

Topological edge states of massless fermions with nonquantized and zero Berry phases

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We investigate the bulk-boundary correspondence for massless Dirac fermions in α - T_3 lattice where the Berry phase can be continuously tuned from π (graphene) to 0 (T_3 or dice lattice) without modifying the energy dispersion. The topological origin of edge states is revealed by the unitary transformation of the Hamiltonian, which maps an α - T_3 zigzag ribbon (ZR) onto a stub Su-Schrieffer-Heeger (SSH) chain. In the Majorana representation of the eigenstates, the topological invariant is manifested by the azimuthal winding number on the Bloch sphere. Particularly, T_3 ZR exemplifies a topologically nontrivial system with zero Berry phase. Contrary to conventional topological insulators, the gap closing in the corresponding stub SSH chain is not accompanied by a topological phase transition.

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Topological matters host robust edge states [1] that are promising for applications in quantum computing and efficient electronic devices [2]. In the Chern insulators [3,4], time-reversal symmetry (TRS) is broken and the number of edge states is given by $\mathbb{Z}$ invariants. In two-dimensional (2D) systems with preserved TRS, $\mathbb{Z}_2$ invariants [5–7] differentiate between topological insulators (TIs) and trivial insulators. The simplest model for TI is the Su-Schrieffer-Heeger (SSH) chain of polyacetylene [8], in which the ratio between the intracell and intercell hopping integrals determines the Berry phase Γ in a 1D Brillouin zone (BZ), also known as the Zak phase [9]. Here, the presence and absence of edge states are indicated by $\Gamma = \pi$ and 0, respectively [10]. The existence of edge states in a graphene zigzag ribbon (ZR) can also be inferred from the Zak phase [10]. By fixing a proper gauge for the Hamiltonian, graphene is mapped onto the SSH model [10–14]. The mapping establishes a relation between the conserved momentum of graphene ZR and the hopping parameter of the SSH chain, and therefore, the system is reduced from 2D to 1D.

Consider a honeycomb lattice comprising A and B sites with a hopping parameter $t_A > 0$. The α - T_3 lattice [15] is constructed by connecting the B sites to the additional C sites at the center of each hexagon with a hopping parameter $t_C = \alpha t_A$ for $\alpha \in [0, 1]$, as depicted in Fig. 1(a). Here, Γ varies continuously between π (graphene, pseudospin $S = 1/2$) and 0 (T_3 lattice, $S = 1$) for $\alpha = 0$ and 1, respectively, without

altering the energy bands when the energy is scaled by $t = \sqrt{t_A^2 + t_C^2}$. A question that naturally arises from this situation is whether the topological edge states remain without the quantized Γ . However, the bulk-boundary correspondence is yet to be formulated, even though the α - T_3 lattice has been extensively studied due to its unconventional properties, e.g., tunable orbital susceptibilities from dia- to paramagnetism [15], enhanced Klein tunneling [16–18], and large Chern numbers equal to ± 2 when the TRS is broken [19–22]. The α - T_3 Hamiltonian can be employed to describe the optical properties of 2D $\text{Hg}_{1-x}\text{Cd}_x\text{Te}$ [23,24] at $x \approx 0.17$. Furthermore, the T_3 lattice is predicted to occur in perovskite-based heterostructures [19,25] and strained blue-phosphorene oxide [26].

In this Letter, an example of α - T_3 ZR [Fig. 1(a)] is mapped onto a stub SSH chain [27,28] [Fig. 1(b)] by a unitary transformation of the Hamiltonian. In the Majorana representation of the eigenstates [29–31], the topological invariant is defined from the azimuthal winding number on the Bloch sphere [27]. Thus, we find that the origin of edge states in the α - T_3 ZR can be understood from the topological phase of the stub SSH chain. In particular, our result shows that T_3 ZR is topologically nontrivial despite $\Gamma = 0$. In contrast to the conventional TIs, the corresponding stub SSH chain is not subject to a topological phase transition by band gap closing.

The bulk Hamiltonian of the α - T_3 lattice in the momentum space $\mathbf{k} = (k_x, k_y)$ is given by

$$h_{\mathbf{k}} = -t \begin{pmatrix} 0 & \cos \vartheta f_{\mathbf{k}} & 0 \\ \cos \vartheta f_{\mathbf{k}}^* & 0 & \sin \vartheta f_{\mathbf{k}} \\ 0 & \sin \vartheta f_{\mathbf{k}}^* & 0 \end{pmatrix}, \quad (1)$$

where $\vartheta = \tan^{-1} \alpha$, and $f_{\mathbf{k}} = \sum_{\ell=1}^3 e^{-i\mathbf{k} \cdot \mathbf{b}_{\ell}} = |f_{\mathbf{k}}| e^{-i\varphi_{\mathbf{k}}}$. $\mathbf{b}_{\ell}$'s are the nearest-neighbor vectors. The eigenvalues of $h_{\mathbf{k}}$ include the energy bands of graphene $\varepsilon_{s,\mathbf{k}} = st|f_{\mathbf{k}}|$, where

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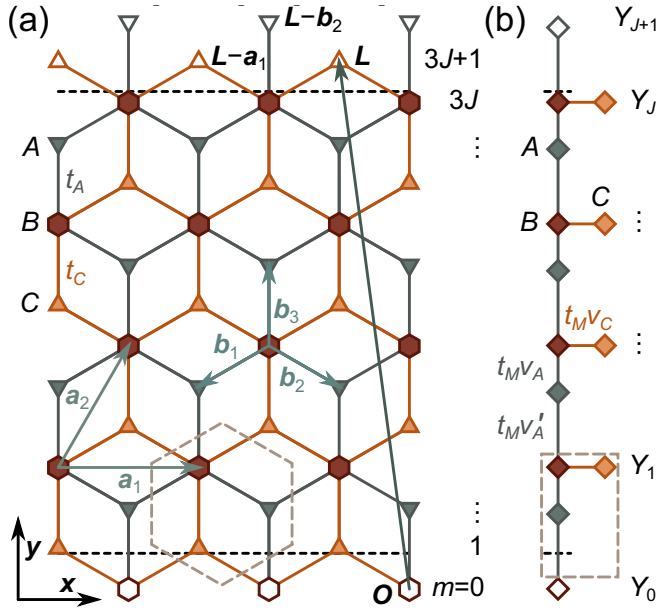


FIG. 1. (a) The α - T_3 ZR, with periodic (open) boundary conditions along the x (y) axis. The open symbols represent the missing sites [similarly for (b)]. The hopping parameter between the B and A (C) sites is t_A (t_C), where $t_C/t_A = \alpha \in [0, 1]$. The dashed hexagon is the unit cell. $\mathbf{b}_1 = -a(1/2, 1/2\sqrt{3})$, $\mathbf{b}_2 = a(1/2, -1/2\sqrt{3})$, and $\mathbf{b}_3 = a(0, 1/\sqrt{3})$ are the nearest-neighbor vectors. $\mathbf{a}_1 = a(1, 0)$ and $\mathbf{a}_2 = a(1/2, \sqrt{3}/2)$ are the primitive vectors. $m = 1, 2, \dots, 3J$ indicates the position of site in the y axis, where J is the number of trimers. $\mathbf{L} = (L_x, L_y)$ connects the missing B and C sites at $m = 0$ and $m = 3J + 1$, respectively, where $L_y = \sqrt{3}(J + 1)a/2 - a/\sqrt{3}$. (b) The stub SSH chain. $t_M v_A$ ($t_M v_A'$) is the intracell (intercell) hopping parameter between the B and A sites. The hopping parameter between the B and C sites is $t_M v_C$. The position of the unit cell (dashed rectangle) is $Y_j = j\sqrt{3}a/2$.

$s = -1$ ($+1$) corresponds to the valence (conduction) band, and a flat band at zero energy $\varepsilon_{0,k} = 0$.

By using the eigenstates $|\psi_{c,k}\rangle$ of h_k for the band $c \in \{s, 0\}$, the Berry phase $\Gamma_c = i \oint d\mathbf{k} \cdot \langle \psi_{c,k} | \nabla_{\mathbf{k}} | \psi_{c,k} \rangle$ is analytically calculated by a contour integral along a path enclosing each Dirac point [32]. The results are $\Gamma_{s,\tau} = \pi\tau \cos(2\vartheta)$ and $\Gamma_{0,\tau} = -2\pi\tau \cos(2\vartheta)$, where $\tau = +1$ (-1) for the K (K') valley [15]. Thus, Γ_c 's are not quantized to π or 0 and differ for each valley for $\alpha \in (0, 1)$, because except for $\alpha \in \{0, 1\}$ the α - T_3 lattice does not possess inversion symmetry that guarantees the quantization of Γ_c 's [9].

The α - T_3 ZR in Fig. 1(a) is constructed by translating the unit cell of the bulk lattice (dashed hexagon) along the x direction with periodicity $\mathbf{a}_1$. The width of the ZR is specified by J , which is the number of trimers along the y axis. The vector $\mathbf{L}$ connects the missing B and C sites at $m = 0$ and $m = 3J + 1$, respectively. The ZR is characterized by the boundary conditions imposed on the wave function $\psi_s = (\psi_s^A \ \psi_s^B \ \psi_s^C)^T$ in the real space $\mathbf{r} = (x, y)$. $\psi_s(\mathbf{r})$ is given by a linear combination of the Bloch state $\psi_{s,k}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} |\psi_{s,k}\rangle$. In Fig. 1(a), the periodic and open boundary conditions along the x and y axes implies

$$\psi_s^B(\mathbf{O} + l\mathbf{a}_1) = 0, \quad (2)$$

$$\alpha[\psi_s^C(\mathbf{L} + l\mathbf{a}_1) + \psi_s^C(\mathbf{L} + (l-1)\mathbf{a}_1)] + \psi_s^A(\mathbf{L} - \mathbf{b}_2 + l\mathbf{a}_1) = 0, \quad (3)$$

where $\mathbf{O} = (0, 0)$ is the origin and $l \in \mathbb{Z}$. Equation (2) describes the missing B sites at $m = 0$. Equation (3) describes the terminations of bonds connecting the B sites at $m = 3J$ to the nearest A and C sites outside the ZR. For the bulk states, we adopt $\psi_s(\mathbf{r}) = \mathcal{A}\psi_{s,k}(\mathbf{r}) + \mathcal{A}'\psi_{s,k'}(\mathbf{r})$ [10]. Equation (2) yields $\mathcal{A}' = -\mathcal{A}$, and $\mathbf{k} \cdot \mathbf{a}_1 = \mathbf{k}' \cdot \mathbf{a}_1$ due to the conservation of k_x . The conservation of energy $\varepsilon_k = \varepsilon_{k'}$ gives $\mathbf{k} \cdot \mathbf{a}_1 = \mathbf{k}' \cdot \mathbf{a}_1 = (\mathbf{k} + \mathbf{k}') \cdot \mathbf{a}_2$, thus $\mathbf{k}' = (k_x, -k_y)$.

Now let us define a unitary matrix

$$M_k = \begin{pmatrix} e^{ik \cdot (\mathbf{b}_2 - \mathbf{a}_1/2)} \cos \gamma & 0 & e^{-ik \cdot \mathbf{b}_3} \sin \gamma \\ 0 & 1 & 0 \\ -e^{ik \cdot (\mathbf{b}_2 - \mathbf{a}_1/2)} \sin \gamma & 0 & e^{-ik \cdot \mathbf{b}_3} \cos \gamma \end{pmatrix}. \quad (4)$$

By setting $\gamma = \tan^{-1}(\alpha\beta)$ with $\beta = 2\cos(k_x a/2) \geq 0$ for $k_x \in [-\pi/a, \pi/a]$, h_k is transformed to the bulk Hamiltonian of the stub SSH chain in Fig. 1(b) as follows,

$$H_k = M_k h_k M_k^\dagger = -t_M \begin{pmatrix} 0 & \mathcal{F}_k & 0 \\ \mathcal{F}_k^* & 0 & v_C \\ 0 & v_C & 0 \end{pmatrix}, \quad (5)$$

where $t_M = t_A \cos \gamma$ and $\mathcal{F}_k = v_A + v_A' e^{-i\sqrt{3}k_y a/2} = |\mathcal{F}_k| e^{-i\Phi_k}$. In each unit cell (dashed rectangle), the positions of all sites in the y axis are identically given by $Y_j = j\sqrt{3}a/2$. $v_A = \beta(1 + \alpha^2)$ ($v_A' = 1 + \alpha^2\beta^2$) is the intracell (intercell) coupling between the B and A sites. $v_C = \alpha(1 - \beta^2)$ is the coupling between the B and C sites. Thus, the unitary transformation reduces the α - T_3 ZR to the stub SSH chain by mapping k_x onto the hopping parameters. The eigenvalues of H_k are $E_{s,k} = st_M \sqrt{|\mathcal{F}_k|^2 + v_C^2}$ and $E_{0,k} = 0$. By defining $\Theta_k = \tan^{-1}(v_C/|\mathcal{F}_k|)$, the eigenstates are given by

$$|\Psi_{s,k}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \Theta_k e^{-i\Phi_k} \\ s \\ \sin \Theta_k \end{pmatrix}, \quad |\Psi_{0,k}\rangle = \begin{pmatrix} \sin \Theta_k e^{-i\Phi_k} \\ 0 \\ -\cos \Theta_k \end{pmatrix}. \quad (6)$$

The stub SSH chain belongs to the BDI class [33] according to the “tenfold way” classification of TIs and superconductors [34], same as the SSH chain. For $v_C \neq 0$, on the other hand, there is no gap closing at $|k_y| = 2\pi/\sqrt{3}a$ to indicate a topological phase transition when $v_A/v_A' = 1$. By denoting Φ_k at $k_y = \pm 2\pi/\sqrt{3}a$ as $\Phi_\pm$, note that

$$\Phi_\pm = \begin{cases} \pm\pi & \text{for } v_A/v_A' < 1, \\ 0 & \text{for } v_A/v_A' > 1, \end{cases} \quad (7)$$

regardless of v_C . Equation (7) will be employed to evaluate the topological invariant for the stub SSH chain.

The wave function of the chain is denoted by $\Psi_s = (\Psi_s^A \ \Psi_s^B \ \Psi_s^C)^T$. For the open boundary conditions in Fig. 1(b), the missing B (A) site at Y_0 (Y_{J+1}) necessitates

$$\Psi_s^B(Y_0) = 0, \quad (8)$$

$$\Psi_s^A(Y_{J+1}) = 0. \quad (9)$$

The equivalence between the α - T_3 ZR in Fig. 1(a) and the stub SSH chain in Fig. 1(b) is established from the boundary conditions. It can be shown that Eqs. (8) and (9) are identical to Eqs. (2) and (3), respectively, because $|\Psi_{s,k}\rangle = M_k|\psi_{s,k}\rangle$. Using $\Psi_s(y) = \Psi_{s,k}(y) - \Psi_{s,k'}(y)$, where $\Psi_{s,k}(y) = e^{iky}|\Psi_{s,k}\rangle$, the quantization condition of k_y is derived from Eq. (9) as follows:

$$\sqrt{3}(J+1)k_y a/2 - \Phi_k = n\pi, \quad \text{for } n = 1, 2, \dots, J. \quad (10)$$

Equation (10) is solved graphically along $k_y \in (0, 2\pi/\sqrt{3}a)$ because $\Psi_s(y) = 0$ for $k_y \in \{0, 2\pi/\sqrt{3}a\}$. The missing solutions correspond to the emergence of edge states [10] when

$$v_A/v'_A < 1 - 1/(J+1). \quad (11)$$

The ratio $(v_A/v'_A)_J = 1 - 1/(J+1)$ is the critical value at which the transition of bulk to edge states occurs in a finite system [10]. It is noted that Eq. (10) also holds for the SSH chain because Φ_k does not depend on v_C . Nevertheless, the inversion symmetry is broken for $v_C \neq 0$ [27]. Subsequently, the presence or absence of edge states cannot be predicted from the Zak phase, which is no longer quantized to π or 0 [27,35].

The existence of edge states in the k_x space of the α - T_3 ZR can be straightforwardly determined from Eq. (11). By inserting the β dependences of v_A and v'_A , we obtain

$$\alpha^2 \beta^2 - \frac{1 + \alpha^2}{1 - (J+1)^{-1}} \beta + 1 > 0. \quad (12)$$

For $\alpha \neq 0$, the solutions of Eq. (12) are $\beta_J^+ < \beta \leq 2$ and $0 \leq \beta < \beta_J^-$, where $\beta_J^\pm = 2 \cos(\chi_J^\pm a/2)$. Here, $\chi_J^\pm$ are the critical values of $|k_x| \in [0, \pi/a]$ at which the bulk states undergo transitions to edge states appearing in the range

$$0 \leq |k_x| < \chi_J^+ \quad \text{and} \quad \chi_J^- < |k_x| \leq \pi/a. \quad (13)$$

In the thermodynamic limit $J \rightarrow \infty$, $\chi_J^\pm$ converge to

$$\chi_\infty^+ \sim (2/a) \cos^{-1}(1/2\alpha^2), \quad \chi_\infty^- \sim 2\pi/3a. \quad (14)$$

For $\alpha \in (0, 1/\sqrt{2})$, the range of existence for the edge states is identical to that of the graphene ZR [10,36,37], because χ_∞^+ is undefined. The edge states in the T_3 ZR ($\alpha = 1$) exist at all $|k_x|$ excluding $\chi_\infty^+ = \chi_\infty^- = 2\pi/3a$.

The energy of the edge states $\tilde{\epsilon}_s$ is deduced as follows. In the SSH chain, $\tilde{E}_s \sim 0$ due to the chiral and inversion symmetries [10,36]. Accordingly, for the stub SSH chain ($v_C \neq 0$), we get $\tilde{E}_s \sim st_M|v_C|$, where the state $s = -1$ (+1) is analogous to the bonding (antibonding) molecular orbital. Thus, the edge state for each s is localized only on the edge that terminates at the B site [top edge in Fig. 1(b)]. Since the eigenvalues are invariant under the unitary transformation, $\tilde{\epsilon}_s = \tilde{E}_s = st_C \cos \gamma |1 - \beta^2|$. Thus, the edge states in the α - T_3 ZR are dispersive due to the dependence on β . At $|k_x| \approx 2\pi/3a$ and $|k_x| = \pi/a$, we get $\tilde{\epsilon}_s \sim 0$ and $\tilde{\epsilon}_s = st_C$, respectively.

Figure 2(a) shows the energy spectra of the ZR as a function of k_x for $\alpha = 0.8$ and $J = 40$. In addition to the dispersive bands, the spectra include a J -fold degenerate flat band at $\epsilon = 0$. The bold red lines depict the edge states, where $\chi_{40}^+ \approx 0.3434\pi/a$ and $\chi_{40}^- \approx 0.6991\pi/a$. The vertical dashed and dotted lines mark $\chi_\infty^+ \approx 0.4292\pi/a$ and $\chi_\infty^- = 2\pi/3a$, respectively. Figures 2(b)–2(e) show the probability density $|\psi|^2$ per site m for the in-gap state at k_x 's indicated in Fig. 2(a).

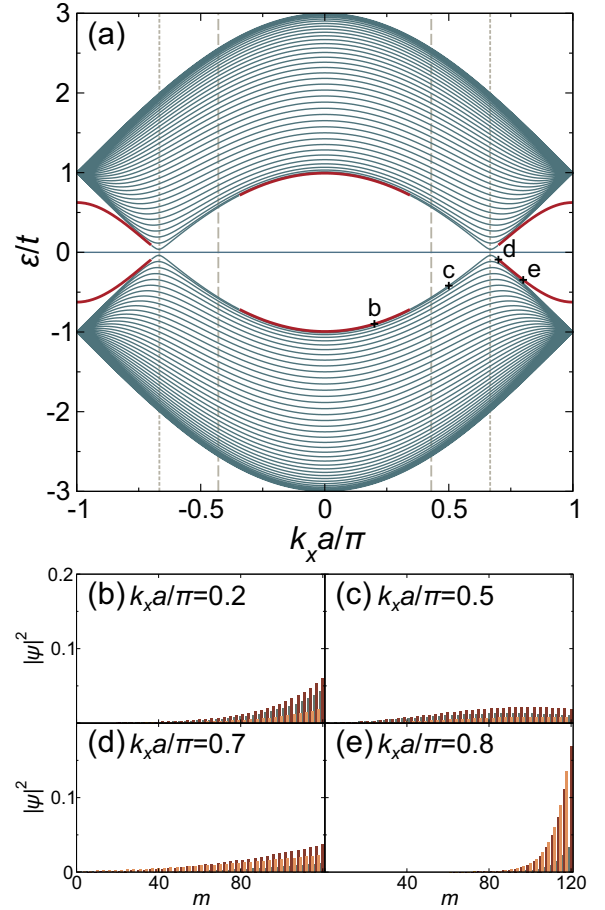


FIG. 2. (a) Energy spectra of the α - T_3 ZR as a function of k_x for $\alpha = 0.8$ and $J = 40$. The bold red lines indicate the edge states. The transitions from bulk to edge states occur at $|k_x| = \chi_{40}^+ \approx 0.3434\pi/a$ and $\chi_{40}^- \approx 0.6991\pi/a$. The vertical dashed and dotted lines mark $\chi_\infty^+ \approx 0.4292\pi/a$ and $\chi_\infty^- = 2\pi/3a$, respectively. Probability density per site m for the in-gap state at (b) $k_x = 0.2\pi/a$, (c) $0.5\pi/a$, (d) $0.7\pi/a$, and (e) $0.8\pi/a$ as indicated by the symbols in (a).

Figures 2(b) and 2(e) clearly show the spatial distribution of an asymmetric edge state, where $|\psi|^2$'s are concentrated around the top edge of the ZR and exponentially decay towards the bottom edge. In Fig. 2(d), the decay of $|\psi|^2$ is not pronounced at k_x close to χ_{40}^- . Figure 2(c) exhibits the profile of a bulk state where $|\psi|^2$ is extended throughout the ZR.

The edge states are topologically characterized using the Majorana representation [27,29,31]. By treating the eigenstate $|\Psi\rangle$ of an $N \times N$ Hamiltonian as a pseudospinor for $S = (N-1)/2$, $|\Psi\rangle$ can be represented by a set of the Pauli spinors $|\zeta_\mu\rangle = [\cos(\eta_\mu/2) \quad \sin(\eta_\mu/2)e^{i\xi_\mu}]^T$, where η and ξ are the polar and azimuthal angles of the Bloch sphere. The trajectory of each Majorana “star” $\zeta_\mu = \langle \zeta_\mu | \sigma | \zeta_\mu \rangle$ thus visualizes the evolution of $|\Psi\rangle$, where σ is the vector of Pauli matrices.

In the Schwinger theory of angular momentum [38], $|\Psi\rangle$ is constructed by the bosonic creation operators $\hat{a}_{\uparrow,\downarrow}^\dagger$ acting on the vacuum state $|\emptyset\rangle$ as follows,

$$|\Psi\rangle = \frac{\mathcal{C}_S}{\sqrt{(2S)!}} \prod_{\mu=1}^{2S} [\hat{a}_\uparrow^\dagger + \lambda_\mu \hat{a}_\downarrow^\dagger] |\emptyset\rangle, \quad (15)$$

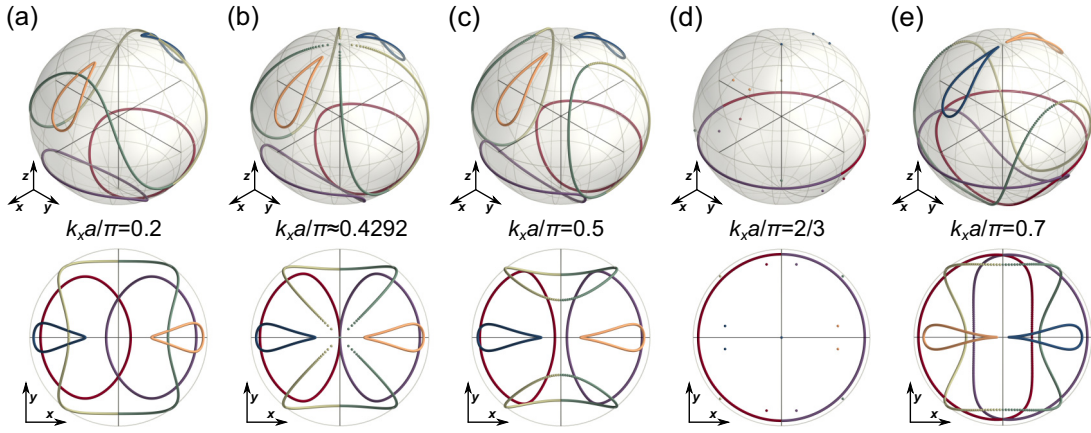


FIG. 3. Trajectories of the Majorana stars on the Bloch sphere (top panels) and the xy plane (bottom panels) for $\alpha = 0.8$, at (a) $k_x = 0.2\pi/a$, (b) $\chi_\infty^+ \approx 0.4292\pi/a$, (c) $0.5\pi/a$, (d) $\chi_\infty^- = 2\pi/3a$, (e) $0.7\pi/a$. The red and yellow (blue and purple) curves correspond to the valence (conduction) band. The lime and green curves correspond to the flat band.

where $\mathcal{C}_S$ is the amplitude of $|\Psi\rangle$. $\lambda_\mu = \tan(\eta_\mu/2)e^{i\xi_\mu}$ is obtained by solving the Majorana polynomial [29]

$$\sum_{\mu=0}^{2S} (-1)^\mu \frac{\mathcal{C}_{S-\mu}}{\sqrt{(2S-\mu)!}\mu!} \lambda^{2S-\mu} = 0. \quad (16)$$

In particular for the stub SSH chain, $S = 1$ and $|\Psi\rangle = (\mathcal{C}_1 \ \mathcal{C}_0 \ \mathcal{C}_{-1})^T$ is given by Eq. (6). For $c = s$ and $c = 0$, Eq. (16) accordingly reduces to

$$\cos \Theta_k e^{-i\Phi_k} \lambda^2 - \sqrt{2}s\lambda + \sin \Theta_k = 0, \quad (17a)$$

$$\sin \Theta_k e^{-i\Phi_k} \lambda^2 - \cos \Theta_k = 0. \quad (17b)$$

Let $\lambda_{s,k}^\mp$ and $\lambda_{0,k}^\mp$ be the roots of Eqs. (17a) and (17b), respectively. From the coefficients of the quadratic equations, it is clear that $\Lambda_{s,k} = \prod_{v=\mp} \lambda_{s,k}^v = \tan \Theta_k e^{i\Phi_k}$, and $\Lambda_{0,k} = \prod_{v=\mp} \lambda_{0,k}^v = -\cot \Theta_k e^{i\Phi_k}$.

The topological invariant is defined by the azimuthal winding number W_c , which counts for the total number of times $\xi_{c,k}^v$'s travel around the z axis of the Bloch sphere. W_c is calculated as follows [27]:

$$W_c = \frac{1}{2\pi} \int_{\text{BZ}} dk_y \frac{\partial}{\partial k_y} \sum_{v=\mp} \xi_{c,k}^v. \quad (18)$$

Since $\lambda_{c,k}^v = \tan(\eta_{c,k}^v/2)\exp(i\xi_{c,k}^v)$ by definition, therefore $\sum_{v=\mp} \xi_{c,k}^v = \Phi_k$. By inserting Eq. (7) into Eq. (18),

$$W_c = \begin{cases} 1 & \text{for } v_A/v'_A < 1, \\ 0 & \text{for } v_A/v'_A > 1. \end{cases} \quad (19)$$

As k_y traverses across the 1D BZ, the trajectories of $\xi_{c,k}^v$'s enclose the z axis at k_x 's that carry the edge states. Thus, W_c gives the number of edge states for each s .

Generally, W_c can take arbitrary $\mathbb{Z}$ values. Therefore, our result is consistent with the tenfold way classification [34] that predicts the $\mathbb{Z}$ invariant for the BDI class in 1D. However, since our model has only nearest-neighbor interactions, $W_c \in \{0, 1\}$. For this reason [12,39,40], hereafter W_c is referred to as a $\mathbb{Z}_2$ invariant.

Figures 3(a)–3(e) show the trajectories of $\xi_{c,k}^v$'s on the Bloch sphere (top panels) and the xy plane (bottom panels) for $\alpha = 0.8$ at $k_x = 0.2\pi/a$ [Fig. 3(a)], $\chi_\infty^+ \approx 0.4292\pi/a$ [Fig. 3(b)], $0.5\pi/a$ [Fig. 3(c)], $\chi_\infty^- = 2\pi/3a$ [Fig. 3(d)], and $0.7\pi/a$ [Fig. 3(e)]. The red/yellow, lime/green, and blue/purple curves correspond to $c = -1, 0$, and $+1$, respectively. In Figs. 3(a) and 3(e), the red, purple, and combined lime/green loops wind around the z axis once, thus $W_c = 1$. Conversely, in Fig. 3(c), none of the loops enclose the z axis, so $W_c = 0$. At $\chi_\infty^\pm$, W_c becomes ill defined for different reasons. In Fig. 3(b), the red, purple, and combined lime/green curves intersect the poles. This behavior indicates a topological phase transition without gap closing [41]. In Fig. 3(d), the trajectories of $\xi_{c,k}^v$'s are discontinuous due to the gap closing at $|k_y| = 2\pi/\sqrt{3}a$. Here, the system is reduced to the semimetallic SSH chain because $\beta = 1$ and consequently $v_C = 0$, $v_A = v'_A$. In the case of T_3 ZR, $W_c = 1$ except at $|k_x| = 2\pi/3a$. Thus, the gap closing in the corresponding stub SSH chain is not accompanied by a topological phase transition.

Previously, Liu and Wakabayashi [42] proposed a 2D SSH model [42,43] exhibiting a topologically nontrivial phase with zero Berry curvature, due to the preserved TRS and inversion symmetry. The $\mathbb{Z}_2$ invariant is given by 2D Zak phase calculated by the integration of the Berry connection over the BZ. In this Letter, we identify T_3 ZR as a topologically nontrivial system with zero Berry phase, where the $\mathbb{Z}_2$ invariant is defined by W_c .

Figure 4 illustrates the topological phase diagram of the α - T_3 ZR. The dark (light) region indicates the nontrivial (trivial) phase, where the presence (absence) of edge states is characterized by $W_c = 1$ ($W_c = 0$). The topological phase transition without (with) gap closing of the corresponding stub SSH chain is marked by the brown (orange) line.

Since the Hamiltonians of the α - T_3 lattice and stub SSH chain are equivalent, the bulk topological property is preserved when the dimension is compacted. Recently, Cáceres *et al.* [28] realized the stub SSH chain in an experiment using a photonic lattice. They demonstrated that for $v_A/v'_A < 1$, the edge states appear at $\tilde{E}_s = s t_M |v_C|$, and persist against

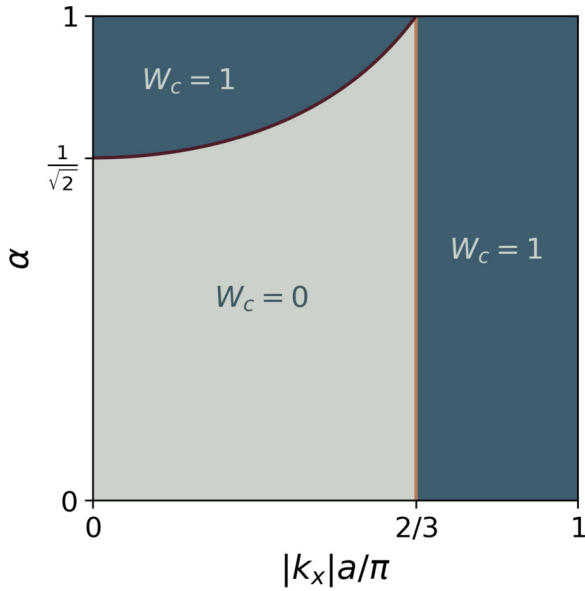


FIG. 4. Topological phase diagram of the α - T_3 ZR. The dark (light) region indicates the nontrivial (trivial) phase characterized by the presence (absence) of edge states. The brown (orange) line marks the phase transition without (with) gap closing of the corresponding stub SSH chain.

disorders that perturb v_A and v'_A . We analytically prove that the edge states are characterized by W_c , which may explain the topological origin of the robustness.

In summary, we formulate the bulk-boundary correspondence for the massless Dirac fermions in the α - T_3 lattice, whose nonquantized Berry phase cannot be utilized as a topological invariant. We found that the α - T_3 lattice can be mapped onto the stub SSH chain by the unitary transformation of the Hamiltonian. In the Majorana representation of the eigenstates, the $\mathbb{Z}_2$ invariant is defined from the azimuthal winding number on the Bloch sphere. This leads to an inference that the T_3 lattice is topologically nontrivial despite a zero Berry phase. Several methods have been proposed to realize α - T_3 or T_3 lattices: by using cold atoms in optical potentials [15,44–47], coupled resonators [48], and a 2D Cu surface decorated by CO molecules [49]. Hopefully, our predictions can be experimentally tested in the foreseeable future. The existence of edge states in distinct α - T_3 ZR configurations is discussed in detail in Ref. [50].

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