in the following $S = 1$. The magnon band structure of the honeycomb ferromagnet at $J_2/J_1 = D/J_1 = 0$ is shown by black dashed lines in Fig. 3(a), reproducing the results of Ref. [92]. The dice ferromagnet exhibits two dispersive bands (lower and upper) and one flat band (middle) for $J_2/J_1 = D/J_1 = 0$. The lower and middle bands touch each other at the K and K' points, while the upper band is separated by a band gap of at least $3J_1 S$ (as the coordination number of sublattice B is twice that of sublattices A and C). $J_2/J_1 \neq 0$ makes the middle band dispersive even though no band gap opens to the lower band. Variation of D/J_1 and α can be used to open a band gap at the K or K' point, or even everywhere in the reciprocal space. At $D/J_1 = \alpha/2\sqrt{3}$ the middle and upper bands touch each other at the K point, at $D/J_1 = (1 - \alpha)/2\sqrt{3}$ the lower and middle bands touch each other at the K' point, and at $D/J_1 = \alpha/9$ the bands do

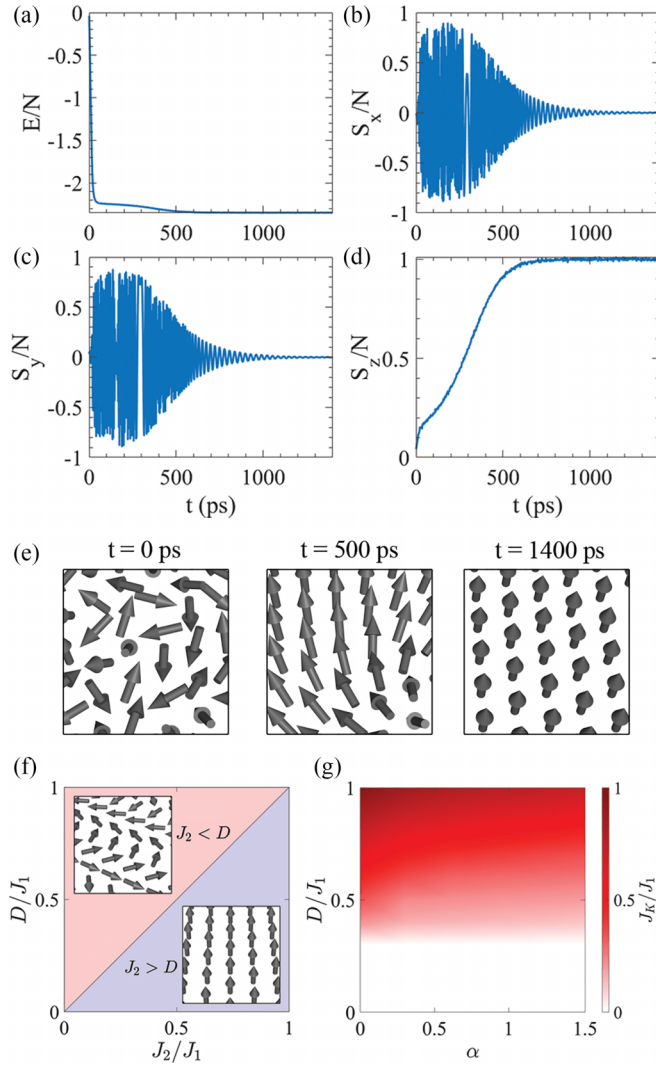


FIG. 2. (a) Energy density and [(b)–(d)] densities of the x , y , and z spin components for $J_2/J_1 = D/J_1 = J_K/J_1 = 0$ and $\alpha = 1.25$. N is the number of lattice sites. (e) Snapshots of the spin configuration at times 0, 500, and 1400 ps. (f) Magnetic phase diagram for $J_K/J_1 = 0$ and $0 \leq \alpha \leq 1.5$, showing ferromagnetic order for $J_2 > D$ and spiral order for $J_2 < D$. (g) Critical value of J_K/J_1 above which the system evolves to ferromagnetic order.

not touch each other, which is demonstrated in Fig. 3(b) for $\alpha = 0.9$. Note that a band gap opens not only for $D/J_1 = \alpha/9$ but for countless relations between D/J_1 and α .

Following Ref. [57], we define the topological phase (C_1, C_2, C_3) through the Chern numbers C_1, C_2 , and C_3 of the lower, middle, and upper magnon bands, see Figs. 4(a)–4(c). The phase diagram in the α - D/J_1 plane is shown in Fig. 4(d). We find one topologically trivial magnon insulator phase (0,0,0) and three magnon Chern insulators phases (0, 1, -1), (2, -1, -1), and (2, -2, 0). While α can open a band gap, the topologically trivial nature is maintained, indicating that the topological properties arise from the DM interaction. To analyze the behavior of κ_{xy} for the four phases, we study the points P1 = (0.5, 0.07), P2 = (0.2, 0.15), P3 = (0.7, 0.25), and P4 = (0.9, 0.20) as well as the rectangular path with

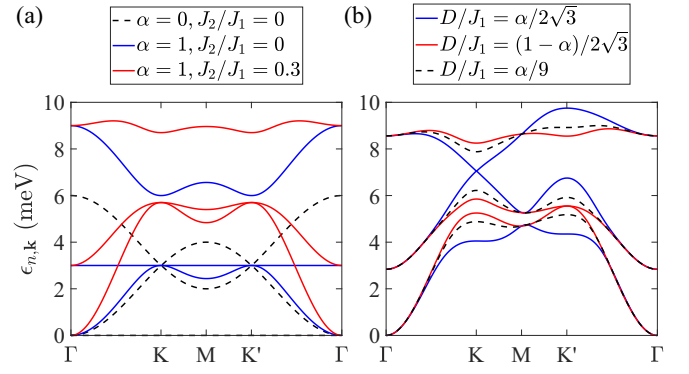


FIG. 3. (a) Magnon band structures at $D/J_1 = 0$ for the honeycomb ferromagnet at $J_2/J_1 = 0$ (black dashed lines) as well as the dice ferromagnet at $J_2/J_1 = 0$ (blue solid lines) and $J_2/J_1 = 0.3$ (red solid lines). (b) Magnon band structures at $J_2/J_1 = 0.3$ and $\alpha = 0.9$ for different values of D/J_1 . The high symmetry points are defined in Fig. 1(c).

corners $\lambda_1 = (0.32, 0.09)$, $\lambda_2 = (0.68, 0.09)$, $\lambda_3 = (0.68, 0.20)$, and $\lambda_4 = (0.32, 0.20)$, see Fig. 4(d).

C. Thermal transport properties

The electric Hall conductivity

$$-\frac{e^2}{h} \sum_n \int \frac{d^2\mathbf{k}}{4\pi^2} \Omega_{n,\mathbf{k}}^z f_F(\epsilon_{n,\mathbf{k}}) \quad (14)$$

of an electronic Chern insulator, where $f_F(\epsilon_{n,\mathbf{k}})$ is the Fermi distribution function, is quantized and linked to the Berry curvatures of the electronic bands below the Fermi energy. As the Fermi energy can be adjusted by doping to lie within the band gap, the topological nature therefore can be probed by measuring the electric Hall conductivity. In contrast, magnons

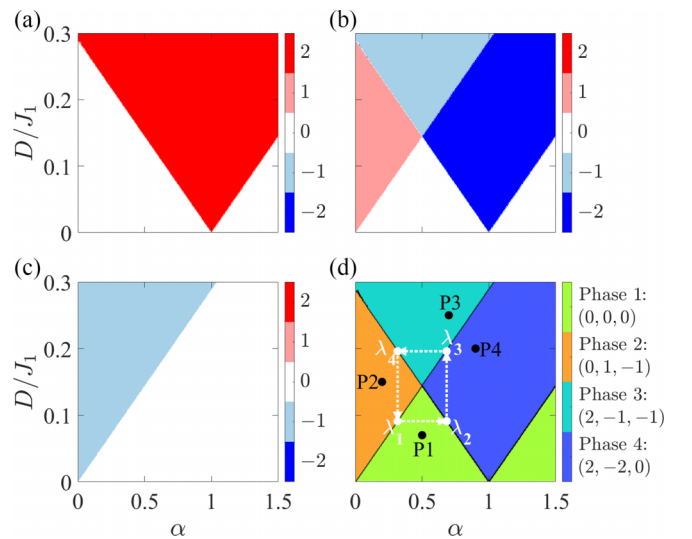


FIG. 4. Chern numbers of the (a) lower, (b) middle, and (c) upper magnon bands. (d) Phase diagram in the α - D/J_1 plane. The points P1-P4 and the rectangular path with corners λ_1 - λ_4 are chosen to analyze the thermal transport properties.

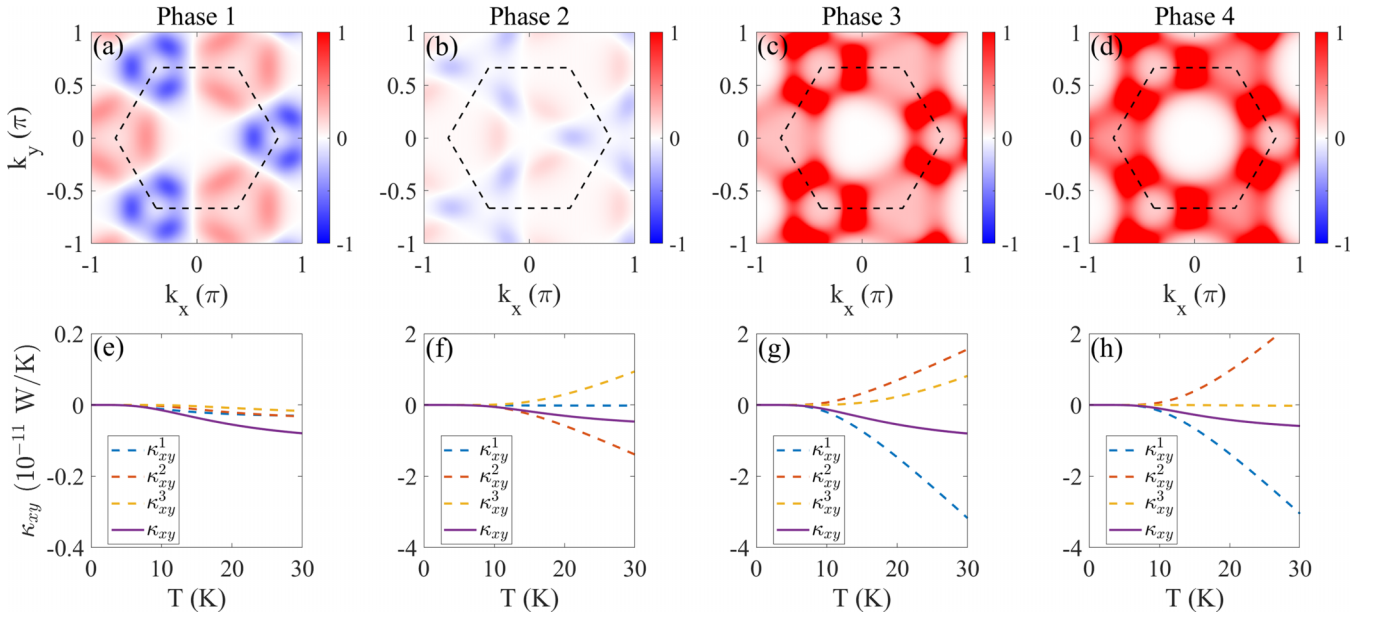


FIG. 5. [(a)–(d)] Berry curvature of the lowest magnon band and [(e)–(h)] magnon thermal Hall conductivity and band contributions as functions of the temperature. The black dashed hexagons in [(a)–(d)] represent the first Brillouin zone.

follow bosonic statistics and thus have no Fermi surface. Nevertheless, topological magnon phases can be explored by measuring κ_{xy} [106,107].

A necessary condition to obtain $\kappa_{xy} \neq 0$ is the absence of effective time-reversal symmetry, $\Omega_{n,\mathbf{k}}^z = -\Omega_{n,-\mathbf{k}}^z$ (combination of time-reversal symmetry and π rotation of the spin) [29]. In our case, we have neither effective time-reversal symmetry nor inversion symmetry due to $D \neq 0$ and therefore can expect $\kappa_{xy} \neq 0$. The magnon thermal conductivity tensor for the magnetic point group $3m'$ (compatible with ferromagnetism) has the form [108]

$$\kappa = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ -\kappa_{xy} & \kappa_{xx} & 0 \\ 0 & 0 & \kappa_{zz} \end{bmatrix}. \quad (15)$$

Note that magnon bands with $C_n = 0$ can result in $\kappa_{xy} \neq 0$ as long as there is no effective time-reversal symmetry and $\Omega_{n,\mathbf{k}}^z \neq 0$.

The Berry curvature of the lowest magnon band, $\Omega_{1,\mathbf{k}}^z$, is shown in Fig. 5 for the four phases together with κ_{xy}^n and κ_{xy} as functions of T . It changes in phases 1 and 2 between negative and positive values across the reciprocal space, resulting in $C_1 = 0$, but is always positive in phases 3 and 4, resulting in $C_1 > 0$. Both κ_{xy}^n and κ_{xy} change monotonously for increasing T . Due to Eq. (10), magnon bands with higher $|C_n|$ contribute more to κ_{xy} . $C_2 = -C_3$ in phase 2 implies that κ_{xy}^2 and κ_{xy}^3 have opposite signs, see Fig. 5(f), and $C_1 = 2$ in phases 3 and 4 results in identical κ_{xy}^1 , see Figs. 5(g) and 5(h). However, κ_{xy} turns out to be negative in all the phases. Furthermore, magnon bands with the same C_n can contribute differently to κ_{xy} . For instance, $C_2 = C_3 = -1$ in phase 3 but $|\kappa_{xy}^2| > |\kappa_{xy}^3|$, see Fig. 5(g). The reason is that Eq. (10) connects κ_{xy}^n to $c_2(\epsilon_{n,\mathbf{k}})$, which decreases for increasing $\epsilon_{n,\mathbf{k}}$ at given T , as shown in Fig. 6, thus acting similar to an activation function, that is, the contributions of higher magnon bands to κ_{xy} increase as T increases.

Figure 7(a) shows κ_{xy} at different T along the rectangular path defined in Fig. 4(d). The path crosses all phases. D/J_1 is constant and α varies from 0.32 to 0.68 in path sections $\lambda_1 \rightarrow \lambda_2$ and $\lambda_3 \rightarrow \lambda_4$, while α is constant and D/J_1 varies from 0.009 to 0.20 in path sections $\lambda_2 \rightarrow \lambda_3$ and $\lambda_4 \rightarrow \lambda_1$. We observe that $|\kappa_{xy}|$ is generally lowest in phase 1 and highest in phase 3. It also is generally higher in phase 3 than in phase 4 (though $C_1 = 2$ in both phases) due to the fact that $C_2 = -1$ in phase 3 results in less reduction than $C_2 = -2$ in phase 4. While κ_{xy} does not change sign at the topological phase boundaries, kinks are evident. They become more pronounced at higher T , as higher magnon bands become activated. Figures 7(b)–7(e) shows κ_{xy}^n and κ_{xy} in the α - D/J_1 plane at $T = 30$ K. κ_{xy}^n exhibits sharp changes at the topological phase boundaries in agreement with the changes of C_n , see Figs. 4(a)–4(c). The topological phase boundaries are less

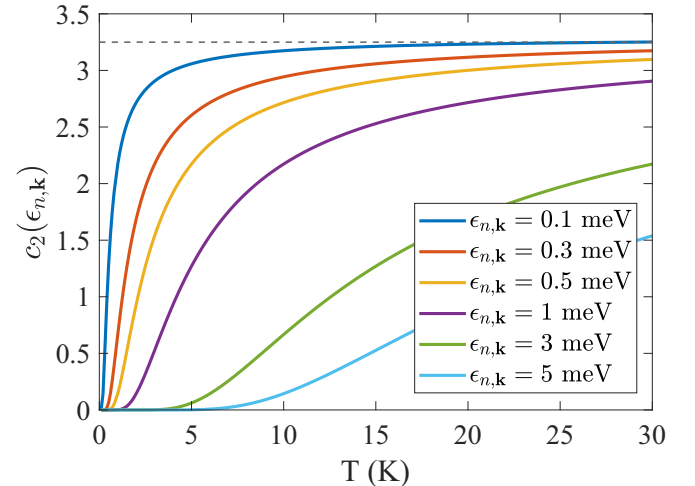


FIG. 6. $c_2(\epsilon_{n,\mathbf{k}})$ as a function of the temperature at different $\epsilon_{n,\mathbf{k}}$. The high temperature limit (dashed line) is $\pi^2/3$.

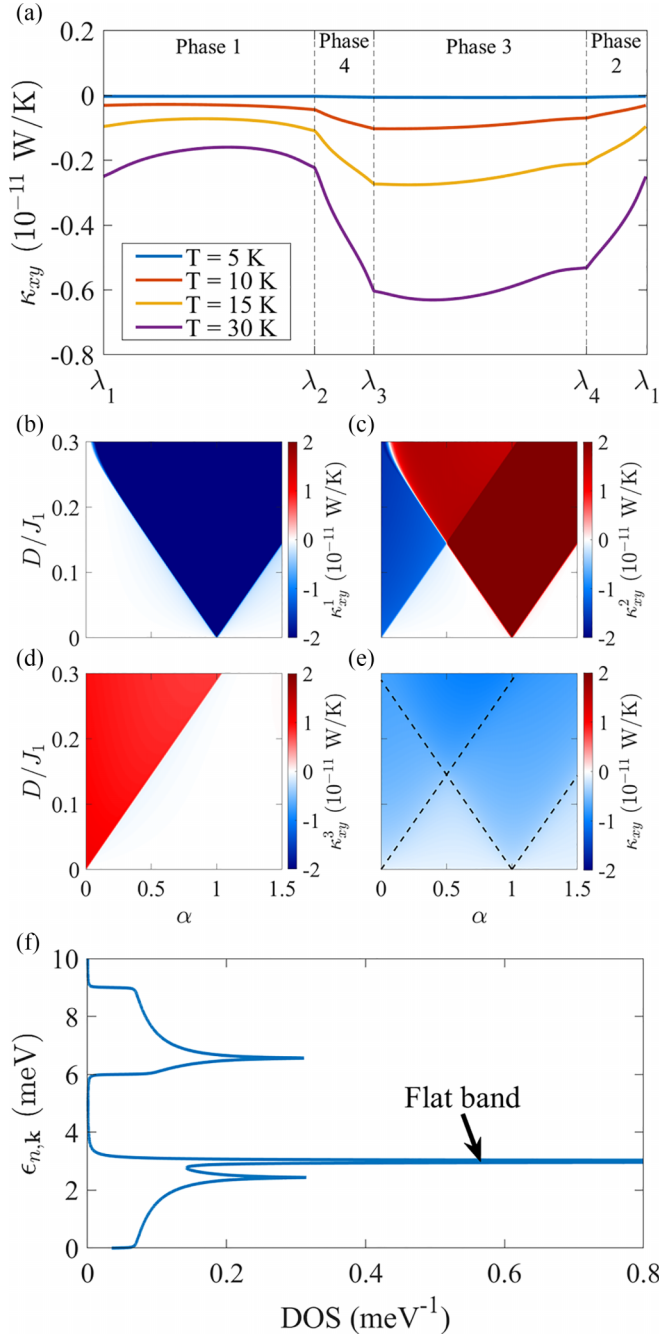


FIG. 7. (a) Magnon thermal Hall conductivity along the rectangular path defined in Figure 4(d) at different temperatures. Magnon thermal Hall conductivities of the (b) lower, (c) middle, and (d) upper magnon bands as well as (e) total magnon thermal Hall conductivity in the α - D/J_1 plane at $T = 30$ K. The black dashed lines in (e) mark the topological phase boundaries. (f) Magnon density of states for $J_2/J_1 = D/J_1 = 0$. The divergence at $\epsilon_{n,\mathbf{k}} = 3$ meV corresponds to the flat band in Fig. 3(a).

sharp in the case of κ_{xy} but still distinctively visible. At the flat band observed in Fig. 3(a) for $J_2/J_1 = D/J_1 = 0$ the density of states diverges, as illustrated in Fig. 7(f) with $\eta = 0.01J_1$. One may assume that the divergent density of states enhances the thermal transport, however, both κ_{xx}^2 and κ_{xy}^2 turn out to be zero, reflecting the vanishing group velocity.

IV. CONCLUSION

We have explored the thermal transport properties of magnons on the α - T_3 lattice. Atomistic spin dynamics simulations in the absence of DM interaction demonstrate always ferromagnetic order. The DM interaction promotes spiral order, while next-nearest neighbor hopping and easy-axis anisotropy promote ferromagnetic order. In the case of ferromagnetic order, we have identified one topologically trivial magnon insulator phase and three magnon Chern insulator phases. Magnon bands with higher $|C_n|$ contribute more to κ_{xy} than those with lower $|C_n|$, but even magnon bands with $C_n = 0$ give finite contributions. The topologically trivial magnon insulator phase exhibits the lowest $|\kappa_{xy}|$ and the magnon Chern insulator phase (2, -1, -1) with the highest $|C_1|$ and lowest $|C_2|$ exhibits the highest $|\kappa_{xy}|$. While κ_{xy} does not change sign at the topological phase boundaries, distinct changes are observed in the magnitude, which can be attributed to the magnon edge states present in the magnon Chern insulator phases.

In flat bands the quasiparticle interaction dominates over the kinetic energy, which may promote exotic many-body phenomena such as strongly correlated phases [109], as observed for the Dice lattice [64], and Bose-Einstein condensation in bosonic flat bands [110]. Our results for noninteracting magnons (at low temperature) show that flat bands do not contribute to κ_{xx} and κ_{xy} due to vanishing group velocities. In contrast, at high temperatures magnons are inherently strongly interacting bosons. For this situation, our results provide a foundation for future research on how strong interaction in bosonic flat bands influences the topology and possible strongly correlated phases.

A ferromagnet with α - T_3 lattice can be experimentally realized using a bilayer of the ferromagnetic honeycomb monolayers CrI_3 [111], VI_3 [112], CrSiTe_3 [113], CrGeTe_3 [114], or Fe_3GeTe_2 [115], arranged in a way that a lattice site of one monolayer aligns with the center of the honeycomb of the other monolayer. Another possibility is a trilayer structure of cubic ferromagnets, such as iron oxides [116], grown along the (111) direction. Nanomagnets [117–120] also offer a promising avenue, as the structure can be flexibly designed.

ACKNOWLEDGMENTS

The authors thanks A. S. Alzahrani for helpful discussions. This work was supported by King Abdullah University of Science and Technology.

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