

Exact quench dynamics from thermal pure quantum states

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We present an exact solution for entanglement entropy for the real-time dynamics following a quench from a thermal pure quantum (TPQ) state in a free-fermion system. In contrast to the usual linear growth and saturation behavior, the entanglement entropy exhibits a characteristic double-plateau structure. We establish this behavior through three complementary approaches: an exact conformal field theory calculation on the Klein bottle, finite-size Gaussian-state simulations, and a quasiparticle picture that becomes quantitatively accurate in the scaling regime.

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Introduction. Understanding the nonequilibrium dynamics of isolated quantum systems is a central challenge in statistical mechanics and quantum gravity [1–5]. Usually, analytical and numerical studies rely on low-entanglement initial states (e.g., ground states or short-range product states), from which the celebrated linear growth and subsequent saturation of entanglement emerge [6–8]. In contrast, thermal pure quantum (TPQ) states are usually constructed from random states and already exhibit volume-law entanglement and are locally indistinguishable from Gibbs ensembles [9]. The entanglement dynamics from such initial states often involves exponential computational complexity, making exactly solvable models for this process very rare. Distinct from the traditional random-state-based TPQ states used for numerical calculations, recent literature has introduced highly structured TPQ states [10,11] that have subsequently been applied in holography to elegantly capture the microstates of single-sided black holes with smooth interiors [12,13].

Such structured TPQ states are constructed by imaginary-time evolving a crosscap state $|C\rangle$ [14–22] as $|\Psi_\beta\rangle \propto e^{-\beta H/4}|C\rangle$ [10–12]. Motivated by the fact that the bipartite entanglement remains constant under chaotic quenches from crosscap states [12,21], in this Letter, we explore the quench dynamics of the bipartite entanglement entropy in the non-chaotic XX spin chain from the state $|\Psi_\beta\rangle$. We provide an exact analytical solution to this problem, where both the state preparation and the time evolution are governed by the same Hamiltonian H .

We study this problem from three complementary perspectives. First, using two-dimensional (2D) conformal field theory (CFT), we derive an analytical formula for the time-

dependent entanglement entropy in terms of Jacobi theta functions by evaluating vertex-operator correlators on the Klein bottle geometry. Second, we obtain an exact numerical benchmark by solving the governing matrix Riccati equation for the TPQ covariance matrix and evolving it in real time. Third, we show that the entire profile is captured by a quasiparticle picture that becomes quantitatively accurate in the scaling regime [6,23–25], where the ballistic propagation of antipodally entangled quasiparticle pairs governs the entropy double-plateau behavior.

Thermal pure quantum state. In 2D CFT, a crosscap state $|C\rangle$ is a boundary state [26] embodying a nonorientable space-time, defined by the constraint $[L_n - (-1)^n \bar{L}_{-n}]|C\rangle = 0$ on the Virasoro generators [14]. While abstract, its physical essence is captured on a lattice of length L as an entangled antipodal pair (EAP) state [11]. For a spin-1/2 chain, this is

$$|C\rangle = \bigotimes_{i=1}^{L/2} \frac{1}{\sqrt{2}} (|\uparrow\rangle_i |\uparrow\rangle_{i+L/2} + |\downarrow\rangle_i |\downarrow\rangle_{i+L/2}). \quad (1)$$

This construction embeds maximal, nonlocal entanglement into the system, profoundly distinguishing it from short-range correlated ground states of local Hamiltonians.

A defining feature of $|C\rangle$ [Eq. (1)] is its entanglement profile. The reduced density matrix ρ_A of any contiguous subsystem A of length $l \leq L/2$ is maximally mixed, $\rho_A = \frac{I_A}{2^l}$. This results in a volume-law entanglement entropy, $S_A(0) = l \log 2$, with a full “Page curve” profile $S_A(0) = \min(l, L - l) \log 2$. This property establishes the crosscap state as an infinite-temperature ($\beta = 0$) TPQ state, as it perfectly mimics a thermal Gibbs state for any local operator.

We generalize this initial condition to a finite temperature by defining a generic TPQ state $|\Psi_\beta\rangle$ via imaginary-time evolution:

$$|\Psi_\beta\rangle \equiv \frac{e^{-\frac{\beta}{4}H}|C\rangle}{\sqrt{\langle C|e^{-\frac{\beta}{2}H}|C\rangle}}. \quad (2)$$

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This state is a deterministic, structured pure state which is, by construction, locally indistinguishable from a canonical Gibbs ensemble at inverse temperature β [10]. The quench dynamics from $|\Psi_\beta\rangle$ thus explore the evolution from a state of “fake” thermal equilibrium, allowing us to investigate how its globally stored quantum information is scrambled and redistributed over time.

The CFT approach. The physical system under investigation is described at low energies by the $c = 1$ free compact boson CFT, which is dual to the massless Dirac fermion. We now outline the analytical calculation of the time-dependent entanglement entropy following the quench from the state $|\Psi_\beta\rangle$. The system is prepared on a Euclidean cylinder of circumference 2π and length $\beta/2$ with crosscap boundary conditions, defining the TPQ state at $t = 0$. This state then evolves in real time t under the massless Hamiltonian H . We introduce the complex coordinates $(y, \bar{y})$ for this spacetime: $y = \tau - i\sigma$, $\bar{y} = \tau + i\sigma$ ($0 \leq \tau \leq \frac{\beta}{2}$, $0 \leq \sigma \leq 2\pi$).

To compute the von Neumann entropy of a subsystem A , $S_A(t)$, we employ the replica trick, requiring the analytic continuation of the moments $\text{Tr}[\rho_A(t)^n]$ to $n \rightarrow 1$, where $\rho_A(t) = \text{Tr}_{\bar{A}}(e^{-iHt}|\Psi_\beta\rangle\langle\Psi_\beta|e^{iHt})$. The quantity $\text{Tr}[\rho_A(t)^n]$ is computed as a path integral on an n -sheeted Riemann surface representing the replicated spacetime. This problem can be mapped to a system of n free Dirac fields on a single cylinder, where the replicas are coupled by twisted boundary conditions at the subsystem’s end points [27].

In the bosonized language, these fermionic replica twists are implemented by the insertion of specific twist fields, $\mathcal{T}_k$ and $\bar{\mathcal{T}}_k$, where k labels the replica mode after a Fourier transform in the replica index. For the massless Dirac fermion, or equivalently for the compact boson at the free-fermion radius ($R = 1$), these fermionic replica twist fields are primary vertex operators $V_{(\pm k, \pm k)}(y, \bar{y}) =: e^{\pm i \frac{k}{n} [\varphi(y) + \bar{\varphi}(\bar{y})]} :.$ The crucial insight is that the geometry of the initial-state preparation (the crosscap) effectively transforms the problem of computing the replicated partition function to that of computing a two-point function of these vertex operators on a Klein bottle $\mathcal{K}$. The total replicated partition function is then the product over all replica modes:

$$\text{Tr}[\rho_A(t)^n] = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} Z_k = \prod_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \langle \mathcal{T}_k(y_1) \bar{\mathcal{T}}_k(y_2) \rangle_{\mathcal{K}}. \quad (3)$$

Here, y_1 and y_2 are the complex coordinates of the subsystem’s end points in the spacetime describing the quench, given by $(y_1, \bar{y}_1) = (\frac{\beta}{4} + it - i\sigma_1, \frac{\beta}{4} + it + i\sigma_1)$ and $(y_2, \bar{y}_2) = (\frac{\beta}{4} + it - i\sigma_2, \frac{\beta}{4} + it + i\sigma_2)$. We denote by $\sigma = \sigma_2 - \sigma_1$ the length of the subsystem.

In Eq. (3), we need to compute the normalized two-point function of vertex operators on a Klein bottle, which is a highly nontrivial task, although similar calculations have been carried out for the conformal boundary state in Ref. [28]. For a subsystem of length σ at time t after the quench from the TPQ state prepared with parameter β , the entanglement entropy is given by

$$S_A(t, \sigma) = \frac{1}{6} \log \frac{\eta\left(\frac{i\beta}{2\pi}\right)^{-6} \left| \theta_1\left(\frac{\sigma}{2\pi} \middle| \frac{i\beta}{2\pi}\right) \theta_2\left(\frac{\beta+4it}{4\pi i} \middle| \frac{i\beta}{2\pi}\right) \right|^2}{\left| \theta_2\left(\frac{\beta+4it+2i\sigma}{4\pi i} \middle| \frac{i\beta}{2\pi}\right) \theta_2\left(\frac{\beta+4it-2i\sigma}{4\pi i} \middle| \frac{i\beta}{2\pi}\right) \right|^2}. \quad (4)$$

See Supplemental Material [29] for the derivation of this formula. Here and in Supplemental Material we follow the string-theory conventions of Ref. [30] with $\alpha' = 2$. This expression involving the Dedekind η function $\eta(\tau)$ and Jacobi θ functions $\theta_{1,2}(z|\tau)$, provides a complete, exact prediction for the entanglement dynamics, revealing the double-plateau structure and its dependence on β . Throughout this work, we adopt the conventions for the θ and η functions as given in Ref. [30]. The derivation uses the vertex-operator representation of fermionic replica twist fields and applies only to the free Dirac fermion. We do not claim that Eq. (4) holds for a generic compact boson at arbitrary radius.

The numerical benchmark. To validate our analytical CFT predictions, we perform exact numerical simulations on the lattice realization of the theory, namely the spin-1/2 XX chain at half filling. The Hamiltonian for a system of $L = 2N$ sites with periodic boundary conditions is

$$H = -J \sum_{j=1}^{2N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y). \quad (5)$$

We take $J > 0$ throughout. After a Jordan-Wigner transformation, $\sigma_j^+ = [\prod_{k=1}^{j-1} (-\sigma_k^z)] c_j^\dagger$, $\sigma_j^- = [\prod_{k=1}^{j-1} (-\sigma_k^z)] c_j$, $\sigma_j^z = 2c_j^\dagger c_j - 1$. The model is described by a Hamiltonian of noninteracting fermions, $H = -2J \sum_{j=1}^{2N} (c_j^\dagger c_{j+1} + \text{H.c.})$, with an antiperiodic boundary condition. The single-particle energy dispersion is given by $E(k) = -4J \cos(k)$. The ground state is formed by filling all negative energy states, which corresponds to half filling. The low-energy excitations occur near the two Fermi points, $k_F = \pm\pi/2$, where the dispersion becomes linear: $E(k_F \pm q) \approx \pm(4J)q$. This linear dispersion is the defining characteristic of the massless (1+1)D Dirac fermion, with an emergent “speed of light” (Fermi velocity) of $v_F = 4J$ (in units where the lattice spacing $a = 1$).

The initial TPQ state $|\Psi_\beta\rangle$ is prepared by evolving the crosscap state $|\mathcal{C}\rangle$ in imaginary time, as described in Eq. (2). For the numerical implementation, we employ a fermionic representation of the system via the Jordan-Wigner transformation. It is important to note a subtle but important technical point regarding the lattice crosscap state itself. As was recently pointed out in Ref. [20], the Jordan-Wigner transformation of the strict spin crosscap state of Eq. (1) results in a superposition of two distinct fermionic Gaussian states. For simplicity and to maintain consistency with the Gaussian state formalism used in our analytical and quasiparticle pictures, our numerical simulation, following the approach in Ref. [21], targets the evolution from a Gaussian version of the crosscap state $|\mathcal{C}\rangle = \bigotimes_{j=1}^{L/2} \frac{1}{\sqrt{2}} (1 + c_j^\dagger c_{j+L/2}) |0\rangle$. We expect that this simplification does not affect the qualitative features of the entanglement dynamics reported here, as the essential non-local entanglement structure is captured by either Gaussian component, although a quantitative investigation of the full superposition remains an interesting direction for future work.

This setting allows us to leverage the powerful formalism of Gaussian states, as the entire quench protocol can be described exactly through the evolution of a fermionic covariance matrix Γ , which is defined in terms of Majorana fermion operators. For a system with $2N$ sites, we have $4N$ Majorana operators. The operators for site j are $\gamma_{2j-1} = c_j + c_j^\dagger$, $\gamma_{2j} =$

$i(c_j^\dagger - c_j)$, and the elements of fermionic covariance matrix Γ are defined as $\Gamma_{mn} = \frac{i}{2} \langle [\gamma_m, \gamma_n] \rangle = i \langle \gamma_m \gamma_n \rangle - i \delta_{mn}$.

The initial crosscap state $|\mathcal{C}\rangle$ is a fermionic Gaussian state, and its properties are fully determined by its two-point correlation matrices. The normal correlator is $C_{ij}(0) \equiv \langle \mathcal{C} | c_i^\dagger c_j | \mathcal{C} \rangle = \frac{1}{2} \delta_{ij}$, and the anomalous correlator is $F_{ij}(0) \equiv \langle \mathcal{C} | c_i c_j | \mathcal{C} \rangle = \frac{1}{2} (\delta_{i,j+N} - \delta_{i,j-N})$. It is more convenient to combine these into the $4N \times 4N$ real, antisymmetric fermionic covariance matrix, $\Gamma_{\mathcal{C}}$. For the crosscap state, $\Gamma_{\mathcal{C}}$ is zero except for 2×2 blocks connecting antipodal sites i and $i + N$, which take the form $\Gamma_{i,i+N} = \sigma_x$.

The imaginary-time evolution operator $e^{-\beta H/4}$ is a Gaussian operator, meaning the initial TPQ state $|\Psi_\beta\rangle$ is also a Gaussian state. Its covariance matrix, Γ_β , can be found by solving the governing matrix Riccati equation [31–33] for the imaginary-time evolution [34,35].

$$\frac{d\Gamma(\tau)}{d\tau} = -\mathcal{H} - \Gamma(\tau)\mathcal{H}\Gamma(\tau). \quad (6)$$

The equation above holds if the Hamiltonian is quadratic: $H = \sum_{i,j=1}^{2N} h_{ij} c_i^\dagger c_j = \frac{i}{4} \sum_{ij} \mathcal{H}_{ij} \gamma_i \gamma_j$. By applying a trick, we can linearize this equation and solve it exactly as

$$\Gamma(\tau) = [\cos(\mathcal{H}\tau)\Gamma_0 - \sin(\mathcal{H}\tau)][\sin(\mathcal{H}\tau)\Gamma_0 + \cos(\mathcal{H}\tau)]^{-1}, \quad (7)$$

where $\mathcal{H}$ is the $4N \times 4N$ matrix representation of the single-particle Hamiltonian in the Majorana basis. The derivation of Eqs. (6) and (7), together with the imaginary-time and real-time covariance-matrix evolution are given in Supplemental Material [29]. For the XX spin chain, $\mathcal{H} = \mathbf{h} \otimes (i\sigma_y)$. The covariance matrix for the TPQ state is then obtained by taking $\tau = \frac{\beta}{4}$ in Eq. (7), i.e., $\Gamma_\beta = \Gamma(\tau = \frac{\beta}{4})$, with $\Gamma_0 = \Gamma_{\mathcal{C}}$.

Having obtained the exact covariance matrix Γ_β for the initial state, the real-time evolution for the quench is given by the following equation:

$$\Gamma(t) = e^{\mathcal{H}t} \Gamma_\beta e^{-\mathcal{H}t}. \quad (8)$$

The entanglement entropy of a subsystem A of length l at any time t can then be computed directly from the eigenvalues $\pm i v_j$ of its restricted covariance matrix $\Gamma_A(t)$, which is the $2l \times 2l$ sub-block of $\Gamma(t)$ corresponding to the sites in A . The von Neumann entropy is a sum of binary entropies:

$$S_A(t) = - \sum_{j=1}^l \left[\frac{1+v_j}{2} \ln \frac{1+v_j}{2} + \frac{1-v_j}{2} \ln \frac{1-v_j}{2} \right]. \quad (9)$$

This procedure provides an exact, numerically efficient method to simulate the dynamics for large systems, serving as a rigorous benchmark for our analytical CFT results.

As shown in Fig. 1, the CFT prediction (symbols) is in excellent quantitative agreement with the exact lattice calculation (solid line), accurately capturing both the double-plateau structure and the characteristic timescales after the rescaling $\beta \rightarrow 2\pi v_F \beta / L$, $\sigma \rightarrow 2\pi l / L$, $t \rightarrow 2\pi v_F t / L$ in Eq. (4).

The quasiparticle picture. The entanglement dynamics can be quantitatively understood through the quasiparticle picture [6,23–25]. In the standard quench paradigm from a low-entanglement state, the quench acts as a source of locally entangled quasiparticle pairs that propagate ballistically, causing entanglement to grow. The TPQ state, however, is

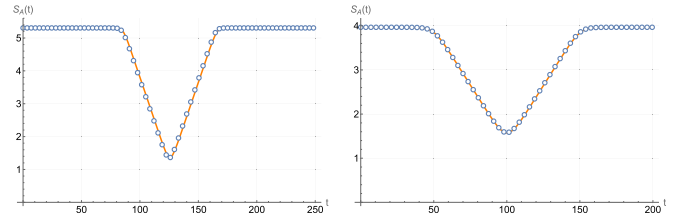


FIG. 1. Comparison between the exact numerical results and the CFT prediction for the entanglement entropy $S_A(t)$. Solid lines: numerical data; circles: CFT prediction. Left: $L = 500$, $l = 80$, $\beta = 20$, $v_F = 1$ ($J = 0.25$). Right: $L = 400$, $l = 100$, $\beta = 40$, $v_F = 1$ ($J = 0.25$).

a high-energy, extensively entangled state whose dynamics require a fundamentally different interpretation. Here, the dominant process is not the creation of entanglement, but the transport and eventual destruction of the preexisting, nonlocal entanglement.

The initial state $|\Psi_\beta\rangle$ can be viewed as a sea of quasi-particle pairs with opposite momenta ($k, -k$). Crucially, the entanglement is nonlocal: a quasiparticle with momentum k at site x is entangled not with its partner at x , but with the particle of momentum $-k$ at the antipodal site $x + L/2$. After the quench at $t = 0$, these pairs propagate ballistically. The entanglement entropy of a subsystem A decreases when both members of an entangled pair become fully contained within A . This process leads to the following prediction for the entanglement dynamics,

$$S_A(t) = S_A(0) - 2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} s(k) \max(0, l - |2v(k)\tau_k - L/2|), \quad (10)$$

which is a straightforward generalization of the formula proposed in Ref. [21]. We now explain this formula term by term. The first term, $S_A(0) = l \int_{-\pi}^{\pi} \frac{dk}{2\pi} s(k)$, is the initial entanglement entropy of the subsystem of length l . The second term is a subtractive correction representing the entropy removed by the quasiparticles. The factor of 2 accounts for the two quasiparticles in a pair. The counting function, $\max(0, \dots)$, is the heart of the picture: it counts the number of antipodal pairs with momentum k that are both inside the subsystem A at time t . The term $|2v(k)\tau_k - L/2|$ is the effective separation of the two members of an antipodal pair, which starts at $L/2$ and decreases as they travel towards each other. Here, $v(k)$ is the velocity of a quasiparticle with momentum k . For the XX chain at half filling, $v(k) = 4J \sin(k)$ and $\tau_k = t \bmod L/|v(k)|$ is a periodic time variable that accounts for revivals in the finite system.

The function $s(k)$ is the effective entanglement density carried by each quasiparticle with momentum k . As shown in Supplemental Material [29], the initial state is equivalent to a BCS-like state,

$$|\Psi_\beta\rangle = \prod_{k>0} [u_k(\beta) + v_k(\beta) c_k^\dagger c_{-k}^\dagger] |0\rangle, \quad (11)$$

with Bogoliubov coefficients

$$|u_k(\beta)|^2 = \frac{1}{1 + e^{-\beta E(k)}}, \quad |v_k(\beta)|^2 = \frac{e^{-\beta E(k)}}{1 + e^{-\beta E(k)}}. \quad (12)$$

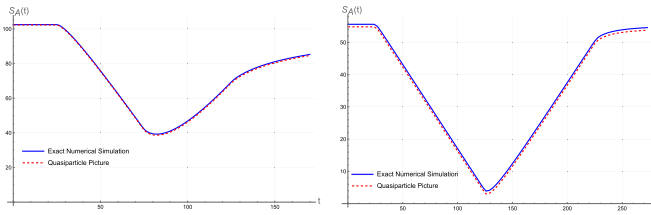


FIG. 2. Comparison between the exact numerical results and the quasiparticle prediction for the entanglement entropy $S_A(t)$. Solid lines: numerical data; dashed line: quasiparticle prediction. Left: $L = 600$, $l = 200$, $\beta = 1$, $J = 0.5$. Right: $L = 1000$, $l = 400$, $\beta = 4$, $J = 0.5$.

This allows for a direct calculation of the entanglement entropy per mode, which is given by the binary entropy of the mode occupation,

$$s(k) = -n_k \ln n_k - (1 - n_k) \ln(1 - n_k), \quad (13)$$

where the occupation n_k is given by the Fermi-Dirac distribution at the inverse temperature β of the initial state:

$$n_k \equiv \langle \Psi_\beta | c_k^\dagger c_k | \Psi_\beta \rangle = \frac{1}{e^{\beta E(k)} + 1}. \quad (14)$$

formula (10) provides a complete quantitative picture that accurately reproduces our exact analytical and numerical results. The initial plateau is the time before any antipodal pairs can fully enter the subsystem. The subsequent decrease is governed by the rate at which pairs enter, weighted by their entanglement density $s(k)$. The rise and second plateau are due to the same pairs exiting the subsystem as they traverse outside the subsystem. This quasiparticle picture thus provides a powerful physical interpretation for the nonmonotonic behavior of entanglement observed in the system. Figure 2 shows a comparison between the exact numerical results and the quasiparticle prediction for the entanglement entropy.

Conclusion and outlook. In this Letter, we study the entanglement evolution following a quench from a volume-law entangled TPQ state. We establish this exotic entanglement dynamics rigorously through three complementary approaches: (i) a 2D CFT derivation yielding an exact analytical formula in terms of Jacobi θ functions on the Klein bottle geometry, (ii) a highly efficient exact numerical evaluation of the restricted covariance matrix, and (iii) a quasiparticle picture that becomes quantitatively accurate in the scaling regime.

Our results pave the way for exploring nonequilibrium dynamics from highly entangled initial states and for extending the CFT quench framework to nonorientable manifolds. The CFT analysis and exact Gaussian numerics apply directly to mutual information and entanglement negativity, while the quasiparticle picture can be adapted with minor modifications. Exploring interacting integrable models in this context would be both interesting and important for future work. This perspective can be generalized to richer scenarios, including inhomogeneous quenches such as the Möbius or sine-square deformation (SSD) [36–41]. The experimental exploration of crosscap-type initial states and the associated entanglement dynamics would be a fascinating direction for future studies, potentially realizable in quantum-simulation platforms such as ultracold atoms or superconducting qubits.

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Data availability. The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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