

Absence of measurement- and unraveling-induced entanglement transitions in continuously monitored one-dimensional free fermions

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Continuous monitoring of one-dimensional free fermionic systems can generate phenomena reminiscent of quantum criticality, such as logarithmic entanglement growth, algebraic correlations, and emergent conformal invariance, but in a nonequilibrium setting. However, whether these signatures reflect a genuine phase of nonequilibrium quantum matter or persist only over finite length scales is an active area of research. We address this question in a free fermionic chain subject to continuous monitoring of lattice-site occupations. An unraveling phase φ interpolates between measurement schemes, corresponding to different stochastic unravelings of the same Lindblad master equation; for $\varphi = 0$, measurements disentangle lattice sites, while for $\varphi = \pi/2$ they act as unitary random noise, yielding volume-law steady-state entanglement. Using replica Keldysh field theory, we obtain a nonlinear sigma model describing the long-wavelength physics. This analysis shows that for $0 \leq \varphi < \pi/2$, entanglement ultimately obeys an area law, but only beyond the exponentially large scale $\ln(l_{\varphi,*}) \sim J/[\gamma \cos(\varphi)]$, where J is the hopping amplitude and γ the measurement rate. Resolving $l_{\varphi,*}$ in numerical simulations is difficult for $\gamma/J \rightarrow 0$ or $\varphi \rightarrow \pi/2$. However, the theory also predicts that critical-like behavior appears below a crossover scale that grows only algebraically in J/γ , making it numerically accessible. Our simulations confirm these predictions, establishing the absence of measurement- or unraveling-induced entanglement transitions in this model.

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I. INTRODUCTION

Continuously monitored free fermions provide a versatile framework for exploring the interplay between unitary dynamics and measurements in quantum many-body systems [1–71]. They display a rich phenomenology while still being amenable to efficient numerical simulations and tractable through field-theoretic analytical methods. Nevertheless, fundamental questions concerning the existence of measurement-induced entanglement transitions in certain models remain a subject of ongoing investigation.

Here, we study a one-dimensional (1D) lattice system of free fermions subject to a continuous measurement process, which can be regarded as a stochastic unraveling of a Lindblad master equation. An *unraveling phase* φ allows us to tune between different measurement schemes, and thus different unravelings of the same master equation. Our analytical and numerical results indicate that in this model, there is no entanglement transition that can be induced by either changing the measurement rate γ or the unraveling phase φ . Therefore, our findings suggest that the apparent Kosterlitz-Thouless (KT) transition to a critical phase, which was observed in numerical

simulations of similar models [4,53], is in fact a finite-size crossover phenomenon.

In view of the ongoing debate about the existence of a critical phase and a measurement-induced phase transition in free fermionic systems with a conserved number of particles, we begin by briefly reviewing previous studies of related models.

A. Previous work

The study of entanglement dynamics in continuously monitored free fermionic systems was initiated in Ref. [1], which considered a 1D tight-binding chain subject to continuous measurements of lattice-site occupation numbers. Based on a phenomenological quasiparticle picture, the authors concluded that no measurement-induced entanglement transition occurs in this model; rather, the stationary entanglement obeys an area law for any nonzero measurement rate γ .

This conclusion was challenged by numerical results from Alberton *et al.* [4], which suggested a KT transition from an area-law phase to a critical phase at low-measurement rates. The critical phase is characterized by logarithmic growth of the entanglement entropy with subsystem size, algebraically decaying correlations, and emergent conformal invariance. These findings were further supported by a field-theoretic analysis of Dirac fermions [8]. Subsequently, several modifications of the model studied in Refs. [1,4] were found to display similar phenomenology. These include long-range hopping [19,21,68], spatial disorder [25,26,54,68,72], Floquet driving [55], as well as tunable measurement schemes that interpolate continuously between different unravelings of the same Lindblad master equation [53].

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In the context of tunable measurements, the quantum state diffusion equation used to model continuous monitoring in Refs. [1,4] can be viewed as one particular stochastic unraveling of the open-system dynamics of the average state, which are governed by a Markovian quantum master equation in Lindblad form. However, the unraveling of a Lindblad equation into quantum trajectories is not unique and depends on the specific type of continuous measurement performed [73–76]. Continuous monitoring can lead to either quantum jumps or quantum state diffusion, described by distinct stochastic Schrödinger equations. In Ref. [4], both types of trajectories were found to yield qualitatively similar results. However, even when restricting to quantum state diffusion, the choice of unraveling remains nonunique. Besides the unraveling considered in Refs. [1,4], where entangling unitary dynamics compete with disentangling measurements, there exists a *unitary unraveling* of the same master equation that describes time evolution under randomly fluctuating onsite potentials [1]. For this unitary unraveling, the steady-state entanglement entropy exhibits volume-law scaling, raising the question of whether an entanglement transition occurs when interpolating between the conventional and unitary unravelings, analogous to unraveling-induced phase transitions observed in quantum circuits [77,78]. This question was answered affirmatively in Ref. [53]: Numerical results indicate that not only changing the measurement rate but also tuning the unraveling can induce a KT transition from an area-law to a critical phase, while volume-law entanglement occurs only at a single point in parameter space.

A surprising new development emerged from the study of yet another measurement protocol [33]: projective measurements of local occupation numbers, performed at random points in space and time, similar to the type of measurements commonly considered in hybrid random circuits [79]. The resulting dynamics are described on large scales by a nonlinear sigma model (NLSM). Such models are familiar from the theory of disordered electronic systems [80]. The connection between 1 + 1D measurement-induced dynamics and the physics of Anderson localization in 2D disordered systems implies that the logarithmic growth of the entanglement entropy with subsystem size is cut off at a length scale l_* that is exponentially large in the inverse measurement rate, $\ln(l_*) \sim \gamma^{-1}$, corresponding to the crossover from weak to strong localization in 2D systems. This crossover occurs for any nonzero measurement rate, indicating that random projective measurements do not induce an entanglement transition. Nevertheless, the exponential dependence of l_* on γ makes the crossover from logarithmic to area-law entanglement appear sharp, which can be misinterpreted as a phase transition in numerical studies. A direct numerical verification of this dependence has only recently been achieved [71].

As stated above, the measurement processes considered in Refs. [4,33] are different: quantum state diffusion and quantum jump trajectories in the former, and random projective measurements in the latter. However, the studied models have the same symmetries, spatial dimensionality, and are restricted to short-range hopping. Therefore, the same large-scale behavior described by the NLSM is expected to apply universally to all of them. This expectation is corroborated by the results of Refs. [50,56], which derived the same NLSM for

yet other types of Hamiltonian dynamics and measurements and clarified the underlying symmetry requirements, in terms of particle-hole symmetry [50] and the role of particle-number conservation [56]. Numerical evidence for the absence of an entanglement transition is also presented in Refs. [12,36,71]. Notably, the description in terms of an NLSM does predict an entanglement transition in higher dimensions [35,41] or if particle-number conservation is broken [37].

The developments outlined above—the accumulation of evidence for a KT transition to a critical phase on the one hand, and for the instability of the critical phase in the thermodynamic limit on the other hand—have progressed in parallel in recent years. By now, the instability of the critical phase has been corroborated by numerical results for specific models. However, is it possible to stabilize the critical phase through one of the modifications of these models mentioned above, such as spatial disorder, Floquet driving, and different unravelings? Addressing that question analytically within the replica Keldysh field theory framework that underlies the NLSM description is in general not straightforward. Demonstrating the instability of the critical phase numerically is equally challenging. Since the scale l_* , beyond which we should expect area-law scaling of the entanglement entropy, is exponentially large in the inverse measurement rate, its direct observation is extremely demanding [71]. An alternative manifestation of the NLSM physics is the logarithmically slow RG flow responsible for the dependence $\ln(l_*) \sim \gamma^{-1}$. This flow is reflected in the weak-localization correction [33,52]. However, to determine the weak-localization correction, detailed knowledge of the Gaussian theory that describes the logarithmic phase in the absence of RG corrections is required.

Signatures of the instability of the critical phase that are accessible more directly and on only algebraically large scales were proposed in Ref. [56]. These are based on the analytical result, derived from the NLSM description, that features characteristic of a critical phase are confined to a finite critical range of length scales, bounded from below by $l_0 \sim \gamma^{-1}$ and from above by $l_c \sim \gamma^{-2}$. In particular, the connected density correlation function shows algebraic decay only up to a crossover scale $l_c \sim \gamma^{-2}$. Moreover, the scale-dependent effective central charge, defined as $c_\ell = 3\partial S_\ell / \partial \ln(\ell)$, where S_ℓ is the von Neumann entanglement entropy of a subsystem of size ℓ , has a maximum at $l_m \sim \gamma^{-3/2}$, between l_0 and l_c . Only in the vicinity of this maximum is the growth of the entanglement entropy approximately logarithmic. The decrease of the effective central charge on scales greater than l_m is a clear indication of slow crossover toward area-law scaling. Importantly, the scaling of l_c and l_m with γ is observable over a wide range of values of the measurement rate, including across the purported critical point [56]. These and further signatures of smooth scaling with γ , in agreement with predictions from the NLSM, provide strong numerical support for the absence of a measurement-induced entanglement transition in the models studied in Ref. [56].

B. Synopsis

In this paper, we use the same ideas to demonstrate the absence of measurement- and unraveling-induced entanglement transitions in free fermions on a 1D lattice under continuous

measurements of lattice-site occupation numbers. Similar to the model of Ref. [53], our model features an unraveling phase φ that allows us to interpolate between conventional quantum state diffusion [1,4] for $\varphi = 0$ and the unitary unraveling [1] for $\varphi = \pi/2$. For any value of $0 \leq \varphi < \pi/2$, the model is described on large scales by the same NLSM as was found previously in Ref. [41] for quantum state diffusion and in Refs. [33,35,50,52,56] for other types of dynamics or measurements, or both. The NLSM description implies that in one spatial dimension, there is no entanglement transition that can be induced by changing either the measurement rate γ or the unraveling phase φ . On large scales, the entanglement entropy always obeys area-law scaling. This indicates that the KT transition to a critical phase observed in Refs. [4,53] is, in fact, a finite-size crossover phenomenon. We validate the analytical predictions obtained from the NLSM description by determining the weak-localization correction numerically, finding good agreement with the universal forms for intact and broken particle-hole symmetry when $\varphi = 0$ and $\varphi \neq 0$, respectively. Furthermore, we demonstrate the scaling $l_c \sim \gamma^{-2}$ and $l_m \sim \gamma^{-3/2}$ for values of the unravelling phase φ below a specific value. The expected scaling is masked by nonuniversal modifications on short scales for larger values of φ .

We find that the effect of increasing φ at fixed γ is qualitatively similar to *decreasing* γ at fixed φ , which is compatible with the observation of an apparent unraveling-induced transition in Ref. [53]. For $\varphi \rightarrow \pi/2$, the scale $l_{\varphi,*}$ that marks the onset of area-law entanglement diverges, $l_{\varphi,*} \rightarrow \infty$. Exactly at $\varphi = \pi/2$, since the unitary unraveling corresponds to classical noise, the system exhibits volume-law entanglement for any value of the measurement rate.

The paper is organized as follows. In Sec. II, we introduce our model and discuss the physical significance of the unraveling phase φ . We present the analytical description of our model using replica Keldysh field theory in Sec. III. In particular, we discuss how our derivation differs from previous ones and how it allows us to classify the resulting Keldysh action in terms of its symmetries. We analyze the field theory within the Gaussian approximation in Sec. IV. The Gaussian theory describes a critical phase with algebraic decay of correlations and logarithmic growth of the stationary entanglement entropy. In Sec. V, we briefly summarize the long-wavelength description in terms of an NLSM, focusing on the case of broken particle-hole symmetry for $0 < \varphi < \pi/2$. We validate the analytical predictions, obtained by incorporating the RG flow of the NLSM into the Gaussian theory, in Sec. VI through a detailed comparison with numerical simulations. Conclusions are presented in Sec. VII. Two appendixes contain technical details of our analytical and numerical studies.

II. MODEL

We consider free fermions on a 1D lattice that are subject to continuous measurements. The Hamiltonian reads

$$\hat{H} = -J \sum_{l=1}^L (\hat{\psi}_l^\dagger \hat{\psi}_{l+1} + \hat{\psi}_{l+1}^\dagger \hat{\psi}_l), \quad (1)$$

where J is the hopping amplitude, L is the length of the 1D chain, and $\hat{\psi}_l$ and $\hat{\psi}_l^\dagger$ are fermionic annihilation and

creation operators, respectively. Periodic boundary conditions are implemented by setting $\hat{\psi}_{L+1} = \hat{\psi}_1$. The time evolution of the system under continuous monitoring is described by a stochastic Schrödinger equation [73–76],

$$|\psi(t + dt)\rangle = \left\{ 1 - i\hat{H}dt - \frac{\gamma}{2} \sum_{l=1}^L [\hat{n}_l - \langle \hat{n}_l(t) \rangle]^2 dt + \sqrt{\gamma} \sum_{l=1}^L [\hat{n}_l - \langle \hat{n}_l(t) \rangle] e^{i\varphi} dW_l(t) \right\} |\psi(t)\rangle. \quad (2)$$

Here, $|\psi(t)\rangle$ is the conditional state that depends on the measurement record, and $\langle \hat{n}_l(t) \rangle = \langle \psi(t) | \hat{n}_l | \psi(t) \rangle$ is the expectation value of $\hat{n}_l = \hat{\psi}_l^\dagger \hat{\psi}_l$ in the conditional state. The randomness of outcomes, which is inherent to quantum measurements, is described by the real stochastic Wiener increments $dW_l(t)$. These are independent for different lattice sites $l \in \{1, \dots, L\}$ and obey the conditions

$$\overline{dW_l(t)} = 0, \quad dW_l(t)dW_{l'}(t) = \delta_{l,l'}dt, \quad (3)$$

where the overline denotes the average over quantum trajectories corresponding to different measurement outcomes. In Eq. (2), the Wiener increment is multiplied by a phase factor that contains the unraveling phase φ . The physical meaning of the unraveling phase becomes apparent from a formal derivation of the stochastic Schrödinger equation, which we sketch in the following and discuss in more detail in Appendix A.

The minimal setting required to introduce the unraveling phase φ is a single lattice site l coupled to an ancilla bosonic mode, which is initially in the vacuum state of the bosonic annihilation operator $\hat{b}$. In a discrete time step of duration Δt , the quantum states of the lattice site and the bosonic ancilla are entangled through the application of the unitary $\hat{U} = e^{\sqrt{\gamma}\hat{n}_l(\hat{b}^\dagger - \hat{b})}$. In other words, information about $\hat{n}_l$ is imprinted on the ancilla through a displacement of the bosonic vacuum. This information is then read out by measuring the phase-dependent field quadrature $\hat{x}_\varphi = (\hat{b}e^{i\varphi} + \hat{b}^\dagger e^{-i\varphi})/\sqrt{2}$. The stochastic Schrödinger equation (2) is obtained by modeling the measurement outcomes as a classical stochastic process, generalizing the dynamics to an extended lattice system, and taking the limit of infinitesimal time steps, $\Delta t \rightarrow dt$. While we focus here on reading out information about $\hat{n}_l$, the derivation of Eq. (2) remains valid for any Hermitian system observable.

The above discussion clarifies the meaning of the unraveling phase φ in a formal measurement performed on an ancilla bosonic mode. How could such a measurement be implemented in practice? A natural physical realization of the model described by Eq. (2) is provided by fermionic atoms trapped in an optical lattice, where the ancillary bosonic mode is represented by an optical field. For example, a continuous measurement of the atomic density can be realized by dispersively coupling the density to the light field of a driven optical cavity and performing homodyne detection on the light leaking out of the cavity [81]. In homodyne detection, the cavity output field is interfered with a coherent phase reference, the local oscillator. The phase of this reference field corresponds precisely to the unraveling phase φ [74].

Interestingly, the unconditional dynamics of the density matrix $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)|$, obtained as the average of the state $|\psi(t)\rangle$ over the ensemble of trajectories, do not depend on the choice of φ and are described by the quantum master equation for dephasing,

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] - \frac{\gamma}{2} \sum_{l=1}^L [\hat{n}_l, [\hat{n}_l, \hat{\rho}(t)]]. \quad (4)$$

The stochastic Schrödinger equation (2) can be regarded as an unraveling of the master equation (4). From this perspective, different choices of φ correspond to different unravelings of the same master equation [1,53,77,78]. For this reason, we refer to φ as the unraveling phase.

The unraveling phase φ allows us to interpolate between conventional quantum state diffusion [1,4] for $\varphi = 0$ and the unitary unraveling corresponding to randomly fluctuating on-site potentials for $\varphi = \pi/2$ [1]. To show this, we perform a time-dependent gauge transformation [74],

$$|\psi(t)\rangle \mapsto e^{-i\phi(t)}|\psi(t)\rangle, \quad d\phi(t) = \sqrt{\gamma} \sin(\varphi) \sum_{l=1}^L \langle \hat{n}_l(t) \rangle dW_l(t). \quad (5)$$

Then, the stochastic Schrödinger equation (2) is recast as

$$|\psi(t+dt)\rangle = \left(1 - i\hat{H}dt - \frac{\gamma}{2} \sum_{l=1}^L \{ \cos(\varphi)^2 [\hat{n}_l - \langle \hat{n}_l(t) \rangle]^2 + \sin(\varphi)^2 \hat{n}_l \} dt + \sqrt{\gamma} \sum_{l=1}^L \{ \cos(\varphi) [\hat{n}_l - \langle \hat{n}_l(t) \rangle] + i \sin(\varphi) \hat{n}_l \} dW_l(t) \right) |\psi(t)\rangle. \quad (6)$$

The terms involving $\cos(\varphi)$ describe continuous monitoring of the field quadrature $\hat{x}_{\varphi=0} = (\hat{b} + \hat{b}^\dagger)/\sqrt{2}$, but with an effective measurement rate $\gamma \cos(\varphi)^2$. In contrast, the terms involving $\sin(\varphi)$ can be regarded as describing onsite potentials and thus add temporally fluctuating disorder to the unitary dynamics governed by the Hamiltonian [1,53]. From this perspective, by increasing the unraveling phase from $\varphi = 0$ to $\varphi = \pi/2$, we interpolate continuously between measurements of local occupation numbers and classical noise [82]. The effect of classical noise is opposite to that of measurements: While local measurements disentangle the quantum state, noise induces heating and thus leads to the growth of entanglement. Note that this observation is not affected by the gauge transformation (5) and remains valid in the original frame. We thus expect increasing φ to have a similar effect as decreasing the measurement rate γ [77].

III. REPLICA KELDYSH FIELD THEORY

Our goal is to understand the dynamics described by Eq. (2) and the dependence on the parameters γ and φ . To this end, we first develop an analytical approach in the framework of replica Keldysh field theory [23,33,35,37,41,50–52]. We later confirm the analytical findings through direct numerical simulations of Eq. (2).

A reformulation of Eq. (2) with $\varphi = 0$ as a fermionic replica Keldysh field theory was derived previously by Chahine and Buchhold [41]. In their approach, the average over measurement outcomes is taken for the R -replica stochastic master equation, and the resulting deterministic master equation is reformulated as a Keldysh functional integral. Here instead, we construct the Keldysh functional integral from the linear stochastic master equation for the unnormalized density matrix; the resulting Keldysh partition function, which still depends on the measurement outcomes, is quadratic in fermionic fields. This form allows for a particularly transparent analysis of symmetries. In particular, we find that for $\varphi = 0$, the theory has a particle-hole symmetry [37,52,56], which is broken for $\varphi \neq 0$. The average over trajectories is taken as a final step.

A. Linear stochastic master equation and replica trick

In deriving a functional-integral description for the dynamics described by Eq. (2), we address two technical issues: (i) The dependence on the expectation values $\langle \hat{n}_l(t) \rangle$ renders the stochastic Schrödinger equation (2) nonlinear in the conditional state $|\psi(t)\rangle$. This applies also to the corresponding stochastic master equation, which is therefore not a suitable starting point for constructing a functional integral description of the dynamics—the resulting action would depend, via the expectation value $\langle \hat{n}_l(t) \rangle$, on the functional integral itself. (ii) Not only the dynamics, but also the observables of interest are nonlinear in the conditional state. Indeed, all linear observables can be calculated from the density matrix $\hat{\rho}(t)$ that obeys the master equation (4). The steady state of the latter is a fully mixed state at infinite temperature, irrespective of the value of γ . Therefore, nontrivial effects of measurements manifest only in nonlinear observables. Important examples, which we focus on in our work and introduce further below, are the connected density correlation function and the entanglement entropy [1,4,33,56].

To overcome (i), we propose to construct the functional integral starting from the linear evolution equation for the unnormalized density matrix $\hat{D}(t)$, which we derive in Appendix A. With the construction of the functional integral in mind, we discretize the evolution of $\hat{D}(t)$ from t_0 to t into N time steps of duration $\Delta t = (t - t_0)/N$. The linear evolution equation for the unnormalized density matrix at discrete times, $\hat{D}_n = \hat{D}(t_n)$ for $t_n = t_0 + n\Delta t$, reads

$$\hat{D}_{n+1} = \hat{D}_n - i[\hat{H}, \hat{D}_n]\Delta t - \frac{\gamma}{2} \sum_{l=1}^L [\hat{n}_l, [\hat{n}_l, \hat{D}_n]]\Delta t + \sqrt{\gamma} \sum_{l=1}^L (\hat{n}_l e^{i\varphi} \hat{D}_n + \hat{D}_n \hat{n}_l e^{-i\varphi}) \Delta W_{l,n}. \quad (7)$$

As the derivation presented in Appendix A shows, this equation is equivalent to a linear evolution equation for the pure state $|v_n\rangle = |v(t_n)\rangle$. Therefore, Eq. (7) preserves the purity of the state $\hat{D}_n = |v_n\rangle\langle v_n|$; however, it does not preserve the norm of the state. According to Born's rule, the norm encodes the probability for a trajectory associated with a sequence of stochastic increments

$$\Delta \mathbf{W} = (\Delta W_{l,n})_{l \in \{1, \dots, L\}, n \in \{0, \dots, N-1\}} \quad (8)$$

to occur. More precisely, the probability of a trajectory is the product of the ostensible distribution of $\Delta \mathbf{W}$ [75], given by

$$Q_{\Delta \mathbf{W}} = \frac{1}{(2\pi \Delta t)^{NL/2}} \exp\left(-\frac{1}{2\Delta t} \sum_{l=1}^L \sum_{n=0}^{N-1} \Delta W_{l,n}^2\right), \quad (9)$$

and the norm of the state, $\| |v_N\rangle \| = \text{tr}(\hat{D}_N)$ at time $t_N = t$.

Having stated the linear evolution Eq. (7) that serves as the starting point for deriving a functional integral description, we now turn to issue (ii) identified above: The question of how to evaluate trajectory averages of nonlinear observables. As we discuss next, this can be achieved by introducing replicas.

The product of the expectation values of two operators $\hat{A}$ and $\hat{B}$ at time t_N provides a basic example of a nonlinear observable. To evaluate the average over trajectories, we introduce R replicas of the system $\hat{D}_{r,n}$ with replica index $r \in \{1, \dots, R\}$. We can thus write

$$\begin{aligned} \overline{\langle \hat{A}_N \rangle \langle \hat{B}_N \rangle} &= \int d\mu_{\Delta \mathbf{W}} \text{tr}(\hat{D}_N)^{-1} \text{tr}(\hat{A} \hat{D}_N) \text{tr}(\hat{B} \hat{D}_N) \\ &= \lim_{R \rightarrow 1} \int d\mu_{\Delta \mathbf{W}} \text{tr}(\hat{D}_N)^{R-2} \text{tr}(\hat{A} \hat{D}_N) \text{tr}(\hat{B} \hat{D}_N) \\ &= \lim_{R \rightarrow 1} \int d\mu_{\Delta \mathbf{W}} \text{tr} \left[(\hat{A}_1 \otimes \hat{B}_2) \left(\bigotimes_{r=1}^R \hat{D}_{r,N} \right) \right], \quad (10) \end{aligned}$$

where $d\mu_{\Delta \mathbf{W}} = d\Delta \mathbf{W} Q_{\Delta \mathbf{W}}$. The assumption underlying the replica trick is that the last equality still holds in the limit $R \rightarrow 1$ even though the formal manipulation of the integrand is valid only for integer $R > 2$ [83].

Generalizing the above to a product of more than two expectation values is straightforward. To calculate any such average, it is convenient to introduce appropriate sources in the Hamiltonian so as to turn the Keldysh partition function for R replicas,

$$Z_R = \int d\mu_{\Delta \mathbf{W}} \text{tr} \left[\bigotimes_{r=1}^R \hat{D}_{r,N} \right] = \int d\mu_{\Delta \mathbf{W}} \prod_{r=1}^R \text{tr}[\hat{D}_r(t)], \quad (11)$$

into a generating functional. Crucially, the right-hand side of Eq. (11) is linear in the unnormalized density matrix of R replicas. Therefore, a functional integral representation of the Keldysh partition function can be obtained via the usual Keldysh construction [83–86], which we summarize below.

B. Replica Keldysh action

According to Eq. (11), before we take the average over $\Delta \mathbf{W}$, the Keldysh partition function factorizes over replicas. Therefore, we consider first a single replica and write Eq. (7) as

$$\hat{D}_{r,n+1} = (1 + \mathcal{L}_n) \hat{D}_{r,n}, \quad (12)$$

where

$$\begin{aligned} \mathcal{L}_n \hat{D}_{r,n} &= -i[\hat{H}, \hat{D}_{r,n}] \Delta t + \frac{\gamma}{2} \sum_{l=1}^L [\hat{n}_l, [\hat{n}_l, \hat{D}_{r,n}]] \Delta t \\ &+ \sqrt{\gamma} \sum_{l=1}^L (\hat{n}_l e^{i\varphi} \hat{D}_{r,n} + \hat{D}_{r,n} \hat{n}_l e^{-i\varphi}) \Delta W_{l,n}. \quad (13) \end{aligned}$$

Next, we represent $\hat{D}_{r,n}$ in a basis of fermionic coherent states,

$$\begin{aligned} \hat{D}_{r,n} &= \int d\psi_{+,r,n}^* d\psi_{+,r,n} d\psi_{-,r,n}^* d\psi_{-,r,n} e^{-\psi_{+,r,n}^* \psi_{+,r,n}} \\ &\times e^{-\psi_{-,r,n}^* \psi_{-,r,n}} \langle \psi_{+,r,n} | \hat{D}_{r,n} | -\psi_{-,r,n} \rangle | \psi_{+,r,n} \rangle \langle -\psi_{-,r,n} |, \quad (14) \end{aligned}$$

where $\psi_{\pm,r,n}$ and $\psi_{\pm,r,n}^*$ are independent Grassmann variables,

$$d\psi_{\pm,r,n} = \prod_{l=1}^L d\psi_{\pm,r,l,n}, \quad | \psi_{\pm,r,n} \rangle = e^{-\sum_{l=1}^L \psi_{\pm,r,l,n} \hat{\psi}_{\pm,r,l,n}^\dagger} | 0 \rangle, \quad (15)$$

and as usual in the construction of fermionic Keldysh functional integrals we flip the sign of Grassmann fields on the backward branch. To obtain a relation between the coherent-state representations of the unnormalized density matrix at times t_n and t_{n+1} , we insert Eq. (14) into Eq. (13). We thus have to evaluate matrix elements of the superoperator $\mathcal{L}_n$,

$$\langle \psi_{+,r,n+1} | \mathcal{L}_n (| \psi_{+,r,n} \rangle \langle -\psi_{-,r,n} |) | -\psi_{-,r,n+1} \rangle. \quad (16)$$

This is done using the fact that coherent ket and bra states are right and left eigenstates of annihilation and creation operators, respectively. The final step of the derivation is to exponentiate the second term in Eq. (12) in the coherent-state representation. Then, using the discrete-time analog of the Itô rule in Eq. (3), $\Delta W_{l,n} \Delta W_{l',n} = \delta_{l,l'} \Delta t$, and taking the product over replicas as in Eq. (11), we obtain the R -replica Keldysh partition function,

$$Z_R = \int d\mu_{\Delta \mathbf{W}} \int D[\psi^*, \psi] e^{i(S_H[\psi^*, \psi] + \gamma S_M[\psi^*, \psi])}. \quad (17)$$

Writing sums over replicas and lattice sites as vector and matrix multiplications, the Hamiltonian contribution to the action is

$$\begin{aligned} S_H[\psi^*, \psi] &= \sum_{n=0}^{N-1} \left[\Delta t \left(-i \frac{\psi_{+,n+1}^\dagger - \psi_{+,n}^\dagger}{\Delta t} \psi_{+,n} \right. \right. \\ &\quad \left. \left. - i \psi_{-,n}^\dagger \frac{\psi_{-,n+1} - \psi_{-,n}}{\Delta t} \right) \right. \\ &\quad \left. - i(\psi_{-,n}^\dagger \psi_{+,n} - \psi_{+,n}^\dagger \psi_{-,n} - \psi_{-,n}^\dagger \psi_{-,n}) \right. \\ &\quad \left. - \Delta t(\psi_{+,n+1}^\dagger H \psi_{+,n} - \psi_{-,n}^\dagger H \psi_{-,n+1}) \right]. \quad (18) \end{aligned}$$

In the continuous-time limit, the discrete differences in the second line can formally be rewritten as derivatives, and the boundary terms in the third line can be neglected. The

elements of the matrix appearing in the last line are given by

$$H_{l,l'} = -J(\delta_{l+1,l'} + \delta_{l,l'+1}). \quad (19)$$

Continuous measurements give rise to the following contribution to the Keldysh action:

$$i\gamma S_M[\psi^*, \psi] = - \sum_{l=1}^L \sum_{n=0}^{N-1} \left[\frac{\gamma \Delta t}{2} v_{l,n+1,n}^0 - \sqrt{\gamma} v_{n+1,n,l}^\varphi \Delta W_{l,n} \right], \quad (20)$$

where

$$v_{l,n,n'}^\varphi = \sum_{r=1}^R (\psi_{+,r,l,n}^* \psi_{+,r,l,n'} e^{i\varphi} + \psi_{-,r,l,n}^* \psi_{-,r,l,n'} e^{-i\varphi}). \quad (21)$$

C. Symmetry classification

The Keldysh partition function in Eq. (17) is similar to that of free fermions in a disordered potential, with the increments $\Delta W_{l,n}$ playing the role of disorder [83,84]. As such, the Keldysh action of our model is amenable to the usual symmetry classification of disordered systems [80,87]. Symmetries of the action have to be taken into account in the derivation of the long-wavelength effective field theory in terms of an NLSM [22,50,52,56].

We consider the continuous-time limit $t_n \rightarrow t$, $\sum_{n=0}^{N-1} \Delta t \rightarrow \int_{t_0}^t dt$, and $\Delta W_{l,n} \rightarrow \xi_l(t) dt$, which allows us to work with more compact expressions. In this limit, the Keldysh action in Eq. (17) can be written as

$$S_H[\psi^*, \psi] + \gamma S_M[\psi^*, \psi] = \int_{t_0}^t dt' L[\psi^*(t'), \psi(t'), t'], \quad (22)$$

with the Lagrangian

$$L[\psi^*, \psi, t] = \psi^\dagger \left[(i\partial_t - H) \sigma_z + \frac{i\gamma}{2} - i\sqrt{\gamma} \Xi(t) e^{i\varphi \sigma_z} \right] \psi, \quad (23)$$

where σ_z is a Pauli matrix and the matrix $\Xi_{l,l'}(t) = \xi_l(t) \delta_{l,l'}$ is diagonal in Keldysh, replica, and real spaces. Following Ref. [52], we introduce a chiral basis,

$$\chi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \chi^\dagger = (-\psi_-^\dagger, \psi_+^\dagger). \quad (24)$$

In this basis, the Lagrangian takes the form $L[\chi^*, \chi, t] = \chi^\dagger A(t) \chi$, where the Lagrangian matrix $A(t)$ is given by

$$A(t) = \begin{pmatrix} 0 & i\partial_t - H - iM^*(t) \\ i\partial_t - H + iM(t) & 0 \end{pmatrix}, \quad (25)$$

with the measurement matrix

$$M(t) = \gamma/2 - \sqrt{\gamma} \Xi(t) e^{i\varphi}. \quad (26)$$

The Lagrangian matrix $A(t)$ has chiral symmetry, $\{A(t), \sigma_z\} = 0$. As discussed by Poboiko *et al.* [52], the Lagrangian matrix additionally has particle-hole symmetry (PHS),

$A(t) = -A^T(t)$, if the following conditions are satisfied: (i) the hopping Hamiltonian H has PHS, meaning that there is a basis in which H is skew-symmetric, $H = -H^T$; and (ii) the measurement matrices $M(t)$ are real in the same basis, $M(t) = M^*(t)$. Let us check whether our model obeys these conditions. The Hamiltonian matrix (19) is rendered skew-symmetric by a local gauge transformation, $H_{l,l'} \mapsto H_{l,l'} i^{l-l'}$. This transformation leaves the measurement matrix Eq. (26), which is diagonal in real space, invariant. Therefore, the Lagrangian matrix (25) has PHS if the measurement matrix (26) is real, which is the case only for $\varphi = 0$. We conclude that for $\varphi = 0$, our model belongs to the chiral orthogonal class BDI; instead, for $\varphi \neq 0$, it belongs to the chiral unitary class AIII. This distinction affects the numerical value of the prefactor of the weak-localization correction; however, it does not modify entanglement properties and correlations qualitatively [52].

D. Average over trajectories and temporal regularization

We proceed with the analysis of the Keldysh partition function (17) in the discrete-time formulation and take the average over stochastic increments with measure $d\mu_{\Delta W} = d\Delta W Q_{\Delta W}$. According to Eqs. (9) and (20), this average amounts to preforming a Gaussian integral. We obtain

$$e^{i\gamma \bar{S}_M[\psi^*, \psi]} = \int d\mu_{\Delta W} e^{i\gamma S_M[\psi^*, \psi]}, \quad (27)$$

where the averaged measurement action is given by

$$i\bar{S}_M[\psi^*, \psi] = \frac{\Delta t}{2} \sum_{l=1}^L \sum_{n=0}^{N-1} \left[(v_{l,n+1,n}^\varphi)^2 - v_{l,n+1,n}^0 \right], \quad (28)$$

with $v_{l,n,n'}^\varphi$ defined in Eq. (21).

The averaged measurement action (28) contains quartic vertices. We wish to decouple these vertices by introducing fermionic bilinears as new variables through a generalized Hubbard-Stratonovich transformation. Furthermore, as detailed in Refs. [33,56], in the continuous-time limit, these bilinears of fermionic fields should describe the symmetrized limit in which both fermionic fields are evaluated at the same time t . Therefore, before taking the continuous-time limit, the measurement action must be recast into a time-local form using the “principal-value” regularization [33,56].

As a first step, we rewrite the measurement action in a form that is strictly local in time. This can be done by using properties of Grassmann integration, which allow us to make the following replacements [88]:

$$\begin{aligned} \psi_{+,r,l,n+1}^* \psi_{+,r,l,n} &\rightarrow \psi_{+,r,l,n}^* \psi_{+,r,l,n} + 1, \\ \psi_{-,r,l,n}^* \psi_{-,r,l,n+1} &\rightarrow \psi_{-,r,l,n}^* \psi_{-,r,l,n} + 1. \end{aligned} \quad (29)$$

Note, however, that these replacements have to be made after expanding all replica-diagonal products of polynomials of fields. We thus obtain

$$\begin{aligned} i\bar{S}_M[\psi^*, \psi] = & \frac{\Delta t}{2} \sum_{l=1}^L \sum_{n=0}^{N-1} \left\{ \sum_{r=1}^R (\psi_{+,r,l,n}^* \psi_{+,r,l,n} + \psi_{-,r,l,n}^* \psi_{-,r,l,n} + 2\psi_{+,r,l,n}^* \psi_{+,r,l,n} \psi_{-,r,l,n}^* \psi_{-,r,l,n}) \right. \\ & + \sum_{r \neq r'} [\psi_{+,r,l,n}^* \psi_{+,r,l,n} e^{i\varphi} + \psi_{-,r,l,n}^* \psi_{-,r,l,n} e^{-i\varphi} + 2\cos(\varphi)] \\ & \left. \times [\psi_{+,r',l,n}^* \psi_{+,r',l,n} e^{i\varphi} + \psi_{-,r',l,n}^* \psi_{-,r',l,n} e^{-i\varphi} + 2\cos(\varphi)] \right\}. \end{aligned} \quad (30)$$

In the replica limit $R \rightarrow 1$, only the first two lines yield a nonvanishing contribution, which corresponds to the unconditional evolution of the average state under dephasing as described by the master equation (4) [88]. Note that the unraveling phase φ appears only in the sum over r and r' with $r \neq r'$. This sum vanishes in the replica limit, reflecting the fact that different unravelings describe the same unconditional dynamics.

The second step toward the Hubbard-Stratonovich transformation is to implement the principal-value regularization [33,56]. To this end, we first perform a Larkin-Ovchinnikov transformation. That is, we introduce new fermionic fields that are defined by [83,84]

$$\psi_{1,2} = \frac{1}{\sqrt{2}}(\psi_+ \pm \psi_-), \quad \psi_{1,2}^* = \frac{1}{\sqrt{2}}(\psi_+^* \mp \psi_-^*). \quad (31)$$

In the action expressed in terms of these fields, the principal-value regularization amounts to adding to each vertex the sum of its partial Wick contractions, with contractions of pairs of fields given by

$$\begin{pmatrix} \langle \psi_{1,r,l,n} \psi_{1,r',l',n'}^* \rangle & \langle \psi_{1,r,l,n} \psi_{2,r',l',n'}^* \rangle \\ \langle \psi_{2,r,l,n} \psi_{1,r',l',n'}^* \rangle & \langle \psi_{2,r,l,n} \psi_{2,r',l',n'}^* \rangle \end{pmatrix} = \frac{1}{2} \delta_{r,r'} \delta_{l,l'} \delta_{n,n'} \sigma_x. \quad (32)$$

After this step, we take the continuous-time limit. Using the shorthand notation $\int d^2\mathbf{x} = \sum_{l=1}^L \int_{t_0}^t dt'$, the averaged measurement action can be written as

$$\bar{S}_M[\psi^*, \psi] = \int d^2\mathbf{x} \mathcal{L}_M[\psi_l^*(t'), \psi_l(t')]. \quad (33)$$

The measurement Lagrangian, expressed in terms of $\psi_r = (\psi_{1,r}, \psi_{2,r})^T$, is given by

$$\begin{aligned} i\mathcal{L}_M[\psi^*, \psi] = & \frac{R}{2} \left[(R-1) \cos(\varphi)^2 - \frac{1}{2} \right] \\ & + (R-1) \cos(\varphi) \sum_{r=1}^R \psi_r^\dagger X_\varphi \psi_r \\ & + \frac{1}{2} \left(\sum_{r=1}^R \psi_r^\dagger X_\varphi \psi_r \right)^2, \end{aligned} \quad (34)$$

where X_φ is a matrix in Keldysh space,

$$X_\varphi = \cos(\varphi) \sigma_x + i \sin(\varphi) = \sigma_x e^{i\varphi \sigma_x}. \quad (35)$$

In this notation, the Hamiltonian action (18) becomes

$$S_H[\psi^*, \psi] = \sum_{r=1}^R \int_{t_0}^t dt' \psi_r^\dagger(t') G_0^{-1} \psi_r(t'), \quad (36)$$

with the free fermion Green's function

$$G_0^{-1} = i\partial_t - H. \quad (37)$$

E. Generalized Hubbard-Stratonovich transformation

We are now in a position to decouple the quartic vertex in the measurement Lagrangian (34). The vertex is local in both space and time, and can thus be decoupled in terms of local fermionic bilinears, which we collect in the matrix $\mathcal{G}_l(t) = -i\psi_l(t)\psi_l^*(t)$ [33,56]. We note that since we have imposed the principal-value regularization, the expectation value $\langle \mathcal{G}(t) \rangle$ is equal to the fermionic Green's function in the symmetrized limit of equal time arguments of the fields $\psi(t)$ and $\psi^*(t)$,

$$\langle \mathcal{G}_l(t) \rangle = -\frac{i}{2} (\langle \psi_l(t) \psi_l^*(t+0^+) \rangle + \langle \psi_l(t) \psi_l^*(t-0^+) \rangle). \quad (38)$$

To perform the Hubbard-Stratonovich transformation of the Keldysh partition function,

$$Z_R = \int D[\psi^*, \psi] e^{i(S_H[\psi^*, \psi] + \gamma \bar{S}_M[\psi^*, \psi])}, \quad (39)$$

we insert the identity

$$1 = \int D[\mathcal{G}, \Sigma] e^{-i \text{Tr}(\mathcal{G}\Sigma) - \psi^\dagger \Sigma \psi}, \quad (40)$$

where $\mathcal{G}$ and Σ are Hermitian $2R \times 2R$ matrix fields. The trace Tr acts in Keldysh, replica, lattice, and time spaces. Integration over Σ yields a delta functional that enforces the identity $\mathcal{G} = -i\psi\psi^\dagger$, which allows us to rewrite the quartic vertex in Eq. (34) in terms of $\mathcal{G}$. In principle, this can be done in several ways, which correspond to decoupling the vertex in different channels. However, as discussed by Poboiko *et al.* [33], for $\mathcal{G}$ to describe long-wavelength fluctuations, the vertex must be decoupled simultaneously in all possible channels. This is achieved by taking the sum over all Wick contractions of the

measurement Lagrangian, which yields

$$\begin{aligned} i\mathcal{L}_M[\mathcal{G}] = & \frac{R}{2} \left[(R-1) \cos(\varphi)^2 - \frac{1}{2} \right] \\ & - i(R-1) \cos(\varphi) \text{tr}(X_\varphi \mathcal{G}) \\ & + \frac{1}{2} \{ \text{tr}[(X_\varphi \mathcal{G})^2] - \text{tr}(X_\varphi \mathcal{G})^2 \}. \end{aligned} \quad (41)$$

As we have noted earlier, in the replica limit $R \rightarrow 1$, corresponding to the unconditional dynamics described by the master equation (4), the unraveling phase φ has to drop out of the measurement Lagrangian. Indeed, in this case, $\mathcal{G}$ reduces to a 2×2 matrix and we find

$$i\mathcal{L}_M[\mathcal{G}] = -\frac{1}{4} + \det_K(\mathcal{G}). \quad (42)$$

Interestingly, this result coincides with the one for random projective measurements of occupation numbers, given in Eq. (23) of Ref. [33], up to a factor of $1/2$.

Now that we have expressed the measurement Lagrangian in terms of $\mathcal{G}$, the remaining integral over fermionic fields is Gaussian and can be performed straightforwardly. We obtain

$$Z_R = \int D[\mathcal{G}, \Sigma] e^{iS[\mathcal{G}, \Sigma]}, \quad (43)$$

where the action is now given by

$$S[\mathcal{G}, \Sigma] = S_0[\mathcal{G}, \Sigma] + \gamma \int d^2\mathbf{x} \mathcal{L}_M[\mathcal{G}_I(t')], \quad (44)$$

with

$$iS_0[\mathcal{G}, \Sigma] = \text{Tr}[\ln(i\partial_t - H + i\Sigma) - i\mathcal{G}\Sigma]. \quad (45)$$

IV. GAUSSIAN THEORY

Our further analysis of the replica field theory proceeds in two steps: First, we study Gaussian fluctuations around a particular saddle point of the action (44); second, in Sec. V below, we extend this analysis to nonlinear fluctuations. The Gaussian approximation is valid for $\gamma \ll J$ and on short to intermediate length scales [33]. On large scales, strong fluctuations of Goldstone modes, which are described by an NLSM, lead to deviations from the Gaussian theory.

A. Saddle points and fluctuation expansion

In the Gaussian approximation, we expand the Keldysh action (44) to second order in fluctuations around a particular saddle point that describes the stationary state, meaning that $\mathcal{G}$ and Σ are spatially homogeneous and time independent. To find such a saddle point, we first take the variational derivative of the Keldysh action with respect to Σ , which yields

$$\mathcal{G} = \int_{-\pi}^{\pi} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{\omega - \xi_q + i\Sigma}, \quad (46)$$

where $\xi_q = -2J \cos(q)$. As it stands, the integral over frequencies is divergent. A regularization that is consistent with the symmetric limit of equal times in Eq. (38) is obtained by inserting the factor $(e^{i\omega t} + e^{-i\omega t})/2$ and taking the limit $t \rightarrow 0^+$ after integration. The integral over the matrix-valued integrand can then be evaluated by writing $\Sigma = \mathcal{V}\lambda\mathcal{V}^{-1}$ where

λ is a diagonal matrix. This leads to

$$\mathcal{G} = -iQ/2, \quad Q = \text{sgn}[\text{Re}(\Sigma)] = \mathcal{V} \text{sgn}[\text{Re}(\lambda)] \mathcal{V}^{-1}. \quad (47)$$

Next, we take the variational derivative of the Keldysh action with respect to $\mathcal{G}$. Setting $\mathcal{G} = -iQ/2$, we obtain

$$\Sigma = -\gamma \left\{ (R-1) \cos(\varphi) + \frac{1}{2} [X_\varphi Q - \text{tr}(X_\varphi Q)] \right\} X_\varphi. \quad (48)$$

Any matrix Q that obeys the condition $Q^2 = 1$ yields a solution of Eqs. (47) and (48). The form of Q can be restricted further on physical grounds: We expect Q to be replica-symmetric, $Q = Q_K \otimes 1_R$; moreover, we expect Q_K to obey the usual causality structure of Green's functions in Keldysh field theory [84]. For a given density $n = N/L$, such a saddle point is given by $Q_K = \Lambda$ with

$$\Lambda = \begin{pmatrix} 1 & 2(1-2n) \\ 0 & -1 \end{pmatrix}. \quad (49)$$

To see that this saddle point yields the correct density of fermions, note that from Eq. (38) and the relation between $\mathcal{G}$ and Q in Eq. (47) it follows that the “classical” Keldysh component of the density for a single replica is given by

$$\rho_r = \frac{1}{4} \text{tr}_K(1 - \sigma_x Q_{r,r}). \quad (50)$$

Therefore, at the replica-symmetric saddle point $Q = \Lambda \otimes 1_R$, the density is indeed $\rho_r = n$. Furthermore, note that $\Lambda = F \sigma_z F$ with

$$F = F^{-1} = \begin{pmatrix} 1 & 1-2n \\ 0 & -1 \end{pmatrix}. \quad (51)$$

As such, Q takes the form given in Eq. (47), with $\mathcal{V} = F \otimes 1_R$ and $\lambda = \sigma_z \otimes 1_R$. Finally, we note that Eq. (48) simplifies further for a replica-symmetric saddle point, as can be seen by using the definition of the matrix X_φ given in Eq. (35). Upon setting $R = 1$ in the numerical prefactor of the first term in Eq. (48), we find that Eqs. (47) and (48) can be written as

$$\mathcal{G} = -iQ/2, \quad \Sigma = \gamma Q/2, \quad Q = \Lambda. \quad (52)$$

Here and in the following, we omit the identity in replica space for the sake of brevity. The fermionic Green's functions corresponding to the saddle point (52) are [33]

$$\begin{aligned} G_q^{R/A}(\omega) &= \frac{1}{\omega - \xi_q \pm i\gamma/2}, \\ G_q^K(\omega) &= (1-2n)[G_q^R(\omega) - G_q^A(\omega)]. \end{aligned} \quad (53)$$

These Green's functions describe fermions at infinite temperature. Gaussian fluctuations around the saddle point (52) can be parametrized as

$$\mathcal{G} = -i(\Lambda + \delta Q_{\mathcal{G}})/2, \quad \Sigma = \gamma(\Lambda + \delta Q_{\Sigma})/2, \quad (54)$$

with Hermitian $2R \times 2R$ matrices $\delta Q_{\mathcal{G}}$ and δQ_{Σ} . The Gaussian theory is obtained by expanding the Keldysh action (44) to second order in $\delta Q_{\mathcal{G}}$ and δQ_{Σ} as detailed in Refs. [33, 56].

B. Connected density correlation function

As explained in Sec. III A, nontrivial effects of measurements manifest only in nonlinear observables. Here, we focus on the equal-time connected density correlation function. Its

definition in the operator formalism and the corresponding expression in terms of field expectation values are given by

$$\begin{aligned} C_{l,l'}(t) &= \frac{1}{2} \langle \{\hat{n}_l(t), \hat{n}_{l'}(t)\} \rangle - \langle \hat{n}_l(t) \rangle \langle \hat{n}_{l'}(t) \rangle \\ &= C_{r,r',l,l'}(t, t) - C_{r,r',l,l'}(t, t), \end{aligned} \quad (55)$$

where the correlation function of density fluctuations is

$$C_{r,r',l,l'}(t, t') = \langle \delta \rho_{r,l}(t) \delta \rho_{r',l'}(t') \rangle. \quad (56)$$

Because of the symmetry of the formalism under permutations of replicas, the specific choices of r and r' in Eq. (55) are irrelevant as long as $r \neq r'$. According to Eq. (50), density fluctuations are related to δQ_G by

$$\delta \rho_{r,l}(t) = -\frac{1}{4} \text{tr}_K[\sigma_x \delta Q_{G,r,r,l}(t)]. \quad (57)$$

The correlation function (55) can be obtained from the Keldysh partition function (43) by introducing sources that couple to density fluctuations [33] and performing the Gaussian integral over δQ_G and δQ_Σ .

The main technical difficulty in calculating the density correlation function (55) is accounting for the boundary condition imposed by stopping the measurement process at a finite time t , which requires us to treat matrices in time space as noncommuting [33]. Without loss of generality, we set the final time to $t = 0$. Then, we define the product of two matrices in time space as

$$(AB)(t, t') = \int_{-\infty}^0 dt'' A(t, t'') B(t'', t'), \quad (58)$$

where we leave Keldysh, replica, and lattice-site indices of A and B unspecified. Using this convention, we obtain the density correlation function (55) in the form

$$C_{l,l'}(t) = \{\mathcal{C}[1 + 2\gamma \cos(2\varphi)\mathcal{C}]^{-1}\}_{l,l'}(t, t). \quad (59)$$

For simplicity, we focus here on a half-filled chain, $n = 1/2$. We do not expect the physics to depend qualitatively on the value of n , apart from the limiting cases $n = 0$ and $n = 1$ in which the model becomes trivial. In Eq. (59), $\mathcal{C}$ is the density correlation function found previously in Refs. [33,56] for different types of measurement processes, with and without a conserved number of particles, but with a fixed mean density of $n = 1/2$. It is given by

$$\mathcal{C} = \frac{1}{2}(\mathcal{B}_+ + \gamma \mathcal{B}_- L \mathcal{B}_-). \quad (60)$$

The quantities $\mathcal{B}_\pm$ and L are most conveniently specified in momentum space,

$$\mathcal{B}_{\pm,q}(t) = \frac{1}{2}[\mathcal{B}_q(t) \pm \mathcal{B}_q(-t)], \quad (61)$$

where

$$\mathcal{B}_q(t) = \theta(t) e^{-\gamma t} J_0[4Jt \sin(q/2)], \quad (62)$$

which is related to the Green's functions (53) via $\mathcal{B}_l(t) = G_l^R(t) G_{-l}^A(-t)$, and where J_0 is a Bessel function of the first kind [33]; furthermore, $L_q(t, t')$ is the solution of a Wiener-Hopf integral equation,

$$L_q(t, t') - \gamma \int_{-\infty}^0 dt'' \mathcal{B}_{+,q}(t - t'') L_q(t'', t') = \delta(t - t'). \quad (63)$$

Through rescaling time, the integral equation (63) can be seen to depend on a single parameter,

$$u = 2l_0 \sin(q/2) \sim ql_0, \quad l_0 = 2J/\gamma. \quad (64)$$

We have solved Eqs. (59), (60), and (63) numerically by discretizing time t such that integrals over time as in Eq. (58) are converted into ordinary matrix multiplications. Details of the numerical procedure are provided in Appendix B.

Interestingly, the leading asymptotic behavior of the momentum-space transform $C_q(t)$ of Eq. (59) for both small and large momenta is correctly reproduced in the bulk approximation, in which the range of integration in Eq. (58) is extended to $t \in (-\infty, \infty)$ —that is, the boundary condition in time is ignored—and the parameter u (64) is replaced by twice its value [33]. We find

$$\frac{C_q}{g_{\varphi,0} q} = \frac{\tilde{c}(u)}{u}, \quad (65)$$

where we define

$$g_{\varphi,0} = \frac{l_0 n(1-n)}{\cos(\varphi)}, \quad (66)$$

and the function $\tilde{c}(u)$ is given by

$$\begin{aligned} \tilde{c}(u) &= \frac{2 \cos(\varphi)}{\pi} \int_0^\infty dv \{ \text{Re}[b(2u, v)] - |b(2u, v)|^2 \} \\ &\quad \times \{ 1 + f^2 |b(2u, v)|^2 - (1 + f^2) \text{Re}[b(2u, v)] \\ &\quad + 4n(1-n) \cos(2\varphi) [\text{Re}[b(2u, v)] - |b(2u, v)|^2] \}^{-1}, \end{aligned} \quad (67)$$

with $f = 1 - 2n$ and

$$b(u, v) = [(1 - iv)^2 + u^2]^{-1/2}. \quad (68)$$

Here and in the following, we omit the time argument of C_q , which becomes time independent in the steady state. The leading asymptotic behavior of $\tilde{c}(u)$ for $n = 1/2$ is

$$\tilde{c}(u) \sim \begin{cases} u, & u \rightarrow 0, \\ \cos(\varphi), & u \rightarrow \infty. \end{cases} \quad (69)$$

We compare the full numerical solution for the density correlation function (59) with the bulk approximation (65) in Sec. VI below. The discrepancies for intermediate values of u indicate that higher orders in asymptotic expansions of the bulk solution (69) do not agree with the full solution. As a result, the bulk approximation lacks the accuracy needed to determine the weak-localization correction, even though the overall agreement between the bulk approximation and the full numerical solution is good.

Transforming the momentum-space correlation function (65) back to real space, the leading asymptotic behavior in Eq. (69) for $u \rightarrow 0$ implies that C_l decays algebraically on large scales as expected for a critical phase [4,53],

$$\frac{l_0^2 C_l}{g_{\varphi,0}} \sim -\frac{1}{\pi y^2}, \quad y = \frac{l}{l_0} \rightarrow \infty. \quad (70)$$

Recall that the Green's functions (53) associated with the replica-symmetric saddle point (52) describe a Fermi gas at infinite temperature, with vanishing correlations (55) for $l \neq l'$.

The result in Eq. (70) shows that including Gaussian fluctuations around the saddle point brings about a first qualitative modification of correlations on scales $l \gtrsim l_0$. A further modification on large scales results from nonlinear fluctuations of Goldstone modes as we discuss in Sec. V.

C. Entanglement entropy

Having discussed the connected density correlation function (55), we now turn to another important example of a nonlinear observable: the entanglement entropy. Note that in quantum trajectories, the system is always in a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$. The reduced density matrix of a subsystem consisting of ℓ contiguous lattice sites is $\hat{\rho}_\ell = \text{tr}_{L-\ell}(\hat{\rho})$, where the trace is taken over the remaining $L - \ell$ sites, and the von Neumann entanglement entropy of the subsystem is defined as

$$S_\ell = -\text{tr}[\hat{\rho}_\ell \ln(\hat{\rho}_\ell)]. \quad (71)$$

To calculate the entanglement entropy, we use another property of the quantum trajectories described by Eq. (2). Since the generator of the stochastic dynamics is quadratic in fermionic operators, an initially Gaussian state remains Gaussian at all times; and for Gaussian states, the entanglement entropy is determined by the fluctuations of the number of particles in the subsystem, $\hat{N}_\ell = \sum_{i=1}^{\ell} \hat{n}_i$, through an expansion in cumulants [89–93],

$$S_\ell = 2 \sum_{k=1}^{\infty} \zeta(2k) C_\ell^{(2k)} = \frac{\pi^2}{3} C_\ell^{(2)} + \frac{\pi^4}{45} C_\ell^{(4)} + \dots \quad (72)$$

The coefficients in this expansion are given in terms of the Riemann zeta function $\zeta(k)$. As demonstrated below, the numerical data for S_ℓ is well approximated by keeping only the contribution from the second cumulant, which is given by

$$C_\ell^{(2)} = \overline{(\hat{N}_\ell - \langle \hat{N}_\ell \rangle)^2}. \quad (73)$$

The second cumulant can be expressed in terms of the conditional density correlation function (55), which leads to

$$S_\ell = \frac{\pi^2}{3} \sum_{l,l'=1}^{\ell} C_{l-l'} = \frac{2\pi}{3} \int_0^\infty \frac{dq}{q^2} C_q [1 - \cos(q\ell)]. \quad (74)$$

Inserting here the Gaussian correlation function in the bulk approximation (65) and using the asymptotic behavior for $n = 1/2$ given in Eq. (69), we find the behavior of S_ℓ for small and large values of the subsystem size ℓ ,

$$S_\ell \sim \begin{cases} \pi \cos(\varphi) \ell / (2l_0), & \ell \rightarrow 0, \\ (2\pi g_{\varphi,0}/3) \ln(\ell/l_0), & \ell \rightarrow \infty. \end{cases} \quad (75)$$

On short scales $\ell \lesssim l_0$, we expect the system to behave as a Fermi gas at infinite temperature. In particular, we expect the entanglement entropy to obey a volume law, $S_\ell \sim \ln(2)\ell$ for $n = 1/2$. This expectation is not borne out by the Gaussian result (75), which indicates that on short scales contributions from higher cumulants in Eq. (72) cannot be discarded [33].

On large scales $\ell \gg l_0$, the Gaussian theory predicts logarithmic growth of the entanglement entropy as characteristic for a 1D conformal field theory (CFT) [94]. To characterize deviations from CFT behavior that occur on short scales and,

as we below, also on large scales because of nonlinear fluctuations, we define the scale-dependent effective central charge [56],

$$c_\ell = 3 \frac{\partial S_\ell}{\partial \ln(\ell)}. \quad (76)$$

According to the asymptotic behavior of the Gaussian entanglement entropy in Eq. (75), on short scales, $c_\ell \sim \ell$ vanishes for $\ell \rightarrow 0$; on large scales, we find

$$c_\ell \sim 2\pi g_{\varphi,0}, \quad \ell \rightarrow \infty, \quad (77)$$

with $g_{\varphi,0}$ given in Eq. (66). As anticipated at the beginning of Sec. IV, we next extend our analysis to include nonlinear fluctuations, described by an NLSM. These fluctuations induce a renormalization group (RG) flow of $g_{\varphi,0}$ that causes the effective central charge to flow to zero on large scales, indicating that logarithmic growth of the entanglement entropy crosses over to area-law scaling.

V. NONLINEAR SIGMA MODEL

According to the symmetry analysis of the Keldysh action in Sec. III C, our model belongs to class BDI for $\varphi = 0$ and to class AIII for $\varphi \neq 0$. References [52,56] present the derivation of the NLSM for class BDI under different measurement schemes, while the derivation for class AIII is detailed in Refs. [33,35,41,56]. Conceptually, the derivation is the same for both symmetry classes. Here, we briefly summarize the key steps, focusing on $\varphi \neq 0$ where particle-hole symmetry is broken, and indicate modifications because of particle-hole symmetry for $\varphi = 0$ where relevant.

A. Target manifold

Because of the continuous symmetries of the action, the saddle point (52) is not isolated but rather embedded in a manifold of saddle points. The NLSM describes long-wavelength fluctuations of the Goldstone modes within this manifold. Consequently, to derive the NLSM, we first have to determine the structure of its target manifold—that is, the manifold of saddle points—and find an explicit parametrization.

To construct the NLSM target manifold, we use the replica-symmetric saddle point (52) as the base point to which we apply transformations $\mathcal{R} \in \text{U}(2R)$. For the time being, we set $n = 1/2$ such that Eq. (49) reduces to $\Lambda = \sigma_z$. We thus obtain

$$\mathcal{G} = -iQ/2, \quad \Sigma = \gamma Q/2, \quad Q = \mathcal{R}\sigma_z\mathcal{R}^{-1}. \quad (78)$$

For this to parametrize the NLSM manifold, we require the transformations $\mathcal{R}$ to obey two conditions: (i) They should be symmetries of the Keldysh action. This guarantees that $Q = \sigma_z$ being a saddle point entails $Q = \mathcal{R}\sigma_z\mathcal{R}^{-1}$ is one as well. (ii) They should transform $\Lambda = \sigma_z$ nontrivially. Regarding condition (i), note that any transformation $\mathcal{R}$ is a symmetry of the contribution to the Keldysh action given in Eq. (45). Therefore, which transformations $\mathcal{R}$ are symmetries is determined solely by the measurement Lagrangian. As can be seen from Eqs. (41) and (42), symmetries of the measurement Lagrangian are different for $R = 1$ and $R > 1$.

We consider first the replica-symmetric case with $R = 1$. Condition (i) does not restrict the NLSM manifold. Arbitrary

transformations $\mathcal{R}_0 \in \text{U}(2)$ are symmetries of Eq. (42). To restrict the group $\text{U}(2)$ to transformations that obey condition (ii), we factor out the group of transformations $\mathcal{R}_0$ that leave the saddle point $\Lambda = \sigma_z$ invariant. These transformations are determined by

$$[\mathcal{R}_0, \sigma_z] = 0 \iff \mathcal{R}_0 = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}. \quad (79)$$

Matrices $\mathcal{R}_0$ with this structure form the group $\text{U}(1) \times \text{U}(1)$. The replica-symmetric NLSM manifold for $n = 1/2$ can thus be parameterized as

$$Q_0 = \mathcal{R}_0 \sigma_z \mathcal{R}_0^{-1}, \quad (80)$$

where $\mathcal{R}_0 \in \text{U}(2)/\text{U}(1) \times \text{U}(1) \simeq \text{S}^2$. The generalization to arbitrary densities $n \neq 1/2$ reads

$$Q_0 = F \mathcal{R}_0 \sigma_z \mathcal{R}_0^{-1} F, \quad (81)$$

with F given in Eq. (51). Note that when Q_0 is inserted in the Keldysh action (44), F drops out owing to $F = F^{-1}$. However, inserting Q_0 in the expression for the density in Eq. (50) shows that Eq. (81) indeed describes a system with density n .

Having discussed the replica-symmetric manifold with $R = 1$, we now turn to the replicon manifold with $R > 1$. The replica-symmetric manifold can be embedded in the replicon manifold by setting

$$\mathcal{G} = -iQ/2, \quad \Sigma = \gamma Q/2, \quad Q = \mathcal{R} Q_0 \mathcal{R}^{-1}, \quad (82)$$

with Q_0 given in Eq. (81). Here, in addition to (i) being a symmetry and (ii) transforming Q_0 nontrivially, the transformations $\mathcal{R}$ should (iii) reduce to the identity for $R = 1$. A necessary condition for $\mathcal{R}$ to obey requirement (iii) is that $\mathcal{R} \in \text{SU}(2R)$, since $\text{U}(1)$ phase factors are already included in the replica-symmetric manifold. To determine the form of the transformations $\mathcal{R} \in \text{SU}(2R)$ that obey conditions (i) and (ii), it is convenient to use the representation $\mathcal{R} = M \mathcal{R}' M$ with $M = M^\dagger = M^{-1} = (\sigma_x + \sigma_z)/\sqrt{2}$. Note that the matrix M describes the Larkin-Ovchinnikov transformation (31) up to a sign flip of ψ_-^* . When we insert this representation of $\mathcal{R}$ into the measurement Lagrangian Eq. (41), the matrix X_φ that occurs in the Lagrangian is transformed into

$$Z_\varphi = M X_\varphi M = \cos(\varphi) \sigma_z + i \sin(\varphi) = \sigma_z e^{i\varphi \sigma_z}. \quad (83)$$

We thus see that $\mathcal{R} = M \mathcal{R}' M$ is a symmetry of the action if

$$[\mathcal{R}', Z_\varphi] = 0 \iff [\mathcal{R}', \sigma_z] = 0. \quad (84)$$

This condition implies that $\mathcal{R}'$ must be block-diagonal in Keldysh space,

$$\mathcal{R}' = \begin{pmatrix} \mathcal{R}'_+ & 0 \\ 0 & \mathcal{R}'_- \end{pmatrix}, \quad (85)$$

with $\mathcal{R}'_\pm \in \text{SU}(R)$, such that $\mathcal{R}' \in \text{SU}(R) \times \text{SU}(R)$. Note that matrices of this form also obey condition (iii). To restrict this form further to transformations $\mathcal{R}$ that obey also condition (ii), we factor out the group of transformations $\mathcal{R}'$ that leave the replica-symmetric NLSM manifold (81) invariant,

$$[\mathcal{R}', M Q_0 M] = 0 \iff \mathcal{R}'_+ = \mathcal{R}'_-. \quad (86)$$

Block-diagonal matrices $\mathcal{R}'$ that obey this condition form the group $\mathcal{R}' \in \text{SU}(R)$. The replicon manifold is thus $\text{SU}(R) \times \text{SU}(R)/\text{SU}(R) = \text{SU}(R)$. An explicit parametrization can be obtained by writing

$$M Q_0 M = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix}, \quad (87)$$

so that

$$Q = M \mathcal{R}' M Q_0 M \mathcal{R}'^{-1} M = M \begin{pmatrix} Q_{++} & Q_{+-} U \\ Q_{-+} U^{-1} & Q_{--} \end{pmatrix} M, \quad (88)$$

with $U = \mathcal{R}'_+ \mathcal{R}'_-^{-1} \in \text{SU}(R)$.

So far, we have focused on class AIII, to which our model belongs for $\varphi \neq 0$. In the derivation of the NLSM in class BDI, realized for $\varphi = 0$, the Keldysh action has to be symmetrized explicitly with respect to PHS by introducing doubled spinors of fields, collecting the original fermionic fields and their charge-conjugated partners [52]. After this step, the construction of the NLSM target manifold is analogous to the one presented above, yielding the replica-symmetric manifold $\text{O}(4)/\text{U}(2) \simeq \text{S}^2$ and the replicon manifold $\text{SU}(2R)/\text{Sp}(R)$, where $\text{Sp}(R)$ is the compact symplectic group [52,56].

We note in passing that the symmetries that we discuss here within the field theory framework have clear analogs in the operator formalism and can also be given an interpretation in terms of particle-number conservation [56]. In particular, the symmetry group $\text{U}(2)$ underlying the replica-symmetric manifold contains a subgroup $\text{U}(1) \times \text{U}(1)$ that corresponds to a strong $\text{U}(1)$ symmetry in the operator formalism [95,96]. This strong symmetry is associated with the conservation of the number of particles in the system alone. In contrast, the symmetry group $\text{SU}(R) \times \text{SU}(R)$ that underlies the replicon manifold and corresponds to a strong $\text{SU}(R)$ symmetry in the operator formalism is present even under a weaker condition: The number of particles does not have to be conserved within the system alone; rather, it has to be conserved within the system and reservoirs that might be needed to realize more general measurement processes [56].

B. Nonlinear sigma model

The expansion of the action in fluctuations around a given saddle point, which forms the basis of the Gaussian theory discussed in Sec. IV, is controlled for $\gamma \ll J$, provided that the fluctuations remain sufficiently small [33]. This condition is satisfied for massive fluctuations. In contrast, massless modes exhibit strong fluctuations that invalidate the Gaussian approximation at large scales. These strong fluctuations occur within the NLSM manifold and are parameterized by Eqs. (81) and (88). A standard procedure to derive the NLSM action for the massless modes is to substitute this parametrization into the action (44), take the spatial continuum limit in which the lattice index l is replaced by a continuous variable x , and perform a gradient expansion [52]. The resulting NLSM extends the Gaussian theory of Sec. IV by avoiding an expansion in the massless modes; however, fluctuations of the massive modes are then neglected. For our purposes, this is insufficient, as it produces an NLSM that fails to capture

the dependence of the physics on the unraveling phase φ . To understand this, note that the action (44) consists of two contributions given in Eqs. (45) and (41). Only the former, which is independent of φ , contributes nontrivially to the NLSM action. The latter depends on φ , but its contribution vanishes on the NLSM manifold and in the limit $R \rightarrow 1$. Consequently, the resulting NLSM is independent of φ .

In principle, the φ dependence of the NLSM coupling constants can be restored by generalizing Eq. (82) to include massive fluctuations outside the NLSM manifold, as incorporated in the Gaussian ansatz (54). Integrating out these massive fluctuations within the Gaussian approximation—which remains controlled for massive modes on all scales—then leads to a renormalization of the NLSM coupling constants. A more direct alternative is to start from the general form of the NLSM obtained via the gradient expansion of Eq. (45), leaving the coefficients undetermined, then linearize the NLSM action and match it to the Gaussian theory after the massive fluctuations have been integrated out in the latter. Following this approach, we find that the NLSM action splits into two contributions,

$$S[Q] = S_0[Q_0] + S_R[U, Q_0], \quad (89)$$

that describe, respectively, the replica-symmetric and replicon sectors of the theory. As noted in Sec. III B, for $R = 1$, the measurement Lagrangian (42) coincides with the one for random projective measurements of occupation numbers [33,52], up to a factor of 1/2 in the definition of γ . Therefore, in the replica-symmetric sector of the theory, the Lagrangian is

$$i\mathcal{L}_0[Q_0] = \frac{1}{2} \text{tr} \left[\sigma_z \mathcal{R}_0^{-1} \partial_t \mathcal{R}_0 - \frac{\nu}{4} (\partial_x Q_0)^2 \right], \quad (90)$$

where $\nu = J^2/\gamma$. For class AIII, we consider here transformations $\mathcal{R}_0 \in \text{U}(2)/\text{U}(1) \times \text{U}(1)$ and Q_0 is given in Eq. (81). The NLSM on the replica-symmetric manifold describes diffusive dynamics on scales larger than l_0 Eq. (64) [33,52].

The NLSM Lagrangian in the replicon sector is

$$i\mathcal{L}_R[U, Q_0] = -\frac{g_\varphi}{2} \text{tr}_R \left(\frac{1}{v_\varphi} \partial_t U^{-1} \partial_t U + v_\varphi \partial_x U^{-1} \partial_x U \right), \quad (91)$$

with, for class AIII, $U \in \text{SU}(R)$ as given below Eq. (88). The parameters appearing in the NLSM Lagrangian are

$$g_\varphi = \frac{l_0 \rho_0 (1 - \rho_0)}{\cos(\varphi)}, \quad v_\varphi = 4J \cos(\varphi) \sqrt{\rho_0 (1 - \rho_0)}, \quad (92)$$

where the replica-symmetric density is defined as in Eq. (50),

$$\rho_0 = \frac{1}{4} \text{tr}_K (1 - \sigma_x Q_0). \quad (93)$$

In the limit $\gamma \ll J$, in which the derivation of the NLSM is valid, we can neglect fluctuations in the replica-symmetric sector and replace the density by its value $\rho_0 = n$ at the saddle point $Q_0 = \Lambda$. Then, the bare coupling constant of the NLSM reduces to the one of the Gaussian theory (66), $g_\varphi = g_{\varphi,0}$. Note that the coupling constant diverges for $\varphi \rightarrow \pi/2$. We comment on the NLSM description for the case $\varphi = \pi/2$ in Sec. VID below.

The NLSM for symmetry class BDI takes the same form as the one for class AIII, but with matrix fields $\mathcal{R}_0$ and U

belonging to different manifolds according to our discussion in Sec. VA. As a result, the one-loop RG flow equations for the two symmetry classes differ only by a numerical prefactor [33,52]. The RG flow of the NLSM coupling constant g_φ reads

$$g_\varphi = g_{\varphi,0} - \frac{1}{4\pi\beta} \ln(l/l_0), \quad (94)$$

where $\beta = 1$ in class AIII for $\varphi \neq 0$, and $\beta = 1/2$ in class BDI for $\varphi = 0$. For $\gamma \ll J$, the bare coupling (66) $g_{\varphi,0} \sim l_0 \sim J/\gamma$ is large, and Eq. (94) describes the slow RG flow toward the strong-coupling regime of the NLSM where $g_\varphi \lesssim 1$. The strong-coupling regime is reached beyond the scale

$$l_{\varphi,*} = l_0 e^{4\pi\beta g_{\varphi,0}} = l_0 \exp \left[\frac{8\pi\beta J n (1 - n)}{\gamma \cos(\varphi)} \right]. \quad (95)$$

As stated at the end of Sec. IV C, the renormalization of g_φ determines the behavior of the effective central charge on large scales. Therefore, the flow of g_φ to small values beyond the scale $l_{\varphi,*}$ implies that the logarithmic growth of the entanglement entropy S_ℓ described by Eq. (75) saturates, and that S_ℓ shows area-law behavior for $\ell \gtrsim l_{\varphi,*}$. This in turn indicates that there is no measurement- or unraveling-induced phase transition between a critical and an area-law phase in our model. Instead, the entanglement entropy obeys an area law beyond the scale $l_{\varphi,*}$ for all values of γ and $0 \leq \varphi < \pi/2$. However, $l_{\varphi,*}$ is exponentially large in $J/[\gamma \cos(\varphi)]$, and therefore we do not observe area-law entanglement in our numerical simulations, which we discuss in detail in Sec. VI below, for large values of $J/[\gamma \cos(\varphi)]$, meaning that either γ is small or φ is close to $\pi/2$. Finally, note that the exponential dependence implies that $l_{\varphi,*}$ drops sharply upon decreasing $J/[\gamma \cos(\varphi)]$ by either increasing γ or decreasing φ . In numerics, this sharp drop might be misinterpreted as a measurement- or unraveling-induced phase transition.

C. RG-corrected Gaussian theory

Because area-law behavior is expected only at exponentially large scales, making it difficult to capture numerically, it is useful to identify signatures of the RG flow (94) that are visible already at shorter scales. To this end, we incorporate the RG correction Eq. (94) into the Gaussian correlation function (65) [33,56]. This is done by replacing the bare NLSM coupling $g_{\varphi,0}$ by its renormalized value (94), with the momentum at which the correlation function is evaluated providing the IR cutoff of the RG flow according to $l = 1/q$. Furthermore, in Eq. (65) we replace $\tilde{c}(u)/u \sim 1$ by its asymptotic value for small momenta given in Eq. (69). We thus obtain

$$C_q \sim C_{q,0} + \frac{q}{4\pi\beta} \ln(q l_0), \quad (96)$$

where $C_{q,0} \sim g_{\varphi,0} q$ describes the asymptotic behavior of the Gaussian correlation function (65) for $q \rightarrow 0$. This form suggests that the “weak-localization correction,” defined as the difference between the exact connected density correlation function and the Gaussian approximation, provides a direct and numerically accessible probe of the RG flow (94) [33,52],

$$\frac{\delta C_q}{q} = \frac{C_q - C_{q,0}}{q} = \frac{1}{4\pi\beta} \ln(q l_0). \quad (97)$$

The slope of $\delta C_q/q$ as a function of $\ln(ql_0)$ is universal, and its value distinguishes between symmetry classes AIII with $\beta = 1$ for $\varphi \neq 0$ and BDI with $\beta = 1/2$ for $\varphi = 0$. Note, however, that the weak-localization correction acquires a nonuniversal shift that depends on the UV cutoff, which we have identified with $1/l_0$ in Eq. (94).

For any value of $\varphi < \pi/2$, the functional form of the renormalized correlation function (96) is the same as that obtained for different models in Refs. [33,56]. Therefore, one might expect the same qualitative behavior to apply: Signatures of emergent conformal invariance as characteristic for a critical phase [4,53] can be observed on a finite critical range of length scales, bounded from below by $l_0 \sim \gamma^{-1}$ and from above by a scale $l_c \sim \gamma^{-2}$. These signatures include algebraic decay of the density correlation function, $C_l \sim l^{-2}$, and logarithmic growth of the entanglement entropy, $S_\ell \sim \ln(\ell)$. Note that these properties are also predicted by the Gaussian theory in Eqs. (70) and (75), respectively. However, the Gaussian value of the central charge, Eq. (77), applies only asymptotically for $\gamma \rightarrow 0$ and $\ell \rightarrow \infty$. For any finite value of γ , approximately logarithmic growth of the entanglement entropy occurs not for $\ell \rightarrow \infty$ but on intermediate scales where c_ℓ is approximately constant, in the vicinity of the maximum of c_ℓ at $\ell = l_m \sim \gamma^{-3/2}$. The scaling of l_m with γ ensures that this scale is within the critical range, $l_0 < l_m < l_c$. Further signatures of conformal invariance, such as a universal functional form of the mutual information, are reproduced with the value of the effective central charge c_ℓ at its maximum [56].

VI. NUMERICAL VALIDATION OF FIELD-THEORY PREDICTIONS

To validate our analytical findings, we compare them to direct numerical simulations of Eq. (2). We anticipate that the expectations formulated at the end of Sec. V C are borne out by our numerics for small values of φ ; for large values of φ , we still find the weak-localization correction as in Eq. (97), indicating that the model shows area-law entanglement on large scales. However, we observe additional structure in short-scale nonuniversal regime $l \lesssim l_0$ that is described by the Gaussian theory. This additional structure masks the scaling behavior $l_c \sim \gamma^{-2}$ and $l_m \sim \gamma^{-3/2}$. Before we discuss these results in detail, we briefly summarize the numerical methods we use.

A. Numerical methods

To numerically simulate the dynamics generated by the stochastic Schrödinger equation (2), we adapt the approach first introduced by Cao *et al.* [1]. In all of our simulations, the initial state is chosen as

$$|\psi_0\rangle = \prod_{l=1}^{L/2} \hat{\psi}_{2l-1}^\dagger |0\rangle, \quad (98)$$

where $|0\rangle$ is the vacuum state. The number of particles is thus $N = L/2$. Both the number of particles and the Gaussianity of the initial state are preserved by Eq. (2). Therefore, at all

times t the state can be written as

$$|\psi(t)\rangle = \prod_{n=1}^N \left[\sum_{l=1}^L U_{l,n}(t) \hat{\psi}_l^\dagger \right] |0\rangle, \quad (99)$$

where the $L \times N$ matrix $U(t)$ is an isometry, $U^\dagger(t)U(t) = 1$, which ensures the normalization of the state. Since the quantum state $|\psi(t)\rangle$ is fully determined by $U(t)$, the stochastic Schrödinger equation (2) can be recast as an evolution equation for this matrix. To obtain this evolution equation, we insert Eq. (99) in Eq. (2), commute the operators $\hat{H}$ and $\hat{n}_l$ to the right using $\hat{H}|0\rangle = \hat{n}_l|0\rangle = 0$, and exponentiate the generator of the evolution of $U(t)$ using the Itô rule in Eq. (3), $dW_l(t)dW_{l'}(t) = \delta_{l,l'}dt$. We thus obtain

$$U(t+dt) = Me^{-iHdt}U(t), \quad (100)$$

where H is the Hamiltonian matrix (19) and the diagonal matrix describing the measurement process is given by

$$M_{l,l'} = \delta_{l,l'} \exp \left\{ \frac{\gamma}{2} [2\langle \hat{n}_l(t) \rangle - 1] (1 + e^{2i\varphi}) dt + \sqrt{\gamma} e^{i\varphi} dW_l(t) \right\}. \quad (101)$$

To calculate expectation values such as $\langle \hat{n}_l(t) \rangle$ in the time-evolved state $|\psi(t)\rangle$, we use Wick's theorem and the fact that the $L \times L$ single-particle density matrix is related to $U(t)$ via

$$D_{l,l'}(t) = \langle \psi(t) | \hat{\psi}_{l'}^\dagger \hat{\psi}_l | \psi(t) \rangle = \sum_{n=1}^N U_{l,n}(t) U_{l',n}^*(t). \quad (102)$$

We simulated the time evolution of $U(t)$ for a system of $L = 1000$ lattice sites and using a time step $J\Delta t = 0.05$, which has been shown to yield sufficient accuracy in the limiting cases $\varphi = 0$ and $\varphi = \pi/2$ [1]. After each time step, a QR decomposition was applied to restore the isometric property of $U(t+dt)$ [1]. For each choice of the parameters γ and φ , we simulated $N_{\text{traj}} = 160$ trajectories.

B. Connected density correlation function

We first consider the connected density correlation function (55). In the steady state, the correlation function is time independent and translationally invariant, $C_{l,l'}(t) = C_{l-l'}$. Using Wick's theorem to express the correlation function in terms of the single-particle density matrix (102), we obtain

$$C_{l-l'} = \overline{\langle \hat{n}_l \hat{n}_{l'} \rangle} - \overline{\langle \hat{n}_l \rangle} \overline{\langle \hat{n}_{l'} \rangle} = \delta_{l,l'}/2 - \overline{|D_{l,l'}|^2}. \quad (103)$$

For our numerical results, the overline denotes averaging over both quantum trajectories and lattice sites l and l' , keeping the distance $l - l'$ constant.

Figures 1(a)–1(c) show the rescaled density correlation function in momentum space, $C_q/(g_{\varphi,0}\tilde{q})$, as a function of $\tilde{q}l_0$ for various values of φ and γ/J . Here, $\tilde{q} = 2 \sin(q/2)$ takes the finite lattice spacing into account [33]. For comparison, the correlation functions obtained from the Gaussian theory, both the bulk approximation (65) and the full numerical solution (59), are also shown in the figure.

As expected, the numerical data are close to the Gaussian results for small values of γ/J and for intermediate to large values of ql_0 . However, we observe qualitatively different behavior at low momenta. The asymptotic behavior

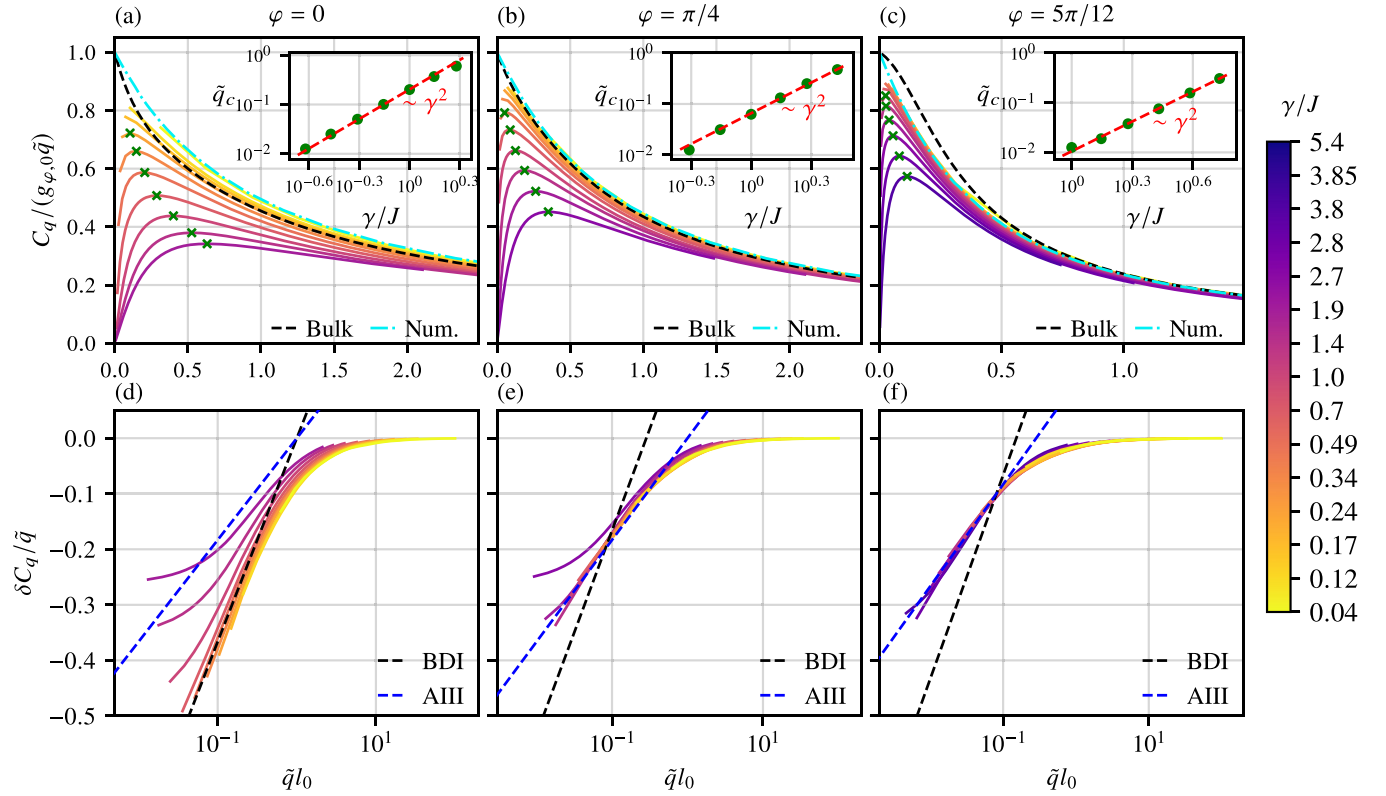


FIG. 1. (a)–(c) Rescaled density correlation function (103) in momentum space and (d)–(f) weak-localization correction (97) for (a), (d) $\varphi = 0$, (b), (e) $\varphi = \pi/4$, and (c), (f) $\varphi = 5\pi/12$. (a)–(c) The Gaussian correlation function [cyan dash-dotted line, full numerical solution (59); black dashed line, bulk approximation (65)] decreases monotonically starting from $C_q/(g_{\varphi,0}q) \rightarrow 1$ for $\tilde{q}l_0 \rightarrow 0$. In contrast, the numerical data attain a maximum at q_c and decrease for $\tilde{q}l_0 \rightarrow 0$. This decrease becomes more pronounced for increasing values of γ/J . [Insets in (a)–(c)] The position of the maximum scales as $q_c \sim \gamma^2$. (d)–(f) The weak-localization correction agrees well with the one-loop RG result (97). In particular, the analytically predicted difference in the slope by a factor of two between symmetry classes BDI for $\varphi = 0$ and AIII for $\varphi \neq 0$ is borne out by the data. Note that the analytical result (97) is shifted for better visual comparison with the numerical data. Such a nonuniversal shift corresponds to a different value of the UV cutoff in the RG flow (94) and does not affect the universal slope of the weak-localization correction.

(69) of the Gaussian correlation function—recall that the bulk approximation and the full numerical solution have the same asymptotic behavior—implies that $C_q/(g_{\varphi,0}q) \rightarrow 1$ for $ql_0 \rightarrow 0$. In contrast, strong nonlinear fluctuations renormalize $C_q/(g_{\varphi,0}q)$ to small values for $ql_0 \rightarrow 0$, as described by the RG correction in Eq. (96). The numerical data clearly follow this trend. We identify the crossover scale q_c between Gaussian behavior at large momenta and the fluctuation-dominated regime at small momenta with the position of the maximum of $C_q/(g_{\varphi,0}q)$, which can be obtained analytically from Eq. (96) by including the quadratic order in the expansion Eq. (69) [56]. As explained below Eq. (69), the coefficient of the quadratic order is not correctly reproduced by the bulk approximation. This coefficient affects the dependence of q_c on φ . However, precise knowledge of the coefficient is not necessary to determine the scaling $q_c \sim \gamma^2$. The numerical data, shown in the inset in Figs. 1(a)–1(c), are in good agreement with this scaling.

Figures 1(d)–1(f) show the weak-localization correction (97), defined as the difference between the numerical data for C_q and the Gaussian correlation function (59). We observe good agreement with the one-loop RG result (97). In particular, the distinction between symmetry classes BDI for $\varphi = 0$

and AIII for $\varphi \neq 0$ is well supported by the data. Values of φ other than those shown in Fig. 1 yield similar results, with a crossover from the behavior expected for symmetry classes BDI and AIII at small values of φ [97]. To obtain these results, we used the full numerical solution for the Gaussian correlation function (59). The bulk approximation is not sufficiently accurate for this purpose.

The agreement between the numerical results for the weak-localization correction and the analytical prediction based on the NLSM provides strong evidence for the absence of an entanglement transition in our model. However, as was found in a variety of similar models, signatures that are characteristic for a critical phase can be observed on intermediate length scales [4,33,53,56]. As examples for such signatures, we focus here on algebraic decay of the density correlation function and logarithmic growth of the entanglement entropy.

Figure 2 shows the rescaled connected density correlation function (103) as a function of the chord length $\tilde{\ell} = (L/\pi) \sin(\pi \ell/L)$, which takes both the finite size of the system and the periodic boundary conditions into account [94]. For comparison, the black dashed line shows the bulk approximation for the Gaussian correlation function, obtained by transforming Eq. (65) to real space. On intermediate

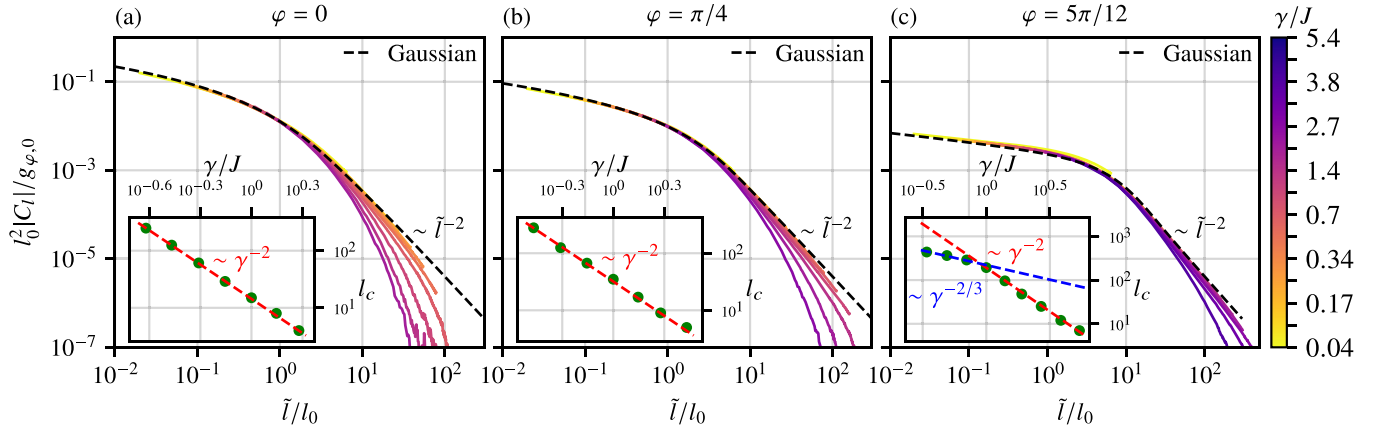


FIG. 2. Rescaled density correlation function (103) for (a) $\varphi = 0$, (b) $\varphi = \pi/4$, and (c) $\varphi = 5\pi/12$. On short scales $\tilde{l} \lesssim l_0$, the numerical data agree well with the Gaussian result [black dashed line, bulk approximation (65)]. Significant deviations occur for $\tilde{l} \gg l_0$, where the Gaussian result decays as a power law, $|C_l| \sim \tilde{l}^{-2}$, while the numerical data exhibit faster decay. (Inset) The crossover scale at which deviations from the Gaussian result become significant scales as $l_c \sim \gamma^{-2}$. For $\varphi = 5\pi/12$, this scaling changes to $l_c \sim \gamma^{-2/3}$ at small values of γ/J .

scales, $l_0 < l < l_c$, the numerical data exhibit approximately algebraic decay, $C_l \sim \tilde{l}^{-2}$, as characteristic for a 1D CFT. Following Ref. [56], we identify l_c with the scale beyond which the deviation between the numerical data and a tangent to the data $\sim \tilde{l}^{-2}$ exceeds 10%. As discussed in more detail below, l_c depends algebraically on γ . A key practical consequence of this algebraic scaling is that l_c remains smaller than the system size $L = 1000$ in our numerics and is therefore accessible over a broad range of γ/J . By contrast, the scale $l_{\varphi,*}$, which signals the onset of the area-law regime, increases exponentially with $J/[\gamma \cos(\varphi)]$ according to Eq. (95). As already noted in Sec. IA, resolving this scale in numerical simulations is therefore extremely challenging [71]. More specifically, for the parameters used in Figs. 2(a)–2(c), corresponding to $\varphi = 0, \pi/4, 5\pi/12$, the scale $l_{\varphi,*}$ exceeds the system size L for $\gamma/J \lesssim 0.6, 1.4, 3.3$, respectively. For the smallest value considered in our numerics, $\gamma/J = 0.04$, we estimate $l_{\varphi,*}$ to be at least tens of orders of magnitude larger than L , placing it well beyond the reach of our numerical approach.

The insets in Fig. 2 show good agreement of the numerical data for l_c with the expected scaling $l_c \sim q_c^{-1} \sim \gamma^{-2}$ for $\varphi = 0$ and $\varphi = \pi/4$. However, for $\varphi = 5\pi/12$, we observe a different scaling $l_c \sim \gamma^{-2/3}$ at small values of γ/J . We attribute the modified scaling of l_c to a change in the shape of the correlation function in the nonuniversal regime $l \lesssim l_0$. In this regime, the numerical data agree well with the Gaussian approximation, and the change in the shape can be seen best in the latter. For small values of φ , the Gaussian correlation function is concave; however, close inspection of Fig. 2(c) reveals that for $\varphi = 5\pi/12$, the Gaussian correlation function exhibits a “dent” between $\tilde{l}/l_0 \approx 10$ and $\tilde{l}/l_0 \approx 10^2$. Recall that our method to determine l_c involves drawing a tangent to the correlation function. Clearly, the point at which the tangent meets the correlation function is affected by this change of shape. This suggests that the change of shape is also the cause for the modified scaling behavior of l_c . We did not attempt to determine precisely the value of φ at which the Gaussian correlation function ceases to be concave; however, the dent

is there also for $\varphi = \pi/3$, which indicates that it appears first between $\varphi = \pi/4$ and $\varphi = \pi/3$ [97].

Note that in Fig. 1(c), the deviation of the data point for the smallest value of γ/J from $q_c \sim \gamma^2$ is compatible with the onset of the scaling $q_c \sim l_c^{-1} \sim \gamma^{2/3}$. However, we have defined q_c as the position of the maximum of $C_q/(g_{\varphi,0}q)$; for the smallest values of γ/J , we do not find a maximum within the range of values of $\tilde{q}l_0$, which is bounded from below by the inverse system size. Therefore, conclusively demonstrating the scaling $q_c \sim \gamma^{2/3}$ would require larger system sizes.

As we discuss next, we find a similar modification of scaling with γ in the effective scale-dependent central charge (76). In this case, the origin of the modification is particularly clear, and the modified scaling can be read off directly from the Gaussian result.

C. Entanglement entropy and effective central charge

To calculate the entanglement entropy (71) numerically, we use the following relation for Gaussian states:

$$S_\ell = - \sum_{l=1}^{\ell} [\lambda_l \ln(\lambda_l) + (1 - \lambda_l) \ln(1 - \lambda_l)], \quad (104)$$

where λ_l are the eigenvalues of the single-particle density matrix (102), restricted to a subsystem of size ℓ ; that is, $D_\ell = (D_{l,l'})_{l,l'=1}^{\ell}$ [98]. As in Eq. (103), the overline denotes spatial and trajectory averaging.

Figures 3(a)–3(c) show the entanglement entropy as a function of subsystem size $\ell \in \{1, \dots, L/2\}$ for three different values of φ . In each case, the numerical data exhibit the same qualitative behavior: For small γ/J , the data closely follow the Gaussian curve, obtained by numerically integrating Eq. (74) with the Gaussian correlation function in bulk approximation (65). In particular, the data exhibit volume-law scaling on short scales $\ell \lesssim l_0$, and apparently logarithmic growth on large scales. For increasing γ/J , the large-scale behavior obeys an area law. The NLSM description implies that area-law scaling occurs for all values of γ and φ , but

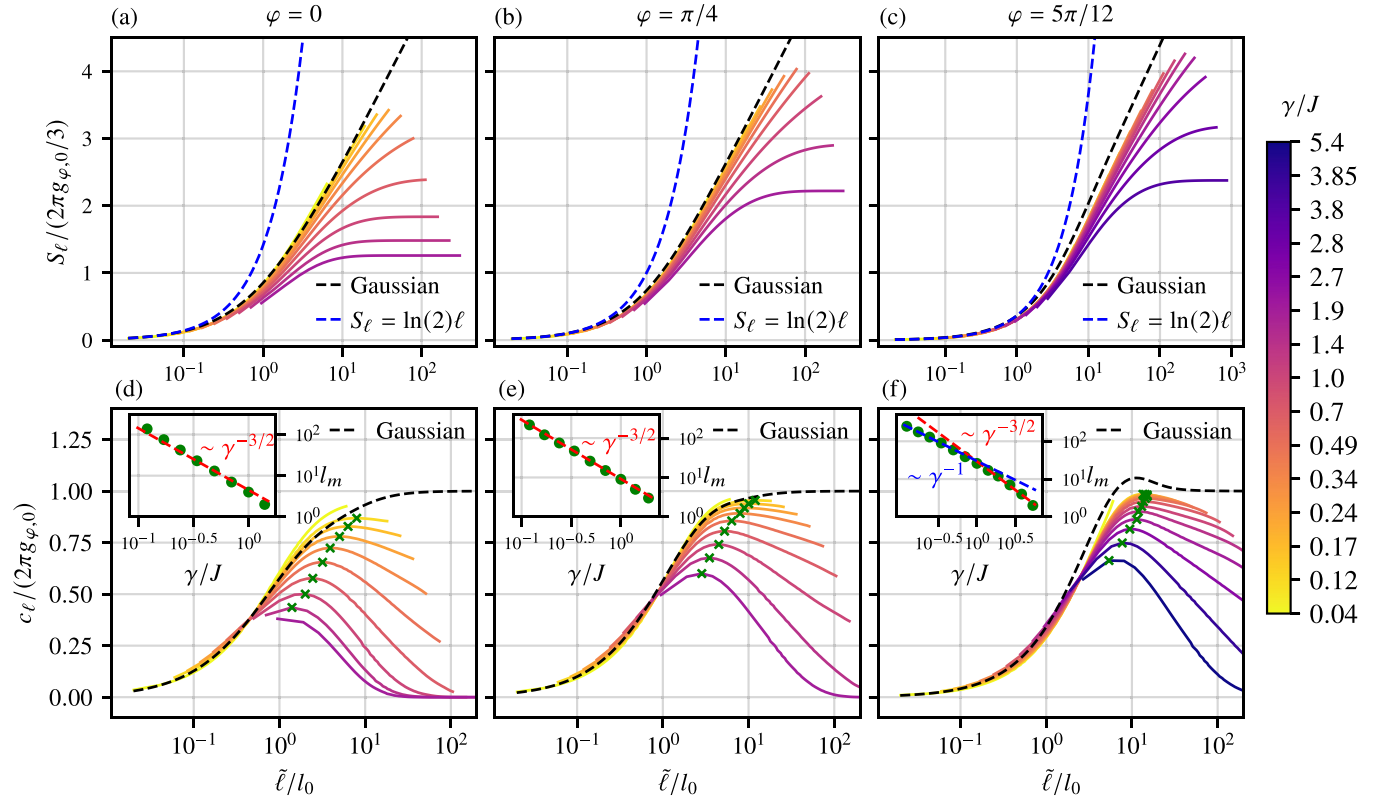


FIG. 3. (a)–(c) Rescaled entanglement entropy and (d)–(f) rescaled scale-dependent effective central charge for (a), (d) $\varphi = 0$, (b), (e) $\varphi = \pi/4$, and (c), (f) $\varphi = 5\pi/12$. (a)–(c) For small values of γ/J , volume-law scaling of the entanglement entropy (blue dashed line) on short scales $\tilde{\ell} \lesssim l_0$ crosses over to apparently logarithmic growth on large scales. At larger values of γ/J , the data exhibit area-law scaling. (d)–(f) Unlike the Gaussian result (black dashed line), the numerical data for the effective central charge do not saturate to a finite value as $\tilde{\ell}/l_0 \rightarrow \infty$. This implies that the growth of the entanglement entropy is, in fact, not logarithmic on large scales. (Inset) The effective central charge attains its maximum at $l_m \sim \gamma^{-3/2}$. For $\varphi = 5\pi/12$, the scaling is modified to $l_m \sim \gamma^{-1}$ at small values of γ/J .

only beyond the exponentially large scale $l_{\varphi,*}$ (95). This is not observable for small values of γ/J and the system size of $L = 1000$ that we have chosen in our numerical simulations. Nevertheless, as we discuss next, the onset of the crossover toward area-law scaling is clearly visible in the scale-dependent effective central charge (76).

The effective central charge is shown in Figs. 3(d)–3(f). Again, for small γ/J , the numerical data are close to the Gaussian result. However, for the central charge, quantitative corrections to the bulk approximation appear to be more pronounced. In particular, for small values of φ , there are clear quantitative discrepancies between the Gaussian result and the numerical data. Aside from this quantitative difference, there is a crucial *qualitative* difference between the Gaussian theory and the full-scale numerics. On large scales, the Gaussian result approaches the constant value given in Eq. (77); a constant value of the scale-dependent effective central charge implies logarithmic growth of the entanglement entropy (75). In contrast, for all but the smallest value of γ/J included in Fig. 3, the central charge obtained from our numerical simulations is approximately constant only in the vicinity of its maximum, and it falls off on large scales. This is a clear signature of the crossover from logarithmic growth towards area-law scaling of the entanglement entropy. However, within the limited range of subsystem sizes in our simulations, the central charge reaches zero—or, equivalently,

the entanglement entropy exhibits area-law scaling—only for large γ/J .

As stated above, we observe a maximum of the effective central charge for all but the smallest value of γ/J . The absence of a maximum for small values of γ/J is caused by the limited system size in our numerical simulations. This conclusion is supported by the data for the position l_m of the maximum of c_ℓ shown in the inset in Figs. 3(d)–3(f). For $\varphi = 0$ and $\varphi = \pi/4$, the data are compatible with the expected scaling $l_m \sim \gamma^{-3/2}$ [56]. Therefore, for a given system size L , the maximum is not observable if γ/J is too small. Note that the scaling $l_m \sim \gamma^{-3/2}$ places the maximum of the central charge and thus the region of approximately logarithmic growth of the entanglement entropy within the critical range, which is bounded from above and below by $l_0 \sim \gamma^{-1}$ and $l_c \sim \gamma^{-2}$, respectively.

For $\varphi = 0$ and $\varphi = \pi/4$, the Gaussian result for c_ℓ grows monotonically, and the maximum of c_ℓ results from the approach to the asymptotic value given in Eq. (77) being counteracted by the renormalization of $g_{\varphi,0}$ according to Eq. (94). In other words, the scaling $l_m \sim \gamma^{-3/2}$ is determined by universal long-wavelength properties of the theory. However, for $\varphi = 5\pi/12$, the data in Fig. 3(f) show the modified scaling $l_m \sim \gamma^{-1}$ at the smallest values of γ/J —similarly to the modified scaling of l_c in Fig. 2(f); and the modified scaling $l_m \sim \gamma^{-1}$ for $\varphi = 5\pi/12$ is a *nonuniversal* feature. To see that

this is the case, note that as expected for very small γ/J and on short and intermediate length scales, the numerical data for c_ℓ agree reasonably well with the Gaussian theory. But for $\varphi = 5\pi/12$, the Gaussian theory c_ℓ exhibits a maximum at a fixed value of ℓ/l_0 , such that $l_m \sim l_0 \sim \gamma^{-1}$. As stated above, the Gaussian result for c_ℓ shown in Fig. 3 is derived from the bulk approximation, and we expect the full numerical solution of the Gaussian theory to exhibit quantitative corrections. However, the key new qualitative feature appearing for increasing φ , which is the maximum of c_ℓ , appears to be reproduced correctly also in the bulk approximation. We observe this maximum also for $\varphi = \pi/3$ but not for $\varphi = \pi/4$, indicating that upon increasing φ the maximum appears first between $\varphi = \pi/4$ and $\varphi = \pi/3$, as is the case for the dent in the connected density correlation function discussed in Sec. VIB [97].

Each of the preceding figures has shown data for a fixed value of φ but varying values of the measurement rate γ/J . This is sufficient to demonstrate that changing either γ/J or φ does not induce an entanglement transition; instead, changing these parameters only modifies the length scales delimiting the critical and area-law regimes. Nevertheless, it is interesting to ask how the effect of changing γ/J compares to that of changing φ . As we have noted already in Sec. VB, the scale $l_{\varphi,*}$, beyond which area-law entanglement is expected, depends exponentially on $J/[\gamma \cos(\varphi)]$, meaning that increasing φ toward $\pi/2$ amounts to decreasing the value of γ/J . Qualitatively, this relation between φ and γ/J is compatible with the observation of an unraveling-induced entanglement transition in a similar model in Ref. [53]. Furthermore, it is supported by the data shown in Fig. 4. In particular, Fig. 4(a) shows the entanglement entropy at a fixed value of $\gamma/J = 0.7$ and different values of φ . Increasing φ appears to induce a transition from area-law scaling to logarithmic growth of the entanglement entropy, similarly to the change induced by increasing γ/J illustrated in Figs. 3(a)–3(c). Nonetheless, as Fig. 4(b) shows, the scale-dependent effective central charge falls off on large scales, indicating that logarithmic growth of the entanglement entropy is only transient. We note, however, that including the unraveling phase in Eq. (2) results not only in a multiplicative renormalization of the measurement rate, replacing γ by $\gamma \cos(\varphi)$. This becomes clear, for example, from the modified scaling of l_c and l_m for $\varphi = 5\pi/12$, shown in the inset in Figs. 2(c) and 3(f), respectively.

D. Unitary unraveling

Finally, we briefly discuss the unitary unraveling at $\varphi = \pi/2$. Numerical results for this case were reported in Refs. [1,53], showing that the entanglement entropy grows as $S_\ell \sim \sqrt{t/\gamma}$ and saturates at a value that exhibits volume-law scaling. Interestingly, the saturation value agrees quantitatively with the time average of the entanglement entropy in the deterministic dynamics at $\gamma = 0$. Our numerical simulations for $\varphi = \pi/2$ further suggest that trajectories at late times approach essentially random Gaussian states. We support this claim with two observations. First, we found that the steady-state density correlation function is consistent with that of a random Gaussian state, which for $N, L \rightarrow \infty$ with

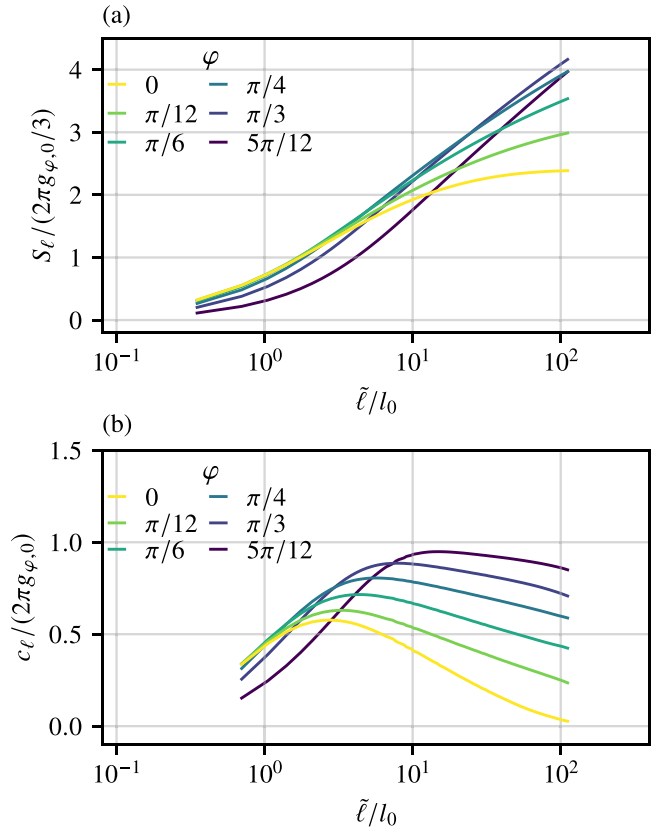


FIG. 4. (a) Rescaled entanglement entropy and (b) rescaled effective central charge for fixed $\gamma/J = 0.7$ and varying values of φ . The effect of increasing φ in the range from 0 to $\pi/2$ is qualitatively similar to that of decreasing γ/J . In the behavior of the entanglement entropy on large scales, there is an apparent transition from area-law scaling to logarithmic growth. Concomitantly, the position of the maximum of the effective central charge shifts to the right.

$N/L = 1/2$ is

$$C_{l-l'} \sim \frac{1}{4} \left(\delta_{l,l'} - \frac{1}{L} \right). \quad (105)$$

This follows by inserting Eq. (102) into Eq. (103) and averaging over random isometries $U(t)$ [99]. Second, as shown in Fig. 5, for any value of γ/J , the entanglement entropy follows the behavior of random Gaussian states given by [100,101]

$$\frac{S_{\text{rand},\ell}}{L} = [\ln(2) - 1] \frac{\ell}{L} + \left(1 - \frac{\ell}{L} \right) \ln \left(1 - \frac{\ell}{L} \right). \quad (106)$$

This form is obtained by averaging over Gaussian states with fixed particle numbers $N \in \{0, \dots, L\}$, allowing N to vary across states. By contrast, in our setting the particle number is fixed at $N = L/2$, yet we still observe good agreement. We attribute the deviations at ℓ close to $L/2$ to our simulations not having fully reached the stationary regime. Because of the slow growth $S_\ell \sim \sqrt{t/\gamma}$ toward volume-law behavior, reaching the steady state requires substantially more time for $\varphi = \pi/2$ than for $0 \leq \varphi < \pi/2$. Note that Eq. (106) implies volume-law behavior, $S_{\text{rand},\ell} \sim \ell$, for any fixed value of the ratio ℓ/L [1,53]. However, in Fig. 5 we keep the system size L fixed while varying the subsystem size ℓ . Then, $S_{\text{rand},\ell} \sim \ell$ only for $\ell \ll L$.

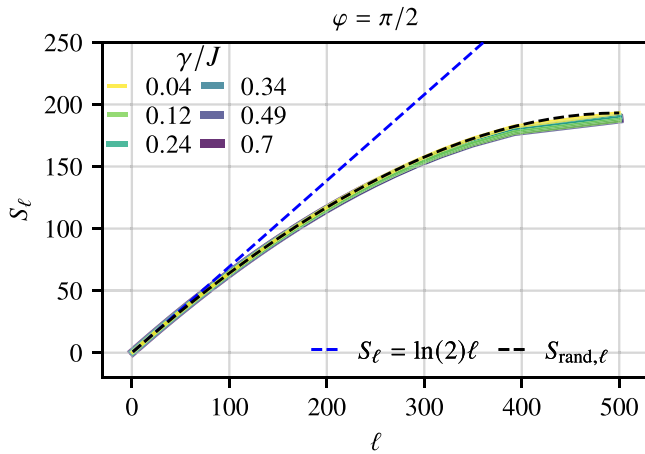


FIG. 5. Entanglement entropy density for $\varphi = \pi/2$. For any value of γ/J , the data are consistent with Eq. (106) for random Gaussian states (black dashed line). We observe deviations for large subsystem sizes ℓ , which are caused by our simulations not having fully reached the stationary regime. For small ℓ , the data exhibit volume-law scaling (blue dashed line), as also found for $\varphi \neq \pi/2$ in Figs. 3(a)–3(c).

It is an interesting question how such an ensemble of random Gaussian states is described analytically by an NLSM. We do not provide a full answer to that question here, but make the following observations. For $\varphi = \pi/2$, the matrix X_φ defined in Eq. (35) reduces to the imaginary unit, such that Eq. (41) simplifies to

$$i\mathcal{L}_M[\mathcal{G}] = -\frac{R}{4} - \frac{1}{2}[\text{tr}(\mathcal{G}^2) - \text{tr}(\mathcal{G})^2]. \quad (107)$$

Thus, unlike the case $\varphi < \pi/2$, there is no distinction in symmetry between $R = 1$ and $R > 1$ for $\varphi = \pi/2$. Both Eqs. (42) and (107) are invariant under arbitrary transformations $\mathcal{G} \mapsto \mathcal{R}\mathcal{G}\mathcal{R}^{-1}$. Consequently, in the construction of the NLSM manifold, the restriction in Eq. (84) does not apply, and the manifold for $\varphi = \pi/2$ is the complex Grassmannian $U(2R)/U(R) \times U(R)$. Another important difference from the case $\varphi < \pi/2$, with NLSM Lagrangian given in Eq. (91), is that the temporal derivative for $\varphi = \pi/2$ is presumably first order. This contrasts with the NLSMs appearing in the theory of 2D disordered systems, where time plays the role of a second spatial dimension and both directions enter on equal footing.

VII. CONCLUSIONS

In this work, we have analyzed continuously monitored one-dimensional free fermions in the presence of an unraveling phase φ , which interpolates between conventional quantum state diffusion and a unitary unraveling corresponding to classical noise. Our main finding is that neither varying the measurement rate γ nor tuning the unraveling phase φ can drive a genuine entanglement transition. Instead, the system ultimately exhibits area-law entanglement beyond the exponentially large scale $l_{\varphi,*}$ (95). Apparent signatures of a critical phase, including logarithmic growth of entanglement and algebraic decay of correlations, persist only below a crossover

scale that grows algebraically with J/γ . Our results thus indicate that the apparent KT transition in related models is a finite-size crossover phenomenon [4,53].

Nevertheless, varying the unraveling phase φ affects the dynamics in nontrivial ways. First, for $\varphi = 0$, our model belongs to the chiral orthogonal symmetry class BDI; instead, for $\varphi \neq 0$, it belongs to the chiral unitary class AIII. This distinction is reflected in the universal slope of the weak-localization correction. Second, increasing φ in the range $0 \leq \varphi < \pi/2$ has a qualitatively similar effect to decreasing γ at fixed φ , in line with the observation of an unraveling-induced transitions reported previously [53]. Third, when φ is increased beyond a certain value between $\pi/4$ and $\pi/3$, novel nonuniversal features appear at short scales, including a maximum of the scale-dependent effective central charge at $l_m \sim \gamma^{-1}$, in contrast to the scaling $l_m \sim \gamma^{-3/2}$ that we observe for smaller values of φ . These features affect also the scaling of the upper boundary of the critical range, which is modified from $l_c \sim \gamma^{-2}$ to $l_c \sim \gamma^{-2/3}$. However, these modifications owing to nonuniversal features are a consequence of our chosen definition of l_c ; we find the same *universal* behavior, as captured by the weak-localization correction, for all values of the unraveling phase in the range $0 \leq \varphi < \pi/2$, and the weak-localization phenomenon implies the absence of an entanglement transition. Only the unitary unravelling at $\varphi = \pi/2$ leads to qualitatively distinct universal behavior. Its description in terms of an NLSM is an interesting subject for future studies.

Our findings indicate that, for free fermionic systems, the choice of unraveling—whether formulated in terms of a quantum jump process or quantum state diffusion, and including further modifications such as the unraveling phase considered here—does not determine the presence or absence of a measurement-induced phase transition. Rather, the absence of such a transition in various models of 1D free fermionic systems with conserved particle number can be traced to a common $SU(R)$ symmetry. As noted in Sec. V A, conservation of particle number within the 1D fermionic chain is, in fact, not required for this symmetry to arise. It also persists when the chain is coupled to reservoirs with which it can exchange particles, provided that the total particle number of the combined system, consisting of the chain and reservoirs, remains conserved [56]. An important question for future research is whether this insensitivity to the choice of unraveling can be established in full generality. It would also be worthwhile to extend the analysis of different unravellings to observables beyond the local occupation numbers considered here. Alternative observables may give rise to qualitatively distinct dynamical behavior. For example, bilocal observables have been shown to induce superdiffusive scaling of the entanglement entropy [58]. How such phenomena depend on the choice of unraveling remains an open question.

Looking further ahead, it will be interesting to explore whether the critical phase can be stabilized by relaxing the constraints of the present model, for instance, by introducing additional ingredients such as disorder [25,26,54,68,72], Floquet driving [55], or nonlocal measurements. An intriguing question is whether the singular point $\varphi = \pi/2$ in our model could, through some mechanism, be extended into a stable phase of nonequilibrium quantum matter that occupies a

finite region of parameter space. Finally, measurement-induced transitions are known to occur in fermionic systems in higher dimensions [35,41], in the presence of interactions [51,52], or in Majorana models with broken particle-number conservation [37]. Elucidating the role of different unravelings in these settings constitutes an important direction for future work.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [97].

APPENDIX A: DERIVATION OF THE STOCHASTIC SCHRÖDINGER EQUATION FOR GENERAL QUADRATURE MEASUREMENTS

In this appendix, we provide a formal derivation of the stochastic Schrödinger equation (2) and the corresponding linear stochastic Schrödinger equation, rewritten in Eq. (7) as a discrete-time linear stochastic master equation. Our approach is not tied to a microscopic model of a specific physical realization; instead, it is intended to clarify the conceptual role of the unraveling phase φ . Along the way, we also highlight connections to alternative formulations of continuous measurements used in Refs. [37,41,50].

1. Setup

We consider a quantum system coupled to a bosonic ancilla on which measurements are performed [74]. Our aim is to demonstrate that the phase factor $e^{i\varphi}$ in Eq. (2) arises naturally from measuring a general quadrature of the ancilla. The derivation of the stochastic Schrödinger equation (2), however, is fully general, and we therefore leave the nature of the system unspecified at this stage.

We derive the evolution equation (2) by considering the dynamics over a finite time step Δt and subsequently taking the limit $\Delta t \rightarrow dt$. The ancilla, which is described by bosonic annihilation and creation operators $\hat{b}$ and $\hat{b}^\dagger$, respectively, is initialized in the vacuum state, $\hat{b}|0\rangle = 0$. In the usual quantum optics treatment, $\hat{b}$ and $\hat{b}^\dagger$ are quantum Itô increments [74,75]. Accordingly, their commutator is determined by the step size,

$$[\hat{b}, \hat{b}^\dagger] = \Delta t. \quad (\text{A1})$$

As stated above, we leave the nature of the system unspecified. We denote the initial state of the system by $|\psi_0\rangle$. The initial state of system and ancilla is thus

$$|\Psi_0\rangle = |\psi_0\rangle \otimes |0\rangle. \quad (\text{A2})$$

During the time step, the system and ancilla are entangled by applying the unitary

$$\hat{U} = e^{\sqrt{\gamma}\hat{c}(\hat{b}^\dagger - \hat{b})}, \quad (\text{A3})$$

where $\hat{c} = \hat{c}^\dagger$ is a system observable. To illustrate the effect of the entangling operation (A3), consider the simple case that the system is prepared in an eigenstate of $\hat{c}$. Then, applying Eq. (A3) to the initial state (A2) displaces the bosonic vacuum. The displacement has magnitude $\langle \hat{c}_0 \rangle = \langle \psi_0 | \hat{c} | \psi_0 \rangle$ and is directed along the quadrature $\hat{x} = (\hat{b} + \hat{b}^\dagger)/\sqrt{2}$. Measuring this quadrature thus provides direct information about $\langle \hat{c}_0 \rangle$.

2. Equivalence between general quadrature measurements and coupling to non-Hermitian system operators

We now generalize the scenario above by either measuring a more general quadrature or, equivalently, by using a non-Hermitian operator $\hat{c}$, obtained by multiplying a Hermitian operator by a complex phase factor. Let us begin by showing that these two approaches are indeed equivalent. To this end, we define the rotated field quadratures

$$\hat{x}_\varphi = \frac{1}{\sqrt{2}}(\hat{b}e^{i\varphi} + \hat{b}^\dagger e^{-i\varphi}), \quad \hat{p}_\varphi = \frac{1}{\sqrt{2}i}(\hat{b}e^{i\varphi} - \hat{b}^\dagger e^{-i\varphi}), \quad (\text{A4})$$

which obey the commutation relation

$$[\hat{x}_\varphi, \hat{p}_\varphi] = i\Delta t. \quad (\text{A5})$$

Quadratures with different values of φ are unitarily related,

$$\hat{x}_\varphi = \hat{R}_\varphi^\dagger \hat{x} \hat{R}_\varphi, \quad \hat{R}_\varphi = e^{i\varphi \hat{b}^\dagger \hat{b}}. \quad (\text{A6})$$

After the entangling operation Eq. (A3), we perform a measurement of $\hat{x}_\varphi$ on the ancilla. We denote the outcome of the measurement by x_m . The measurement projects the ancilla onto the corresponding eigenstate $|x_\varphi = x_m\rangle$ of $\hat{x}_\varphi$; the effect of the measurement on the system can be expressed through the Kraus operator $\hat{K}_{x_m}$ that is defined by

$$\hat{K}_{x_m}|\psi_0\rangle = \langle x_\varphi = x_m | \hat{U} | \Psi_0 \rangle \Leftrightarrow \hat{K}_{x_m} = \langle x_\varphi = x_m | \hat{U} | 0 \rangle. \quad (\text{A7})$$

From Eq. (A6) it follows that $|x_\varphi = x_m\rangle = \hat{R}_\varphi^\dagger |x = x_m\rangle$; that is, eigenvectors of $\hat{x}_\varphi$ and $\hat{x}$ are related by $\hat{R}_\varphi^\dagger$. Using this and the fact that the transformation $\hat{R}_\varphi$ leaves the bosonic vacuum invariant, $\hat{R}_\varphi|0\rangle = |0\rangle$, we obtain

$$\hat{K}_{x_m} = \langle x = x_m | \hat{U}_\varphi | 0 \rangle, \quad (\text{A8})$$

where

$$\hat{U}_\varphi = \hat{R}_\varphi \hat{U} \hat{R}_\varphi^\dagger = e^{\sqrt{\gamma}(\hat{c}_\varphi \hat{b}^\dagger - \hat{c}_\varphi^\dagger \hat{b})}, \quad \hat{c}_\varphi = \hat{c}e^{i\varphi}. \quad (\text{A9})$$

Compare Eqs. (A7) and (A8): The former describes the measurement of $\hat{x}_\varphi$ after coupling the system to the bath via Eq. (A3), with a Hermitian system operator $\hat{c}$; the latter describes the measurement of $\hat{x}$ after coupling the system to the bath via Eq. (A9), where the phase factor $e^{i\varphi}$ has been absorbed in the definition of the non-Hermitian system operator $\hat{c}_\varphi$. Both equations provide equivalent definitions of the Kraus operator. Therefore, without loss of generality, we can restrict ourselves to measurements of $\hat{x}$ while allowing for an entangling operation with a non-Hermitian system operator $\hat{c} \neq \hat{c}^\dagger$. We thus consider

$$\hat{K}_x = \langle x | \hat{U} | 0 \rangle, \quad (\text{A10})$$

with

$$\hat{U} = e^{\sqrt{\gamma}(\hat{c}\hat{b}^\dagger - \hat{c}^\dagger\hat{b})}. \quad (\text{A11})$$

Here and in the following, we omit the subscript m of the measurement outcome.

3. Kraus operators

We now aim to determine the explicit form of the Kraus operators $\hat{K}_x$. As discussed above, from the ancilla's perspective, $\hat{U}$ in Eq. (A11) essentially acts as a displacement operator, with the complication that $\hat{c}$ and $\hat{c}^\dagger$ may not commute. This noncommutativity, however, affects only higher-order terms in Δt . To demonstrate this, one can, for instance, expand $\hat{U}$ in Δt , as done in the derivation of the stochastic Schrödinger equation in Ref. [74]. Here, we instead follow the alternative approach of factorizing $\hat{U}$ using the Zassenhaus formula [102]. Equation (A1) implies $\hat{b} \sim \sqrt{\Delta t}$. Therefore, to leading order in Δt , we obtain

$$\hat{U} = e^{\sqrt{\gamma}\hat{c}\hat{b}^\dagger} e^{-\sqrt{\gamma}\hat{c}^\dagger\hat{b}} e^{\frac{\gamma}{2}\hat{c}\hat{c}^\dagger\hat{b}^\dagger\hat{b}} e^{-\frac{\gamma}{2}\hat{c}^\dagger\hat{c}\hat{b}\hat{b}^\dagger}. \quad (\text{A12})$$

Then, using $\hat{b}\hat{b}^\dagger|0\rangle = [\hat{b}, \hat{b}^\dagger]|0\rangle = \Delta t|0\rangle$, we find

$$\hat{K}_x = M_x(\hat{c})e^{-\frac{\gamma\Delta t}{2}\hat{c}^\dagger\hat{c}}, \quad M_x(\hat{c}) = \langle x|e^{\sqrt{\gamma}\hat{c}\hat{b}^\dagger}|0\rangle. \quad (\text{A13})$$

To evaluate $M_x(\hat{c})$, we can treat $\hat{c}$, which is the only system operator appearing in this expression, as a number $c \in \mathbb{C}$. We introduce a displacement operator $\hat{V}$,

$$\hat{V} = e^{\sqrt{\gamma}(c\hat{b}^\dagger - c^*\hat{b})} = e^{\sqrt{\gamma}c\hat{b}^\dagger} e^{-\sqrt{\gamma}c^*\hat{b}} e^{-\frac{\gamma\Delta t}{2}|c|^2}, \quad (\text{A14})$$

in terms of which we can write

$$M_x(c) = e^{\frac{\gamma\Delta t}{2}|c|^2} \langle x|\hat{V}|0\rangle = \frac{1}{(\pi\Delta t)^{1/4}} e^{-\frac{1}{2\Delta t}(x - \sqrt{2\gamma}\Delta t c)^2 + \frac{\gamma\Delta t}{2}c^2}. \quad (\text{A15})$$

Note that the representation of the coherent state $\hat{V}|0\rangle$ in the basis $|x\rangle$ is modified as compared to the usual one owing to the factor Δt in the commutation relations in Eqs. (A1) and (A5). Replacing c by the operator $\hat{c}$ yields the desired explicit form of the Kraus operators,

$$\hat{K}_x = \frac{1}{(\pi\Delta t)^{1/4}} e^{-\frac{x^2}{2\Delta t} + \sqrt{2\gamma}x\hat{c} - \frac{\gamma\Delta t}{2}(\hat{c} + \hat{c}^\dagger)\hat{c}}, \quad (\text{A16})$$

where we again keep only the leading order in Δt , taking into account that $x \sim \sqrt{\Delta t}$ because of the first term in the exponent. The Kraus operators obey the completeness relation

$$\int_{-\infty}^{\infty} dx \hat{K}_x^\dagger \hat{K}_x = 1. \quad (\text{A17})$$

4. Quantum trajectories

The formulation of the measurement process in terms of the Kraus operators (A16) provides the basis for defining quantum trajectories. We iterate the measurement process described above n times, corresponding to a time evolution of total duration $T = n\Delta t$. After each measurement, the ancilla is reinitialized in the vacuum state. We denote the resulting sequence of measurement outcomes by $\mathbf{x} = (x_0, \dots, x_{n-1})$. According to Born's rule, the probability $P_{\mathbf{x}}$ of observing this

sequence of outcomes is given by the squared norm of the resulting state,

$$|u_n\rangle = \hat{K}_{x_{n-1}} \cdots \hat{K}_{x_0} |\psi_0\rangle, \quad (\text{A18})$$

where we leave the dependence of the state on the measurement outcomes implicit; that is,

$$P_{\mathbf{x}} = \| |u_n\rangle \|^2. \quad (\text{A19})$$

The average of a function of the normalized state, $|\psi_n\rangle = |u_n\rangle / \| |u_n\rangle \|$, can be calculated as

$$\overline{f(|\psi_n\rangle)} = \int d\mathbf{x} P_{\mathbf{x}} f(|\psi_n\rangle). \quad (\text{A20})$$

Important examples include the connected density correlation function (55) and the entanglement entropy (71).

Equations (A18) and (A19) provide a complete description of the system dynamics under repeated measurements. We proceed to rephrase this description, first as a linear evolution equation for an unnormalized state, and further below as a nonlinear evolution equation for a normalized state.

5. Linear stochastic Schrödinger equation

Instead of labeling measurement outcomes by the quadrature x , we use the variable $\Delta W = \sqrt{2}x$ in the following. We define the ostensible distribution of ΔW as [75]

$$\mathcal{Q}_{\Delta W} = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{\Delta W^2}{2\Delta t}\right), \quad (\text{A21})$$

and we factor out the ostensible distribution from Eq. (A16) to define a new set of Kraus operators,

$$\hat{J}_{\Delta W} = \frac{1}{\sqrt{\mathcal{Q}_{\Delta W}}} \frac{\hat{K}_x}{2^{1/4}} \Big|_{x=\Delta W/\sqrt{2}} = e^{\sqrt{\gamma}\Delta W\hat{c} - \frac{\gamma\Delta t}{2}(\hat{c} + \hat{c}^\dagger)\hat{c}}. \quad (\text{A22})$$

These operators obey the completeness relation

$$\int_{-\infty}^{\infty} d\Delta W \mathcal{Q}_{\Delta W} \hat{J}_{\Delta W}^\dagger \hat{J}_{\Delta W} = 1. \quad (\text{A23})$$

Consider now a trajectory, defined as in Eq. (A18) but with the Kraus operators $\hat{K}_x$ replaced by $\hat{J}_{\Delta W}$,

$$|v_n\rangle = \hat{J}_{\Delta W_{n-1}} \cdots \hat{J}_{\Delta W_0} |\psi_0\rangle. \quad (\text{A24})$$

As above, we leave the dependence of the state on the sequence of measurement outcomes $\Delta \mathbf{W} = (\Delta W_0, \dots, \Delta W_{n-1})$ implicit. We denote the norm of the state by

$$R_{\Delta \mathbf{W}} = \| |v_n\rangle \|^2 = P_{\mathbf{x}} / (2^{n/2} \mathcal{Q}_{\Delta \mathbf{W}}) \Big|_{\mathbf{x}=\Delta \mathbf{W}/\sqrt{2}}, \quad (\text{A25})$$

where the ostensible probability of a the full sequence of measurement outcomes is the product of probabilities of the outcomes of individual measurements,

$$\mathcal{Q}_{\Delta \mathbf{W}} = \prod_{n'=0}^{n-1} \mathcal{Q}_{\Delta W_{n'}}. \quad (\text{A26})$$

However, the *true* probability of a trajectory is the product of the ostensible probability $\mathcal{Q}_{\Delta \mathbf{W}}$ and the norm of the state

$R_{\Delta W}$. Therefore, the average of a function of the normalized state $|\psi_n\rangle = |v_n\rangle/\|v_n\|$ should be computed as

$$\overline{f(|\psi_n\rangle)} = \int d\Delta W Q_{\Delta W} R_{\Delta W} f(|\psi_n\rangle) = \int d\mathbf{x} P_{\mathbf{x}} f(|\psi_n\rangle), \quad (\text{A27})$$

which correctly reproduces Eq. (A20).

The formulation of the stochastic dynamics in terms of the Kraus operators $\hat{J}_{\Delta W}$ leads to the usual form of the linear stochastic Schrödinger equation [73,76]. To see this, we expand Eq. (A22) up to first order in Δt , using $\Delta W^2 = \Delta t$ [73],

$$\hat{J}_{\Delta W} \sim 1 - \frac{\gamma}{2} \hat{c}^\dagger \hat{c} \Delta t + \sqrt{\gamma} \hat{c} \Delta W. \quad (\text{A28})$$

We can generalize the Kraus operators to include a Hamiltonian contribution that acts on the system alone. This does not affect the above derivation. The resulting linear evolution equation for the unnormalized state (A24) reads

$$|v_{n+1}\rangle = \left(1 - i\hat{H}\Delta t - \frac{\gamma}{2} \hat{c}^\dagger \hat{c} \Delta t + \sqrt{\gamma} \hat{c} \Delta W_n\right) |v_n\rangle, \quad (\text{A29})$$

and the corresponding linear master equation for the unnormalized density matrix $\hat{D}_n = |v_n\rangle\langle v_n|$ is given by

$$\begin{aligned} \hat{D}_{n+1} = & \hat{D}_n - i[\hat{H}, \hat{D}_n] + \gamma(\hat{c} \hat{D}_n \hat{c}^\dagger - \frac{1}{2} \{\hat{c}^\dagger \hat{c}, \hat{D}_n\}) \Delta t \\ & + \sqrt{\gamma}(\hat{c} \hat{D}_n + \hat{D}_n \hat{c}^\dagger) \Delta W_n. \end{aligned} \quad (\text{A30})$$

Finally, we specialize this equation to the fermionic lattice system considered in the main text. This entails performing, during each time step of duration Δt , measurements on all lattice sites with $\hat{c} = \hat{n}_l e^{i\varphi}$, rather than a single measurement. To leading order in Δt , the measurement outcomes satisfy $\Delta W_{l,n} \Delta W_{l',n} = \delta_{l,l'} \Delta t$, yielding Eq. (7). An alternative derivation of the stochastic master equation, based on general results from stochastic process theory, is given in Ref. [50].

6. Other formulations of linear stochastic dynamics

We find it useful to briefly digress and discuss the relation between our approach and other formulations of linear stochastic dynamics that have been employed to derive replica field theories for Hermitian observables, $\hat{c} = \hat{c}^\dagger$ [37,41,50]. To this end, we first introduce the parameter

$$\alpha = \frac{x}{\sqrt{2\gamma}\Delta t}, \quad (\text{A31})$$

and define new Kraus operators

$$\hat{K}_\alpha = (2\gamma\Delta t^2)^{1/4} \hat{K}_x|_{x=\sqrt{2\gamma}\Delta t} = \left(\frac{2\gamma\Delta t}{\pi}\right)^{1/4} e^{-\gamma\Delta t(\alpha - \hat{c})^2}. \quad (\text{A32})$$

A formulation of continuous measurements using Kraus operators of the form Eq. (A32) was employed in Ref. [41] to derive a replica master equation, which was then rewritten as a Keldysh field theory. A slightly different but equivalent formulation is used in Refs. [37,50]. As in Eq. (A22), these works factor out the ostensible distribution of α ,

$$p_\alpha = \sqrt{\frac{2\gamma\Delta t}{\pi}} e^{-2\gamma\Delta t\alpha^2}. \quad (\text{A33})$$

7. Nonlinear stochastic Schrödinger equation

We now return to the case of non-Hermitian operators, $\hat{c} \neq \hat{c}^\dagger$, and derive the nonlinear stochastic Schrödinger equation for the normalized state. This follows a standard procedure [74], which we summarize here for completeness. The probability of obtaining the measurement outcome ΔW_n in time step n is

$$\begin{aligned} P_{\Delta W_n} &= Q_{\Delta W_n} \langle \psi_n | \hat{J}_{\Delta W_n}^\dagger \hat{J}_{\Delta W_n} | \psi_n \rangle \\ &= \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{1}{2\Delta t} (\Delta W_n - \sqrt{\gamma} \langle \hat{a}_n \rangle \Delta t)^2}, \end{aligned} \quad (\text{A34})$$

where $\hat{a} = \hat{c} + \hat{c}^\dagger$ and $\langle \hat{a}_n \rangle = \langle \psi_n | \hat{a} | \psi_n \rangle$. This suggests shifting $\Delta W_n \rightarrow \Delta W_n + \sqrt{\gamma} \langle \hat{a}_n \rangle \Delta t$, where the shifted ΔW_n is a Wiener increment with statistics given by Eq. (A21). With this substitution, Eq. (A29) becomes

$$|v_{n+1}\rangle = \left(1 - i\hat{H}\Delta t - \frac{\gamma}{2} \hat{c}^\dagger \hat{c} \Delta t + \gamma \langle \hat{a}_n \rangle \hat{c} \Delta t + \sqrt{\gamma} \hat{c} \Delta W_n\right) |v_n\rangle. \quad (\text{A35})$$

The corresponding evolution equation for the normalized state, $|\psi_n\rangle = |v_n\rangle/\|v_n\|$, reads

$$\begin{aligned} |\psi_{n+1}\rangle = & \left[1 - i\hat{H}\Delta t - \frac{\gamma}{2} \left(\hat{c}^\dagger \hat{c} - \langle \hat{a}_n \rangle \hat{c} + \frac{1}{4} \langle \hat{a}_n \rangle^2\right) \Delta t\right. \\ & \left. + \sqrt{\gamma} \left(\hat{c} - \frac{1}{2} \langle \hat{a}_n \rangle\right) \Delta W_n\right] |\psi_n\rangle. \end{aligned} \quad (\text{A36})$$

A time-dependent gauge transformation, $|\psi_n\rangle \mapsto e^{-i\phi_n} |\psi_n\rangle$ with $\phi_n = \sum_{n'=0}^n \Delta\phi_{n'}$ and

$$\Delta\phi_n = -\frac{\gamma}{4} \langle \hat{a}_n \rangle \langle \hat{b}_n \rangle \Delta t - \frac{\sqrt{\gamma}}{2} \langle \hat{b}_n \rangle \Delta W_n, \quad (\text{A37})$$

where $\hat{b} = -i(\hat{c} - \hat{c}^\dagger)$, yields

$$\begin{aligned} |\psi_{n+1}\rangle = & \left[1 - i\hat{H}\Delta t - \frac{\gamma}{2} (\hat{c}^\dagger \hat{c} - 2\hat{c} \langle \hat{c}_n^\dagger \rangle + |\langle \hat{c}_n \rangle|^2) \Delta t\right. \\ & \left. + \sqrt{\gamma} (\hat{c} - \langle \hat{c}_n \rangle) \Delta W_n\right] |\psi_n\rangle. \end{aligned} \quad (\text{A38})$$

In the continuous-time limit, $\Delta t \rightarrow dt$ and $\Delta W_n \rightarrow dW(t)$, this reduces to the standard nonlinear stochastic Schrödinger equation [74]. To obtain Eq. (2), which describes continuous measurements carried out simultaneously on all lattice sites, we set $\hat{c} = \hat{n}_l e^{i\varphi}$ and sum over l .

APPENDIX B: NUMERICAL SOLUTION FOR THE DENSITY CORRELATION FUNCTION

Here, we detail the numerical procedure we use to calculate the density correlation function. In Eq. (59), the density correlation function is expressed in terms of $\mathcal{C}$, which in turn is expressed in Eq. (60) in terms of L , the solution of the Wiener-Hopf integral equation (63). To find L , we rewrite Eq. (63) in terms of the dimensionless momentum $u = 2l_0 \sin(q)$ and time $\tau = \gamma t$ as

$$\tilde{L}_u(\tau, \tau') - \int_{-\infty}^0 d\tau'' B_{+,u}(\tau - \tau'') \tilde{L}_u(\tau'', \tau') = \delta(\tau - \tau'), \quad (\text{B1})$$

where $\tilde{L}_u(\tau, \tau') = L_q(t, t')/\gamma$. A substitution of integration variables $\tau = 1 - 1/s$ maps the range of integration from $\tau \in (-\infty, 0]$ to $s \in (0, 1]$, leading to

$$\tilde{L}_u(s, s') - \int_0^1 ds'' \frac{\mathcal{B}_{+,u}(s-s'')}{(s'')^2} \tilde{L}_u(s'', s') = s^2 \delta(s-s'). \quad (\text{B2})$$

This integral equation can be recast as a matrix equation that can be solved numerically by approximating the integral through a Riemann sum. To that end, we use the partition of the interval $(0, 1]$, which is given by the sequence $s_n = n\Delta s$

with $\Delta s = 1/N$ such that $s_0 = 0$ and $s_N = 1$. The discretized form of Eq. (B2) thus reads

$$\tilde{L}_u - \Delta s \mathcal{B}_{+,u} D^{-1} \tilde{L}_u = D/\Delta s, \quad (\text{B3})$$

which is written here in terms of matrices with elements $\tilde{L}_{u,n,n'} = \tilde{L}_u(s_n, s_{n'})$, $\mathcal{B}_{+,u,n,n'} = \mathcal{B}_{+,u}(s_n - s_{n'})$, and $D_{n,n'} = s_n^2 \delta_{n,n'}$. Solving this equation for $\tilde{L}_u$ yields

$$\tilde{L}_u = [\Delta s(1 - \Delta s \mathcal{B}_{+,u} D^{-1})]^{-1} D. \quad (\text{B4})$$

We obtain the density correlation function by discretizing equations (59) and (60) in a similar fashion and inserting (B4).

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