

Finite-temperature properties of antiferromagnetic quantum spin chains with random long-range couplings

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(Received 10 November 2025; revised 23 February 2026; accepted 31 March 2026; published 22 April 2026)

We study excited states and finite temperature properties of chains of randomly placed spins with long range interactions. To this end we extend the recently introduced strong disorder renormalization group method in real space, well suited to study bond disordered antiferromagnetic power law coupled quantum spin chains. For bond disordered, short range coupled spin chains, defined by keeping only interactions between adjacent spins, we show that the distribution of the absolute value of the couplings is the infinite randomness fixed point distribution also for the excited states. Moreover, we derive the distribution function of the sign of the couplings and find that the number of negative couplings increases with temperature T . For power law long range interaction with power exponent α we derive and solve the Master equation for the distribution function of couplings for $\alpha > 2$. The coupling amplitude is found to have the strong disorder distribution with finite width 2α , with small corrections. We derive resulting finite temperature properties of both short and power law long ranged spin systems for $\alpha > 2$. The magnetic susceptibility is dominated by paramagnetic $S = 1/2$ -spin, resulting in a Curie law susceptibility. The distribution function of pair lengths, and the spin correlation function are found to be the same as in the ground state. We find that the entanglement entropy increases logarithmically with the partition length n and has the same functional form as for the ground state of critical quantum spin chains with temperature dependent effective central charge $\bar{c}(T)$, which converges at high temperature to $\frac{1}{2} \ln 2$, half the value it has in the ground state. We confirm that the entanglement entropy grows with time t after a quantum quench logarithmically as $S(t) \sim \ln(t)/(2\alpha)$.

DOI: [10.1103/9d9b-mpfm](https://doi.org/10.1103/9d9b-mpfm)

I. INTRODUCTION

Magnetic properties of a wide range of materials are governed by randomly placed quantum spins, coupled by long range interactions [1–6]. Tunable interactions have been realized between atoms trapped near photonic crystals [7], of trapped ions [8,9] and by coupling Rydberg states with opposite parity [10,11]. Single nitrogen-vacancy centers in diamond allow to probe the dynamics of disordered spin ensembles at a diamond surface [12,13].

It remains a challenge to derive thermodynamic and dynamic properties of such systems. The strong disorder renormalization group (SDRG) method was introduced to study disordered quantum spin chain models [14–19]. The ground state of short range antiferromagnetically coupled disordered spin chain models was thereby shown to be composed of a product state of randomly placed spin pairs in singlet states, the *random singlet phase* with couplings distributed according to the infinite randomness fixed point (IRFP) distribution of infinite width. The SDRG method has been applied

to other short range random quantum spin chains, like the transverse field Ising model [18]. Recently, the SDRG method was extended to study disordered spin $S = 1/2$ - chains with antiferromagnetic power law XX interactions. The ground state was shown for sufficiently large power exponent α to be still composed by a product state of random singlets, with a fixed point distribution of finite width [20–23]. This was confirmed by numerical exact diagonalization and extensions of the DMRG method [22]. A similar fixed point distribution was found for long range coupled transverse field Ising chains [24,25].

It remains to derive excited states and finite temperature properties of long range bond disordered antiferromagnetic quantum spin chains. Here, we introduce a real space representation of the SDRG-X method. That method was introduced to study properties of excited states [26–28] and recently applied to study excitations [29] and entanglement dynamics after global quantum quenches of long range coupled spin chains [30].

Model. We study long range antiferromagnetic XX-coupled quantum spin chains of $NS = 1/2$ spins, randomly placed at positions $\mathbf{r}_i$ on a line of length L , as shown in Fig. 1 and defined by

$$H = \sum_{i < j} J_{ij} (S_i^x S_j^x + S_i^y S_j^y). \quad (1)$$

We take N to be even and take open boundary conditions. The couplings between spins at sites i, j are antiferromagnetic and

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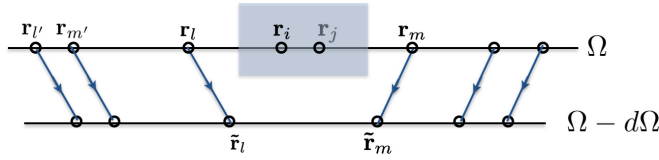


FIG. 1. SDRG step for bond disordered short range coupled spin chains: decimation of the strongest coupled spin pair (i, j) (shaded), whose coupling defines the RG scale Ω . It induces a renormalization of the spin positions $\mathbf{r}_l \rightarrow \tilde{\mathbf{r}}_l$ while the RG scale is reduced to $\Omega - d\Omega$.

long ranged, decaying with a power law with distance $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ with exponent α ,

$$J_{ij} = J_0 |\mathbf{r}_i - \mathbf{r}_j|^{-\alpha}, \quad (2)$$

with $J_0 > 0$, and $|\mathbf{r}_i - \mathbf{r}_j| > a$ for all i, j , where $a \ll L/N$ is the smallest possible distance between spins.

II. SDRG-X METHOD

The SDRG-X method was introduced in Ref. [26], assuming that all many body Eigen states of the Hamiltonian Eq. (1) can be written as tensor products of pair states, one starts by identifying the strongest coupled pair of spins (i, j) for a given initial distribution of couplings $P(J, \Omega_0)$, where Ω_0 is the largest energy scale. One rewrites the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{V}$, where the Hamiltonian of the most strongly coupled pair of spins (i, j) is given by $\hat{H}_0 = J_{ij}^x (S_i^x S_j^x + S_i^y S_j^y)$ and $\hat{V}$ models the interaction of spins (i, j) with all other $N - 2$ spins and between themselves. Diagonalizing $\hat{H}_0$, one finds its four eigenstates $|s\rangle$, $s \in \{0, 1, 2, 3\}$, with eigen energies E_s . Projecting that pair (i, j) on one of the pair states s , we construct the effective Hamiltonian of the remaining $N - 2$ spins $\hat{H}_{\text{eff}}$, such that it commutes with $\hat{H}_0$ and is thereby diagonal in its eigenstates $|s\rangle$. Thereby, the effective Hamiltonian can be written as

$$\hat{H}_{\text{eff}} = \exp(i\hat{S})\hat{H}\exp(-i\hat{S}). \quad (3)$$

Next one expands $\hat{H}_{\text{eff}}$ to second order in $\hat{S}$ and enforces commutation with $\hat{H}_0$. Denoting subspaces D_u such that all states $s \in D_u$ have the same eigen energy E_s , $\hat{H}_{\text{eff}}$ is to second order in $\hat{V}$ given by

$$\begin{aligned} \hat{H}_{\text{eff}} \approx & \sum_u \sum_{s, s' \in D_u} |s\rangle\langle s'| \left[E_s \delta_{s, s'} + \langle s|\hat{V}|s'\rangle + \frac{1}{2} \sum_{s'' \notin D_u} \right. \\ & \times \langle s|\hat{V}|s''\rangle \langle s''|\hat{V}|s'\rangle \left(\frac{1}{E_s - E_{s''}} + \frac{1}{E_{s'} - E_{s''}} \right) \Big]. \end{aligned} \quad (4)$$

The last terms define the renormalized couplings and local fields for all remaining $N - 2$ spins. Even if the form of the Hamiltonian remains unchanged, the renormalized couplings differ from the initial ones, and the distribution function of couplings is changed to $P_E(J, \Omega_0 - d\Omega)$, with $\Omega_0 - d\Omega$ the largest energy scale in the reduced system of $N - 2$ spins. E is the total energy of the system. Repeating this procedure until

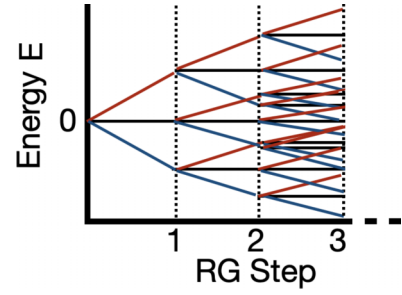


FIG. 2. Schematic SDRG-X procedure: at each RG step, four possible pair states are indicated by blue, black, and red lines with pair energies $E = -J/2, 0, +J/2$, respectively. After $N/2$ RG steps the many body eigenstates with total energy E is obtained, following a specific SDRG path.

all N spins formed pairs, we find the distribution of effective couplings $P_E(J, \Omega)$ in the limit of $\Omega \rightarrow 0$. This allows one to derive thermodynamic and dynamic properties of the spin chain. The total energy E is equal to the sum over all pair energies E_{s_n} , as obtained in the n th RG step, $E = \sum_{n=1}^{N/2} E_{s_n}$. Thus a specific total energy E corresponds to a specific RG path, sketched in Fig. 2.

Instead of using a micro canonical ensemble at energy E , it is often more convenient to consider a canonical ensemble at bath temperature T . Then, any of the four pair states $s \in \{0, \dots, 3\}$ of the strongest coupled pair can be occupied at RG scale Ω with probability

$$p_s(E_s(\Omega), T) = \exp[-\beta E_s(\Omega\sigma)]/Z(\Omega), \quad (5)$$

where $\beta = 1/(k_B T)$ and the partition sum is given by $Z(\Omega) = \sum_s \exp[-\beta E_s(\Omega)]$.

III. NEAREST NEIGHBOUR COUPLING

We first apply the SDRG-X procedure to study excited states of randomly placed spins on a chain, with power law XX coupling, Eq. (1), keeping only the coupling between adjacent spins. We identify the strongest coupling $J_{ij} = \Omega$, as highlighted in the example of randomly placed spins in Fig. 1. While the ground state of the spin pair (i, j) is the singlet state $|0_{ij}\rangle$, the excited states are one of three triplet states, the two unentangled states $|1_{ij}\rangle = |\uparrow_i \uparrow_j\rangle$ and $|2_{ij}\rangle = |\downarrow_i \downarrow_j\rangle$, and the entangled triplet state $|3_{ij}\rangle = (|\uparrow_i \downarrow_j\rangle + |\downarrow_i \uparrow_j\rangle)/\sqrt{2}$. The corresponding eigenenergies of spin pair i, j are given by $E_0 = -J_{ij}/2$, $E_1 = E_2 = 0$, and $E_3 = J_{ij}/2$.

A. SDRG-X rules

Insertion in Eq. (4) yields no local fields, $\langle s|\hat{V}|s\rangle = 0$, for all $s \in \{0, 1, 2, 3\}$. But a new coupling is generated between spins adjacent to i, j , denoted l, m in Fig. 1. Thereby, we find the following renormalization rules.

Singlet state. If (i, j) forms a singlet state, $s = 0$, $(|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle)/\sqrt{2}$, the generated coupling is given by

$$\tilde{J}_{lm} = \frac{J_{il}J_{jm}}{J_{ij}}. \quad (6)$$

Unentangled triplet states. If the pair (i, j) is projected onto one of the unentangled triplet states, $s = 1, |\uparrow\uparrow\rangle$ or $s = 2, |\downarrow\downarrow\rangle$, a coupling between spins at sites l, m is generated with

$$\tilde{J}_{lm} = -\frac{J_{il}J_{jm}}{J_{ij}}. \quad (7)$$

Thus the coupling acquires a minus sign, generating ferromagnetic couplings, even though all original couplings are antiferromagnetic.

Entangled triplet state. If (i, j) is in the entangled triplet state, $s = 3, (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ or in the singlet state, then the generated coupling is

$$\tilde{J}_{lm} = \frac{J_{il}J_{jm}}{J_{ij}}. \quad (8)$$

Thus, while the amplitude of the generated couplings is independent on the state which the paired spins form, the nonentangled states generate a different sign of the coupling. Thus we can summarize all SDRG rules as

$$\tilde{J}_{lm} = \sigma_s \frac{J_{il}J_{jm}}{J_{ij}}, \quad (9)$$

where $\sigma_s = 1$ for $s = 0, 3$ and $\sigma_s = -1$ for $s = 1, 2$.

Since the magnitude of the adjacent couplings depends on their distances, it is convenient to represent the renormalized couplings $\tilde{J}$ in terms of renormalized distances $\tilde{r}$ as $\tilde{J}_{lm} = J_0/\tilde{r}_{lm}^\alpha$. Now, the RG rules can be recast for the renormalized distance $\tilde{r}_{lm} = \tilde{r}$, as sketched in Fig. 1. At the RG length scale $\rho = (\Omega/J_0)^{-1/\alpha}$, the renormalized distance is thus

$$\tilde{r} = \frac{r_{li}r_{jm}}{\rho}. \quad (10)$$

In order to trace the sign of the coupling σ , we introduce the distribution function of distances $P_\sigma(\tilde{r}, \Omega)$ with sign of coupling σ . This is related to the distribution of couplings with amplitudes $\tilde{J} \leq \Omega$ having sign σ , $P_\sigma(\tilde{J}, \Omega)$ by $P_\sigma(\tilde{r}, \Omega) = (\alpha\tilde{J}/\tilde{r})P_\sigma(\tilde{J}, \Omega)|_{\tilde{J}=\Omega_0\tilde{r}^{-\alpha}}$. The normalization condition is given by $\sum_{\sigma=\pm} \int_0^\Omega d\tilde{J} P_\sigma(\tilde{J}, \Omega) = 1$. Accordingly, the normalization for the distribution of distances is given by $\sum_{\sigma=\pm} \int_\rho^\infty \tilde{r} dP_\sigma(\tilde{r}, \Omega) = 1$.

B. Finite temperature distribution function

The distribution functions for excited states can be derived from a generalized Master equation, employing the real space formulation introduced in Ref. [23]. To obtain that Master equation, we note that when the spin pair (i, j) forms a state $|s\rangle$ at RG scale $\Omega = J_{ij}$, the couplings between the spin pairs (l, i) and (j, m) are taken away while a coupling between spin pair (l, m) is newly created with renormalized coupling $\tilde{J}_{lm}$. In the representation of distances this corresponds to take the edges between spin pairs (l, i) and (j, m) with distances $r_{l,i} = R_L$ and $r_{j,m} = R_R$ away and to create an edge between spin pair (l, m) with renormalized distance $\tilde{r}_{lm} = \tilde{r}$, as shown in Fig. 1, thereby replacing the bare distance $r_{lm} = r$.

We derive the Master equation for the distribution function of distances at temperature T , $P_{\sigma,T}(\tilde{r}, \Omega)$ for the short ranged

model in Appendix A, where we find

$$\begin{aligned} & -\frac{d}{d\Omega} P_{\sigma,T}(\tilde{r}, \Omega) \\ &= \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega) \sum_{\sigma_L=\pm} \int_\rho^\infty dR_L \\ & \times \sum_{\sigma_R=\pm} \int_\rho^\infty dR_R P_{\sigma_L,T}(R_L, \Omega) P_{\sigma_R,T}(R_R, \Omega) \\ & \times \sum_{s=0}^3 p_s(E_s(\sigma'\Omega), T) \delta\left(\tilde{r} - \frac{R_LR_R}{\rho}\right) \delta_{\sigma,\sigma_L\sigma_R\sigma's}. \end{aligned} \quad (11)$$

Here, the state with energy $E_s(\sigma\Omega)$ at RG scale Ω with coupling sign σ is occupied with probability Eq. (5), where the partition sum is given by $Z(\Omega) = 2 + 2 \cosh(\beta\Omega/2)$. We make the product Ansatz

$$P_{\sigma,T}(\tilde{r}, \Omega) = P_{\sigma,T}(\Omega) P_T(\tilde{r}, \Omega), \quad (12)$$

where $P_{\sigma,T}(\Omega)$ is the probability at temperature T that the sign of any coupling $\tilde{J} \leq \Omega$ is σ .

Let us first find the distribution $P_T(\tilde{r}, \Omega) = \sum_{\sigma=\pm} P_{\sigma,T}(\tilde{r}, \Omega)$, summing over the sign of couplings σ . Performing the integral over one of the distances R_L , and with the normalization condition $\sum_{\sigma=\pm} P_{\sigma,T}(\Omega) = 1$ it follows that it is governed for $d\Omega \rightarrow 0$ by the Master equation

$$\begin{aligned} & -\frac{d}{d\Omega} P_T(\tilde{r}, \Omega) = P_T(\Omega, \Omega) \int_\rho^\infty dR_R \frac{\rho}{R_R} \\ & \times P_T(R_R, \Omega) P_T(\rho\tilde{r}/R_R, \Omega). \end{aligned} \quad (13)$$

This is the same Master equation, as was previously derived for the distribution of distances in the ground state of the short ranged AFM coupled spin chain [23]. It is independent of temperature T , and we recover the infinite disorder fixed point distribution

$$P_T(\tilde{r}, \Omega) = \frac{c(\Omega)}{\tilde{r}} \left(\frac{\rho}{\tilde{r}}\right)^{c(\Omega)} \theta(\tilde{r}/\rho - 1), \quad (14)$$

with $\theta(x)$ the unit step function, $\theta(x > 0) = 1$, and $\theta(x < 0) = 0$, and $c(\Omega) = \alpha/\Gamma_\Omega$, with $\Gamma_\Omega = \ln(\Omega_0/\Omega)$. Transforming back to the distribution function of couplings $P(\tilde{J}, \Omega) = \tilde{r}/(\alpha\tilde{J})P(\tilde{r}, \Omega)$, we find

$$P_T(\tilde{J}, \Omega) = \frac{1}{\Omega\Gamma_\Omega} \left(\frac{\Omega}{\tilde{J}}\right)^{1-1/\Gamma_\Omega} \theta(\Omega/\tilde{J} - 1). \quad (15)$$

which is the infinite randomness fixed point distribution function with width $\Gamma_\Omega = \ln(\Omega_0/\Omega)$, diverging to infinity for $\Omega \rightarrow 0$. Here, we find that it applies to the excited states of the XX chains, as well, and is independent of temperature T . Thus, with the Ansatz for $P_{\sigma,T}(\tilde{r}, \Omega)$, given by Eq. (12) it remains only to find the Bernoulli distribution for the sign of couplings $P_{\sigma,T}(\Omega)$. Insertion of Eq. (12) with the solution for $P_T(\tilde{r}, \Omega)$, given by Eq. (14), yields the Master equation for $P_{+,T}(\Omega)$ as

$$\begin{aligned} & \frac{d}{d\Omega} P_{+,T}(\Omega) = g_\alpha(\Omega)(1 - p_{+,T}(\Omega) \\ & \times P_{+,T}(\Omega)^2)(2P_{+,T}(\Omega) - 1). \end{aligned} \quad (16)$$

Here, we defined the function

$$g_\alpha(\Omega) = \frac{1}{\Omega} \left(\frac{1}{2\alpha} - \frac{1}{\ln \Omega_0/\Omega} \right). \quad (17)$$

The distribution $p_{+,T}$ is the probability, that the sign is not switched during an RG step, which is equal to the sum of occupation probabilities Eq. (5) of the two states, which do not yield a sign change under renormalization,

$$p_{+,T}(\Omega) = \cosh\left(\frac{\Omega}{2T}\right) / \left[1 + \cosh\left(\frac{\Omega}{2T}\right) \right]. \quad (18)$$

For large RG scale Ω , we thus find $p_{+,T}(\Omega \gg T) = 1$. In the high temperature limit, on the other hand, all states are occupied with equal probability $p_{+,T}(\Omega \ll T) = 1/2$. We know that initially all couplings are antiferromagnetic, so that $P_{+,T}(\Omega \gg T) = 1$. As the RG scale is lowered to $\Omega < T$, the sign changes can occur more frequently, when triplet states become occupied. When $\Omega \rightarrow 0$, both signs of the couplings become equally likely, $P_{+,T}(\Omega \rightarrow 0) = 1/2$. Indeed, we find that Eq. (16) has for $\Omega \leq 2T$ the solution for $\sigma \in \{+, -\}$

$$P_{\sigma,T}(\Omega) = \frac{1}{2} + \sigma \frac{1}{2} \left[\frac{\Omega}{2T} \right]^{\frac{7}{8\alpha}} \left[\frac{\ln(\frac{\Omega_0}{\Omega})}{\ln(\frac{\Omega_0}{2T})} \right]^{\frac{7}{4}}, \quad (19)$$

while $P_{+,T}(\Omega) = 1$ for $\Omega \geq 2T$, $P_{-,T}(\Omega) = 0$ for $\Omega \geq 2T$.

Thus the distribution of couplings smaller than RG scale Ω is given by

$$P_{\sigma,T}(\tilde{J}, \Omega) = \frac{P_{\sigma,T}(\Omega)}{\Omega \Gamma_\Omega} \left(\frac{\Omega}{\tilde{J}} \right)^{1-1/\Gamma_\Omega} \theta(\Omega/\tilde{J} - 1), \quad (20)$$

where $P_{\sigma,T}(\Omega)$ as given by Eq. (19) is the probability that the coupling $\tilde{J} < \Omega$ has the sign σ . For $\Omega \rightarrow 0$ that probability approaches $P_{\sigma,T}(\Omega \rightarrow 0) \rightarrow 1/2$.

Having the full distribution function of couplings Eq. (20), allows us to derive finite temperature properties of this model. Since the spectrum of the pair states of XX coupled spins is symmetric in a sign change of exchange coupling, see Fig. 2, the sign of the coupling does not enter physical properties in that case, so that it is sufficient to have the distribution function of coupling amplitudes

$$\begin{aligned} P_T(\tilde{J}, \Omega) &= \sum_{\sigma=\pm} P_{\sigma,T}(\tilde{J}, \Omega) \\ &= \frac{1}{\Omega \Gamma_\Omega} \left(\frac{\Omega}{\tilde{J}} \right)^{1-1/\Gamma_\Omega} \theta(\Omega/\tilde{J} - 1), \end{aligned} \quad (21)$$

with $\Gamma(\Omega) = \ln \Omega_0/\Omega$, which is the infinite randomness fixed point distribution. We have shown that it is valid for this short ranged model for any temperature T .

IV. LONG RANGE COUPLING

Next, we apply the SDRG-X procedure to spin chains with power law XX couplings, Eq. (1).

A. SDRG-X rules

The strongest coupled spin pair (i, j) , highlighted in Fig. 3, forms one of four pair states $|s_{ij}\rangle$. The couplings between all remaining spin pairs (l, m) are renormalized according to

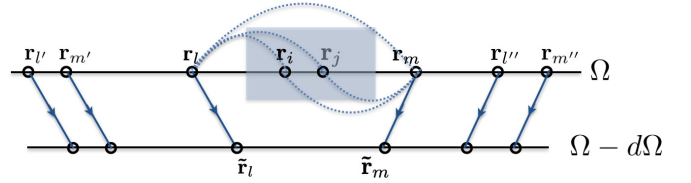


FIG. 3. Strong disorder RG step for bond disordered long range coupled spin chains: Decimation of the strongest coupled spin pair (i, j) (shaded), whose coupling defines the RG scale Ω . It is followed by renormalization of spin positions $\mathbf{r}_l \rightarrow \tilde{\mathbf{r}}_l$, while the RG scale is reduced to $\Omega - d\Omega$. The initial couplings are indicated by blue dashed lines.

Eq. (4), depending on that state $|s_{ij}\rangle$. When it is in the singlet state $|0_{ij}\rangle$ the renormalized coupling is given by [29]

$$\tilde{J}_{lm}[0_{ij}] = J_{lm} - \frac{(J_{li} - J_{lj})(J_{im} - J_{jm})}{J_{ij}}. \quad (22)$$

The Hilbert space of unentangled triplet states $|1_{ij}\rangle = |\uparrow_i \uparrow_j\rangle$ and $|2_{ij}\rangle = |\downarrow_i \downarrow_j\rangle$, denoted in Eq. (4) by D_1 , is degenerate with eigenenergy $E_1 = E_2 = 0$. Deriving all terms to second order in V given in Eq. (4), we find

$$\hat{H}_{\text{eff}}[D_1] \approx \sum_{s=1}^2 |s\rangle \langle s| \sum_{l < m} \tilde{J}_{lm}[s] (S_l^x S_m^x + S_l^y S_m^y) + \hat{H}_{\text{new}}[D_1], \quad (23)$$

where the first term contains all terms which do not change the form of the interaction, resulting in the renormalized interaction [29]

$$\tilde{J}_{lm}[s] = J_{lm} - \frac{J_{li}J_{jm} + J_{lj}J_{im}}{J_{ij}}, \quad (24)$$

for $s = 1, 2$. The remaining terms have a different form. These were not taken into account in the implementation of SDRG-X for the long range coupled AFM XX-spin model presented in Ref. [29], but were included recently in Ref. [31]. These terms are given by

$$\begin{aligned} \hat{H}_{\text{new}}[D_1] &= \sum_{l < m} [(h_l S_l^z + h_m S_m^z) \Sigma^z \\ &\quad + \tilde{J}_{lm}[12] (S_l^x S_m^x - S_l^y S_m^y) \Sigma^x \\ &\quad + (S_l^y S_m^x + S_l^x S_m^y) \Sigma^y], \end{aligned} \quad (25)$$

where we introduced a pseudospin Σ acting on the Hilbert space of degenerate two levels, D_1 with components

$$\Sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (26)$$

The z components of the spins and the pseudospins of the degenerate two-level system are coupled by

$$h_l = -\frac{J_{li}J_{lj}}{J_{ij}}, \quad h_m = -\frac{J_{lm}J_{jm}}{J_{ij}}, \quad (27)$$

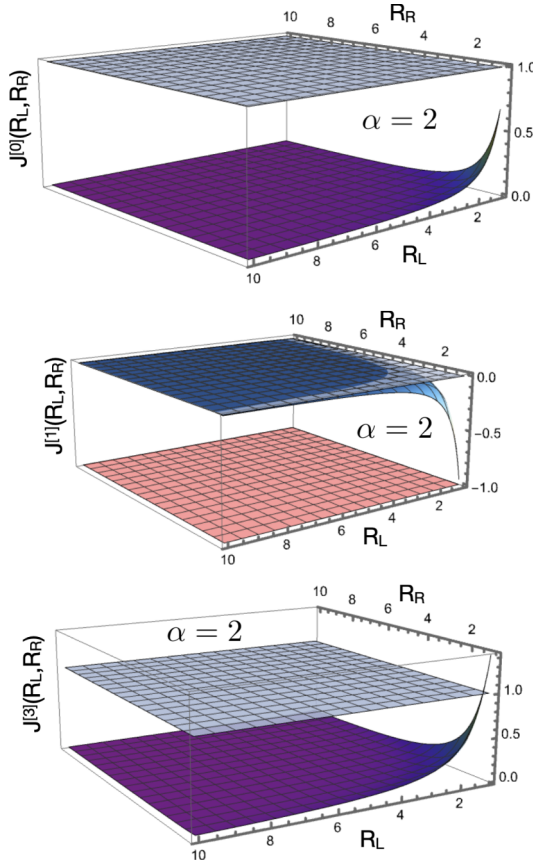


FIG. 4. Renormalized couplings $\tilde{J}[s]$ for different projection states $s = 0$ (top), $s = 1, 2$ (middle), and $s = 3$ (bottom) in units of RG scale Ω as a function of distances $R_L = r_{li}$ and $R_R = r_{jm}$ in units of RG distance $\rho = (\Omega/\Omega_0)^{-1/\alpha}$ for $\alpha = 2$.

The three-point interaction between the transverse components of two spins and the pseudospin is

$$\tilde{J}_{lm}[1, 2] = -\frac{J_{li}J_{im} + J_{lj}J_{jm}}{J_{ij}}. \quad (28)$$

Note that when keeping only the couplings between adjacent spins, all these renormalized interaction terms vanish. Thus these terms only emerge when taking into account longer range interactions.

When the pair is in the entangled triplet state $|3_{ij}\rangle$, the renormalized coupling is given by [29]

$$\tilde{J}_{lm}[3_{ij}] = J_{lm} + \frac{(J_{li} + J_{lj})(J_{im} + J_{jm})}{J_{ij}}. \quad (29)$$

Let us next explore these SDRG rules in more detail. The renormalized couplings between spin pair (l, m) depend on the initial coupling between spins (l, m) , the couplings between the removed spins (i, j) , and the four couplings between spins (l, m) , and the removed spins (i, j) , shown schematically by the blue dashed lines in Fig. 3. These couplings are very different for different projection states $s \in \{0, 1, 2, 3\}$ and depend highly nonlinearly on their distances. We plot them (in units of RG scale Ω) in Fig. 4 for $\alpha = 2$, in Fig. 5 for $\alpha = 1/2$ as a function of distances $R_L = r_{li}$ and

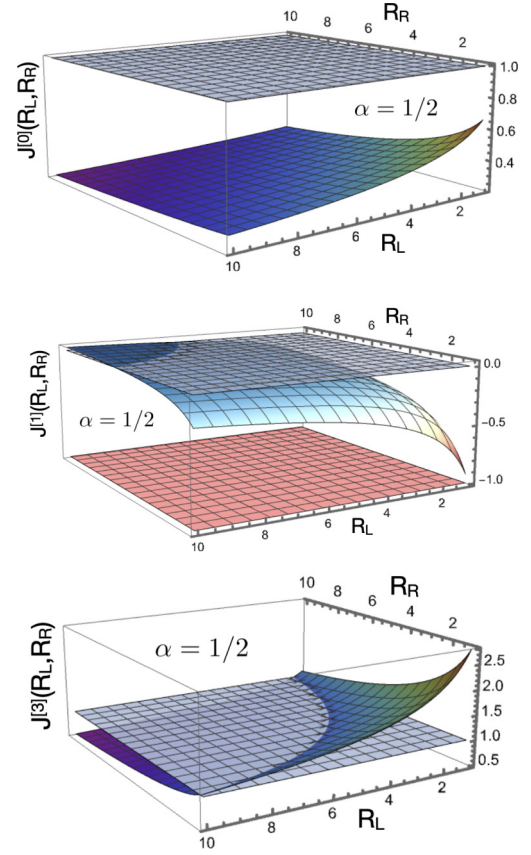


FIG. 5. Same as Fig. 4 but for $\alpha = 1/2$.

$R_R = r_{jm}$ (in units of distance $\rho = (\Omega/\Omega_0)^{-1/\alpha}$ between the removed spin pair).

Singlet state. We see that, when the removed pair is in a singlet state, the renormalized pairing $\tilde{J}[0_{ij}]$, Eq. (22) is smaller than the RG scale Ω for all possible distances, see Figs. 4 (upper), 5 (upper), so that this SDRG step is well defined, for any $\alpha > 0$ [23].

Unentangled triplet states. When the removed pair is in one of the unentangled triplet states, the renormalized pairing $\tilde{J}[1_{ij}] = \tilde{J}[2_{ij}]$ switches sign for a range of small distances, but its amplitude remains smaller than the RG scale Ω for all possible distances, see Figs. 4(bottom) and 5(middle). For decreasing power α , the range of distances for which the sign of the coupling changes becomes larger, so that the probability of ferromagnetic renormalized couplings increases. In Fig. 6, we plot the probability that the sign changes in an RG step in which an unentangled triplet state is formed as a function of α . We see that it is finite for any α and approaches one for $\alpha \ll 1$. Therefore ferromagnetic couplings can be encountered in subsequent renormalization steps.

However, as we reviewed above, when the pair is in the degenerate space of the unentangled triplet space, also interactions between spins and pseudo spins h_l, h_m and three-point couplings $\tilde{J}_{lm}[1, 2]$ are generated [31]. We plot the ratio of the renormalized couplings which preserve the form of the couplings $\tilde{J}_{lm}[1]$ with the sum of the local fields $h_l + h_m$ and with the new couplings $\tilde{J}_{lm}[1, 2]$ in Fig. 7 top and bottom, respectively. We see that for $R_L, R_R \rightarrow \rho$, where the couplings

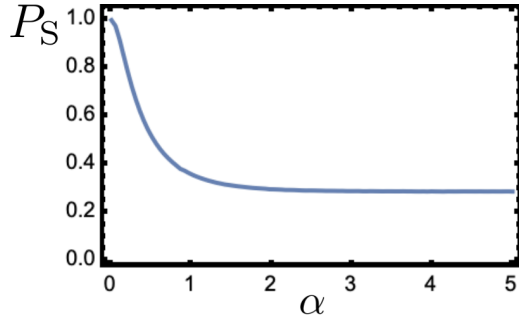


FIG. 6. The probability of sign change, when an unentangled triplet state forms, as a function of α .

are strongest, that ratio is maximal. It turns out that at $R_L = R_R = \rho$ both ratios are identical and exceed 1 for $\alpha \gg 1$, so that the conventional coupling dominates both the generated couplings h_l, h_m and $\tilde{J}_{lm}[1, 2]$, as seen in Fig. 8.

Entangled triplet state. When the removed pair is in the entangled triplet state, the renormalized pairing $\tilde{J}[3_{ij}]$ remains positive for all possible distances. However, it can exceed the RG scale Ω for a finite range of distances R_L, R_R as seen in Figs. 4 and 5 (bottom), thus violating the consistency condition for the SDRG. In Fig. 9, we plot the probability that the renormalized distance $\tilde{r}$ is smaller than the distance of the removed pair ρ , as a function of α , when the removed pair is in the entangled triplet state, $|3_{ij}\rangle$. We find that a violation of the SDRG condition can occur for any α , and its probability increases sharply for $\alpha < 2$ as seen in Fig. 9. Indeed, in Ref. [29], we found by numerical solution of the SDRG equation and numerical exact diagonalization that at small power exponent, $\alpha \ll 2$ the excited states are no longer random pair states but rather become (imperfect) rainbow states, where a finite number of subsequent pairs overarch previous ones. This is in accordance with our finding above, that, when the removed pair is in the entangled triplet state, the renormalized coupling can be larger than the RG scale Ω for $\alpha < 2$,

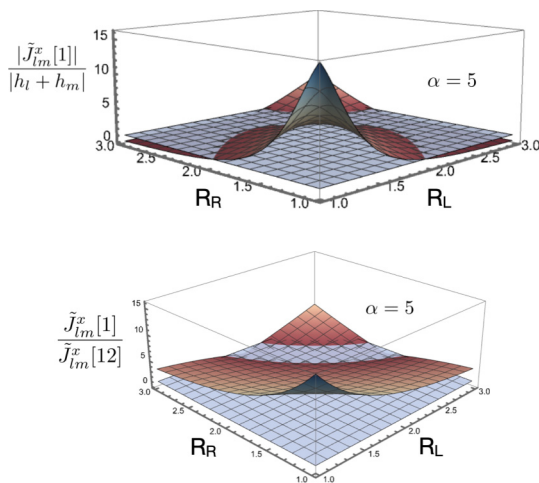


FIG. 7. The ratio of the renormalized coupling $|\tilde{J}_{lm}[1]|$ with the sum of the RG generated fields $(h_l + h_m)$ (top) and the new coupling strength $|\tilde{J}_{lm}[1, 2]|$ (bottom), respectively, as a function of R_L and R_R in units of ρ .

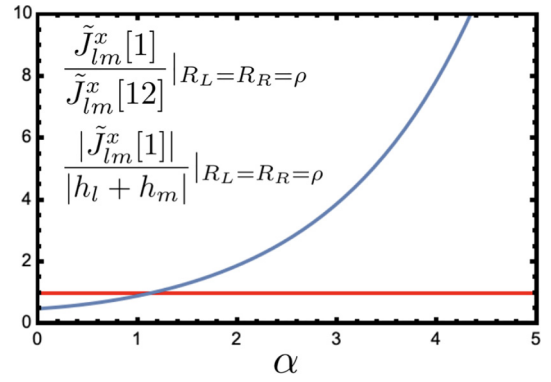


FIG. 8. Ratio of renormalized couplings $|\tilde{J}_{lm}[1]|$ with the sum of the RG generated couplings $|h_l + h_m|$, and the 3-point coupling strength $|\tilde{J}_{lm}[1, 2]|$, respectively, for $R_L = R_R = \rho$.

and accordingly, the renormalized distance smaller than the previous one $\tilde{r} < \rho$, for a finite range of distances of removed pairs R_L, R_R . Then, in the next RG step, the following pair is forced to *overarch* the previous one, as in a rainbow state.

Since the sign of the couplings can switch with finite probability during an RG step, Fig. 6, it has to be traced as a function of the RG scale Ω . Therefore we need to define the distribution function $P_{\sigma,T}(\tilde{r}, \Omega)$, the pdf of the renormalized distances between adjacent spins $\tilde{r}$ at RG scale Ω with coupling sign σ at temperature T . As the renormalization proceeds to lower RG scales, the sign of the coupling may switch frequently, as all renormalized couplings in Eqs. (22), (24), and (29) may switch their sign. When the couplings have either sign, all pair states can result according to Eqs. (22), (24), and (29) for sufficiently small α in effective couplings equal or exceeding the RG scale Ω for a finite range of distances R_L, R_R , so that the SDRG condition is violated and the formulation of the SDRG in terms of pairs of spins is only valid for sufficiently large $\alpha \gg 1$. For smaller α the SDRG rules need to be adopted, for example by considering larger clusters of spins at each RG step. In the following, we therefore focus on the regime $\alpha \gg 1$, where we have established above that the pair SDRG-X scheme is valid for all excited states.

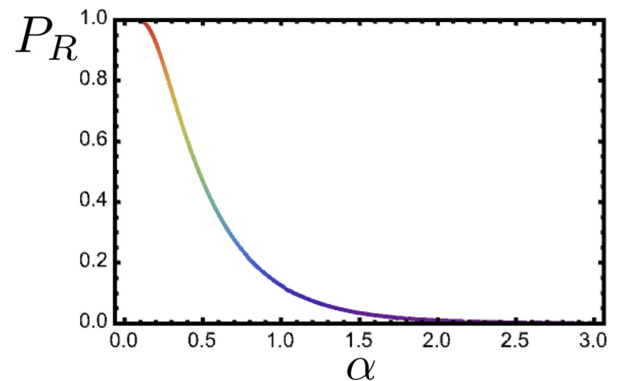


FIG. 9. The probability that the renormalized distance $\tilde{r}$ becomes smaller than the distance of the removed pair ρ as a function of α when the removed pair is in the entangled triplet state, $|3_{ij}\rangle$ for $\gamma = 0$, the XX-model.

B. Finite temperature distribution function

The distribution function $P_{\sigma,T}(\tilde{r}, \Omega)$ for $\alpha \gg 1$ for $\tilde{r} \gtrsim \rho$ is governed by the Master equation

$$-\frac{d}{d\Omega}P_{\sigma,T}(\tilde{r}, \Omega) = P_T(\Omega, \Omega)P_{\sigma,T}(\tilde{r}, \Omega) + \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega)C_{\sigma,\sigma',T}(\tilde{r}, \Omega, \alpha), \quad (30)$$

see Appendix B for its derivation. We use the notation $P_T(\Omega, \Omega) = \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega)$ and define the function

$$C_{\sigma,\sigma',T}(\tilde{r}, \Omega, \alpha) = \sum_{s=0}^3 p_{s,T}(E_s(\sigma'\Omega)) \sum_{\sigma_L, \sigma_R, \sigma_{lm}} P_{\sigma_{lm},T}(\Omega) \int_{\rho}^{\infty} dR_L dR_R P_{\sigma_L,T}(R_L, \Omega) P_{\sigma_R,T}(R_R, \Omega) [\delta(\tilde{r} - f[s](R_L, R_R, \rho))_{\sigma_{lm}\sigma'\sigma_L\sigma_R} \\ \times (\delta_{\sigma,\sigma_{lm}}|_{R_L, R_R \in A_s} + \delta_{\sigma,\sigma_s\sigma_L\sigma_R\sigma'}|_{R_L, R_R \in A_s^C}) - \delta(\tilde{r} - (R_L + \rho + R_R))\delta_{\sigma,\sigma_{lm}}], \quad (31)$$

which contains all terms arising due to the renormalization of the couplings, when the removed pair is in one of the states $s = 0, 1, 2, 3$, with energy $E_s(\sigma\Omega)$ with probability $p_{s,T}(E_s(\sigma\Omega))$. In order to trace the sign of the renormalized coupling, when transforming to RG rules for the renormalized distance $\tilde{r}$ we introduced the region A_s of distances R_L, R_R , where the bare coupling is larger than the renormalization correction when the pair is in state s , allowing no sign change. The complementary region is A_s^C . Aiming for an iterative solution, we first solve the equation without the renormalization terms

$$-\frac{d}{d\Omega}P_{\sigma,T}^0(\tilde{r}, \Omega) = P_T^0(\Omega, \Omega)P_{\sigma,T}^0(\tilde{r}, \Omega), \quad (32)$$

which has the solution

$$P_{\sigma,T}^0(\tilde{r}, \Omega) = P_{\sigma,T}^0(\Omega) \frac{1}{2\tilde{r}} \left(\frac{\rho}{\tilde{r}}\right)^{1/2} \theta(\tilde{r} - \rho), \quad (33)$$

where the coupling signs are distributed as $P_{\sigma,T}^0 = \delta_{\sigma,+}$, since all bare couplings are antiferromagnetic. This distribution is identical to the strong disorder fixed point distribution of width 2α [20–23]. In order to derive a correction to distribution (33), we insert it into the correction term Eq. (31), aiming for an iterative solution. Thereby, the Master equation Eq. (30) simplifies to

$$-\frac{d}{d\Omega}P_{\sigma,T}(\tilde{r}, \Omega) = P_T(\Omega, \Omega)P_{\sigma,T}(\tilde{r}, \Omega) + P^0(\Omega, \Omega)C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha), \quad (34)$$

where the function $C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha)$ is defined by replacing all distribution functions in Eq. (31) by Eq. (33). The integral over one of the distances in $C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha)$ can now be performed, as outlined in Appendix C.

We insert the occupation probability $p_{s,T}(E_s(\sigma'\Omega))$ Eq. (5) into the resulting equation for $C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha)$, Eq. (C1) and perform the integrals numerically. The result for $C_{\sigma+,T}^0(\tilde{r}, \Omega \gg T, \alpha)$ is plotted as a function of $\tilde{r}$ for various α in Fig. 10 for $\sigma = +$ (upper figure) and $\sigma = -$ (lower figure). We see that $C_{+,T}^0(\tilde{r}, \Omega, \alpha)$ is for $\alpha > 2$ vanishing at $\tilde{r} = \rho$. It is maximal at $\tilde{r} \approx 3\rho$. For $\tilde{r} > 3\rho$, it decays, changing sign and converging to zero for $\tilde{r} \gg 3\rho$. $C_{-,T}^0(\tilde{r}, \Omega, \alpha)$ is for $\alpha > 2$ vanishing at $\tilde{r} = \rho$, maximal at $\tilde{r} \approx 2\rho$ and decaying to zero for $\tilde{r} \gg 2\rho$. For the ground state (corresponding to $T = 0$ K), we found in Ref. [23] that the correction to the Master equation is vanishing exactly at $\tilde{r} = \rho$ for all α . We find here that at finite temperature T for $\alpha \gg 2$ the functions

$C_{+,T}^0(\tilde{r}, \Omega, \alpha)$ and $C_{-,T}^0(\tilde{r}, \Omega, \alpha)$, defined in Eqs. (C2) and (C3), are vanishing at $\tilde{r} = \rho$. Rewriting $C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha) = \frac{1}{\rho} \tilde{C}_{\sigma+,T}(x = \tilde{r}/\rho, \alpha)$ we solve the Master equation, a 1st order inhomogeneous, ordinary differential equation, as outlined in Appendix C, and find that its solution is given by

$$P_{\sigma}(\tilde{r}, \Omega) = \frac{1}{2\tilde{r}} \left(\frac{\rho}{\tilde{r}}\right)^{1/2} \theta(\tilde{r} - \rho) \times \left[\delta_{\sigma,+} - \int_1^{\tilde{r}/\rho} dx' \sqrt{x'} \tilde{C}_{\sigma+,T}(x', \alpha) \right]. \quad (35)$$

The normalization condition $\sum_{\sigma=\pm} \int_{\rho}^{\infty} d\tilde{r} P_{\sigma,T}(\tilde{r}, \Omega) = 1$ is fulfilled exactly. This can be seen directly by performing the integral over $\tilde{r}$ with integration by parts inserting $\tilde{C}_{\sigma+,T}(x, \alpha)$, using Eq. (31), performing the integral over the Dirac-delta functions and then the sum over σ . In Fig. 11 we plot the ratio of the correction to the distribution function and the SDRG distribution $P^0(\tilde{r}, \Omega)$, $\delta P(\tilde{r}, \Omega) = [\sum_{\sigma} P_{\sigma}(\tilde{r}, \Omega) - P^0(\tilde{r}, \Omega)]/P^0(\tilde{r}, \Omega)$ by inserting Eq. (35) for $\Omega = 2T$ as a function of $\tilde{r}$ for various values of power α .

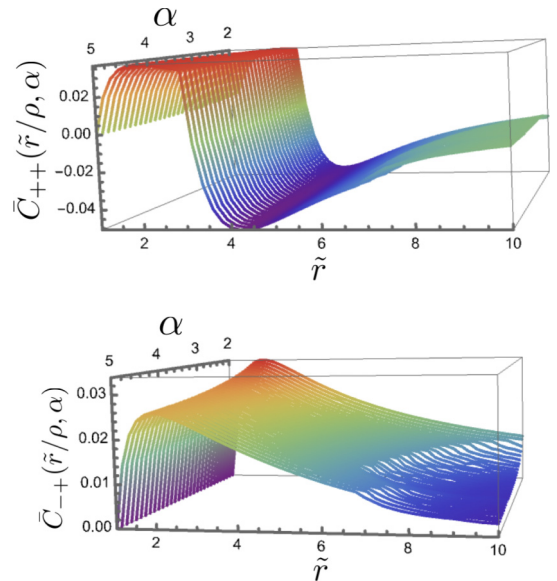


FIG. 10. Line plot of the correction term in the Master equation $\tilde{C}_{\sigma+,T}(x = \tilde{r}/\rho, \alpha) = \rho C_{\sigma+,T}^0(\tilde{r}, \Omega, \alpha)$ (a) for $\sigma = +$, Eq. (C2) and (b) $\sigma = -$, Eq. (C3) at finite temperature for $\Omega = 2T$ as a function of $\tilde{r}$, for various values of power α .

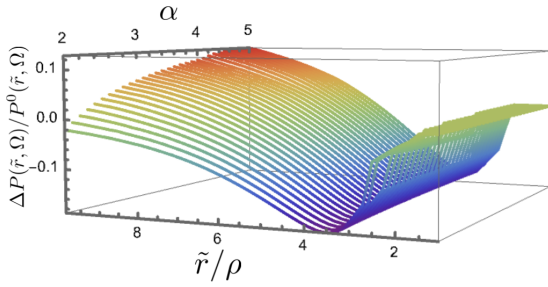


FIG. 11. Line plot of the ratio of the correction to the distribution function and the SDRG distribution for $\Omega = 2T$ as a function of $\tilde{r}$, for various values of power α .

Transforming back to the distribution of couplings $P^0(\tilde{J}, \Omega)$, using $P_{\sigma,T}(\tilde{J}, \Omega)|_{\tilde{J}=\Omega_0\tilde{r}^{-\alpha}} = P_{\sigma,T}(\tilde{r}, \Omega) \times \tilde{r}/(\alpha\tilde{J})$, we recover the SDRG fixed point distribution Eq. (15) with finite width $\Gamma = 2\alpha$ with small corrections for $\tilde{J} < \Omega$. Thus we find that the power law couplings result in the SDRG distribution with finite width, not only for the ground state $s = 0$, as derived in Ref. [22], but also for finite temperature T for $\alpha \gg 2$. These results imply that for $\alpha \gg 2$.

$$P_{\sigma T}(\Omega, \Omega) = \frac{1}{2\alpha\Omega} \delta_{\sigma,+}, \quad (36)$$

which is the SDRG fixed point result, previously obtained for the ground state [21,22]. Here, we derived that result to be valid for any finite temperature T , as long as $\alpha \gg 1$.

V. PHYSICAL PROPERTIES

Having derived the distribution function of couplings at finite temperature T $\alpha \gg 1$, we are all set to derive thermodynamic and dynamic properties of bond disordered spin chains, Eq. (1), as summarized in the following.

Magnetic susceptibility. It has at finite temperature two terms; (1) spin pairs whose coupling J is smaller than temperature T are broken up into two free $S = 1/2$ - spins contributing a paramagnetic Curie law susceptibility, and (2). Spin pairs in unentangled triplet states, which are not broken up when $J > T$, act as free paramagnetic spins with $S = 1$. Thus the total magnetic susceptibility is

$$\chi(T) = n_{S=1/2}(T) \frac{1}{T} + n_{S=1}(T) \frac{4}{T}. \quad (37)$$

The density of paramagnetic spins with $S = 1/2$, $n_{S=1/2}(T)$ is governed by the differential equation

$$\frac{dn_{S=1/2}(\Omega)}{d\Omega} = 2P(J = \Omega, \Omega)n_{S=1/2}(\Omega). \quad (38)$$

Integration yields

$$n_{S=1/2}(T) \sim \exp \left[2 \int^T d\Omega P(\Omega, \Omega) \right]. \quad (39)$$

The density of spins with $S = 1$, $n_{S=1}(T)$ is

$$n_{S=1}(T) \sim \int_T d\Omega P(\Omega, \Omega) [p_{1,T}(\Omega) + p_{2,T}(\Omega)], \quad (40)$$

where the occupation probability of states $|s\rangle$, $s = 1, 2$ is $p_{1,T}(\Omega) = p_{2,T}(\Omega) = 1/\{2 + 2 \cosh[\Omega/(2T)]\}$.

Short range coupling. Inserting the IRFP distribution Eq. (20), noting that $P(\Omega, \Omega) = \sum_{\sigma} P_{\sigma,T}(\Omega, \Omega) = 1/[\ln(\Omega_0/\Omega)\Omega]$, we find

$$\chi(T) \sim \frac{1}{\ln(\Omega_0/T)^2} \frac{1}{T} + \frac{c}{\ln(\Omega_0/T)} \frac{1}{T}, \quad (41)$$

where c is a constant. Thus the magnetic susceptibility is dominated by paramagnetic $S = 1$ pairs, the second term in Eq. (41).

Long range coupling. Insertion of $P(\Omega, \Omega) = 1/(2\alpha\Omega)$, and performing the integration over Ω yields

$$n_{S=1/2}(T) \sim (T/\Omega_0)^{1/\alpha}. \quad (42)$$

Thereby, we find the magnetic susceptibility

$$\chi(T) \sim T^{\frac{1}{\alpha}-1} + \frac{c}{\alpha} T^{-1}, \quad (43)$$

where c is a constant. Thus, we conclude that the magnetic susceptibility is dominated by paramagnetic $S = 1$ - pairs, yielding the second term in Eq. (43), whose weight is decreasing with increasing α .

Distribution of singlet lengths. The distribution of distances l between spins which are bound into pairs $P(l)$ is determined by

$$P(l) \sim \frac{n_{S=1/2}(\Omega)}{n_0} P(\Omega, \Omega)|_{\Omega=\Omega_0 l^{-\alpha}} \left| \frac{d}{dl} \Omega_0 l^{-\alpha} \right|. \quad (44)$$

Inserting Eqs. (39) and (36), we find

$$P(l) \sim l^{-2}, \quad (45)$$

as previously derived for the ground state of the XX model [22]. Here we find that that it is valid also at finite temperature T for $\alpha \gg 1$.

Spin correlation function, concurrence. When the spins in the chain are in a pure state, such as the random pair state, the concurrence between spins at site i and site j is given by the correlation function $C_n = |\langle \psi | \sigma_i^x \sigma_j^y | \psi \rangle|_{n=|i-j|}$. The ensemble average yields both for short range [16,33] and long range [22] disordered antiferromagnetic quantum spin chains for odd n by $\langle C_n \rangle \sim P(n) \sim 1/n^2$. The typical correlation function is dominated by deviations from the random pair state, and is known for the short range model to be given by $C_{\text{typ}}(n) = \langle \exp(\ln C_n) \rangle \sim \exp(-k\sqrt{n})$ [16], while for the long range model, the typical value decays more slowly as $C_{\text{typ}}(n) \sim n^{-2-\alpha}$ [22]. Since the distribution function for the pair length $P(l) \sim l^{-2}$ remains valid for the random pair state at finite temperature, these results for the correlation function remain valid at finite temperature T .

Entanglement entropy. The entanglement entropy of a subsystem of length n with the rest of the chain is for a specific random pair state given by $S_n = M_T \ln 2$, where M_T is the number of singlets $s = 0$ or entangled triplet states $s = 3$ at temperature T , crossing the partition of the subsystem. The ensemble average of the entanglement entropy is accordingly given by $\langle S_n \rangle = \langle M_T \rangle \ln 2$. The average of M_T can be derived from the distribution of singlet lengths $P_s(l)$. For the ground state $T = 0$, the leading term was found to be given by [22,32,33] $S_n \sim \frac{1}{2} \ln 2 \int_0^n dl l P(l)$. At finite temperature

T , we accordingly get $S_n \sim \frac{1}{2} \ln 2 \int_0^n dl l P(l) p_T(\Omega_l)$, where $p_T(\Omega_l) = \cosh[\Omega_l/(2T)] / \{1 + \cosh[\Omega_l/(2T)]\}$ is the probability that the pair with length l is in an entangled state. Here, $\Omega_l = \Omega_0 l^{-\alpha}$. This yields with Eq. (45) a logarithmic growth of entanglement entropy with subsystem length n with a correction term

$$S_n(T) \sim \frac{1}{6} \ln 2 \left[\ln n - \frac{2}{\alpha} \int_{z_n}^{z_0} dz \frac{1}{(1+z)^2 \ln z} \right], \quad (46)$$

where $z_n = \exp[\Omega_n/(2T)]$ and $z_0 = \exp[\Omega_0/(2T)]$. The first term has the functional form of the entanglement entropy of critical quantum spin chains [34] with effective central charge $\bar{c} = \ln 2$ [32], as was found previously in Ref. [22] for the ground state of the XX model with power law interactions, as confirmed there by numerical exact diagonalization. Here, we found a correction term at finite temperature T for $\alpha \gg 1$. For $T \rightarrow \infty$, the last term in the brackets becomes $-(1/2) \ln n$, so that the effective central charge halves to $\bar{c}_{T \rightarrow \infty} = \frac{1}{2} \ln 2$.

Entanglement entropy growth after a quantum quench. Preparing the system in an unentangled state, such as a Néel state $|\psi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\ldots\rangle$, the entanglement dynamics can be monitored by the time dependent entanglement entropy of a subsystem with the rest of the system $S(t)$. When entanglement is generated by singlet or entangled triplet state across the partition, the entanglement entropy at time t after the global quench is proportional to the number of such pairs formed at RG-scale $\Omega \sim 1/t$ [35–37]. Neglecting the history of previously formed pairs, the number of newly formed pairs at RG scale Ω , n_Ω is $dn_\Omega = P(J = \Omega, \Omega) d\Omega$ [32]. Substituting Eq. (36) we find $n_\Omega = \frac{1}{2\alpha} \ln(\Omega)$. Inserting $\Omega \sim 1/t$, the entanglement entropy increases with time as

$$S(t) = S_p \frac{1}{2\alpha} \ln(t), \quad (47)$$

with the time-averaged contribution of pairs of spins $S_p = 2 \ln 2 - 1$, when the initial state is a Néel state [35]. Then, only singlet and entangled triplet states are populated in the RSRG-t flow, contributing equally. This coincides with the result found in Ref. [30]. For the nearest neighbor XX spin chain with random bonds, the growth after a global quench is slower, $S(t) \sim \ln[\ln(t)]$ [35].

VI. CONCLUSIONS

We extend the recently introduced real space representation of the strong disorder renormalization group for disordered short range and long range coupled quantum spin chains [23] to study excited states and finite temperature properties. For interactions cutoff at short range, the infinite randomness fixed point distribution is found to hold also for the excited states and thus at finite temperature. For long range interactions, we find for $\alpha \gg 1$ small corrections to the strong disorder fixed point distribution, which depend on power exponent α . The value of the distribution at $J = \Omega$ is found to be independent of temperature there. We derive the resulting temperature dependence of the magnetic susceptibility, which turns out to be dominated by paramagnetic $S = 1$ spins resulting in a

Curie law susceptibility. While the distribution function of pair lengths, and the spin correlation function are found to be the same as in the ground state, we find that the entanglement entropy is diminished as the temperature is increased saturating at high temperature at half its value in the ground state.

For small values $\alpha < 2$, however, corrections to the SDRG are found to diverge and new coupling and field terms emerge. Then, the SDRG method has to be extended to include the renormalization flow of these new couplings [31]. The three-spin interactions arising there, Eq. (25), can possibly be treated with an extension of the SDRG method to multi-spin interactions, recently introduced in Ref. [38]. We found in Ref. [29] with numerical methods that for $\alpha < 2$, excited states are no longer random pair states, but rather (imperfect) rainbow states. Here, we found evidence that the probability that the renormalized distance is smaller than their distance at the previous RG step increases strongly for $\alpha < 2$, as seen in Fig. 9. When that occurs the next pair is enforced to overarch the previous one, initializing a rainbow state. However this amounts to a violation of the SDRG condition, according to which the renormalized couplings should be smaller than the bare couplings. Therefore, for $\alpha < 2$, the SDRG-X method has to be extended to take into account the formation of larger clusters of spins, forming overarching rainbow states, which we leave for future research. The real space representation of the SDRG-X method, introduced here, may thereby be applied to situations where larger clusters of spins than pairs need to be considered, like in spin chains at small $\alpha < 2$ and in higher dimensional disordered spin systems, where frustrations may occur.

ACKNOWLEDGMENTS

I acknowledge gratefully the hospitality of the division of condensed matter at the Max-Planck institute for physics of complex systems and of the advanced study group on *strongly correlated extreme fluctuations*, lead by Abbas Saberi.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF THE FINITE TEMPERATURE MASTER EQUATION FOR SHORT RANGE COUPLINGS

Following the argumentation given in Ref. [18] and adopting it to the distribution function of distances at temperature T , $P_{\sigma,T}(r, \Omega)$, the distribution at RG scale $\Omega - d\Omega$, lowered by an infinitesimal amount $d\Omega$, is related to the distribution

function at RG scale Ω by

$$P_{\sigma,T}(\tilde{r}, \Omega - d\Omega) = \left\{ P_{\sigma,T}(\tilde{r}, \Omega) + d\Omega \sum_{\sigma'=\pm} P_{\sigma',T}(\Omega, \Omega) \sum_{s=0}^4 p_s(E_s(\sigma'\Omega), T) \int_{\rho}^{\infty} dR_L dR_R \sum_{\sigma_L, \sigma_R=\pm} P_{\sigma_L,T}(R_L, \Omega) P_{\sigma_R,T}(R_R, \Omega) \right. \\ \left. \times \left[\delta\left(\tilde{r} - \frac{R_L R_R}{\rho}\right) \delta_{\sigma, \sigma_L \sigma_R \sigma'} - \delta(\tilde{r} - R_L) \delta_{\sigma, \sigma_L} - \delta(\tilde{r} - R_R) \delta_{\sigma, \sigma_R} \right] \right\} / \left[1 - 2d\Omega \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega) \right]. \quad (\text{A1})$$

Here, the second term on the right-hand side of Eq. (A1) accounts for the addition of a renormalized bond at distance $\tilde{r}$, determined by the RG rule Eq. (10) with coupling sign σ , as given by the product of the signs of the three couplings which enter the RG rule multiplied by σ_s , accounting for the sign introduced by the RG rule Eq. (9). The addition of that bond occurs with probability $d\Omega \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega)$, the probability to add a bond of either sign σ' in the RG step of width $d\Omega$ at temperature T . The following two terms take into account the removal of the two bonds with distance R_L , R_R , and sign σ_L , σ_R , respectively. These terms are integrated over all distances R_L , R_R exceeding ρ , which is by definition the smallest distance at RG step Ω . In order to normalize the distribution function, we need to divide the right side of Eq. (A1) by $1 - 2d\Omega \sum_{\sigma} P_{\sigma,T}(\Omega, \Omega)$, the probability that bonds are not removed during the RG step $d\Omega$. Next, performing the integrals over the last two delta functions and using the normalization condition $\int_{\rho}^{\infty} dr \sum_{\sigma} P_{\sigma,T}(r, \Omega) = 1$, performing in each term the sum over sign σ , we find in the limit $d\Omega \rightarrow 0$ the Master equation for the short ranged model, Eq. (11).

APPENDIX B: DERIVATION OF THE MASTER EQUATION WITH LONG RANGE COUPLINGS

In deriving the Master equation for the distribution function $P_{\sigma,T}(\tilde{r}, \Omega)$, at RG scale Ω with coupling sign σ at temperature T , we focus on the parameter regime $\alpha \gg 1$, where we established that the SDRG condition $\tilde{r} \geq \rho$ holds. We derive $P_{\sigma,T}(\tilde{r}, \Omega)$ for $\tilde{r} \gtrsim \rho$, where the renormalized coupling strength has the conventional form $\tilde{J}_{lm}^x$ [1], Eq. (24), as it always exceeds the generated unconventional couplings $\tilde{J}_{lm}^x$ [1, 2], Eq. (28) and h_m , h_l , Eq. (27). As noted in Ref. [23], even though all spin pairs are interacting with each other, the distribution of the $N - 1$ nearest neighbor distances in the chain contains all information on all couplings, since the distances between all nonadjacent spins in the chain are functions of those $N - 1$ adjacent distances. When a spin pair (i, j) forms a state $|s\rangle$, $s \in \{0, 1, 2, 3\}$ at RG scale $\Omega = J_{ij}$, corresponding to a distance $\rho = (\Omega_0/\Omega)^{1/\alpha}$, the two bonds between the two adjacent spin pairs (l, i) and (j, m) , shown in Fig. 3, with distances $r_{l,i} = R_L$ and $r_{j,m} = R_R$ are taken away. The bare coupling J_{lm} is then renormalized into the coupling $\tilde{J}_{lm}$. In the representation of distances, this corresponds to creating an adjacent bond with renormalized distance $\tilde{r}_{lm} = \tilde{r}$, replacing their previous distance $r_{lm} = r$, as indicated in Fig. 3. For other adjacent spin pairs like l', m' in Fig. 3, where both spins l', m' are located on the same side of the singlet (i, j) , their bare coupling is renormalized into the coupling

$\tilde{J}_{l'm'}$. In the representation of distances, this corresponds to the creation of an adjacent bond with renormalized distance $\tilde{r}_{l'm'} = \tilde{r}$, replacing their previous distance $r_{l'm'} = \tilde{r}$. However, for such pairs, the renormalization is small, of the order of $(\rho/R)^{2\alpha+2}$, where R is the distance between the pair l', m' and the removed pair i, j . Therefore only the renormalization of the distance of adjacent spins (l, m) needs to be taken into account, in good approximation, in the derivation of the Master equation for the distribution function $P_{\sigma,T}(\tilde{r}, \Omega)$. Transforming Eqs. (22), (24), and (29) to RG rules for renormalized distance $\tilde{r}$, when the spin pair at sites i, j forms the state $|s\rangle$, we obtain the RG rules listed in the following. Since signs of nearby couplings are not independently distributed, we assume that the sign of close-by couplings between spins at sites l, i and the ones at sites l, j in Fig. 3 are the same, $\sigma_L = \sigma_{li} = \sigma_{lj}$, and likewise $\sigma_R = \sigma_{jm} = \sigma_{im}$, while the signs of all other couplings shown in Fig. 3 are distributed independently. When the pair is in the pair state $|s_{ij}\rangle$ the renormalized distance $\tilde{r}$ is given by

$$f[s](R_L, R_R, \rho)_{\sigma_{lm}\sigma\sigma_L\sigma_R} = r|1 + \sigma_{lm}\sigma\sigma_L\sigma_R g[s](R_L, R_R, \rho)|^{-1/\alpha}, \quad (\text{B1})$$

with bare distance $r = (R_L + \rho + R_R)$. Here, $\sigma, \sigma_{lm}, \sigma_L, \sigma_R$ are signs of the couplings $J_{ij} = \sigma\Omega, J_{lm}, J_L$ and J_R , respectively. We introduced the function $g[s](R_L, R_R, \rho)$ defining the renormalization of distance r , when the spin pair taken away is in state $|s\rangle$. It can be rewritten as

$$g[s](R_L, R_R, \rho) = \left(\frac{r\rho}{R_L R_R} \right)^{\alpha} h[s](R_L, R_R, \rho), \quad (\text{B2})$$

where the function $h[s](R_L, R_R, \rho)$ depends on state s , chosen by pair (i, j) : When the pair is in the singlet state $|0_{ij}\rangle$, it is given by

$$h[0](R_L, R_R, \rho) = \left[1 - \left(1 + \frac{\rho}{R_L} \right)^{-\alpha} \right] \left[1 - \left(1 + \frac{\rho}{R_R} \right)^{-\alpha} \right]. \quad (\text{B3})$$

When it is in one of the states $|1_{ij}\rangle$ or $|2_{ij}\rangle$ then

$$h[1](R_L, R_R, \rho) = 1 - \left(1 + \frac{\rho}{R_L} \right)^{-\alpha} \left(1 + \frac{\rho}{R_R} \right)^{-\alpha}, \quad (\text{B4})$$

When it is in state $|3_{ij}\rangle$, then

$$h[3](R_L, R_R, \rho) = \left[1 + \left(1 + \frac{\rho}{R_L} \right)^{-\alpha} \right] \left[1 + \left(1 + \frac{\rho}{R_R} \right)^{-\alpha} \right]. \quad (\text{B5})$$

Now, we can derive the Master equation for the distribution function of $\tilde{r}$ with coupling sign σ at temperature T noting that at the reduced energy scale $\Omega - d\Omega$ we get

$$P_{\sigma,T}(\tilde{r}, \Omega - d\Omega) = \left\{ P_{\sigma,T}(\tilde{r}, \Omega) + d\Omega \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega) \sum_{s=0}^4 p_{s,T}[E_s(\sigma'\Omega)] \sum_{\sigma_L, \sigma_R, \sigma_{lm}} \int_{\rho}^{\infty} dR_L dR_R P_{\sigma_L,T}(R_L, \Omega) \right. \\ \times P_{\sigma_R,T}(R_R, \Omega) \left[\delta(\tilde{r} - f[s](R_L, R_R, \rho))_{\sigma_{lm}\sigma\sigma_L\sigma_R} (\delta_{\sigma,\sigma_{lm}}|_{R_L, R_R \in A_s} + \delta_{\sigma,\sigma_s\sigma_L\sigma_R\sigma'}|_{R_L, R_R \in A_s^c}) \right. \\ \left. \left. - \delta(\tilde{r} - (R_L + \rho + R_R))_{\sigma,\sigma_{lm}} - \delta(\tilde{r} - R_L)_{\sigma,\sigma_L} - \delta(\tilde{r} - R_R)_{\sigma,\sigma_R} \right] \right\} \frac{1}{1 - 3d\Omega \sum_{\sigma'} P_{\sigma',T}(\Omega, \Omega)}. \quad (\text{B6})$$

The delta functions in the bracket account for the fact that one adjacent edge is created between sites l and m with distance $\tilde{r}$, removing the one with distance r , and removing the two adjacent bonds with distances R_L and R_R . We defined σ_s with $\sigma_0 = 1, \sigma_1 = \sigma_2 = -1$ accounting for the minus sign in $h[1](R_L, R_R, \rho)$, Eq. (B4) and $\sigma_3 = 1$. In order to trace the sign of the renormalized coupling correctly, when transforming the RG rules Eqs. (22), (24), and (29) to RG rules for the renormalized distance $\tilde{r}$ we introduced the region A_s of distances R_L, R_R where the bare coupling is larger than the renormalization correction allowing no sign change, and its complement A_s^c . The last factor on the right side of Eq. (B6) is needed for normalization of the pdf, since in total three edges are taken away. The proper normalization can be checked, by integrating both sides of Eq. (B6) over $\tilde{r}$ from ρ to infinity, Taylor expanding in $d\Omega$ and using the normalization condition $\sum_{\sigma=\pm} \int_{\rho}^{\infty} d\tilde{r} P_{\sigma}(\tilde{r}, \Omega) = 1$ with $\rho = (\Omega_0/\Omega)^{1/\alpha}$. Taking the limit $d\Omega \rightarrow 0$, we subtract and add another term $d\Omega P(\Omega, \Omega)P(\tilde{r}, \Omega)$, in order to be able to cancel the normalization factor in Eq. (B6) in the first term. Thereby, we find in the limit $d\Omega \rightarrow 0$, the Master equation Eq. (30) with correction term given by Eq. (31).

APPENDIX C: SOLUTION OF THE MASTER EQUATION WITH LONG RANGE COUPLINGS

In the correction term to the Master equation Eq. (31) we perform the integral over distance R_R , and obtain

$$C_{\sigma,\sigma',T}(\tilde{r}, \Omega, \alpha) = \sum_{s=0}^3 p_{s,T}[E_s(\sigma'\Omega)] \sum_{\sigma_L, \sigma_R, \sigma_{lm}} P_{\sigma_{lm},T}(\Omega) \int_{\rho}^{\infty} dR_L P_{\sigma_L,T}(R_L, \Omega) P_{\sigma_R,T}(R_R, \Omega) \\ \times \left| \frac{df[s]}{dR_R} \right|_{f[s]=\tilde{r}}^{-1} |_{R_R=R[s](R_L, \tilde{r}, \rho)}_{\sigma_{lm}\sigma'\sigma_L\sigma_R} (\delta_{\sigma,\sigma_{lm}}|_{R_L, R[s] \in A_s} + \delta_{\sigma,\sigma_s\sigma_L\sigma_R\sigma'}|_{R_L, R[s] \in A_s^c}) \\ - P_{\sigma,T}(\Omega) \theta(\tilde{r} - 3\rho) \sum_{\sigma_L, \sigma_R} \int_{\rho}^{\tilde{r}-2\rho} dR_L P_{\sigma_L,T}(R_L, \Omega) P_{\sigma_R,T}(\tilde{r} - R_L - \rho, \Omega). \quad (\text{C1})$$

Here we denoted with $R[s](R_L, \tilde{r}, \rho)_{\sigma_{lm}\sigma'\sigma_L\sigma_R}$ the solution for R_R of the equation $\tilde{r} = f[s](R_L, R_R, \rho)_{\sigma_{lm}\sigma'\sigma_L\sigma_R}$. Aiming for an iterative solution we insert the distribution without renormalization corrections, Eq. (33), into Eq. (C1), yielding

$$C_{+,+,T}^0(\tilde{r}, \Omega, \alpha) = \sum_{s=0}^3 p_{s,T}(E_s(\Omega)) \frac{\rho}{4} \int_{\rho}^{\infty} dR_L (R_L R_R)^{-3/2} \left| \frac{df[s]}{dR_R} \right|_{R_R, f[s]=\tilde{r}}^{-1} |_{R_R=R[s](R_L, \tilde{r}, \rho)_+} \\ \times (1|_{R_L, R[s] \in A_s} + \delta_{+, \sigma_s}|_{R_L, R[s] \in A_s^c}) - \theta(\tilde{r} - 3\rho) \int_{\rho}^{\tilde{r}-2\rho} dR_L \frac{\rho}{4} R_L^{-3/2} (\tilde{r} - R_L - \rho)^{-3/2}, \quad (\text{C2})$$

and when the sign of the coupling is changing

$$C_{-+,T}^0(\tilde{r}, \Omega, \alpha) = \sum_{s=0}^3 p_{s,T}(E_s(\Omega)) \frac{\rho}{4} \int_{\rho}^{\infty} dR_L (R_R R_L)^{-3/2} \left| \frac{df[s]}{dR_R} \right|_{R_R, f[s]=\tilde{r}}^{-1} |_{R_R=R[s](R_L, \tilde{r}, \rho)_+, \Omega} \delta_{-, \sigma_s}|_{R_L, R[s] \in A_s^c}, \quad (\text{C3})$$

It remains to perform the integral over R_L in Eqs. (C2) and (C3). That can analytically be done for the last term in Eq. (C2), giving $-\theta(\tilde{r} - 3\rho)(\tilde{r} - 3\rho)/(\sqrt{\tilde{r} - 2\rho}(\tilde{r} - \rho)^2)$. We integrate the other terms numerically. Inserting that result for $C_{\sigma,\sigma',T}^0(\tilde{r}, \Omega \gg T)$ into the Master equation, and using the Ansatz $P_{\sigma,T}(\tilde{r}, \Omega \gg T) = c_{\sigma} f_{\sigma}(x = \tilde{r}/\rho)/\tilde{r}$, we find that the Master equation reduces to a 1st order inhomogeneous ordinary differential equation for the function $f(x = \tilde{r}/\rho)$. Solving it and transforming back to $P_{\sigma,T}(\tilde{r}, \Omega)$, we obtain Eq. (35).

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