

Non-Hermitian phase-biased Josephson junctions

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(Received 7 August 2023; revised 24 May 2024; accepted 16 October 2024; published 12 November 2024)

We study non-Hermitian Josephson junctions formed by superconductors with a finite phase difference under non-Hermiticity, which naturally appears due to coupling to normal reservoirs. Depending on the structure of non-Hermiticity, captured in terms of retarded self-energies, the low-energy spectrum hosts topologically stable exceptional points either at zero or finite real energies as a function of the superconducting phase difference. Interestingly, the corresponding phase-biased supercurrents may acquire divergent profiles at such exceptional points. This instance is a natural and unique non-Hermitian effect that signals a possible way to enhance the sensitivity of Josephson junctions. Our work opens a way for realizing unique non-Hermitian phenomena due to the interplay between non-Hermitian topology and the Josephson effect.

DOI: [10.1103/PhysRevB.110.L201403](https://doi.org/10.1103/PhysRevB.110.L201403)

Non-Hermitian (NH) systems have recently attracted enormous interest due to their potential for realizing unexpected states of matter that are impossible in the Hermitian realm [1–7]. NH systems exhibit intriguing properties, being the most relevant to their complex spectrum with degeneracies known as exceptional points (EPs), where eigenvalues and eigenvectors coalesce [8–17]. EPs have proven to be topological objects with topology defined in the complex spectrum, thus introducing the concept of point gaps unlike Hermitian topology [18–21]. Also, the NH spectrum is very sensitive to perturbations, causing strong responses at EPs and offering a way for sensing devices with no Hermitian analog [6,22–24].

NH physics has been shown to emerge in open systems [3,4], with well-established mechanisms in photonics and optics where non-Hermiticity can be controlled almost at will [12,15,16,25–29]. Another less explored but promising route is offered by material junctions where non-Hermiticity naturally arises when coupling a closed system to normal reservoirs [30]. These ideas have recently been explored in normal systems [31–35] and a few examples also exist in setups with superconductors [36–43]. While these initial studies have shed some light on the interplay between non-Hermiticity and superconductivity, they have overlooked the impact of non-Hermiticity on the phase of the superconducting order parameter.

The most straightforward systems where the superconducting phase is crucial are Josephson junctions (JJs), formed by two superconductors with their order parameters exhibiting a finite phase difference [44,45]. This enables the formation

of Andreev bound states (ABSs) and the transfer of Cooper pairs, giving rise to a dissipationless supercurrent across the JJ [46–52]. These properties make JJs excellent devices for revealing the type of superconductivity [53–63] and also essential superconducting building blocks for future quantum applications [64–79]. Thus, even though JJs offer a natural ground for NH physics, the response of ABSs and supercurrents to non-Hermiticity is still unknown, especially the response to the presence of EPs.

In this work, we consider NH phase-biased JJs formed by conventional superconductors coupled to normal leads (Fig. 1) and discover topologically robust EPs and EP-enhanced supercurrents fully controlled by the phase difference. We find that the ABSs reveal the formation of EPs at finite and zero real energies. Interestingly, their phase-dependent supercurrents become sharply enhanced at EPs, but only the EPs at zero real energy produce sizable signals in the total supercurrents. Our work thus demonstrates the potential of NH topology for realizing EPs and enhanced supercurrents in phase-biased JJs.

NH Josephson junctions. We consider JJs under non-Hermiticity induced by normal leads, as shown in Fig. 1. We

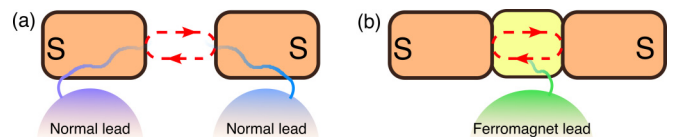


FIG. 1. (a) A NH JJ formed by coupling two superconductors (orange) without any mediating region to different normal leads. (b) A NH JJ formed by coupling a very short mediating normal region (yellow) to a ferromagnet lead. In both cases, the emerging ABSs (red dashed loop between superconductors) feel the presence of non-Hermiticity, indicated by shaded lines inside the orange and yellow regions.

model these JJs by an effective Hamiltonian given by

$$H_{\text{eff}} = H_{\text{JJ}} + \Sigma^r(\omega = 0), \quad (1)$$

where H_{JJ} describes the Hermitian JJ with superconductors (Ss) having a finite phase difference (ϕ) between their order parameters [80,81]. $\Sigma^r(\omega = 0)$ is the retarded zero-frequency self-energy originating from the normal (ferromagnet) leads, which renders the JJ NH, as derived in the Supplemental Material (SM) [82]. In the Hermitian regime, the spectrum of H_{JJ} acquires a strong dependence on ϕ and reveals the ABSs within the induced superconducting gap, producing a dissipationless supercurrent $I(\phi)$ [83]. NH effects arise from $\Sigma^r(\omega = 0)$ [30], which in general has real (Re) and imaginary (Im) parts, $\Sigma^r(\omega = 0) = \text{Re}\Sigma^r - i\text{Im}\Sigma^r$ [82]. While $\text{Re}\Sigma^r$ only causes a shift in the energies of H_{JJ} , $\text{Im}\Sigma^r$ makes H_{eff} NH, inducing changes in the spectrum that are entirely due to the NH mechanism.

Before going further, we stress the relation between ABSs and $I(\phi)$. The spectrum, and hence ABSs, can be obtained from $\det(\omega - H_{\text{eff}}) = 0$, which corresponds to the poles of the retarded Green's function [30,81,84] $G^r(\omega) = (\omega - H_{\text{eff}})^{-1}$, where H_{eff} is given by Eq. (1). Due to non-Hermiticity, the spectrum becomes complex with eigenvalues coming in pairs due to particle-hole symmetry [36–41], namely, $(E_n, -E_n^*)$, $E_n(\phi) = \text{Re}E_n(\phi) - i\text{Im}E_n(\phi)$. The Re part represents the quasiparticle energy while the Im part characterizes its lifetime [30]. Since the Josephson current is an equilibrium flow in the junction, the equilibrium spectrum of H_{eff} determines the Josephson current. Thus, in the same spirit as in Hermitian JJs, where the Andreev spectrum determines the supercurrent [47–49], it is natural here to define a supercurrent associated with the complex spectrum of NH JJs as [85] $I = -(e/\hbar) \sum_{n \geq 0} dE_n(\phi)/d\phi$, which leads to

$$I(\phi) = \frac{e}{\hbar} \sum_{n \leq 0} \left[\frac{d\text{Re}E_n(\phi)}{d\phi} - i \frac{d\text{Im}E_n(\phi)}{d\phi} \right], \quad (2)$$

where we take the summation over n for the occupied Andreev states with $\text{Re}E_n \leq 0$ [86]. As shown in the SM [82], the physical supercurrent is $\text{Re}I(\phi)$. Below, we consider two types of NH JJs with non-Hermiticity due to coupling to normal and ferromagnet leads and study EPs in the complex Andreev spectrum and their impact on the supercurrent $I(\phi)$.

NH JJ with superconductors coupled to different normal leads. To begin, we consider the NH JJ with single-site superconductors (Ss) coupled to distinct normal (N) leads, as shown in Fig. 1. Equation (1) with

$$H_{\text{JJ}}^{(1)} = \begin{pmatrix} H_L & V \\ V^\dagger & H_R \end{pmatrix}, \quad \Sigma^{(1)} = \text{diag}(\Sigma_L, \Sigma_R), \quad (3)$$

describes ABSs in this NH JJ, where $H_\alpha = \varepsilon_\alpha \tau_z + \text{Re}(\Delta_\alpha) \tau_x - \text{Im}(\Delta_\alpha) \tau_y$ for $\alpha = L$ (R) is the Bogoliubov–de Gennes Hamiltonian of the left (right) S with an s -wave pair potential $\Delta_\alpha = \Delta e^{i\phi_\alpha}$ and the on-site normal energy ε_α , while $V = t\tau_z$ is the coupling between Ss [87]. The self-energy term gives non-Hermiticity $\Sigma_{L(R)} = -i\Gamma_{L(R)}\tau_0$ [88], where the coupling of the left (right) S to the left (right) N lead determines the decay rate $\Gamma_{L(R)}$ [40–42]. (τ_i is the i th Pauli matrix in the Nambu space.) For $\varepsilon_\alpha = 0$, $\phi_L = 0$, and $\phi_R = \phi$,

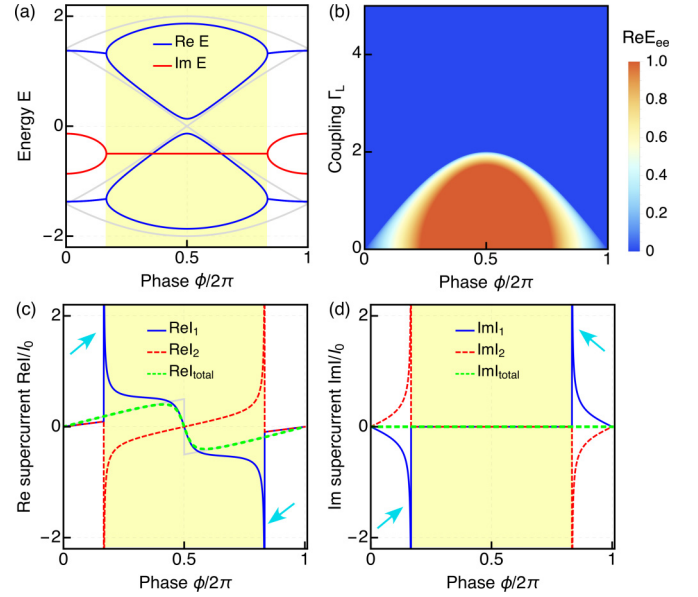


FIG. 2. (a) Re (blue) and Im (red) parts of the eigenvalues given by Eq. (4) as a function of ϕ . (b) Re part of the difference between the two positive eigenvalues as a function of Γ_L and ϕ , with the blue region borders marking the EPs. (c), (d) Total (green) and individual (blue and red) supercurrents $I(\phi)$ due to occupied ABSs as a function of ϕ . Cyan arrows indicate the divergence of $I(\phi)$ at the EPs. Gray curves in (a) and (c) correspond to eigenvalues and total $I(\phi)$ at $\Gamma_{L,R} = 0$. Parameters: $\Delta = 1$, $t = 1$, $\varepsilon_\alpha = 0$, $\Gamma_L = 1$, $\Gamma_R = 0$, $I_0 = e\Delta/\hbar$.

the eigenvalues of H_{eff} read

$$E_j = -i\Gamma \pm \sqrt{\delta^2 - \gamma^2 \pm 2\Delta\sqrt{t^2\sin^2(\phi/2) - \gamma^2}}, \quad (4)$$

where $\delta^2 = t^2 + \Delta^2$, $\gamma = (\Gamma_L - \Gamma_R)/2$, $\Gamma = (\Gamma_L + \Gamma_R)/2$, and $j = 1, \dots, 4$ labels the eigenvalues. Figure 2(a) shows the Re and Im parts of E_j as a function of ϕ without and with non-Hermiticity. For $\Delta = t$, E_j without non-Hermiticity shows level crossings at $\phi = 0, \pi$, which qualitatively reproduces the standard ABSs in transparent JJs [82]. Notably, once turning on $\gamma \neq 0$, the modes crossing at $\phi = 0$ coalesce, with their Re parts merging at finite energy, signaling the formation of EPs [3]. See yellow region ends in Fig. 2(a). From Eq. (4), EPs appear when $t^2\sin^2(\phi/2) = \gamma^2$, and thus, they exist even for small but nonzero γ . One can also confirm this feature in Fig. 2(b), which shows the Re part of the difference between the two positive eigenvalues ($\text{Re} E_{ee}$) as a function of γ and ϕ . The border of the blue region ($\text{Re} E_{ee} = 0$) marks the EPs, which persist for $|\gamma| < |t|$. In contrast, the modes crossing at $\phi = \pi$ do not evolve into EPs. The system has parity-time, spin-rotation, and particle-hole symmetries. An NH topology protected by these symmetries ensures the presence and absence of these EPs [82]; thus, the discussed EP feature is universal.

We next analyze the current-phase curve $I(\phi)$ for the ABSs in Eq. (4), obtained from Eq. (2). See Figs. 2(c) and 2(d). In the Hermitian regime, $I(\phi)$ exhibits a sawtooth profile at $\phi = \pi$ due to the zero-energy crossing in the ABSs, as the gray curve indicates. In the NH case, with $\gamma \neq 0$, the $\text{Re}I(\phi)$ and $\text{Im}I(\phi)$ of each ABS develop sharp

divergences at the EP since each ABS carries $I_{1(2)}(\phi) \sim \pm \sin(\phi)/\sqrt{t^2 \sin^2(\phi/2) - \gamma^2}$, whose denominator vanishes at the EPs [82]. See red and blue curves in (a) and (b), and cyan arrows. However, the divergences of $I_1(\phi)$ and $I_2(\phi)$ cancel out in the total $\text{Re}I(\phi)$, leading to a smooth curve across EPs with an overall sine-like profile, as shown by the green dotted curve [82]; similarly, the total $\text{Im}I(\phi)$ acquires vanishing values [Fig. 2(d)]. Nevertheless, it is worth noting that, even though no EP effect is evident in the total $I(\phi)$, the Hermitian sawtooth profile at $\phi = \pi$ smooths out by $\gamma \neq 0$. In summary, NH JJs can host tunable EPs at finite Re energies as a pure NH topological effect, but the total supercurrent is insensitive to such EPs.

NH JJ with a middle N region coupled to a ferromagnet lead. Here we consider spinful NH JJs with a short mediating N region coupled to a ferromagnet lead, as shown in Fig. 1(b). For simplicity, we focus on single-site S and N regions [89]. The NH JJ is then modeled by

$$H_{\text{JJ}}^{(2)} = \begin{pmatrix} H_L & V & 0 \\ V & H_N & V \\ 0 & V & H_R \end{pmatrix}, \quad \Sigma^{(2)} = \text{diag}(0, \Sigma, 0), \quad (5)$$

where $H_\alpha = \varepsilon_\alpha \tau_z + \text{Re}(\Delta_\alpha) \sigma_y \tau_y - \text{Im}(\Delta_\alpha) \sigma_y \tau_x$ for $\alpha = L$ (R) describes the left (right) S with an s -wave pair potential $\Delta_\alpha = \Delta e^{i\phi_\alpha}$, $H_N = \varepsilon_N \tau_z$ presents the N region, and $V = t\sigma_0 \tau_z$ is the hopping between N and S [90]. (σ_i and τ_i are the Pauli matrices in the spin and Nambu spaces.) The coupling between N and a ferromagnet lead induces a spin-dependent self-energy $\Sigma = \text{diag}(\Sigma_e^r, \Sigma_h^r)$ where $\Sigma_{e,h}^r(\omega=0) = -i\Gamma\sigma_0 - i\gamma\sigma_z$, $\Gamma = (\Gamma_\uparrow + \Gamma_\downarrow)/2$, $\gamma = (\Gamma_\uparrow - \Gamma_\downarrow)/2$; Γ_σ is the coupling of spin σ to the lead [91]. The eigenvalues of H_{eff} are not simple, but it is still possible to explore them as solutions of $\det(\omega - H_{\text{JJ}}^{(2)} - \Sigma^{(2)}) = 0$. Taking up to second order in ω , the solutions for $\varepsilon_\alpha = 0$, $\phi_L = 0$, $\phi_R = \phi$ read

$$E_\pm = -i\Gamma \frac{\Delta^2(2t^2 + \Delta^2)}{A} \pm \frac{\Delta}{A} \sqrt{C + 2t^4 A \cos\phi}, \quad (6)$$

where $A = 4t^4 + \Delta^4 + 2\Delta^2(-\gamma^2 + \Gamma^2 + 2t^2)$, and $C = 2t^4 A - 4t^2 \gamma^2 \Delta^4 + 2(\gamma^2 + \Gamma^2)^2 \Delta^4 - \gamma^2 \Delta^6$. From Eq. (6), we find that the ABSs are degenerate when $C + 2t^4 A \cos\phi = 0$. In the absence of non-Hermiticity ($\Gamma = \gamma = 0$), it holds that $C = 2t^4 A > 0$; thus, the degeneracy appears at $\phi = \pi$ as a level crossing at zero energy, as illustrated by gray curves in Fig. 3(a). Then, in the presence of non-Hermiticity, the degeneracy evolves into a pair of EPs at $\phi_{\text{EP}} = \pm \arccos(-C/2t^4 A)$ once we have $|C| < 2t^4 |A|$ by non-Hermiticity [92]. Intriguingly, the Re part of ABSs sticks to zero energy between the EPs, unlike the EPs in the previous model. We note that a weak but nonzero spin-dependent non-Hermiticity γ suffices to realize the EPs because C decreases as $C = 2t^2 A - 4t^2 \gamma^2 \Delta^4$ for $0 < |\gamma|, |\Gamma| \ll |t|, |\Delta|$ so to meet the condition $|C| < 2t^4 |A|$. The tunability of the EPs is also evident in Fig. 3(b), which shows the Re part of the difference between the eigenvalues $E_\pm$ as a function of $\Gamma_\uparrow$ and ϕ ; EPs appear at weak $\Gamma_\uparrow$. The border of the blue region marks the EPs. Contrary to the previous system, this JJ breaks parity-time and spin-rotation symmetries and only keeps particle-hole symmetry intrinsic

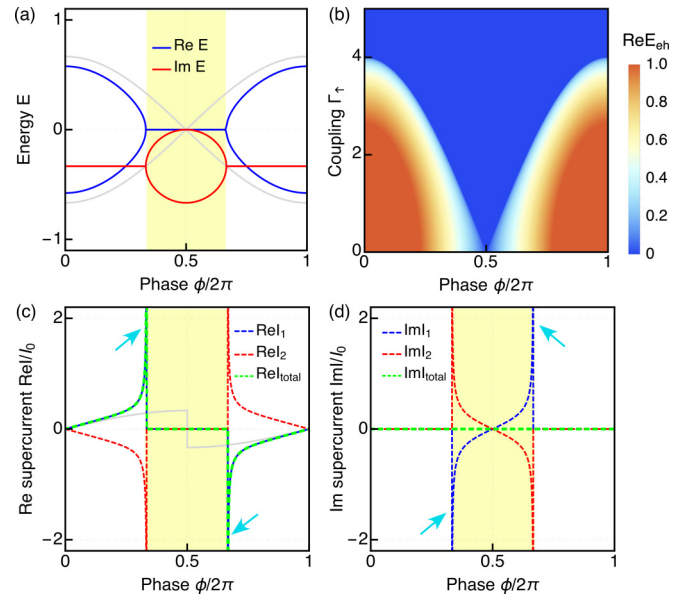


FIG. 3. (a) Re (blue) and Im (red) parts of the eigenvalues given by Eq. (6) as a function of ϕ . (b) Re part of the difference between the two lowest positive and lowest negative eigenvalues as a function of $\Gamma_\uparrow$ and ϕ , with the blue region borders marking the EPs. (c), (d) Total (green) and individual (blue and red) $I(\phi)$ due to both ABSs as a function of ϕ . Cyan arrows show the enhancement of $I(\phi)$ at the EPs. Gray curves in (a) and (c) correspond to eigenvalues and $I(\phi)$ at $\Gamma_{\uparrow,\downarrow} = 0$. Parameters: $\Delta = 1$, $t = 1$, $\varepsilon_\alpha = 0$, $\Gamma_\uparrow = 2$, $\Gamma_\downarrow = 0$, $I_0 = e\Delta/\hbar$.

to superconductors. This difference enables the topologically stable EPs at zero energy, as argued in the SM [82].

When it comes to the supercurrent $I(\phi)$ due to ABSs in Eq. (6), Figs. 3(c) and 3(d) show its Re and Im parts as a function of ϕ . In the Hermitian case, $I(\phi)$ develops a sawtooth profile at $\phi = \pi$ due to the zero-energy crossing in the ABSs; see the gray curve. In the NH regime, the Re and Im parts of $I(\phi)$ interestingly develop divergences at the EPs of the ABSs; see the cyan arrows. Here, $I(\phi) \sim \sin(\phi)/\sqrt{C + 2t^4 A \cos\phi}$, whose denominator thus vanishes at the EPs [82]. At first sight, this seems similar to $I(\phi)$ from EPs at finite Re energy in Figs. 2(c) and 2(d), but there is a key difference. In the present case, $\text{Re}I(\phi)$ is entirely determined by only the non-negative ABS, which then implies that the enhancement at the EPs could be robust in contrast to $I(\phi)$ from EPs at finite Re energy [93]. Moreover, while the total $\text{Im}I(\phi)$ vanishes because the two ABSs merging into a single level at zero Re energy (yellow region) necessarily involves taking both Im ABSs, the $\text{Re}I(\phi)$ intriguingly vanishes between EPs as a result of the zero-energy lines in the Re ABSs. The fact that $I(\phi)$ has robust enhancement at EPs fully controlled by ϕ reveals that this effect has a pure NH origin.

NH JJs in superconductor-semiconductor hybrids. Having found EPs and EP-enhanced $I(\phi)$ in simple NH JJs, here we show their realization in experimentally feasible NH Rashba JJs. The Hermitian part is modeled by $H_R = \xi_k \tau_z + iak\sigma_y \tau_z + B\sigma_x \tau_z + \Delta\sigma_y \tau_y$, where $\xi_k = (\hbar^2 k^2/2m - \mu)$ is the kinetic term with wave number k along x , μ is the chemical potential, α is the Rashba spin-orbit coupling (SOC), and Δ is

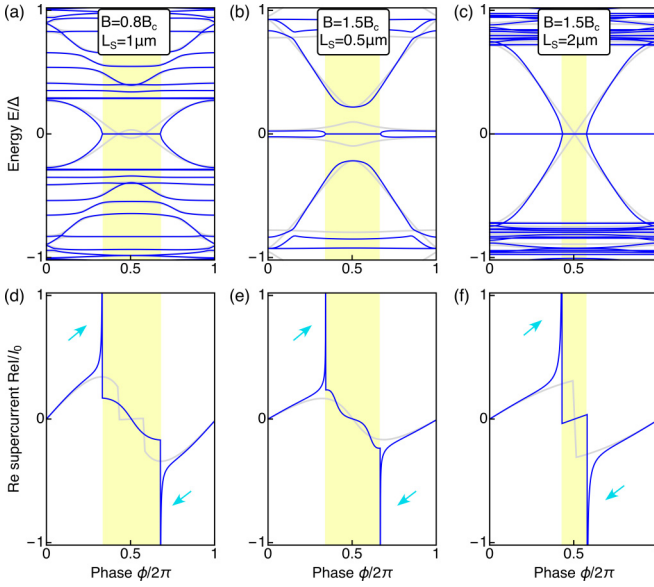


FIG. 4. (a)–(c) Re spectrum (blue) of a short NH Rashba JJ as a function of ϕ for different B and L_S . In (a) $\Gamma_{\uparrow} = 1.5$ meV, while for (b), (c) $\Gamma_{\uparrow} = 3.5$ meV. Gray curves correspond to $\Gamma_{\uparrow,\downarrow} = 0$. The yellow region ends mark the EPs at zero energy. (d)–(f) $\text{Re} I(\phi)$ for (a)–(c) as a function of ϕ with (blue) and without (gray) non-Hermiticity. Cyan arrows indicate the $\text{Re} I(\phi)$ at EPs. Parameters: $\alpha = 40$ meV nm, $\Delta = 0.5$ meV, $\mu_{N(S)} = 0.5$ meV, $L_N = 20$ nm, $\Gamma_{\downarrow} = 0$, $I_0 = e\Delta/\hbar$.

the spin-singlet s -wave pair potential. Since SOC is intrinsic in semiconductors, we consider it to be finite [94]. B is a Zeeman field along x inducing a Hermitian topological phase transition at $|B_c| = \sqrt{\Delta^2 + \mu^2}$ that predicts Majorana bound states (MBSs) [60–63,95–99]. Then, H_R is discretized into a tight-binding lattice with spacing $a = 10$ nm [100] and divided into three finite regions: The left and right S regions host a finite pair potential Δ with a finite phase difference ϕ , while the central N region has $\Delta = 0$; each region has length $L_{N,S}$ and the chemical potential $\mu_{N,S}$. We take realistic parameters: $\alpha_R = 40$ meV nm and $\Delta = 0.5$ meV, in the range of values for InSb and InAs nanowires, and Nb and Al Ss [60]. We finally model a short NH JJ by coupling the middle N region to a ferromagnet lead as in Eq. (5) [Fig. 1(b)], and obtain the Andreev spectrum and $I(\phi)$.

In Fig. 4, we present the Re Andreev spectrum and $\text{Re} I(\phi)$ as a function of ϕ . For $B < B_c$ in the Hermitian regime, the JJ develops zero-energy crossings, which evolve into stable zero-energy lines at finite spin-dependent non-Hermiticity $\gamma \neq 0$ whose ends mark the presence of EPs. See the gray and blue curves and also the yellow region in Fig. 4(a). These EPs do not depend on the length of the S regions [101]. Above B_c , the Hermitian JJ hosts four (nearly zero energy) MBSs at $\phi = \pi$ (two at the inner side of the junction and two at the outer ends) but only two at $\phi = 0$ (two at the outer), as revealed in the gray curves in (b) or (c) [100,102–105]. Interestingly, for the short S case $L_S < 2\xi_M$, with ξ_M the Majorana localization length, a finite amount of non-Hermiticity induces EPs between lowest positive and highest negative ABSs (outer MBSs) and forms a zero-energy line between them [Fig. 4(b)]. For $L_S > 2\xi_M$, the

outer MBSs acquire zero energy for all ϕ and the inner MBSs develop a zero-energy crossing at $\phi = \pi$ in the Hermitian regime (gray curves), which, at $\gamma \neq 0$, transforms into a pair of EPs between inner MBSs connected by a stable zero-energy line [Fig. 4(c)]. This EP regime is analogous to the case found in Fig. 3(a).

In Figs. 4(e) and 4(f), we show $I(\phi)$ of Eq. (2) for the Andreev spectrum in Figs. 4(a)–4(c), which includes the contribution from ABSs and the quasicontinuum. In all the cases, we find divergent profiles at EPs, which remain robust even under the presence of a phase-dependent quasicontinuum, as indicated by cyan arrows in Figs. 4(d)–4(f). Note that EPs between inner MBSs produce larger supercurrents for $L_S > 2\xi_M$ since the quasicontinuum is weakly dependent on ϕ so the MBSs carry almost all the contribution. As argued above, this surprising behavior, the enhancement of $\text{Re} I(\phi)$ at EPs, originates from the anomalous contributions of ABSs occurring at zero real energy and represents a unique and robust NH effect [106].

Impact of the imaginary energy on the supercurrent. We have shown that the intrinsic contribution of EPs to the Josephson current comes from the Re energy and develops a divergent profile. While this aspect reveals the intrinsic EP contribution, the Im part of the energy results in the band broadening and obscures each energy band's distinction. In particular, the band broadening mixes an occupied state with a negative real energy and its particle-hole partner empty state with a positive real energy near the exceptional point. Since an occupied state and its particle-hole partner contribute to the Josephson current with an opposite sign, the mixing suppresses the divergence in the Josephson current. Still, we can show that the exceptional point enhances the Josephson current. To show the enhancement, let us consider a typical occupied Andreev state near an EP,

$$E_n(\phi) = -\sqrt{s(\phi)} - i\Gamma, \quad (7)$$

where $s(\phi)$ is given by

$$s(\phi) = a^2 \cos^2(\phi/2) - b\gamma^2 \quad (a > 0, b \geq 0). \quad (8)$$

Here a has the same dimension as γ and $b \geq 0$ is dimensionless. At $\phi = \phi_0$ with $s(\phi_0) = 0$, $E_n(\phi)$ degenerates with the particle-hole partner $-E_n^*(\phi)$. For $b = 0$, the degeneracy is a simple level crossing, while for $b > 0$, the degeneracy is an EP with a branch cut of square root. Because the mixing becomes significant when $|\text{Re} E_n(\phi)|$ is comparable with the imaginary part $|\text{Im} E_n(\phi)| = \Gamma$, the current growth near the exceptional point terminates at ϕ_1 satisfying $\sqrt{s(\phi_1)} = \Gamma$. Thus, for $\gamma, \Gamma \ll a$, we can estimate the maximal current as

$$I(\phi)|_{\text{max}} \sim \frac{e}{\hbar} \frac{d\text{Re} E_n(\phi)}{d\phi} \Big|_{\phi=\phi_1} = \frac{e}{\hbar} \frac{a}{2} \sqrt{1 + b(\gamma/\Gamma)^2}, \quad (9)$$

where $\phi_1 = \pi - (2\Gamma/a)\sqrt{1 + b(\gamma/\Gamma)^2}$ [107]. We compare this estimation with the current without the EP. Since we can obtain the latter current by putting $b = 0$ in Eq. (9), we conclude that the exceptional point enhances the Josephson current by a factor of $\sqrt{1 + b(\gamma/\Gamma)^2}$. We confirm this enhancement numerically in Sec. S7 of Ref. [82].

In conclusion, we studied non-Hermitian phase-biased Josephson junctions and demonstrated the formation of

exceptional points at finite and zero real energies in the phase-dependent Andreev spectrum. We discovered that the Andreev exceptional points at zero real energy give rise to drastically enhanced phase-dependent supercurrents, thus boosting the sensing capability of Josephson junctions. Our findings hold experimental relevance because similar non-Hermitian junctions have already been fabricated, including Josephson junctions based on superconductor-semiconductor hybrids [53–55,58,59,108–113] and a few Kitaev chains [114,115], where coupling to normal leads seems to be a feasible task. Hence, these recent works place our findings within experimental reach. Our work thus opens a route for realizing exceptional points and enhanced supercurrents entirely by the interplay of non-Hermitian topology and the Josephson effect.

Note added. Recently the preprint Ref. [116] was posted online, partially overlapping with our results but using a

different method and distinct setups. We also note that a recent calculation by Shen *et al.* shows a similar enhancement of the Josephson currents by EPs [117], consistent with our results as we demonstrate in Secs. 5, 6, and 7 of Ref. [82].

Acknowledgments. We thank R. Aguado and Y. Tanaka for their insightful discussions. J.C. acknowledges financial support from the Swedish Research Council (Vetenskapsrådet Grant No. 2021-04121), the Royal Swedish Academy of Sciences (Grant No. PH2022-0003), the Carl Trygger Foundation (Grant No. 22:2093), and the Japan Society for the Promotion of Science via the International Research Fellow Program. M.S. was supported by JST CREST Grant No. JPMJCR19T2. The computations were enabled by resources provided by the National Academic Infrastructure for Supercomputing in Sweden (NAISS), partially funded by the Swedish Research Council through Grant Agreement No. 2022-06725.

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- [83] Hermitian JJs formed by conventional superconductors exhibit $I(\phi) \sim \sin(\phi)$ [44,51] but more exotic ABSs can induce anomalous $I(\phi)$ [120–130]. We also note that Ss forming JJs can be coupled directly or by a mediating region [Figs. 1(a) and 1(b)]: when the length of the mediating region is shorter than the superconducting coherence length, only a pair of ABSs appears within the gap which then fully determines $I(\phi)$ [49].
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- [87] The parameter t controls the transmission across the junction, enabling reaching vanishing transmission and transmission equal to 1 which, respectively, characterize the tunnel and fully transparent regimes [30,49,80].
- [88] Different couplings $\Gamma_{L(R)}$ give a finite asymmetry in the amount of non-Hermiticity, shown before to be the key for inducing EPs [40,41].
- [89] JJs with a very small N region belong to the short junction regime where the length of the N region is smaller than the superconducting coherence length [49]. The same applies to the JJ studied in Eq. (3).
- [90] Here, t controls the transmission across the junction [30,49,80].
- [91] The choice of non-Hermiticity here is different in comparison to the previous section, where the asymmetry therein originated from coupling Ss to different N leads.
- [92] We have verified that the corresponding wave functions also coalesce at these EPs.
- [93] While reaching the EPs in real systems might be challenging, being close to them already induces a drastically enhanced $I(\phi)$.
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