

Incommensurate gapless ferromagnetism connecting competing symmetry-enriched deconfined quantum phase transitions

Anthony Rey¹,², Ömer M. Aksoy², Daniel P. Arovas³, Claudio Chamon^{4,5} and Christopher Mudry^{1,6}

¹Condensed Matter Theory Group, *PSI Center for Scientific Computing, Theory and Data*, 5232 Villigen PSI, Switzerland

²Department of Physics, *Massachusetts Institute of Technology*, Cambridge, Massachusetts 02139, USA

³Department of Physics, *University of California at San Diego*, La Jolla, California 92093, USA

⁴Department of Physics, *Boston University*, Boston, Massachusetts 02215, USA

⁵Department of Physics and Astronomy, *Purdue University*, West Lafayette, Indiana 47907, USA

⁶Institut de Physique, *EPF Lausanne*, CH-1015 Lausanne, Switzerland



(Received 19 March 2025; revised 10 June 2025; accepted 19 September 2025; published 7 October 2025)

We present a scenario in which a gapless extended phase serves as a “hub” connecting multiple symmetry-enriched deconfined quantum critical points. As a concrete example, we construct a lattice model with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry for quantum spin-1/2 degrees of freedom that realizes four distinct gapful phases supporting antiferromagnetic long-range order and one extended incommensurate gapless ferromagnetic phase. The quantum phase transition between any two of the four gapped and antiferromagnetic phases goes through either a (deconfined) quantum critical point, a quantum tricritical point, or the incommensurate gapless ferromagnetic phase. In this phase diagram, it is possible to interpolate between four deconfined quantum critical points by passing through the extended gapless ferromagnetic phase. We identify the phases in the model and the nature of the transitions between them through a combination of analytical arguments and density matrix renormalization group studies.

DOI: [10.1103/71kq-8lc8](https://doi.org/10.1103/71kq-8lc8)

Introduction. In one-dimensional space, when periodic boundary conditions (PBCs) hold, there is a single quantum phase of bosonic matter that is described by a nondegenerate and gapped ground state [1]. For example, any local Hamiltonian hosting quantum spin degrees of freedom, obeying PBCs, and supporting a nondegenerate gapped ground state can be adiabatically connected without gap closing, up to adding or removing ancilla degrees of freedom, to a trivial quantum paramagnet whose ground state is a product state of the quantum spins. This picture changes in the presence of symmetries, namely, when the adiabatic evolution is restricted to be invariant under a symmetry transformation from a group G . There are then degenerate and gapped ground states that cannot be connected to a G -symmetric product state without gap closing or spontaneous breaking of G . This possibility leads to the concept of symmetry-protected topological (SPT) phases [1–6].

Just like gapped phases, quantum critical points or gapless phases can also be enriched by symmetries [7–44]. For example, when enriched by a global $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, there are two kinds of Ising conformal field theories (CFT) [12]. It is believed that any smooth path in the coupling space of a microscopic, local, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric Hamiltonian that interpolates between these two Ising CFT must go through either a multicritical point or a gapped phase.

In this work we offer a scenario where a gapless extended phase serves as a “hub” connecting multiple symmetry-enriched deconfined quantum critical points (DQCPs). We study the interplay of four symmetry-enriched DQCPs, with each one describing a continuous quantum phase transition between two gapped phases of matter supporting distinct long-range orders. While two of the four DQCPs separate two gapped phases realizing the same SPT phase, the other two DQCPs separate those phases that realize distinct SPT gapped phases of matter. Hence, the latter two DQCPs can also be interpreted as examples of quantum topological phase transitions (QTPTs). We construct a local, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ -symmetric lattice Hamiltonian [Eq. (1)] for quantum spin-1/2 degrees of freedom that depends on two dimensionless couplings and that interpolates between all four DQCPs. We show that any two of the four DQCPs cannot be smoothly connected in the parameter space of Fig. 1 without encountering a multicritical point or a gap opening. Surprisingly, we find that the competition between the four symmetry-enriched DQCPs leads to an extended gapless phase in the center of the phase diagram (see Fig. 1), which is connected to all four symmetry-enriched DQCPs through four tricritical points. This phase (i) supports a pair of degenerate ground states that are ferromagnetically ordered along the quantization Z axis in spin space; (ii) is incommensurate in that the longitudinal spin-spin correlation function is long-ranged with, for both periodic and open boundary conditions (OBCs), an oscillatory factor $e^{iq^z s}$, whereby the wave vector q^z is not commensurate with the lattice and s is the separation between two spins; and (iii) is gapless in that the transverse spin-spin correlation functions decay algebraically with the wave vector π . We call this phase

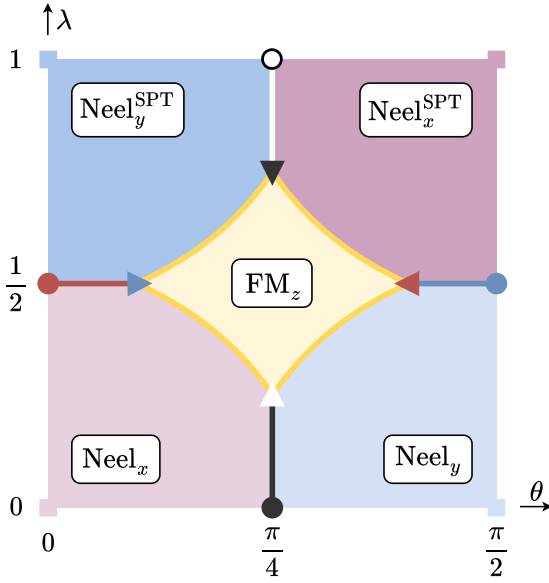


FIG. 1. Phase diagram of Hamiltonian (1). Squares (circles) on the boundary denote stable (unstable) fixed points (DQCPs).

the incommensurate gapless ferromagnetic phase. Our work thus provides an explicit example of an exotic gapless phase resulting from the competition between different symmetry-enriched DQCPs.

Model and its symmetries. We define the family of Hamiltonians parametrized by two dimensionless couplings $\theta \in [0, \pi/2]$ and $\lambda \in [0, 1]$ by

$$\hat{H}_b(\theta, \lambda) := (1 - \lambda) \sum_{j=1}^{2N-b} \hat{h}_j(\theta) + \lambda \sum_{j=1}^{2N-3b} \hat{Z}_j \hat{h}_{j+1}(\theta) \hat{Z}_{j+3}, \quad (1a)$$

with

$$\hat{h}_j(\theta) := \cos \theta \hat{X}_j \hat{X}_{j+1} + \sin \theta \hat{Y}_j \hat{Y}_{j+1}. \quad (1b)$$

Here, $\hat{X}_j$, $\hat{Y}_j$, and $\hat{Z}_j$ are on-site Pauli operators that commute on unequal sites and satisfy the Pauli algebra on the same site, e.g., $\hat{X}_j \hat{Y}_j = i\hat{Z}_j$. Periodic boundary conditions follow from choosing $b = 0$ under the assumption that $\hat{X}_{j+2N} \equiv \hat{X}_j$ and $\hat{Y}_{j+2N} \equiv \hat{Y}_j$. Open boundary conditions follow from choosing $b = 1$. The number of sites $2N$, with N being a positive integer, is explicitly even.

When $\theta \in [0, \pi/2]$ and $\lambda = 0$, Hamiltonian (1) reduces to that of the anisotropic, nearest-neighbor, and antiferromagnetic quantum spin-1/2 XY model. As the coupling λ is tuned from 0 to 1, the two-body XY interactions in Hamiltonian (1) compete with the four-body interactions obtained by dressing each two-body XY interaction with a two-body third-neighbor ZZ interaction. We call the limit $\lambda = 1$ of Hamiltonian (1) the “dressed” quantum spin-1/2 XY model.

For any pair $\theta \in [0, \pi/2]$ and $\lambda \in [0, 1]$, Hamiltonian (1) is invariant under a $\mathbb{Z}_2^X \times \mathbb{Z}_2^Y \times \mathbb{Z}_2^T$ symmetry generated by the operators

$$\hat{X} := \prod_{j=1}^{2N} \hat{X}_j, \quad \hat{Y} := \prod_{j=1}^{2N} \hat{Y}_j, \quad \hat{T} := \hat{Y} \hat{K}, \quad (2)$$

respectively, where $\hat{K}$ denotes complex conjugation. The subgroup $\mathbb{Z}_2^X \times \mathbb{Z}_2^Y$ implements global π rotations around the X , Y , and Z axes in spin space, while the subgroup $\mathbb{Z}_2^T$ implements the reversal of time. Because the number of sites $2N$ is even, the symmetry generators in Eq. (2) commute pairwise.

Gapped phases. In the coupling space (θ, λ) from Fig. 1, Hamiltonian (1) realizes four gapped phases, which we label as Néel_x , Néel_y , $\text{Néel}_y^{\text{SPT}}$, and $\text{Néel}_x^{\text{SPT}}$. The properties of each gapped phase are captured by the exactly solvable point in the corresponding corner of the phase diagram, marked by a square in Fig. 1.

The $\text{Néel}_{x(y)}$ phase is controlled by the exactly solvable point at $\theta = 0$ ($\pi/2$) and $\lambda = 0$, i.e., the lower left (right) corner in Fig. 1. At this point, Hamiltonian (1) reduces to the classical Ising antiferromagnet with XX (YY) couplings. Consequently, the $\text{Néel}_{x(y)}$ phase is long-range ordered with spontaneously broken $\hat{Y}$ ($\hat{X}$) and $\hat{T}$ symmetries. In the thermodynamic limit, the ground states are twofold degenerate for both phases independently of the choice between PBCs or OBCs.

The $\text{Néel}_y^{\text{SPT}}$ phase is controlled by the exactly solvable stable fixed point at $\theta = 0$ ($\pi/2$) and $\lambda = 1$, i.e., the upper left (right) corner in Fig. 1. At this point, Hamiltonian (1) reduces to that of a cluster model [45] with $ZXXZ$ ($ZYYZ$) couplings. When PBCs are chosen, the $\text{Néel}_y^{\text{SPT}}$ phase describes the same long-range order as the $\text{Néel}_{y(x)}$ phase does in that the same internal symmetries, $\hat{X}$ ($\hat{Y}$) and $\hat{T}$, are spontaneously broken by the selection of one out of two degenerate ground states. However, each one of the two degenerate ground states in the $\text{Néel}_y^{\text{SPT}}$ phase realizes a nontrivial SPT state protected by the unbroken $\hat{Y}$ and $\hat{X} \hat{T}$ ($\hat{X}$ and $\hat{Y} \hat{T}$) symmetries [46–51]. Hence, when OBCs are chosen, the $\text{Néel}_y^{\text{SPT}}$ phase supports two decoupled quantum spin-1/2 degrees of freedom localized on the left and right boundaries, respectively; i.e., the ground states are now eightfold (instead of twofold) degenerate in the thermodynamic limit. While the $\text{Néel}_y^{\text{SPT}}$ and $\text{Néel}_x^{\text{SPT}}$ phases support distinct long-range orders, their symmetry-breaking ground states realize the same SPT phases.

Symmetry-enriched DQCPs. The four corners describing the gapped phases in the coupling space in Fig. 1 are pairwise connected by four edges. When restricted to any of these edges, Hamiltonian (1) describes a linear interpolation between two exactly solvable models that support distinct long-range orders. The midpoint of each edge, marked by a circle in Fig. 1, is then a DQCP that separates two gapped phases with competing long-range orders, as we now explain.

At the midpoint $(\pi/4, 0)$ of the lower horizontal edge in Fig. 1, Hamiltonian (1) reduces to the well-known isotropic spin-1/2 quantum XY model, which is the DQCP between the Néel_x and Néel_y phases as shown in Refs. [52–54]; a $\mathfrak{c} = 1$ CFT with a nondegenerate gapless ground state [55]. The two exactly solvable corners controlling the Néel_x and Néel_y phases are related to each other by the unitary transformation $\hat{U}_R = \hat{U}_{\pi/2} \hat{X}$, where $\hat{U}_{\pi/2}$ and $\hat{X}$ implement rotations in spin space by $\pi/2$ and π around the Z and X axes, respectively; i.e., the only nontrivial action of $\hat{U}_R$ on the Pauli operators is given by $(\hat{X}_j, \hat{Y}_j) \mapsto (\hat{Y}_j, \hat{X}_j)$ for any site j .

The two midpoints $(0, 1/2)$ and $(\pi/2, 1/2)$ of the left and right vertical edges in Fig. 1 are the two DQCPs

between the $\text{Néel}_{x(y)}$ and $\text{Néel}_{y(x)}^{\text{SPT}}$ phases, respectively. These DQCPs are also QTPTs between distinct topological phases [51]. Indeed, when PBCs are chosen, the two corners in the phase diagram of Fig. 1 that control the $\text{Néel}_{x(y)}$ and $\text{Néel}_{y(x)}^{\text{SPT}}$ phases are related to each other by the unitary transformation $\hat{U}_E = \hat{U}_{\text{CZ}} \hat{U}_{\pi/2} \hat{X}$. Here, $\hat{U}_E$ is the composition of $\hat{U}_R$ with the global controlled-Z unitary, whose only nontrivial action on the Pauli operators is defined by $\hat{A}_j \mapsto \hat{Z}_{j-1} \hat{A}_j \hat{Z}_{j+1}$, with $A = X$ and Y for any site j . This unitary equivalence breaks down for OBCs. The unitary $\hat{U}_E$ is an example of an entangler, that exchanges distinct SPT phases when PBCs are chosen [56,57]. Here, it becomes an anomalous $\mathbb{Z}_2$ symmetry of the two DQCPs at the midpoints of the vertical edges [51].

The midpoint $(\pi/4, 1)$ of the top horizontal edge in Fig. 1 is the DQCP between the $\text{Néel}_y^{\text{SPT}}$ and $\text{Néel}_x^{\text{SPT}}$ phases. When PBCs are chosen, this DQCP is unitarily equivalent to that at $(\pi/4, 0)$ under the action of the entangler $\hat{U}_E$. Hence, it is also described by the $\mathfrak{c} = 1$ CFT and has the same bulk properties. However, when OBCs are chosen, this unitary equivalence no longer holds. As opposed to the spin-1/2 quantum XY model at $(\pi/4, 0)$, the dressed spin-1/2 quantum XY model at $(\pi/4, 1)$ has four degenerate ground states associated with zero modes localized on the boundaries [51].

Finally, the point $(\pi/4, 1/2)$ is invariant under the action of $\hat{U}_R$ for both periodic and open boundary conditions, while it is only invariant under the action of $\hat{U}_E$ for periodic boundary conditions. Remarkably this point displays the phenomenon of Hilbert-space fragmentation [58], is Bethe Ansatz integrable, realizes a CFT with central charge 1, and supports ferromagnetic long-range order as shown in Refs. [59–62].

Phase diagram. We now turn our attention to the fate of the four gapped phases labeled by squares and the associated four DQCPs labeled by circles in Fig. 1. Each corner of the phase diagram in Fig. 1 realizes a gapped phase of matter supporting long-range order. Associated with the long-range order is a degeneracy of 2 that originates from time-reversal symmetry and that is independent of the choice between PBCs and OBCs. Additionally, the degeneracy of the upper corners increases by a factor of 4 upon choosing OBCs. The locus of points in coupling space along which such a degeneracy changes defines the phase boundary of each one of these four phases. An estimate of the location of these phase boundaries is achieved by diagonalizing Hamiltonian (1) for $2N = 10$ sites and counting the number of eigenenergies within a distance of $\Delta\varepsilon = 0.3 > 1/(2N)$ from the ground-state energy, as is illustrated in Fig. 2. In doing so, we find a fifth phase centered around the point $(\pi/4, 1/2)$ that is shaped like a diamond with a degeneracy of two for both PBCs and OBCs. We show using density matrix renormalization group (DMRG) studies that, remarkably, this fifth phase is gapless, while it supports long-range ferromagnetic order along the quantization Z axis in spin space.

Incommensurate gapless ferromagnetic phase. The DMRG [63] numerical study was performed using our own code as well as the open-source Julia library ITensor [64,65]. While confirming the existence of the nonvanishing value taken by the antiferromagnetic order parameter (the staggered

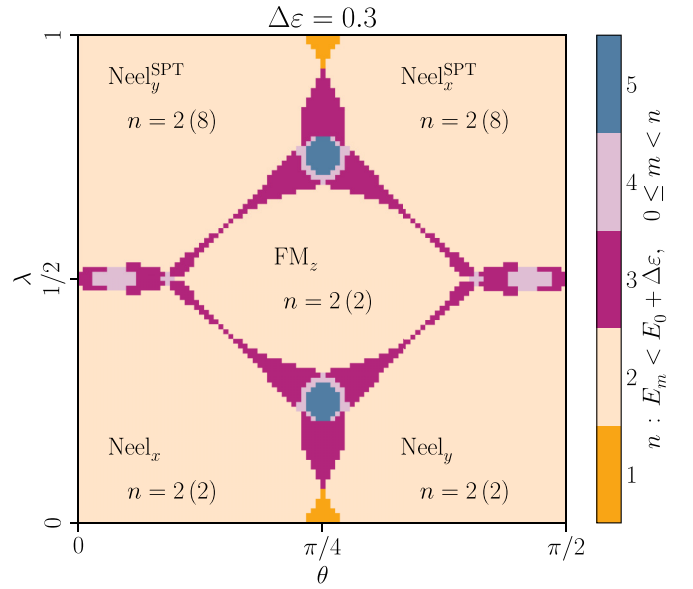


FIG. 2. Number n of eigenenergies E_m with an energy smaller than $E_0 + \Delta\varepsilon$, where E_0 is the ground-state energy calculated using exact diagonalization with PBCs for $2N = 10$ sites, $\Delta\varepsilon = 0.3$, and the integer $0 \leq m < n$. We also quote the value of n for the five distinct phases computed with OBCs in parentheses.

magnetization) [66]

$$m_{\text{sta}}^x := \lim_{B_{\text{sta}} \rightarrow 0} \lim_{\beta \rightarrow \infty} \lim_{2N \rightarrow \infty} \frac{1}{2N} \sum_{j=1}^{2N} (-1)^j \langle \hat{X}_j \rangle_{\beta, B_{\text{sta}}} \quad (3)$$

in the Néel_x phase from Fig. 1, we establish with Fig. 3 the nonvanishing value taken by the ferromagnetic (FM) order parameter (the uniform magnetization) [66]

$$m_{\text{uni}}^z := \lim_{B_{\text{uni}} \rightarrow 0} \lim_{\beta \rightarrow \infty} \lim_{2N \rightarrow \infty} \frac{1}{2N} \sum_{j=1}^{2N} \langle \hat{Z}_j \rangle_{\beta, B_{\text{uni}}} \quad (4)$$

inside the FM_z phase from Fig. 1. The maximum value taken by m_{uni}^z inside the FM_z phase from Fig. 1 is 4/10, i.e., 40% of

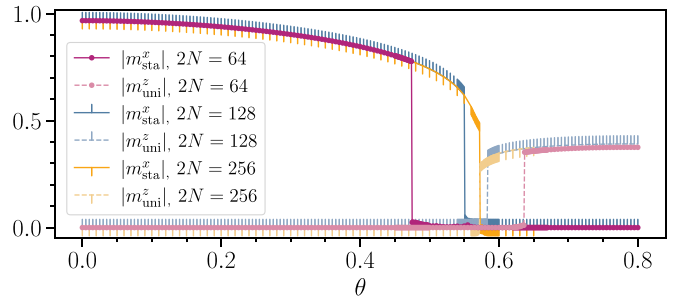


FIG. 3. Magnitudes of the staggered magnetization per site along the X axis (3) and the uniform magnetization per site along the Z axis (4), obtained along the line $\lambda = 0.4$ using DMRG with OBCs for $2N = 64, 128$, and 256 sites. The sharp suppression of the order parameters on both sides of the quantum phase transition around $\theta = 0.57$ indicates that the correlation length is larger than the length of the chain.

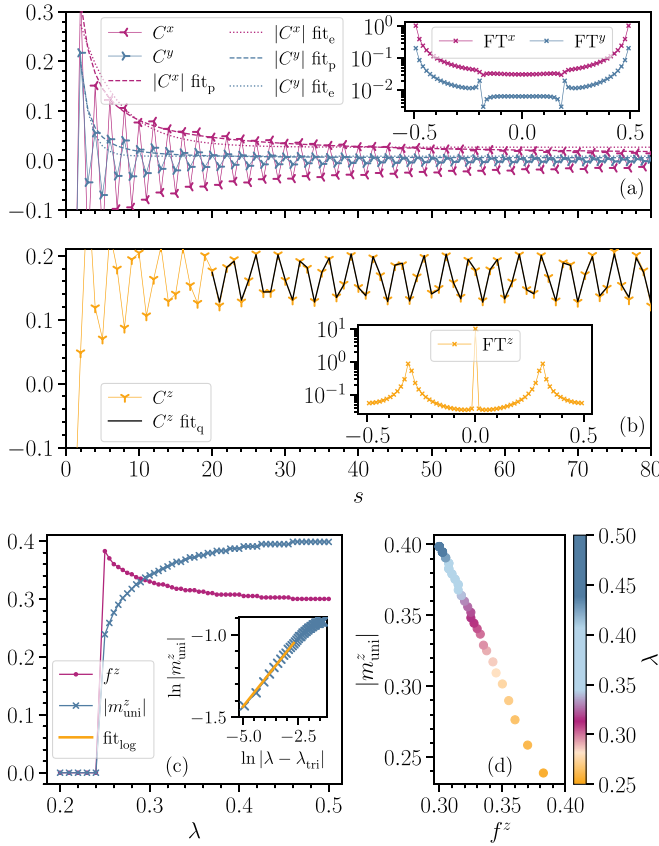


FIG. 4. Spin-spin correlation functions C^α defined in Eq. (5) computed using DMRG with OBCs, for $2N = 128$, $\lambda = 0.4$, and $\theta = \pi/4 - 0.1$ with (a) C^x and C^y and (b) C^z . Fits used are labeled “p” for power law, “e” for exponential, and “q” for cosine form. We find $q^z \simeq 2\pi \times 3/10$. The goodness of fit for p is always greater than that for e. The insets show the Fourier transforms of the respective correlations, where the abscissa is the frequency $f^\alpha \equiv q^\alpha/(2\pi)$ and the ordinate is the expansion coefficient $A(f^\alpha)$, $\alpha = x, y$, and z . (c) Dependence on λ of the uniform magnetization m^z_{uni} defined in Eq. (4) and the frequency f^z of the Fourier transform of the spin-spin correlations C^z in the bulk of a chain with $2N = 512$ sites obtained using DMRG with OBCs at $\theta = \pi/4$. The inset shows the $\text{fit}_{\log}$ ansatz $\ln |m^z_{\text{uni}}(\lambda)| \sim 1/6 \ln |\lambda - \lambda_{\text{tri}}|$, where $\lambda_{\text{tri}} \approx 0.25$. (d) Proportionality between $f^z(\lambda)$ and $m^z_{\text{uni}}(\lambda)$ on a chain of $2N = 512$ sites obtained using DMRG with OBCs at $\theta = \pi/4$. The different values of λ are specified by different colors.

the classical uniform magnetization per site and greater than the lower bound $1/3 \leq |m^z_{\text{uni}}|$ found in Ref. [59], independently of $2N > 32$ and the choice between PBCs and OBCs.

We argue that the FM_z phase in Fig. 1 is critical as follows [51]. First, the spin-spin correlation functions

$$C^\alpha_{j,j+s} := \lim_{\beta \rightarrow \infty} \lim_{2N \rightarrow \infty} \langle \hat{A}_j \hat{A}_{j+s} \rangle_\beta, \quad (5)$$

with $A = X, Y$, and Z , decay algebraically as $(-1)^s s^{-\eta^\alpha(\theta, \lambda)}$ for $\alpha = x$ and y with both PBCs and OBCs, if $s \gg 1$ [67]. The exponents η^x and η^y are identical on the $O(2)$ symmetric line $\theta = \pi/4$, while they differ when this symmetry is explicitly broken for $\theta \neq \pi/4$, as is illustrated in Fig. 4. We quote the values $\eta^x = \eta^y = 1$ at $(\pi/4, 1/2)$. Second, any point from the FM_z phase in Fig. 1 realizes a $\mathfrak{c} = 1$ CFT, where the central

charge $\mathfrak{c}$ is obtained from the bipartite entanglement entropy [68]. Finally, any energy spacing between two consecutive nondegenerate eigenenergies scales as $1/(2N)$ in the thermodynamic limit.

The incommensurability of the FM_z phase is corroborated by the form $C^z_{j,j+s} \sim (m^z_{\text{uni}})^2 + B \cos(q^z s + \phi)$, with $s \gg 1$ [67], of the correlation function, as is illustrated in Figs. 4(a) and 4(b). We find that the dependence on λ of q^z is symmetric with respect to the $\lambda = 1/2$ line, while it is independent of θ [51]. Moreover, q^z is a monotonically decreasing function of increasing λ in the interval from the tricritical point up to $\lambda = 1/2$. The wave vector q^z saturates to the approximate value $q^z \simeq 2\pi \times 3/10$ for $\lambda \in [0.45, 0.5]$ at $\theta = \pi/4$. We could not establish if this dependence on λ is continuous or displays a nonanalytic behavior, reminiscent of disorder points [69]. Hereto, the amplitude B is independent of θ , but is sensitive to the choice between OBCs and PBCs [70] and is a decreasing function of λ upon approaching the phase boundary of the FM_z phase in Fig. 1. The uniform magnetization m^z_{uni} along the vertical cut at $\theta = \pi/4$ is a function of λ that behaves in the vicinity of the tricritical value $\lambda_{\text{tri}} \approx 0.25$ as $(\lambda - \lambda_{\text{tri}})^{1/6}$ [51] [see Fig. 4(c)]. Remarkably, Fig. 4(d) shows that $q^z(\lambda)$ is proportional to $m^z_{\text{uni}}(\lambda)$ along the vertical cut at $\theta = \pi/4$. This proportionality extends to the scaling exponent $\eta^x(\lambda)$ and to the scaling exponent $\eta^{z,c}(\lambda)$ of the connected two-point function for $\hat{Z}_j$, as [51]

$$q^z = \pi(1 - |m^z_{\text{uni}}|) \propto \eta^x \propto \eta^{z,c}. \quad (6)$$

As such this proportionality is reminiscent of the critical phase called the Tomonaga-Luttinger-Liquid phase with central charge one (TLL1) in the study done in Ref. [71] of the quantum spin-1/2 chain with nearest- and next-nearest-neighbor Heisenberg exchange couplings in a uniform magnetic field.

Phase boundaries. The phase boundary of any one of the four gapped phases in Fig. 1 terminates at two of the four DQCPs located at $(\pi/4, 0)$, $(\pi/2, 1/2)$, $(\pi/4, 1)$, and $(0, 1/2)$ through straight segments emanating from the four tricritical points. The DQCPs $(\pi/4, 0)$ and $(\pi/4, 1)$ are known analytically to realize a $\mathfrak{c} = 1$ CFT. Whereas it was possible to confirm numerically the value of the central charge $\mathfrak{c} = 1$ at the two DCQPs $(\pi/4, 0)$ and $(\pi/4, 1)$ with a systematic uncertainty of less than 3%, the same estimate of $\mathfrak{c} = 1$ at the two DCQPs $(0, 1/2)$ and $(\pi/2, 1/2)$ suffers a systematic uncertainty of order 30% due to strong finite-size effects [51]. The same systematic uncertainty of order 30% also plagues the numerical estimate of $\mathfrak{c} = 1$ upon exiting the FM_z phase [51].

Spin-spin correlations $C^x_{j,j+s}$ and $C^y_{j,j+s}$ at $(\pi/4, 0)$ and $(\pi/4, 1)$ are observed numerically to decay as $(-1)^s s^{-1/2}$, as expected from the Jordan-Wigner transformation to a free fermion theory [51]. Strong finite-size effects prevented us from estimating the rate of decay of correlation functions at either $(0, 1/2)$ or upon exiting the FM_z phase [51].

The resolution of the phase boundaries in Fig. 2 can be improved by measuring the competing order parameters along a cut in the phase diagram, as is done in Fig. 3. For example, we locate in this way the two tricritical points attached to the Néel_x phase at $(0.3 \pm 0.04, 1/2)$ and $(\pi/4, 0.25 \pm 0.005)$. Exiting the FM_z to any one of the four gapped phases is done

through a smooth concave curve [72] that connects a pair of tricritical points as is qualitatively captured by Fig. 1.

The location of the latter tricritical point is consistent with the following exact diagonalization study. The dependence on $2N$ of the spacing $\delta E_{1,0}$ between the first excited energy E_1 above the ground-state energy E_0 at the point $(\pi/4, 0)$ is computed for increasing values of λ . The position of the tricritical point is the value of λ at which $\delta E_{1,0}$ changes from a power to an exponential suppression as a function of $2N$ [51].

Degeneracies at the phase boundaries with OBCs. With PBCs, the degeneracy of the ground-state energy is onefold at the four DQCPs. This fact has been verified numerically with DMRG [51]. The SPT nature of the gapped phases at the upper-left and right corners of the phase diagram in Fig. 1 has a measurable signature at the DQCP $(\pi/4, 1)$ as its ground-state degeneracy with OBCs is fourfold [51]. The vertical boundary separating the $\text{Néel}_y^{\text{SPT}}$ and $\text{Néel}_x^{\text{SPT}}$ phases with $\lambda < 1$ exhibits a twofold ground-state degeneracy in the thermodynamic limit. Indeed, the energy difference between the first excited state and the ground state scales like $(2N)^{-\kappa}$ with $\kappa \approx 6.6$ when computed using the Lanczos algorithm with $2N = 14, 16, 18, 20$, and 22 [51]. We have also verified using DMRG that the finite-size spacing between the second and the first energies above the ground-state energy scales as $\alpha_2/(2N)$ with $\alpha_2 \sim (1 - \lambda)^2$ in the vicinity of $\lambda = 1$ along $\theta = \pi/4$ [51]. The same analysis shows that the spacing between the fourth and third excited energies scales as $\alpha_4/(2N)$ with $\alpha_4 - 4.66 \sim (\lambda - 1)^1$, which is consistent with a mean-field analysis. Strong finite-size effects did not allow a reliable measurement of the ground-state degeneracies under OBCs along the phase boundaries $(\theta, 1/2)$ and $(\pi/2 - \theta, 1/2)$, where $0 \leq \theta \leq \theta_{\text{tri}}$ with the tricritical value $\theta_{\text{tri}} \approx 0.3$ [51].

Dynamical exponent z . For any point in Fig. 1 realizing a gapless phase of matter, the dynamical critical exponent

can be computed by measuring the first nonvanishing finite-size spacing between low-lying excitation energies above the ground state's energy. Using DMRG, we observe the scaling $(2N)^{-z}$ [73] with $z = 1$ in the FM_z phase as well as at the points $(\pi/4, 0)$, $(\pi/2, 1/2)$, $(\pi/4, 1)$, and $(0, 1/2)$ [51].

Discussion. We introduced a Hamiltonian that interpolates between four DQCPs enriched by $\mathbb{Z}_2^X \times \mathbb{Z}_2^Y \times \mathbb{Z}_2^T$ symmetry. We numerically obtained the phase diagram inside the coupling space, showing that the four symmetry-enriched DQCPs cannot be connected to one another without gap-opening transitions or passing through multicritical points. Importantly, we found that the four DQCPs are all connected to an extended gapless phase that is ferromagnetically ordered with incommensurate oscillatory longitudinal correlations. In closing, we conjecture that the topology of the phase diagram we found for the one-dimensional model (1) would be similar for its natural generalization in higher dimensions: a combination of an XY model and its version dressed with products of $\hat{Z}$ on all neighbors to sites sharing a bond. We note that such generalization of the model, in any dimension, does not have a sign problem for $\lambda \leq 1/2$ (and its dual for $\lambda \geq 1/2$) and could therefore be studied via quantum Monte Carlo simulations (see also Dupont *et al.* in Refs. [74,75], who found using Monte Carlo simulations an extended gapless phase separating two topologically ordered phases on the triangular lattice).

Acknowledgments. We thank Akira Furusaki, Toshiya Hikihara, Saran Prembabu, Subir Sachdev, and Rahul Sahay for discussion. This work is supported by SNSF Grant No. 219339 (A.R.), by NSF Grant No. DMR-2022428 (Ö.M.A.), and by DOE Grant No. DE-FG02-06ER46316 (C.C.). We thank one of the referees for pointing out that the center $(\pi/4, 1/2)$ of the phase diagram was treated in Refs. [58–62].

Data availability. The data that support the findings of this article are openly available [76].

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