

Multiplicative Chern insulator: Generalized bulk-boundary correspondences, 4π Aharonov-Bohm effect, and adiabatic evolution to topological skyrmion phase with symmetry breaking

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We study multiplicative Chern insulators (MCIs) as canonical examples of multiplicative topological phases of matter. Constructing the MCI Bloch Hamiltonian as a symmetry-protected tensor product of two topologically nontrivial parent Chern insulators (CIs), we study two-dimensional (2D) MCIs and introduce three-dimensional mixed MCIs, constructed by requiring the two 2D parent Hamiltonians share only one momentum component. We study the 2D MCI response to time-reversal symmetric flux insertion, observing a 4π Aharonov-Bohm effect, relating these topological states to fractional quantum Hall states via the microscopic field theories of the quantum skyrmion Hall effect. As part of this response, we observe evidence of quantization of a proposed topological invariant for compactified many-body states to a rational number, suggesting higher-dimensional topology may also be relevant. Finally, we study effects of bulk perturbations breaking the symmetry-protected tensor-product structure of the child Hamiltonian, finding the MCI evolves adiabatically into a topological skyrmion phase.

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I. INTRODUCTION

The quantum Hall effect (QHE) [1,2] revealed the powerful role of topology in modern condensed matter physics [3–10]. Its lattice counterpart, the Chern insulator (CI) [11], is the prototypical topological insulator [12–15] and still the only one with quantization precision of transport signatures comparable to that of the QHE itself [16–29]. The CI is the foundation of rich classification of band topology [30–32] but also the foundation for many strongly correlated states [33–49].

The CI therefore represents a dichotomy between noninteracting and correlated states, motivating the present study of multiplicative topological phases (MTPs) [50] within the framework of the quantum skyrmion Hall effect (QSkHE) [50–54], as these phases are characterized by symmetry-protected tensor-product Hilbert spaces similar to the Fock space of multiple particles. We study the MCI as a canonical example [50]. In addition to introducing higher-dimensional MCIs, we study MCI response signatures and robustness of the MCI against symmetry-breaking perturbations. We observe a 4π Aharonov-Bohm (AB) effect consistent with Chern-Simons (CS) theories similar to those of the fractional quantum Hall effect (FQHE) [7–10,55–57]. We also find MCIs can evolve adiabatically into topological skyrmion

phases of matter upon breaking the bulk product structure [51,58–60].

II. MODEL HAMILTONIANS

Multiplicative topological phases are realized in Hamiltonians, which are quadratic in creation and annihilation operators. We consider those of the form [50]

$$\mathcal{H}_c = \sum_{\mathbf{k}_1, \mathbf{k}_2} \Psi_{\mathbf{k}_1, \mathbf{k}_2}^\dagger \mathcal{H}^c(\mathbf{k}_1, \mathbf{k}_2) \Psi_{\mathbf{k}_1, \mathbf{k}_2}, \quad (1)$$

which are four-band child Bloch Hamiltonians, writing the basis vector as $\Psi_{\mathbf{k}_1, \mathbf{k}_2} = (c_{\mathbf{k}_1, \mathbf{k}_2, \uparrow}, c_{\mathbf{k}_1, \mathbf{k}_2, \downarrow}, c_{-\mathbf{k}_1, -\mathbf{k}_2, \downarrow}^\dagger, c_{-\mathbf{k}_1, -\mathbf{k}_2, \uparrow}^\dagger)^\top$, where $c_{\mathbf{k}_1, \mathbf{k}_2, \sigma}^\dagger$ creates a fermion labeled by two momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and spin σ . Due to the class *D* particle-hole symmetry of each parent Hamiltonian, the child Bloch Hamiltonian $\mathcal{H}_c$ also possesses a particle-hole symmetry [50], meaning the presence of both creation and annihilation operators in $\Psi_{\mathbf{k}_1, \mathbf{k}_2}$ is a reflection of interpreting the child Hamiltonian as a Bogoliubov–de Gennes (BdG) mean-field Hamiltonian for a superconductor [30,31]. The multiplicative child Bloch Hamiltonian is expressed in terms of two parent Bloch Hamiltonians $\mathcal{H}_{p1}(\mathbf{k}_1)$ and $\mathcal{H}_{p2}(\mathbf{k}_2)$ combined by a Kronecker product. This product structure is symmetry protected [50]. The parent Hamiltonians $\mathcal{H}_{p1}(\mathbf{k}_1)$ and $\mathcal{H}_{p2}(\mathbf{k}_2)$ are given as

$$\mathcal{H}_{p1}(\mathbf{k}_1) = \mathbf{d}_1(\mathbf{k}_1) \cdot \boldsymbol{\tau}, \quad \mathcal{H}_{p2}(\mathbf{k}_2) = \mathbf{d}_2(\mathbf{k}_2) \cdot \boldsymbol{\sigma}, \quad (2)$$

where $\mathbf{d}_1(\mathbf{k}_1)$ and $\mathbf{d}_2(\mathbf{k}_2)$ are momentum-dependent, three-component vectors of scalar functions, each of $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ is a vector of Pauli matrices, and $\mathbf{k}_{1,2}$ are each momentum vectors

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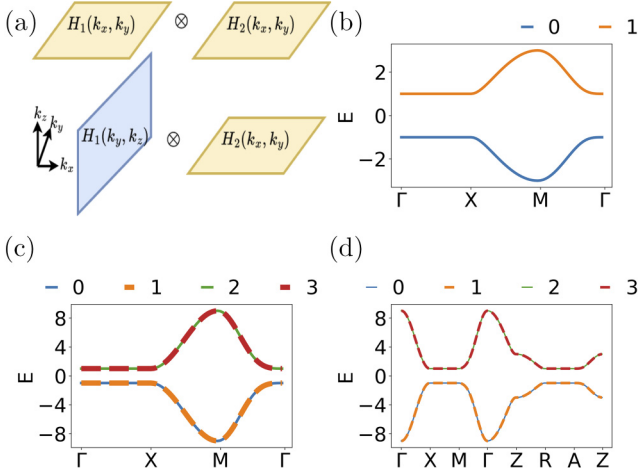


FIG. 1. (a) Schematic depicting construction of 2D (3D) MCI as tensor product of two parent Hamiltonians sharing two (one) momentum component. (b)–(d) Bulk band structure for (b) a parent Hamiltonian with $(m, t, \Delta) = (+1, +0.5, -1)$, with 0 denoting the bulk valence band and 1 denoting bulk conduction band, (c) the 2D MCI, with both parents sharing $(m, t, \Delta) = (+1, +0.5, -1)$, (d) the 3D MCI with both parents sharing the parameters $(m, t, \Delta) = (+1, -0.5, +1)$. Solid and dashed lines 0, 1 correspond to degenerate bulk valence bands, whereas solid and dashed lines 2, 3 correspond to degenerate bulk conduction bands in (c) and (d).

with two components. We take the $\mathbf{d}$ vector of parent 1, $\mathbf{d}_1 = \langle d_{11}, d_{21}, d_{31} \rangle$, to be that of the Qu–Wu–Zhang (QWZ) model [61], $d_{11} = \Delta \sin k_y$, $d_{21} = \Delta \sin k_x$, $d_{31} = m - 2t(\cos(k_x) + \cos(k_y))$. We take $\mathcal{H}_{p2}(\mathbf{k}_2)$ to be $\mathcal{H}_{p1}(\mathbf{k}_1)$ subjected to time reversal, using the time-reversal operator $\mathcal{T} = i\sigma_y \mathcal{K}$, where σ_y is the second Pauli matrix and $\mathcal{K}$ is complex conjugation, as $\mathcal{H}_{p2}(\mathbf{k}) = \mathcal{T} \mathcal{H}_{p1}(-\mathbf{k}) \mathcal{T}^{-1}$ (here the effect of time reversal on momentum, taking $\mathbf{k} \rightarrow -\mathbf{k}$, is explicitly denoted by dependence of $\mathcal{H}_{p1}$ on $-\mathbf{k}$). The child Bloch Hamiltonian is then expressed as $\mathcal{H}^c(\mathbf{k}) = (d_{11}, d_{21}, d_{31}) \cdot \boldsymbol{\tau} \otimes (-d_{11}, d_{21}, d_{31}) \cdot \boldsymbol{\sigma}$. We will consider both a two-dimensional (2D) *parallel* MCI ($\parallel$ MCI) by taking $\mathbf{k}_2 = \mathbf{k}_1 = (k_x, k_y)$, and also a generalization to a three-dimensional (3D) *mixed* MCI, by taking $\mathbf{k}_1 = (k_x, k_y)$ and $\mathbf{k}_2 = (k_y, k_z)$.

Concerning the variety of possibilities in defining momentum dependence for Hamiltonians realizing multiplicative topological phases, we note that the established symmetry protection of multiplicative topological phases requires only that each parent Hamiltonian be in class D of the tenfold way [50], meaning it possesses a particle-hole symmetry but not time-reversal symmetry. Combining the two parent Hamiltonians—each individually without time-reversal symmetry—in a Kronecker product then yields the remaining symmetries that protect the Kronecker product structure, including time-reversal symmetry. So the momentum dependence of the child Hamiltonian considered here satisfies symmetry constraints of multiplicative topological phases.

The tensor-product structure guarantees that the eigenvalues of the child Hamiltonian, $E_{12}^c(\mathbf{k})$, are products of eigenvalues of $\mathcal{H}_{p1}(\mathbf{k})$, i.e., $E_{p1}(\mathbf{k})$, and $\mathcal{H}_{p2}(\mathbf{k})$, i.e., $E_{p2}(\mathbf{k})$, respectively, or $E_{12}^c(\mathbf{k}) = \pm E_{p1}(\mathbf{k}) E_{p2}(\mathbf{k})$, and therefore also at least doubly degenerate as shown in Figs. 1(c) and 1(d).

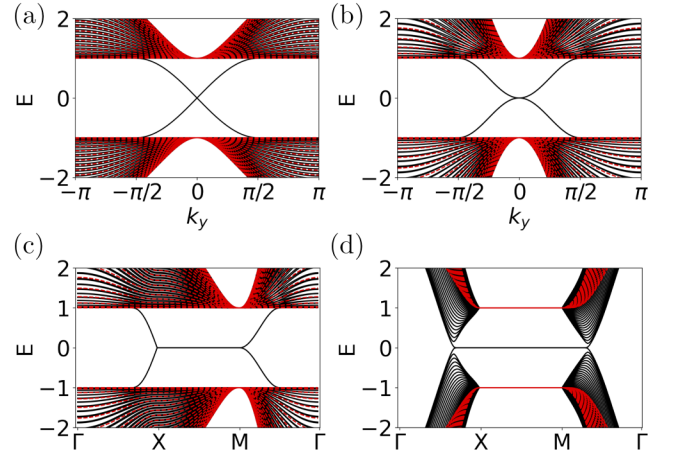


FIG. 2. (a) Slab spectrum for a single parent Bloch Hamiltonian with $(m, t, \Delta) = (+1, +0.5, -1)$ for OBCs in the $\hat{x}$ direction, with system size $L_x = 50$. (b) Slab spectrum for $\parallel$ MCI as shown, with both parents having $(m, t, \Delta) = (+1, +0.5, -1)$. (c) Slab spectrum for mixed MCI for OBCs in $\hat{x}$ (shared) direction, with two parents sharing the parameters $(m, t, \Delta) = (+1, -0.5, +1)$. (d) Slab spectrum for 3D mixed MCI for OBC in $\hat{z}$ (unshared direction) with two parents sharing the parameters $(m, t, \Delta) = (+1, -0.5, +1)$.

Indeed, comparison between the bulk spectrum of a parent Hamiltonian $\mathcal{H}_{p1}(\mathbf{k})$ along high-symmetry lines in the Brillouin zone (BZ) in Fig. 1(b) and the bulk spectrum of the $\parallel$ MCI shown in Fig. 1(c) shows the bandwidth of bands in the child Hamiltonian is the square of corresponding parent bands for the case we consider.

In Fig. 1(d) we observe both high-symmetry lines along which bands of the child Hamiltonian disperse like those of an individual parent [compare the band structure along the XZ line for the child Hamiltonian in Fig. 1(d) with the band structure along the MX line for the parent Hamiltonian in Fig. 1(b)], as well as lines along which the dispersion is the square of that for an individual parent [compare the Γ Z line in Fig. 1(d) with the MX line in Fig. 1(b)].

III. BULK-BOUNDARY CORRESPONDENCE

We now characterize bulk-boundary correspondence of the $\parallel$ and mixed MCIs, with results shown in Fig. 2. The slab spectrum for a single parent is shown in Fig. 2(a) for direct comparison with the slab spectrum for the $\parallel$ MCI with open boundary conditions (OBCs) in the $\hat{x}$ direction in Fig. 2(b). While the edge modes disperse linearly for an individual parent and are nondegenerate, the edge modes of the $\parallel$ MCI disperse quadratically, reflecting a multiplicative dependence of eigenvalues for the child Hamiltonian on parent eigenvalues as in the bulk. These edge states are exponentially localized. For $k_y = -(4/9)$ and system size in the direction of OBCs of $L = 50$, we observe boundary modes with a probability density peak at layer indices $x = 0$, $x = 1$, $x = 49$, and $x = 50$, respectively. Similar edge-state probability density peak locations are also observed for the parallel multiplicative Kitaev chain [62]. For the 3D mixed MCI, we consider both OBCs in the $\hat{x}$ - (shared) direction, given that each of parents 1 and 2 depend on k_x , as well as OBCs in the $\hat{z}$ - (unshared) direction,

given that only one parent depends on k_z . Slab spectra for these two cases are shown in Figs. 2(c) and 2(d), respectively. In Fig. 2(c) the surface bands are everywhere at least twofold degenerate and traverse the bulk gap. There is a zero-energy manifold of states along the XM line, which disperses linearly from X toward Γ and quadratically from M toward Γ . These states are exponentially localized, with probability density peaks located in the first or last layers of the slab. For the case of Fig. 2(d), we again observe an extended zero-energy manifold, primarily along high-symmetry line XM. Strikingly, while the system is gapped in the bulk for PBCs, OBCs for an unshared direction yield additional *bulk* states not present for PBCs. The edge states emerge from bulk gap closings, *which are only present for OBCs*. That is, in this scenario the system is a topological insulator in the bulk for PBCs but a *topological semimetal in the bulk for OBCs*.

IV. TOPOLOGICAL RESPONSE TO EXTERNAL MAGNETIC FIELD

We now study response signatures of the MCI, focusing on the case of the ||MCI. In this case, we consider a time-reversal symmetric (TRS) flux insertion at two locations in the lattice for system size 50×50 unit cells and PBCs. A schematic of this flux insertion for a smaller system size is shown in Fig. 3(a). The TRS flux insertion procedure is motivated by the time-reversal symmetry of the ||MCI: such a TRS flux insertion procedure preserves this symmetry and is a standard probe of the topological response for TRS systems such as the quantum spin Hall insulator [63].

The flux insertion processes considered here involve inserting flux specifically through the lattice of the child Hamiltonian without any assumption of its symmetry-protected Kronecker product structure. While one could instead study the consequences of flux insertion through the lattices of the parent Hamiltonians, the former procedure is the canonical approach used for other TRS states and furthermore, the more experimentally accessible, while it is currently unclear how to perform the latter procedure experimentally. Additionally, we note that the topological response first reported in the present work is now well understood at the level of a microscopic field theory for the QSkHE [64]. We will discuss the relevant microscopic field theory in more detail as part of analysis of the topological response signatures, but it is useful to mention now that flux insertion at the level of the child lattice is also motivated by these microscopic quantum field theories, a point which we will later revisit as part of our analysis.

Evolution of the spectrum vs magnetic flux ϕ is shown in Fig. 3(b), for both TRS flux insertion via two bond-centered flux tubes at two locations of the lattice detailed schematically in (a) (black dashed lines) as well as for a second TRS flux insertion procedure detailed in the Supplemental Material (red lines) [65]. This latter procedure yields ϕ_0 -periodic features (ϕ_0 is one flux quantum) for PBCs, which are also observed for TRB flux insertion at a single location in the lattice for OBCs, as is also shown in the Supplemental Material [65]. These ϕ_0 -periodic features correspond to response signatures of the individual parent Hamiltonians with Chern numbers $C = \pm 1$, with entanglement between the parent

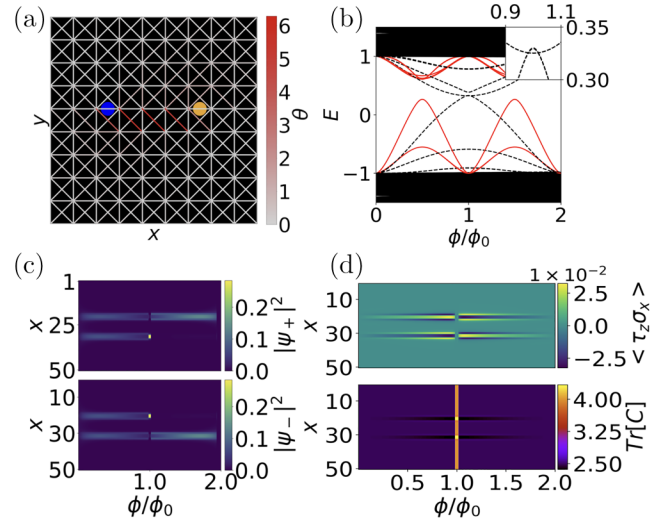


FIG. 3. (a) TRS flux insertion scheme for a 10×10 square lattice: TRS pair of flux tubes are located at points marked in orange and blue. The amplitude of Peierls phases for nearest-neighbor and diagonal hopping integrals are represented by the color bar for the case of $\phi = \phi_0$. (b) Energy spectrum vs magnetic flux ϕ of the child ||MCI with $(m, t, \Delta) = (+1, +0.5, -1)$ with PBCs for the TRI flux insertion procedure shown schematically in (a) (shown in dashed black lines) and for a second flux insertion procedure detailed in the Supplemental Materials [65] (red lines). Inset of (b) shows level crossings between twofold degenerate states near $\phi = \phi_0$ for the former flux insertion procedure, which are absent for the latter. (c) and (d) characterize evolution of the system as a function of magnetic flux ϕ in units of flux quantum ϕ_0 inserted in time-reversal symmetric fashion as shown in (a), with (c) probability density for states $\psi_{\pm}$ vs real-space coordinate x ($y_0 = 25$) and magnetic flux ϕ . (d) Ground-state expectation value for operator $\tau_z \sigma_x$ (top) and topological charge $\text{Tr}[C]$ (bottom).

degrees of freedom (DOFs), yielding gaps near $\phi = \phi_0/2$ and $\phi = 3\phi_0/2$. This result is expected for multiplicative topological phases based on earlier work [50], with many topological features of the child Hamiltonian identifiable with counterpart features of the parents, along with some compromise due to entanglement. However, the flux insertion process shown in Fig. 3(a) instead yields a $2\phi_0$ -periodic feature. Additionally, the bulk gap closes in the vicinity of $\phi = \phi_0$, as highlighted in the inset of Fig. 3(b). By further characterizing the degenerate states involved in the level crossing and properties of the states below the energy of the level crossing, we will further characterize this result in the following discussion to identify it as part of a generalized 4π Aharonov-Bohm effect, which can be understood as an additional feature of entanglement in the system at the level of an effective Abelian Chern-Simons theory and, more concretely, via the microscopic field theories of the QSkHE [64].

For the time-reversal invariant flux-tube insertion procedure only, we denote the two degenerate eigenstates that enter the bulk gap from the bulk valence bands, outputted by numerical diagonalization to be $\phi_{1,2}$, and we define particular linear combinations of these states as $\psi_{\pm} = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2)$. We show evolution of the probability density for $\psi_{\pm}$ for $y_0 = 25$ vs x and ϕ in Fig. 3(c). We observe even distribution of $\psi_{\pm}$

between the sites of flux insertion for $\phi < \phi_0$, and probability density becomes strongly localized at just one site of flux insertion for each state for $\phi > \phi_0$.

We note that numerical diagonalization is performed here such that at each value of the flux ϕ , one indeed obtains a normalized yet arbitrary superposition of the two degenerate states. However, in our work, to establish coherence from one point in flux to the other, we use a gauge-fixing procedure. Suppose the two states obtained from numerical diagonalization are $|\phi_1\rangle = \alpha|1\rangle + \beta|2\rangle$, $|\phi_2\rangle = \gamma|1\rangle + \zeta|2\rangle$ at value of flux ϕ and two states $|\phi'_1\rangle = \alpha'|1'\rangle + \beta'|2'\rangle$, $|\phi'_2\rangle = \gamma'|1'\rangle + \zeta'|2'\rangle$ at value of flux $\phi' = \phi - d\phi$, $d\phi$ being the discretization used in the numerics, and $|1(1')\rangle$, $|2(2')\rangle$ are some orthonormal basis vectors for the degenerate subspace. The two sets of states will generally be related by a $U(2)$ transformation. However, we see that correlations such as $\langle\phi_1|\phi'_2\rangle$ and $\langle\phi_2|\phi'_1\rangle$ are zero. Then the only possibility is a $U(1)$ gauge freedom at each value of the flux in determining either $|\phi_i\rangle$ or $|\phi'_i\rangle$, which we fix to be unity.

We further characterize evolution of the system under time-reversal invariant flux insertion by computing the expectation value of $\tau_z\sigma_x$ vs flux ϕ and position x for $y_0 = 25$, shown in Fig. 3(d) (top), observing a net change in sign. $\tau_x\sigma_z$ yields the same result up to a sign reversal. As well, we compute a topological invariant associated with the parent isospin DOFs [54]. To do so, we construct a spin representation associated with the τ (σ) DOF for the occupied states by computing the density matrix from the one-point correlator [66] for a given site in the lattice as $\rho(x, y)$. We then compute a spin representation for the occupied states as $S_i^{\text{occ}}(x, y) = \rho(x, y)\tau_i\sigma_0$ ($\tau_0\sigma_i$ yields the same result). We finally compute the topological charge associated with a compactified many-body state as $\text{Tr}[\mathbf{C}(x, y)] = \text{Tr}[[S_x^{\text{occ}}(x, y), S_y^{\text{occ}}(x, y)](S_z^{\text{occ}}(x, y))^{-1}]$ [54]. Figure 3(d) (bottom) shows $\text{Tr}[\mathbf{C}]$ vs magnetic flux ϕ and layer index in the $\hat{x}$ direction, x , for $y_0 = 25$. At sites of flux insertion for ϕ near ϕ_0 , $\text{Tr}[\mathbf{C}]$ deviates from 4, the value expected for the spin representation without projection to the occupied subspace, by very close to $1/3$. This value converges toward $1/3$ with increasing system size and occurs for an extended region of phase space [65].

The 4π AB effect corresponds to a minimal CS field theory within the framework of the QSkHE [53,54]. It possesses two types of gauge fields, those coupling to the quasiparticle species of each individual parent DOF, written as a $U(1)$ gauge field A_1 or $U(1)$ gauge field A_2 , respectively, as well as a gauge field coupling to the total charge of a quasiparticle pair encoded in the isospin DOFs, a , with $U(1) \times U(1)$ structure. This quasiparticle pair is a quantum skyrmion, which can be modeled as a severely fuzzified Landau level (LL_F) [53,54]. This corresponds to a minimal effective action with two terms:

$$S_{c,\text{eff}} = \frac{C_1}{2\pi} \int d^2x dt \varepsilon^{\mu\nu\rho} A_{1/2,\mu} \partial_\nu a_\rho + \frac{Q}{4\pi} \int d^2x dt \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \quad (3)$$

Here, $A_{1/2,\mu}$ acts on the composite particles by the $A_{1(2),\mu}$ component acting on the parent 1 (2) particle species. The gauge field a may be expressed in terms of the parent gauge fields as $a_\mu = A_{1,\mu} + T[A_{2,\mu}]$, where $T[A_{2,\mu}]$ is a chiral

transformation on $A_{2,\mu}$. With this definition of the emergent gauge field, $Q = 2$ if $C_1 = 1$. Here, Q is the skyrmion number previously used to characterize topological skyrmion phases of matter [51,53]. For 2 + 1 D systems, it takes the form

$$Q = \frac{1}{4\pi} \int_{BZ} d\mathbf{k} [\langle \hat{\mathbf{S}}(\mathbf{k}) \rangle \cdot (\partial_{k_x} \langle \hat{\mathbf{S}}(\mathbf{k}) \rangle \times \partial_{k_y} \langle \hat{\mathbf{S}}(\mathbf{k}) \rangle)], \quad (4)$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is a (pseudo)spin representation specific to the DOFs of a given system, and $\langle \hat{\mathbf{S}}(\mathbf{k}) \rangle$ is the normalized expectation value of the spin for occupied states. Here $\langle S_i(\mathbf{k}) \rangle = \sum_{n \in \text{occ}} \langle n, \mathbf{k} | S_i | n, \mathbf{k} \rangle$, with $i \in \{1, 2, 3\}$ and $|n, \mathbf{k}\rangle$ the Bloch state associated with the n th band. We can compute Q analytically due to the tensor-product structure of the bulk Bloch Hamiltonian as well, finding $Q = 2$ for parent Chern numbers of ± 1 [65]. Then the CS theory has minimal terms for the $\nu = 1/2$ FQH plateau [8] and 4π AB effect. However, results for $\text{Tr}[\mathbf{C}]$ in Fig. 3(d) suggest an additional topological invariant for compactified many-body states [54], and potentially higher-dimensional terms in the CS theory are needed for deeper understanding of the MCI.

Since the present work first became publicly available, we have completed a second work introducing microscopic field theories of the QSkHE [64], which provide considerable insight into how the TRS flux insertion process yields the reported response signatures. Most importantly, the $1/3$ contribution to $\text{Tr}[\mathbf{C}]$ occurring in combination with a 4π AB effect for the entire system—a signature typically associated with a $\nu = 1/2$ FQH state—is formulated microscopically in terms of a matrix gauge theory [64] for an array of coupled unit cells, each unit cell consisting of a matrix gauge theory for a quantum Hall droplet at filling fraction $1/3$ with $N = 4$ spinless electrons. This N is the matrix dimension of the gauge fields and is notably the same as that of each unit cell of the MCI Hamiltonian presented here.

The $N = 4$ droplet at filling fraction $1/3$ is then understood within this microscopic field theory of the QSkHE [64] as a consequence of a pair of $N = 2$ droplets—one for each 2×2 matrix contributed by a parent Hamiltonian to the unit cell—merging into a single $N = 4$ droplet due to entanglement introduced by the TRS flux insertion process reported here. This merging corresponds to breaking the Kronecker product structure of the MCI Hamiltonian for general flux value ϕ . Flux insertion through the child Hamiltonian lattice as performed here can furthermore be understood as effective flux insertion through this $N = 4$ quantum Hall droplet, which by incompressibility of the $\nu = 1/3$ state yields a quasihole or quasiparticle excitation of charge $1/3$, low-lying charged excitations collectively identified as the quantum skyrmions of the QSkHE [64]. The quantization of $\text{Tr}[\mathbf{C}]$ first reported here is furthermore derived from first principles as a consequence of previously unidentified fusion rules of the $N = 4$ quantum Hall droplet.

Minimal microscopic field theories for the QSkHE are then constructed as arrays of coupled small N quantum Hall droplets described above. Coupling specifically of many $N = 4$ droplets, each individually at filling fraction $\nu = 1/3$, yields a filling fraction of $\nu = 1/2$ for the entire array due to a quantum level shift [64,67], consistent with the 4π AB effect for the entire MCI system first reported here. It is now clear that the response signature of the MCI first reported in this work

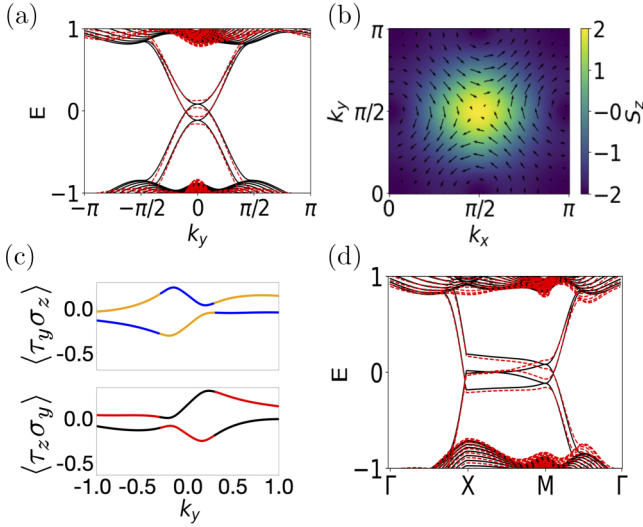


FIG. 4. (a) Slab spectrum of ||MCI [with both parents sharing $(m, t, \Delta) = (+1, +0.5, -1)$], with $m' = 0.1$ and $t' = 0$ (black) and $m' = 0.1$ and $t' = 0.05$ (red), with OBCs (PBCs) in the $\hat{x}$ - ($\hat{y}$ -) direction and system size in the $\hat{x}$ direction of $L = 50$. (b) BZ spin texture with skyrmion number 1 for spin representation $\tau_i \sigma_0$. (c) $\langle \tau_y \sigma_z \rangle$ and $\langle \tau_z \sigma_y \rangle$ for the 99th (blue and red) and 100th (orange and black) lowest-energy eigenstates of the ||MCI for $m' = 0.1$ and $t' = 0.05$ vs k_y . (d) Slab spectrum of 3D mixed MCI along high-symmetry line in the slab BZ for OBCs in the x direction (also with $L = 50$) and PBCs in other directions $[(m, t, \Delta) = (+1, -0.5, +1)]$ with $m' = 0.1$, for $t' = 0$ (black) and $t' = 0.05$ (red).

has played an essential role in development of the microscopic field theories of the QSkHE and is expected to be a useful tool in developing various extensions of the microscopic field theories to, for instance, supersymmetric variants.

V. EFFECTS OF SYMMETRY-BREAKING PERTURBATIONS

We now consider effects of weak symmetry-breaking terms, which break the tensor-product structure—but do not close the bulk energy gap—of the child Bloch Hamiltonian. We first consider a symmetry-breaking term, which is constant in momentum, of the form $\mathcal{H}'_{\text{const}} = m'(\tau_z \sigma_z + \tau_z \sigma_0 + \tau_0 \sigma_z + \tau_x \sigma_z + \tau_y \sigma_0)$, with free parameter m' . Finite m' reduces the symmetry of the Hamiltonian to class A [30,31]. Rather than gapping out the slab spectrum, this constant term shifts quadratic band-touchings on each edge up and down in energy, respectively, as shown in Fig. 4(a) in black. The quadratic band-touchings are not gapped out due to differences in location of probability density peaks for the two in-gap states localized on a given edge, similar to those observed for the multiplicative Kitaev chain [62].

We therefore consider an additional momentum-dependent symmetry-breaking term $\mathcal{H}'(\mathbf{k}) = 2t' \cos(k_x) \tau_y \sigma_y$, with free parameter t' . This term gaps out the quadratic band-touchings at each edge, but linear band-crossings remain as shown in Fig. 4(a) (dashed red lines). These remaining band-crossings are realized by a pair of nondegenerate bands of states localized on opposite edges.

This bulk-boundary correspondence of the MCI with additional finite $\mathcal{H}'_{\text{const}}$ and $\mathcal{H}'(\mathbf{k})$ is that of a topological skyrmion phase of matter [53,59]. Indeed, computing the expectation value texture over the 2D BZ for spin representation $S_i = \tau_i \sigma_0$ (or $\tau_0 \sigma_i$), corresponding to one parent DOF (or the other), yields a skyrmion over the BZ with topological charge of 1. Combining contributions from individual parents then yields the topological charge of $\mathcal{Q} = 2$ [65]. $\langle \tau_z \sigma_y \rangle$ (or $\langle \tau_y \sigma_z \rangle$) for individual edge states is transferred between edges over the interval in k_y bounded by the in-gap band crossings, as shown in Fig. 4(c), with some deviations due to class-A symmetry. This is a signature of topological skyrmion phases [53]. Finally, we note that the symmetry-breaking terms, $\mathcal{H}'_{\text{const}}$ and $\mathcal{H}'(\mathbf{k})$, also preserve gaplessness in the mixed MCI, as shown in Fig. 4(d).

VI. DISCUSSION AND CONCLUSION

We study multiplicative topological phases (MTPs) of the broader quantum skyrmion Hall effect (QSkHE) framework, focusing on multiplicative Chern insulators (MCIs). We introduce three-dimensional (3D) mixed MCIs and study topological response signatures and robustness of MCIs. Time-reversal symmetric flux insertion reveals a 4π Aharonov-Bohm (AB) effect of the MCI understood in terms of a Chern-Simons theory within the QSkHE framework [54], similarly to the FQHE and related states [55]. This indicates the MCI exhibits a pairing mechanism yielding a composite compactified many-body state generalizing Jain's notion of a composite particle [9]. We also observe evidence of an additional topological invariant associated with compactified, topologically nontrivial many-body states [54], however, suggesting richer physics of the QSkHE. Upon breaking the product structure of the bulk Hamiltonian for the MCI, the system evolves adiabatically into more general topological states of the QSkHE [51,53,54]. These results are also now well supported from first-principles derivations within the microscopic field theories of the QSkHE [64]. Our work therefore reveals MTPs and the QSkHE as a potential framework for unifying understanding of noninteracting and interacting systems.

Concerning physical relevance of the momentum dependence of the child Hamiltonian, while it remains unclear which physical systems most easily realize the symmetry protection of multiplicative topological phases, the present work is the first step in demonstrating that multiplicative topological phases are not trivialized by breaking the symmetries ensuring Kronecker product structure of the child Bloch Hamiltonian in general. The multiplicative topological phases simply generalize to the topological skyrmion phases, which are very experimentally relevant as they can occur even for class A of the tenfold way (sometimes called the class of “no symmetry”) [51,53].

We also note that there is separate work studying topological skyrmion phases of the Bernevig-Hughes-Zhang model for the quantum spin Hall insulator, showing these phases exhibit signatures consistent with past experiments on HgTe quantum wells [68]. From the perspective of the recently introduced microscopic field theories of the quantum skyrmion Hall effect, which explain many distinctive signatures of the

topological response of the multiplicative Chern insulator [64], non-Abelian quantum Hall states are also currently some of the most promising candidates for experimental observation of the generalized Aharonov-Bohm effect first reported here, motivating development of experimental probes for detecting the topological invariant first applied in this work, $\text{Tr}[\mathbf{C}]$. As $\text{Tr}[\mathbf{C}]$ characterizes the charge of quantum skyrmions [64], it may be possible to generalize existing probes of charge accumulation and transport, first by isolating such charges, and future work will explore development of such experimental probes.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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