

Noninteracting tight-binding models for Fock parafermions

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We model p -state Fock parafermions on a lattice in one dimension (with occupation per orbital of $0, 1, \dots, p-1$). For p a composite number, they may be mapped to q_m -state parafermions where q_m are the prime factors of p . For a Hamiltonian with a single-particle spectrum, the parafermions decompose into q_m -state parafermions. When p is a power of two, the decomposition is into fermions. We use this to construct a parafermionic Hamiltonian for $p=4$ with a single-particle spectrum using a fermionic tight-binding model which is bilinear in creation and annihilation operators. The single-particle levels may be determined by diagonalizing a square matrix whose order scales linearly with system size, and they are the same as those of the fermionic model. We show that the intermediate statistics of the thermodynamic distribution function for the occupation numbers (known as Gentile statistics) are consistent with the mapping to fermions, and we provide an example calculation of the internal energy and heat capacity for a simple linear chain.

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I. INTRODUCTION

Models of single-particle fermions are important in providing an intuitive approximate description of many physical systems as well as being the basis for perturbative calculations [1,2]. Parafermions are fascinating generalizations of fermions with exclusion statistics intermediate between fermions and bosons [1,3–5], and they may provide a platform for topological quantum computing [5–8]. However, parafermions typically arise in strongly correlated systems and generally do not have a single-particle description [1,9–22]. There are exceptions [1,4,23–31], most notably Baxter’s clock model [1,4,23–25,30] which is a non-Hermitian generalization of the Ising model in one dimension (1D) with a complex, single-particle spectrum. For p -state parafermions in a system with L sites, there are p^L many-body energies, each of the form

$$E = \omega^{n_1}\epsilon_1 + \omega^{n_2}\epsilon_2 + \dots + \omega^{n_L}\epsilon_L, \quad (1)$$

where $\omega = \exp(2\pi i/p)$, $(n_1, n_2, \dots, n_L)$ are parafermion occupation numbers $n_j = 0, 1, \dots, p-1$, and ϵ_j are real single-particle energy levels found by diagonalizing a square matrix whose order scales linearly with the system size L . Moreover, the levels ϵ_j for arbitrary p are related to those of the Ising model (with $p=2$) [1,4,23–25].

In this paper, we consider noninteracting tight-binding models for four-state ($p=4$) Fock parafermions [3] in 1D with a real single-particle spectrum. Four-state Fock parafermions [3] are indistinguishable quantum particles with

an associated Fock space, and creation and annihilation operators where each orbital [32] may have an occupancy of 0, 1, 2, or 3. For a system with L orbitals, there are 4^L many-body energies, each of the form

$$E = n_1\epsilon_1 + n_2\epsilon_2 + \dots + n_L\epsilon_L, \quad (2)$$

where $(n_1, n_2, \dots, n_L)$ are parafermion occupation numbers $n_j = 0, 1, 2, 3$, and ϵ_j are real single-particle energy levels found by diagonalizing a square matrix whose order scales linearly with the system size L . At finite temperature, the mean occupation number of a single-particle level $\langle n_\ell \rangle$ for p -state parafermions with spectrum (2) is an example of statistics intermediate between the Fermi-Dirac and Bose-Einstein distributions, also known as Gentile statistics [35,36].

Single-particle tight-binding models of fermions are solvable because the number operator and the Hamiltonian are bilinear functions of creation and annihilation operators [2,37,38]. For Fock parafermions [3], however, there is a nonlinear relation between the number operator and the creation and annihilation operators, as well as nonlinear on-site commutation relations. Hence, it is not possible to solve an arbitrary tight-binding model for Fock parafermions [3,18,39–43], even if the Hamiltonian is bilinear, e.g., with single-particle hopping only.

However, there are mappings from spin-1/2 fermions to four-state ($p=4$) parafermions [40,44–54]. Here, we modify a previously considered mapping [40] so that the parafermion number operator $\hat{N}_j$ can be written as a linear combination $\hat{N}_j = \hat{n}_{j\uparrow} + 2\hat{n}_{j\downarrow}$ of number operators for spin-up $\hat{n}_{j\uparrow}$ and spin-down $\hat{n}_{j\downarrow}$ fermions. This allows the occupations of 0, 1, 2, or 3 to be described by fermions with occupations of 0 or 1. Thus, energies in a single-particle spectrum (2) for four-state parafermions may be written as a linear combination of single-particle spectra for fermions, namely

$$E = n_{1\uparrow}\epsilon_1 + n_{2\uparrow}\epsilon_2 + \dots + n_{L\uparrow}\epsilon_L + 2(n_{1\downarrow}\epsilon_1 + n_{2\downarrow}\epsilon_2 + \dots + n_{L\downarrow}\epsilon_L). \quad (3)$$

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Four-state parafermions with such a spectrum can be considered to decompose into two species of fermions.

With this realization, it is possible to construct Hamiltonians for four-state parafermions with the single-particle spectrum (2) using any single-particle tight-binding model for fermions which is bilinear in fermionic creation and annihilation operators. Here, we describe an example in 1D with arbitrary on-site energies and nearest-neighbor hopping parameters. Such a Hamiltonian conserves the number of spin-up and of spin-down fermions as well as the number of parafermions. We show that thermodynamic properties of the noninteracting four-state parafermions, particularly the intermediate statistics of the occupation distribution function (also known as Gentile statistics [35,36]), are consistent with the mapping to spin-1/2 fermions. Finally, we describe the generalization to p -state parafermions [3,55,56]: When p is a composite number, occupation numbers may be written as a linear combination of occupation numbers for q_m -state parafermions, where q_m are the prime factors of p . Particularly, when p is a power of two, parafermions with a single-particle spectrum decompose into fermions.

II. TIGHT-BINDING MODEL

We consider a tight-binding model consisting of L sites, arbitrary on-site energies u_j for $j = 1, 2, \dots, L$, and nearest-neighbor hopping characterized by arbitrary parameters t_j for $j = 1, 2, \dots, L-1$. For simplicity, we consider real tight-binding parameters, giving an Hermitian Hamiltonian for four-state ($p=4$) parafermions with open boundary conditions,

$$H = \sum_{j=1}^L u_j \hat{N}_j + \sum_{j=1}^{L-1} t_j [c_j^\dagger \hat{\Gamma}_j \hat{\Gamma}_{j+1} c_{j+1} (-i)^{\hat{N}_j} + 2c_j^{\dagger 2} c_{j+1}^2 (-1)^{\hat{N}_j} + \text{H.c.}], \quad (4)$$

where $\hat{\Gamma}_j = 1 - c_j^\dagger c_j + c_j^{\dagger 2} c_j^2$. Parafermion creation $c_j^\dagger$ and annihilation c_j operators act on Fock states [3] as

$$c_j^\dagger |n_1, \dots, n_j, \dots, n_L\rangle = (-i)^{\sum_{\ell < j} n_\ell} |n_1, \dots, n_j + 1, \dots, n_L\rangle, \\ c_j |n_1, \dots, n_j, \dots, n_L\rangle = i^{\sum_{\ell < j} n_\ell} |n_1, \dots, n_j - 1, \dots, n_L\rangle,$$

where $n_j = 0, 1, 2, 3$ is an eigenvalue of the number operator $\hat{N}_j$ [3],

$$\hat{N}_j |n_1, \dots, n_j, \dots, n_L\rangle = n_j |n_1, \dots, n_j, \dots, n_L\rangle, \quad (5)$$

which is given by

$$\hat{N}_j = c_j^\dagger c_j + c_j^{\dagger 2} c_j^2 + c_j^{\dagger 3} c_j^3. \quad (6)$$

Operators on different sites commute up to a phase [3],

$$c_\ell^\dagger c_j^\dagger = i c_j^\dagger c_\ell^\dagger, \quad c_\ell c_j = i c_j c_\ell, \quad c_\ell c_j^\dagger = -i c_j^\dagger c_\ell \quad \text{for } \ell < j,$$

whereas, on the same site,

$$c_j^{\dagger m} c_j^m + c_j^{4-m} c_j^{\dagger(4-m)} = 1 \quad \text{for } m = 1, \dots, 3.$$

In the Hamiltonian (4), the combination of terms, including string phase factors such as $(-i)^{\hat{N}_j}$, ensures that it is local in

space [3,40], i.e., H could be written in terms of parafermionic versions of hard-core bosons [3] without any string factors. The operator $\hat{\Gamma}_j$ restricts single-particle hopping so that it is not possible to perform a single hop off a site that is doubly occupied or for a single hop to create a doubly occupied site. However, these states can generally be accessed by coherent hopping of two particles [the second hopping term in Eq. (4)].

The Hamiltonian (4) may be solved by mapping [57] the parafermions to itinerant spinful fermions, $\sigma = \{\uparrow, \downarrow\}$, with creation and annihilation operators $f_{j\sigma}^\dagger$ and $f_{j\sigma}$,

$$f_{j\uparrow} = i^{\sum_{\ell < j} \hat{N}_\ell} \hat{\Gamma}_j c_j, \quad (7)$$

$$f_{j\downarrow} = (-1)^{\sum_{\ell > j} \hat{N}_\ell} (-1)^{\sum_{\ell < j} \hat{N}_\ell} c_j^2, \quad (8)$$

$$c_j = (-i)^{\sum_{\ell < j} \hat{N}_\ell} f_{j\uparrow} + i^{\sum_{\ell < j} \hat{N}_\ell} (-1)^{\sum_{\ell > j} \hat{N}_\ell} f_{j\uparrow}^\dagger f_{j\downarrow}, \quad (9)$$

where $\hat{n}_{j\sigma}$ is the number operator for fermions on site j ,

$$\hat{n}_{j\uparrow} = f_{j\uparrow}^\dagger f_{j\uparrow} = c_j^\dagger c_j - c_j^{\dagger 2} c_j^2 + c_j^{\dagger 3} c_j^3, \quad (10)$$

$$\hat{n}_{j\downarrow} = f_{j\downarrow}^\dagger f_{j\downarrow} = c_j^{\dagger 2} c_j^2. \quad (11)$$

The annihilation operators $f_{j\sigma}$ for spinful fermions act on Fock states with $n_{j\sigma} = 1$ as

$$f_{j\uparrow} |\dots, n_{j\uparrow}, \dots\rangle = (-1)^{\sum_{\ell < j} \hat{n}_{\ell\uparrow}} |\dots, n_{j\uparrow} - 1, \dots\rangle, \\ f_{j\downarrow} |\dots, n_{j\downarrow}, \dots\rangle = (-1)^{\sum_{\ell=1}^L \hat{n}_{\ell\downarrow}} (-1)^{\sum_{\ell < j} \hat{n}_{\ell\downarrow}} \\ \times |\dots, n_{j\downarrow} - 1, \dots\rangle,$$

where $n_{j\sigma}$ is an eigenvalue of the number operator $\hat{n}_{j\sigma}$. These operators obey the usual fermionic anticommutation relations including anticommutation of different spins, $\{f_{j\sigma}, f_{\ell\sigma'}^\dagger\} = \delta_{j\ell} \delta_{\sigma\sigma'}$. The mapping relates the on-site parafermion basis $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ to the action of spinful fermions on the vacuum,

$$|0\rangle \equiv |0\rangle, \quad |1\rangle \equiv f_{j\uparrow}^\dagger |0\rangle, \quad |2\rangle \equiv f_{j\downarrow}^\dagger |0\rangle, \quad |3\rangle \equiv f_{j\uparrow}^\dagger f_{j\downarrow}^\dagger |0\rangle,$$

which is a modification of the mapping described in Ref. [40].

With the fermionic number operators (10) and (11), the parafermionic number operator (6) may be expressed as

$$\hat{N}_j = f_{j\uparrow}^\dagger f_{j\uparrow} + 2f_{j\downarrow}^\dagger f_{j\downarrow}. \quad (12)$$

This mapping allows the Hamiltonian to be written as a sum of quadratic terms,

$$H = \Phi_\uparrow^\dagger \mathcal{H} \Phi_\uparrow + 2\Phi_\downarrow^\dagger \mathcal{H} \Phi_\downarrow, \quad (13)$$

where $\Phi_\sigma^\dagger = (f_{1\sigma}^\dagger \ f_{2\sigma}^\dagger \ \dots \ f_{L\sigma}^\dagger)$ and $\mathcal{H}$ is a $L \times L$ matrix,

$$\mathcal{H} = \begin{pmatrix} u_1 & t_1 & 0 & 0 & \dots \\ t_1 & u_2 & t_2 & 0 & \dots \\ 0 & t_2 & u_3 & t_3 & \dots \\ 0 & 0 & t_3 & u_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (14)$$

The linear relationship (12) between the parafermion number operator and the fermion number operators is the reason why the model is solvable with a single-particle spectrum. On diagonalizing the matrix $\mathcal{H}$, linear combinations of $f_{j\sigma}^\dagger$ and

$f_{j\sigma}$ create number operators for fermion eigenstates which are combined, by the form of the Hamiltonian (13), into number operators for parafermion eigenstates. Thus, the many-body energy spectrum of Hamiltonian (4) consists of 4^L energies, each of which may be expressed as in Eq. (2) where ϵ_ℓ are single-particle energy levels with $\ell = 1, 2, \dots, L$, i.e., eigenvalues of the matrix $\mathcal{H}$, and $n_\ell = 0, 1, 2, 3$ are parafermion occupation numbers [57].

For simplicity, we have considered a model with nearest-neighbor hopping only. However, the construction of parafermion models with a single-particle spectrum (2) may be generalized to any fermion model which is bilinear in fermion creation and annihilation operators. An example is given in the Supplemental Material [57] with next-nearest-neighbor hopping, giving a nonlocal parafermion Hamiltonian. A second example in the Supplemental Material [57] generalizes the approach to mean-field models of a superconductor that may be solved using a Bogoliubov transformation. Specifically, we construct a parafermion counterpart of the Kitaev superconducting chain [64], and we show that the ground state in the topological phase is fourfold degenerate, with each ground state distinguished by the fourfold parafermion occupation numbers of Majorana edge modes.

III. TRANSLATIONAL INVARIANCE

When the parameters in a Hamiltonian have translational invariance, a Fourier transformation of parafermion operators is generally not useful because of complicated commutation relations in k space [9,65]. However, with the mapping (12) and (13), we can write parafermion eigenvalues and eigenstates in terms of known fermion ones. As an example, consider a simple linear chain in 1D with one orbital per unit cell, $u_j = u$ and $t_j = t$ for all j in the Hamiltonian (4) and the matrix (14). The eigenvalues for a fermion system with open boundary conditions are $\epsilon_k = u + 2t \cos(ka)$ where $ka = m\pi/(L+1)$ with index $m = 1, 2, \dots, L$. The matrix has eigenstates $\psi_k(j) = \sqrt{2/(L+1)} \sin(kaj)$, and the fermionic Hamiltonians may be diagonalized as $H_\sigma = \sum_k \epsilon(k) \tilde{f}_{k\sigma}^\dagger \tilde{f}_{k\sigma}$ where $\tilde{f}_{k\sigma} = \sum_j \psi_k(j) f_{j\sigma}$. With $H = H_\uparrow + 2H_\downarrow$, then the parafermion Hamiltonian may be written as $H = \sum_k \epsilon(k) \hat{N}_k$ where $\hat{N}_k = \hat{n}_{k\uparrow} + 2\hat{n}_{k\downarrow}$ for $\hat{n}_{k\sigma} = \tilde{f}_{k\sigma}^\dagger \tilde{f}_{k\sigma}$.

IV. PARAFERMION DISTRIBUTION FUNCTION

Given a single-particle energy spectrum (2), one would like to know the occupation of levels as a function of temperature T and chemical potential μ . The mean occupation number of a single-particle level $\langle n_\ell \rangle$ for four-state parafermions with spectrum (2) is an example of statistics intermediate between the Fermi-Dirac and Bose-Einstein distributions, also known as Gentile statistics [35,36]. The grand canonical partition function Z may be represented as $Z = \prod_\ell Z_\ell$ where the partition function for a given single-particle level is $Z_\ell = \sum_{n_\ell} \exp[\beta(\mu - \epsilon_\ell)n_\ell]$, and $\beta = 1/k_B T$ is the inverse temperature [66–68]. For four-state parafermions,

$$Z_\ell = 1 + x^{-1} + x^{-2} + x^{-3}, \quad x = e^{\beta(\epsilon_\ell - \mu)}. \quad (15)$$

The mean occupation number is given [35,36] by

$$\langle n_\ell \rangle = \frac{1}{\beta} \frac{\partial \ln Z_\ell}{\partial \mu} = \frac{3 + 2x + x^2}{(1+x)(1+x^2)}. \quad (16)$$

This may be expressed as a sum of two fermionic terms,

$$\langle n_\ell \rangle = \langle n_{\ell\uparrow} \rangle + 2\langle n_{\ell\downarrow} \rangle, \quad (17)$$

in agreement with the expectations of the mapping from parafermions to spin-1/2 fermions, Eq. (12). Here, $\langle n_{\ell\sigma} \rangle = 1/[e^{\beta_\sigma(\epsilon_\ell - \mu)} + 1]$ where $\beta_\uparrow = \beta$ and $\beta_\downarrow = 2\beta$, i.e., the down-spin component $\langle n_{\ell\downarrow} \rangle$ has a Fermi-Dirac distribution, but at a lower effective temperature of $T/2$. At low temperature, $\langle n_\ell \rangle \approx 3$ for $\epsilon_\ell \ll \mu$, and $\langle n_\ell \rangle \approx 0$ for $\epsilon_\ell \gg \mu$ [35,36,68]. Owing to the lower effective temperature of the down-spin component, $\langle n_\ell \rangle \approx \exp[-\beta(\epsilon_\ell - \mu)]$ for $\beta(\epsilon_\ell - \mu) \gg 1$, recovering Maxwell-Boltzmann statistics.

The variance $\langle (\Delta n_\ell)^2 \rangle$ in the occupation number of a single-particle level may be determined with $\langle (\Delta n_\ell)^2 \rangle = (1/\beta) \partial \langle n_\ell \rangle / \partial \mu$ [66],

$$\langle (\Delta n_\ell)^2 \rangle = \langle n_{\ell\uparrow} \rangle (1 - \langle n_{\ell\uparrow} \rangle) + 4\langle n_{\ell\downarrow} \rangle (1 - \langle n_{\ell\downarrow} \rangle). \quad (18)$$

As with the Fermi-Dirac distribution, $\langle (\Delta n_\ell)^2 \rangle \approx 0$ for $\epsilon_\ell \ll \mu$ and for $\epsilon_\ell \gg \mu$ at low temperature, and it has its maximal value at $\epsilon_\ell = \mu$ where $\langle (\Delta n_\ell)^2 \rangle_{\max} = 5/4$. This is larger than that of the Fermi-Dirac distribution, $\langle (\Delta n_\ell)^2 \rangle_{\max} = 1/4$, as expected for statistics intermediate between those of fermions and bosons.

The mean parafermionic occupation number (17) determines thermodynamic properties of parafermionic systems with single-particle spectra (2) such as the internal energy per unit length $u_E = \int_{-\infty}^{\infty} \langle n(\epsilon) \rangle g(\epsilon) \epsilon d\epsilon$ and the specific heat capacity $c_V = \partial u_E / \partial T$ [66]. Here, $\langle n(\epsilon) \rangle$ is the parafermionic occupation number (17) in the limit of continuous energy ϵ , and $g(\epsilon)$ is the energy density of states per unit length. With the parafermion occupation number (17), $u_E(T) = u_0(T) + 2u_0(T/2)$, where $u_0(T)$ is the internal energy of a system of nondegenerate fermions. Specifically,

$$u_E = \int_{-\infty}^{\infty} \frac{g(\epsilon) \epsilon d\epsilon}{e^{\beta(\epsilon_\ell - \mu)} + 1} + 2 \int_{-\infty}^{\infty} \frac{g(\epsilon) \epsilon d\epsilon}{e^{2\beta(\epsilon_\ell - \mu)} + 1}. \quad (19)$$

For $T = 0$, $u_E = 3 \int_{-\infty}^{\epsilon_F} g(\epsilon) \epsilon d\epsilon$, where $\epsilon_F = \mu(T = 0)$, which is three times that of nondegenerate fermions [66]. The specific heat of parafermions is $c_V(T) = c_0(T) + c_0(T/2)$, where $c_0(T)$ is the specific heat of nondegenerate fermions. At low temperature, the linear-in-temperature part of the specific heat $c_V \approx (\pi^2/2) k_B^2 T g(\epsilon_F)$ is 3/2 that of nondegenerate fermions [66].

As a specific example, consider the simple linear chain with one orbital per unit cell, zero on-site energy ($u = 0$), nearest-neighbor hopping t , and a single band $\epsilon_k = 2t \cos(ka)$ for $L \gg 1$. The density of states is $g(\epsilon) = 1/(\pi a \sqrt{4t^2 - \epsilon^2})$ for $|\epsilon| < 2t$. Figure 1(a) shows $u_E(T)$ and Fig. 1(b) shows $c_V(T)$ when the chemical potential is at the center of the band, $\mu = 0$, determined by numerical evaluation of Eq. (19) and $c_V = \partial u_E / \partial T$. The properties of parafermions (solid lines) are compared to those of fermions with degeneracies ranging from one to three. The internal energy of parafermions, Fig. 1(a), coincides with that of threefold-degenerate fermions at $T = 0$, but the temperature dependence is different accord-

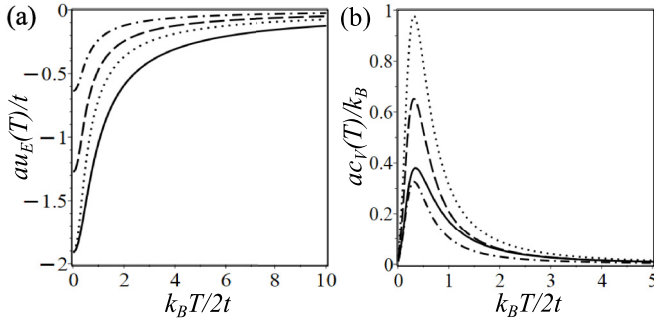


FIG. 1. (a) Internal energy per unit length u_E and (b) specific heat capacity c_V plotted as a function of temperature T , for a linear chain with nearest-neighbor hopping t with dispersion $\epsilon(k) = 2t \cos(ka)$ where a is the lattice constant. Curves show the contribution of four-state parafermions (solid), nondegenerate fermions (dashed-dotted), twofold degenerate fermions (dashed), and threefold degenerate fermions (dotted). For all plots, the chemical potential $\mu = 0$. Data are obtained by numerical evaluation of $u_E = \int_{-\infty}^{\infty} \langle n(\epsilon) \rangle g(\epsilon) \epsilon d\epsilon$ and $c_V = \partial u_E / \partial T$.

ing to $u_E(T) = u_0(T) + 2u_0(T/2)$ with the magnitude of the parafermion energy being larger for $T > 0$. The heat capacity of parafermions, Fig. 1(b), is also distinct from that of fermions for $T > 0$ according to $c_V(T) = c_0(T) + c_0(T/2)$. All the curves show a linear-in- T dependence at low T , but with different gradients.

The parafermion Hamiltonian (4) conserves the total number of spin-up fermions and of spin-down fermions, and, as a consequence, it is block diagonalized in the many-body basis of atomic orbitals $|n_1, n_2, \dots, n_L\rangle$. Each energy E in the single-particle parafermion energy spectrum (2) may be written as a sum of two parts corresponding to two separate fermionic species (3). Here, we show that interpreting the spectra as consisting of two separate fermionic species is consistent with the thermodynamics of single-particle parafermions, specifically we consider the internal energy (19). The mean occupation numbers $\langle n_{\uparrow} \rangle$ and $\langle n_{\downarrow} \rangle$ follow the Fermi-Dirac distribution with chemical potentials $\mu_{\uparrow}$ and $\mu_{\downarrow}$, respectively. We relate the chemical potentials at zero temperature to the number of particles, assuming that $N \approx 3N_{\uparrow} \approx 3N_{\downarrow}$ in the thermodynamic limit, where N is the total number of parafermions and $N_{\uparrow}$ ($N_{\downarrow}$) is the total number of spin-up (spin-down) fermions. For the spin-up fermions, $g_{\uparrow}(\epsilon) = g(\epsilon)$ where $g(\epsilon)$ is the parafermion density of states [as they have the same spectrum according to Eq. (13)], and $N_{\uparrow} = L \int_{-\infty}^{\mu_{\uparrow}} g(\epsilon) d\epsilon$. For spin down, $g_{\downarrow}(\epsilon) = (1/2)g(\epsilon/2)$ and $N_{\downarrow} = L \int_{-\infty}^{\mu_{\downarrow}/2} g(\epsilon) d\epsilon$. Finally, for the parafermions with chemical potential μ , $N = 3L \int_{-\infty}^{\mu} g(\epsilon) d\epsilon$. Comparing these expressions with $N \approx 3N_{\uparrow} \approx 3N_{\downarrow}$ gives $\mu_{\uparrow} \approx \mu_{\downarrow}/2 \approx \mu$. Thus, the internal energy is given by

$$\begin{aligned} u_E &= \int_{-\infty}^{\infty} \langle n_{\uparrow}(\epsilon) \rangle g_{\uparrow}(\epsilon) \epsilon d\epsilon + \int_{-\infty}^{\infty} \langle n_{\downarrow}(\epsilon') \rangle g_{\downarrow}(\epsilon') \epsilon' d\epsilon', \\ &= \int_{-\infty}^{\infty} \frac{g(\epsilon) \epsilon d\epsilon}{e^{\beta(\epsilon-\mu)} + 1} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{g(\epsilon'/2) \epsilon' d\epsilon'}{e^{\beta(\epsilon'-2\mu)} + 1}. \end{aligned} \quad (20)$$

Substituting $\epsilon' = 2\epsilon$ recovers the parafermion result (19).

V. GENERALIZATION TO p -STATE PARA-FERMIONS

When p is a composite number, we consider the factorization of p into prime factors q_m (including repeated factors) as

$$p = q_1 q_2 \cdots q_k, \quad (21)$$

where the prime factors are placed in ascending order, $q_m \leq q_{\ell}$ for $m < \ell$. By counting the possible occupations $0, 1, \dots, p-1$, an occupation number for p -state parafermions $n^{(p)}$ may be written as a linear combination of occupation numbers for q_m -state parafermions,

$$n^{(p)} = n_1^{(q_1)} + \sum_{m=2}^k \left(\prod_{\ell=1}^{m-1} q_{\ell} \right) n_m^{(q_m)}, \quad (22)$$

where the subscript is used to distinguish different species (which is needed for repeated factors), and k is the number of prime factors including repeated factors. We expect there is a corresponding mapping of creation and annihilation operators and of number operators, and such mappings for $p = 6$ and $p = 8$ are given in the Supplemental Material [57]. For p composite and a Hamiltonian with a single-particle spectrum (2), the parafermions decompose into q_m -state parafermions. Hence, a p -state parafermion satisfying Gentile statistics [35,36] is a composite particle when p is a composite number. For the special case when p is a power of two, the parafermions decompose into fermions, and it is possible to construct parafermionic Hamiltonians with a single-particle spectrum, as shown here for $p = 4$ using Eq. (13). Note that one could consider any solvable fermionic model, i.e., not necessarily of the form Eq. (13), and use the mapping to write it in a four-state parafermion representation with a known energy spectrum. However, it generally would not have the single-particle parafermionic spectrum (2).

VI. CONCLUSIONS

For p -state Fock parafermions, when p is a power of two, and for a Hamiltonian with a single-particle spectrum, the parafermions decompose into fermions. We use this to construct a parafermionic Hamiltonian for $p = 4$ with a single-particle spectrum using a fermionic tight-binding model which is bilinear in creation and annihilation operators. The thermodynamic distribution function for noninteracting four-state parafermions (17), known as Gentile statistics [35,36], is consistent with the mapping, but yields a down-spin component with a Fermi-Dirac distribution at a lower effective temperature $T/2$.

The mapping and decomposition may be extended to p -state Fock parafermions when p is a composite number, and mappings for $p = 6$ and $p = 8$ are given in the Supplemental Material [57]. We speculate that the mapping may provide an avenue for experimental simulation of parafermionic systems, e.g., using separate fermionic systems. Further theoretical studies should consider the properties of weakly interacting parafermions, for which the decomposition would break down. Another open question is whether it is possible to realize models with the single-particle spectrum (2), and, hence, satisfying Gentile statistics [35,36], when p is prime.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [69].

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