

Ferrimagnetic Blume-Emery-Griffiths sawtooth chain: Spin ice and quasiordering

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(Received 5 January 2026; revised 30 March 2026; accepted 1 June 2026; published 16 June 2026)

The ferrimagnetic Blume-Emery-Griffiths one-dimensional model on the sawtooth chain with bilinear and biquadratic interactions between spins is studied exactly. Conditions at which the characteristic feature of spin ices, the remnant entropy at zero magnetic field appears, are studied. On the other hand, we determine at which set of parameters the model manifests quasiordering. That quasiordering reveals itself in almost nondecaying correlation functions below the quasicritical temperature, the jumplike temperature behavior of the entropy, and the quasipeculiarity in the temperature behavior of the specific heat at the quasicritical temperature. We show that the temperature behavior of magnetic moments of spins and the total magnetization drastically change its properties at the quasicritical temperature. The magnetic Grüneisen parameter manifests the most dramatic changes at the temperature of the quasiordering if the latter exists and in the ground state otherwise.

DOI: [10.1103/PhysRevB.113.214440](https://doi.org/10.1103/PhysRevB.113.214440)

I. INTRODUCTION

Phase transformations are generic features of many-body systems [1]. They have been studied by physicists both experimentally and theoretically for many years [2]. Phase transitions are usually related to thermodynamic characteristics of matter. Notice, though, so-called dynamical phase transitions, which explore the analogy between the temperature and the imaginary time [3]. States of matter, which can exist simultaneously in the thermodynamic equilibrium with each other, are usually called phases of matter. The phase transition point is the special point in the space of thermodynamic quantities [1]. It is commonly accepted that the phase transition point divides the phase with some nonzero (order) parameter from the one in which that parameter is zero. For instance, for the liquid-gas phase transition the order parameter is the density, and for the ferromagnets the order parameter is the magnetization. By definition, the order parameter is the value, the correlation function of which in two different points remains finite even for the infinite distance between those points. This is why phase transitions can exist only in many-body systems and the interactions between particles play the principal role. From the experimental viewpoint, studies of phase transitions are very attractive because in the vicinity of the critical point properties of matter are often changed drastically with relatively small changes of the external governing parameters. Such a feature of phase transitions determines

also the interest of theoreticians to the investigations of phase transitions.

A magnetically ordered ferrimagnet contains two (or more) magnetic sublattices with different moments [4]. The vector sum of magnetic moments of sublattices for a ferrimagnet is, in general, nonzero [5]. It can be achieved due to different values of spins of magnetic ions or different values of effective magnetons (g factors) for spins of the same values. The magnetic order parameter of the ferrimagnet is more complicated than the one of the antiferromagnet with equal magnetic moments of magnetic sublattices. Both the magnetization and the vector of antiferromagnetism (the vector difference between the magnetic moments of sublattices) characterize phases of a ferrimagnet, being components of the ferrimagnetic order parameter. On the other hand, one can use the magnetizations of sublattices as equivalent components of the ferrimagnetic order parameter. For the simplest isotropic two-sublattice ferrimagnet in an external magnetic field, three equilibrium magnetic configurations (magnetic phases) exist. In strong magnetic fields, the spin-flipped phase with magnetic moments of sublattices directed along the magnetic field takes place. The nonzero component of the order parameter is the total magnetization. On the other hand, in weak magnetic fields the antiparallel configuration exists. There magnetic moments of sublattices are antiparallel to each other. That phase has the net magnetization since values of magnetic moments of sublattices are different. Finally, the noncollinear phase takes place in the intermediate magnetic field region. The compensation point is characteristic for ferrimagnets: At some values of the exchange constant, the external magnetic field, and the temperature magnetic moments of sublattices can be turned in such a way that the total magnetization becomes zero. The component of the order parameter, which is nonzero in that point, is the vector of antiferromagnetism.

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As a rule, theoretical models of ferrimagnets are not integrable [1,5]. However, there exist several one-dimensional models which describe the behavior of ferrimagnets (with different values of effective magnetons of different spins). One of the known exactly solvable quantum models is the dimerized spin-1/2 (XX) chain with different values of effective magnetons [6], see also, e.g., [7]. Such one-dimensional ferrimagnet is characterized by zero net magnetic moment (in the absence of the external magnetic field), even in the ground state. It is different from the case of magnetically ordered ferrimagnets. Only quantum phase transitions (which exist in the ground state only) take place in that model. It is characteristic for one-dimensional models [7]. Magnetic phases, which exist in the ground state, are reminiscent of those in magnetically ordered ferrimagnets. In a strong magnetic field, above the quantum critical point, a phase with the maximal values of the projections of the expectation value of the order parameter exists. In a weak field, below the other critical point, the phase reminiscent to the antiparallel one of the ordered ferrimagnet exists. Notice, however, that such a phase exists only for dimerized exchange constants. Here the expectation value of the projection of the total spin is zero, while the projection of the magnetic moment is nonzero. Finally, between two quantum critical points the intermediate phase exists [6]. The role of the next-nearest neighbor (NNN) interactions for that ferrimagnetic model was studied in [8]. Similar magnetic field behavior was studied for a quantum spin dimer [9].

Recently, several one-dimensional models of ferrimagnets on so-called decorated lattices with different effective magnetons for spins with finite radius of interactions between particles, exact solutions of which manifest phenomena at nonzero temperatures, reminiscent of phase transitions, were studied [10–18]. It was shown that for some parameters of models, the entropy manifests the jumplike behavior in the narrow temperature region. Therefore, the specific heat reveals the narrow maximum in that region. Such a behavior can be called quasiordering (or pseudo-ordering). It is different from the quasicritical phenomena. The latter is related to a region near the phase transition. Quasicritical phenomena refers to features, such as scaling laws and universality, without being exactly at the critical point and its vicinity. So far, mostly models of that group with Ising spins 1/2, which determine the onset of quasiordering, were studied (see, though, [13,19–23] for more complicated models). However, it is natural that spins of realistic magnetic systems can be larger than 1/2. In that situation, the single-ion magnetic anisotropy is present in many magnetic materials due to the crystalline electric field of ligands and the spin-orbit interaction [24]. Also, for larger values of site spins, not only bilinear spin-spin couplings but also biquadratic ones are possible [25,26].

In this paper, we consider exactly the one-dimensional ferrimagnetic Blume-Emery-Griffiths model (which, unlike the Ising model, permits the existence of the single-ion magnetic anisotropy) for spins 1 on the sawtooth chain with nearest-neighbor (NN) and NNN bilinear and biquadratic spin-spin interactions (the NNN coupling only between odd spins). Properties of sawtooth spin chains were observed in a number of magnetic materials [27–31]. The Blume-Emery-Griffiths model was introduced in [32]. This model is the generalization of the Blume-Capel model introduced in [33]

and independently in [34]. The Blume-Emery-Griffiths model was successfully applied for theoretical descriptions of various physical systems, e.g., multicomponent fluids, ternary alloys, $^3\text{He} - ^4\text{He}$ mixtures, and magnetic systems. Good agreement with the experimental results, obtained for site spins (total moment for rare-earth compounds) larger than 1/2 with biquadratic exchange interactions, for ferrimagnetic cases, including one-dimensional situation [35–39], determines theoretical interest in these kinds of systems. For instance, the model on the sawtooth chain with larger values of total moments of magnetic ions can explain some features of the behavior of magnetic compounds, such as the olivines ZnR_2S_4 ($R = \text{Er, Tm, Yb}$) [40].

We show that there is no real magnetic ordering in that model (except in the ground state). Since the geometrical frustration [41,42] exists for such a chain, we show that for some parameter ranges the remnant entropy in zero magnetic field appears due to the high degeneracy of eigenstates. In that case, we show that the magnetic field behavior of the magnetization is similar to other models of frustrated magnets like spin ices [43,44], see, e.g., [45]. From that viewpoint, the considered model presents the rare example of exact solution for spin ices. On the other hand, we show that for antiferromagnetic NN bilinear spin-spin interactions, the correlation function between odd spins of the chain manifests a behavior very similar to the one for the phase transition: That correlator almost does not decay for temperatures below the quasiordering temperature and decays above that temperature. Such a situation is reminiscent of the behavior of the order parameter. The entropy manifests the jumplike behavior in the narrow region of temperatures—the specific heat shows the sharp maximum there. We prove that the temperature range, in which drastic changes of those observables exist, decreases with the growth of the NNN ferromagnetic bilinear spin-spin couplings. We also show that the magnetic field behavior of expectation values of magnetic moments of spins manifests the most dramatic changes below the quasiordering temperature. Finally, we show that unlike the cases without quasiordering, the magnetic Grüneisen parameter [46,47] (which characterizes the magnetocaloric effect) shows maximal changes not at zero temperature, like for other one-dimensional models, but instead at the quasiordering temperature.

II. ONE-DIMENSIONAL BLUME-EMERY-GRIFFITHS FERRIMAGNETIC MODEL ON THE SAW-TOOTH CHAIN

Let us consider the Blume-Emery-Griffiths chain with the Hamiltonian

$$\begin{aligned} \mathcal{H} = & - \sum_n [J_1 (S_{1,n}^z S_{2,n}^z + S_{1,n+1}^z S_{2,n}^z) + D_1 (S_{1,n}^z)^2 \\ & + D_2 (S_{2,n}^z)^2 + H (\mu_1 S_{n,1}^z + \mu_2 S_{n,2}^z) + J S_{1,n}^z S_{1,n+1}^z \\ & + K_1 [(S_{1,n}^z)^2 (S_{2,n}^z)^2 + (S_{1,n+1}^z)^2 (S_{2,n}^z)^2] \\ & + K (S_{1,n}^z)^2 (S_{1,n+1}^z)^2], \end{aligned} \quad (1)$$

where $S_{1,2,n}^z$ are the operators of spin-1 z projections of the chain, J_1 describes the bilinear coupling between NN spins, J is the bilinear coupling between NNN (odd) spins of the chain,

K_1 and K are the parameters of the biquadratic NN and NNN (for odd spins) interspin couplings, respectively, H is the external magnetic field directed along the z axis, $\mu_{1,2} \equiv g_{1,2}\mu_B$ are the effective magnetons ($g_{1,2}$ are the effective g factors and μ_B is the Bohr magneton; in what follows, we use the units in which $\mu_B = 1$ for brevity) of odd and even spins, respectively, and $D_{1,2}$ are the parameters of the single-ion magnetic anisotropy. For the Blume-Emery-Griffiths model of any space dimension and any radii of couplings, one does not need to look for solutions of the stationary Schrödinger equation for the Hamiltonian because the local projections of spins, $S_{1,2,n}^z$, commute with the Hamiltonian and eigenfunctions are trivial. However, the Blume-Emery-Griffiths model possesses a discrete spectrum, the property of quantum systems. For $K = K_1 = 0$, the model is reduced to the Blume-Capel model on the sawtooth chain. For the cases $J_1 = K_1 = 0$ or $J = K = 0$ with $\mu_1 = \mu_2$ and $D_1 = D_2$, the model is reduced to the standard one-dimensional Blume-Emery-Griffiths model.

We use periodic boundary conditions, $S_{1,2,L+1}^z = S_{1,2,1}^z$, $2L$ is the number of spins in the chain. The partition function of the considered Blume-Emery-Griffiths chain can be written in a form $Z = \sum_{s_{1,n}} \sum_{s_{2,n}} \exp(-\beta\mathcal{H})$, where $\beta = 1/k_B T$, k_B is the Boltzmann constant (in what follows, we use units in which $k_B = 1$ for brevity), T is the temperature, and $s_{1,2,n}$ are the eigenvalues of the operators $S_{1,2,n}^z$ (equal to 0, ± 1 for spins 1). Parts of the Hamiltonian containing $S_{1,n}^z$, $S_{2,n}^z$, and $S_{1,n+1}^z$ in each triangle of the sawtooth chain commute. Hence, one can write the partition function as

$$Z = \prod_{n=1}^L \sum_{s_{1,n}=0,\pm 1} \sum_{s_{2,n}=0,\pm 1} \times \exp \left(\beta \left[J(s_{1,n}s_{2,n} + s_{2,n}s_{1,n+1}) + J_1 s_{1,n}s_{1,n+1} + K(s_{1,n}^2 s_{2,n}^2 + s_{2,n}^2 s_{1,n+1}^2) + K_1 s_{1,n}^2 s_{1,n+1}^2 + \frac{D_1}{2}(s_{1,n}^2 + s_{1,n+1}^2) + D_2 s_{2,n}^2 + H \left(\frac{g_1}{2}[s_{1,n} + s_{1,n+1}] + g_2 s_{2,n} \right) \right] \right). \quad (2)$$

The sum over all states of $s_{2,n}$ in each triangle can be taken independently from the others. Hence, we can get the product of the terms corresponding to different n ,

$$Z = \sum_{s_{1,n}, s_{1,n+1}} \prod_{n=1}^L (z_1 + z_2 + z_3) \equiv \text{Tr}(\hat{\tau})^L, \quad (3)$$

where we have summed over all eigenstates of $S_{n,2}^z$ and have introduced the transfer matrix $\hat{\tau}_{\mu,\nu}$ (with $\mu, \nu = 1, 2, 3$). Here we use the notations

$$z_1 = \exp \left[-\frac{J_1}{T}(s_{1,n} + s_{1,n+1}) + \frac{J}{T}s_{1,n}s_{1,n+1} + \frac{D_2}{T} + \frac{D_1}{2T}(s_{1,n}^2 + s_{1,n+1}^2) + \frac{g_1 H}{2T}(s_{1,n} + s_{1,n+1}) - \frac{g_2 H}{T} + \frac{K_1}{T}(s_{1,n}^2 + s_{1,n+1}^2) + \frac{K}{T}s_{1,n}^2 s_{1,n+1}^2 \right],$$

$$z_3 = \exp \left[\frac{J_1}{T}(s_{1,n} + s_{1,n+1}) + \frac{J}{T}s_{1,n}s_{1,n+1} + \frac{D_2}{T} + \frac{D_1}{2T}(s_{1,n}^2 + s_{1,n+1}^2) + \frac{g_1 H}{2T}(s_{1,n} + s_{1,n+1}) + \frac{g_2 H}{T} + \frac{K_1}{T}(s_{1,n}^2 + s_{1,n+1}^2) + \frac{K}{T}s_{1,n}^2 s_{1,n+1}^2 \right],$$

$$z_2 = \exp \left[\frac{J}{T}s_{1,n}s_{1,n+1} + \frac{D_1}{2T}(s_{1,n}^2 + s_{1,n+1}^2) + \frac{g_1 H}{2T}(s_{1,n} + s_{1,n+1}) + \frac{K}{T}s_{1,n}^2 s_{1,n+1}^2 \right]. \quad (4)$$

Here index 1 corresponds to $s_{2,n} = -1$, index 2 is related to $s_{2,n} = 0$, and index 3 is connected with the state with $s_{2,n} = 1$. The elements of the transfer matrix are

$$\begin{aligned} \tau_{11} &= e^{(2K_1/T)+(K/T)+(J/T)+(D_1/T)+(g_1 H/T)} [1 + 2e^{D_2/T} \cosh([2J_1 + g_2 H]/T)], \\ \tau_{12} &= \tau_{21} = e^{(K_1/T)+(D_1/2T)+(g_1 H/2T)} [1 + 2e^{D_2/T} \cosh([J_1 + g_2 H]/T)], \\ \tau_{13} &= \tau_{31} = e^{(2K_1/T)+(K/T)-(J/T)+(D_1/T)} [1 + 2e^{D_2/T} \cosh(g_2 H/T)], \\ \tau_{22} &= 1 + 2e^{D_2/T} \cosh(g_2 H/T), \\ \tau_{23} &= \tau_{32} = e^{(K_1/T)+(D_1/2T)-(g_1 H/2T)} [1 + 2e^{D_2/T} \cosh([J_1 - g_2 H]/T)], \\ \tau_{33} &= e^{(2K_1/T)+(K/T)+(J/T)+(D_1/T)-(g_1 H/T)} [1 + 2e^{D_2/T} \cosh([2J_1 - g_2 H]/T)]. \end{aligned} \quad (5)$$

Here the first index 1 of τ_{ij} corresponds to $s_{1,n} = 1$, the first index 2 is related to $s_{1,n} = 0$, and the first index 3 is connected with the state with $s_{1,n} = -1$. The second index 1 corresponds to $s_{1,n+1} = 1$, the second index 2 is related to $s_{1,n+1} = 0$, and the second index 3 is connected with the state with $s_{1,n+1} = -1$. One can see that at nonzero temperatures all matrix elements are positive. The transfer matrix is symmetric and irreducible (except of some special cases, see below). Hence, according to the Perron-Frobenius theorem [48,49], the transfer matrix has a real maximal eigenvalue. The property of equal nondiagonal elements $\tau_{12} = \tau_{21}$, $\tau_{13} = \tau_{31}$, and $\tau_{23} = \tau_{32}$ is characteristic for other known models on decorated lattices since $S_{1,n}^z$ and $S_{1,n+1}^z$ are present equivalently in the Hamiltonians, including the considered case Eq. (1).

Let us introduce the notations

$$\begin{aligned} r &= -\tau_{11} - \tau_{22} - \tau_{33}, \\ s &= -\tau_{12}^2 - \tau_{13}^2 - \tau_{23}^2 \\ &\quad + \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11}, \\ t &= -\tau_{11}\tau_{22}\tau_{33} - 2\tau_{12}\tau_{23}\tau_{13} \\ &\quad + \tau_{11}\tau_{23}^2 + \tau_{12}^2\tau_{33} + \tau_{13}^2\tau_{22}, \\ p &= \frac{-r^2 + 3s}{3}, \quad q = \frac{2r^3 - 9rs + 27t}{27}, \quad d = \frac{p^3}{27} + \frac{q^2}{4}, \\ u &= \left(-\frac{q}{2} + \sqrt{d}\right)^{1/3}, \quad v = \left(-\frac{q}{2} - \sqrt{d}\right)^{1/3}. \end{aligned} \quad (6)$$

Then the eigenvalues of the transfer matrix $\hat{\tau}$ are (written in the Cardano form)

$$\begin{aligned}\tau_1 &= u + v - \frac{r}{3}, \\ \tau_2 &= -\frac{u+v}{2} + i\frac{u-v}{2}\sqrt{3} - \frac{r}{3}, \\ \tau_3 &= -\frac{u+v}{2} - i\frac{u-v}{2}\sqrt{3} - \frac{r}{3}.\end{aligned}\quad (7)$$

Since $p < 0$ and $d < 0$, all eigenvalues are real. It is possible to show that τ_1 is the largest eigenvalue of the transfer matrix. To see it, we can rewrite Eqs. (7) in the Viette form as

$$\begin{aligned}\tau_1 &= 2R^{1/3} \cos(\phi/3) - \frac{r}{3}, \\ \tau_{2,3} &= 2R^{1/3} \cos[(\phi \pm 2\pi)/3] - \frac{r}{3},\end{aligned}\quad (8)$$

where $R = -p^3/27$, $\phi = \cos^{-1}(-q/2R)$. That form manifests their real values and clarifies the eigenvalue ordering. For the considered Hamiltonian, one has $-\pi \leq \phi \leq \pi$, and, therefore, $\tau_1 \geq \tau_3 \geq \tau_2$. It is easy to prove that in the thermodynamic limit $L \rightarrow \infty$ the Helmholtz free energy of the considered model (here and below we work with thermodynamic characteristics per spin, i.e., the total free energy is divided by $2L$) is $f = -T \ln(\tau_1)$. Then all thermodynamic characteristics of the model per spin can be calculated by simple differentiations. For example, the entropy is $S = -\partial f / \partial T$, the specific heat is $c = T \partial S / \partial T$, and the expectation values of the operators of the z projection of spins are $\langle S_{1,2,n}^z \rangle = -\partial f / \partial (g_{1,2} H)$. Expectation values of z projections of magnetic moments are $M_{1,2} = g_{1,2} \langle S_{1,2,n}^z \rangle$, and the z projection of the total magnetic moment per spin is $M = (M_1 + M_2)/2$. The magnetic Grüneisen parameter $\Gamma = (\partial M / \partial T) / c$ characterizes the magnetocaloric effect [47,50].

Since all matrix elements of the transfer matrix of the model are positive for $T \neq 0$, the free energy is the analytic function of T , and there are no real phase transitions at nonzero temperatures. However, in the ground state, quantum phase transitions can exist. For the special case of the Blume-Capel antiferromagnetic chain $K = K_1 = 0$ and $\mu_1 = \mu_2$, $D_1 = D_2$, and $J_1 = J$, such quantum phase transitions were considered in [51].

The correlation function for odd spins of the studied Blume-Emery-Griffiths chain is

$$\begin{aligned}\langle S_{1,n}^z S_{1,n+x}^z \rangle &= \frac{1}{Z} \sum S_{1,n}^z S_{1,n+x}^z e^{-\beta \mathcal{H}} \\ &= \sum_{s_{1,n}, s_{1,n+x}, s_{2,n}=0, \pm 1} S_{1,n}^z \hat{\tau}_{n,n+x}^{L-x} S_{1,n+x}^z \hat{\tau}_{n,n+x}^x,\end{aligned}\quad (9)$$

where in the first line the sum over all possible eigenstates is performed, and $\hat{\tau}_{n,n+x}$ is the $n, n+x$ element of the 3×3 matrix $\hat{\tau}$. Such matrices can be written in terms of eigenvalues and eigenfunctions of the transfer matrix. It is possible to show that the correlation length ξ is characterized by the inverse logarithm of the ratio of the largest and next-largest eigenvalues of the transfer matrix $\xi = -1 / \ln(\tau_3/\tau_1)$.

Unfortunately, the expressions for mentioned thermodynamic characteristics are too long to be presented in the paper, and in the figures below we show the main features of their

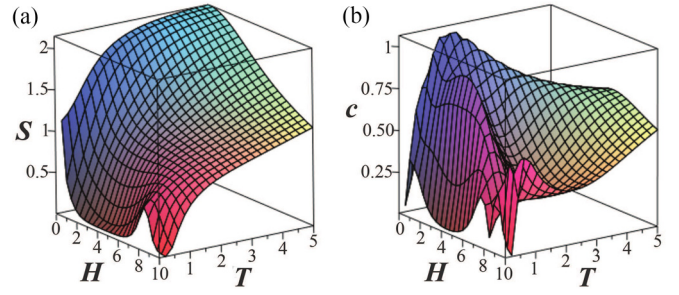


FIG. 1. (a) Temperature and magnetic field dependence of the entropy S of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the specific heat c of that chain.

behavior, which are characteristic for the considered model. These figures were obtained using exact analytical formulas.

III. RESULTS

A. Antiferromagnetic NN and NNN coupling: Spin ice behavior

For the antiferromagnetic NN ($J_1 < 0$) and NNN ($J < 0$) spin-spin bilinear couplings, the geometric frustration of the sawtooth chain manifests itself. Figures 1–3 show the magnetic field and temperature behavior of the entropy, specific heat, magnetic moments of even and odd spins, total magnetic moment, and magnetic Grüneisen parameter. We use the values of the NN and NNN bilinear spin-spin couplings $J_1 = J = -1$ (for equal NN and NNN couplings the magnetic frustration is maximal), the parameters of the single-ion anisotropy $D_1 = 1$, $D_2 = 2$, the constants of the biquadratic NN and NNN exchanges $K_1 = -0.5$, and $K = 0.5$, and the effective g factors $g_1 = 0.5$ and $g_2 = 1.5$. In what follows, we refer to the case with $K, K_1 > 0$ as for ferromagnetic couplings, and the case with $K, K_1 < 0$ as the antiferromagnetic ones for brevity (though biquadratic spin-spin coupling connects spin quadrupole operators, and it is not directly related to the total spin of the system).

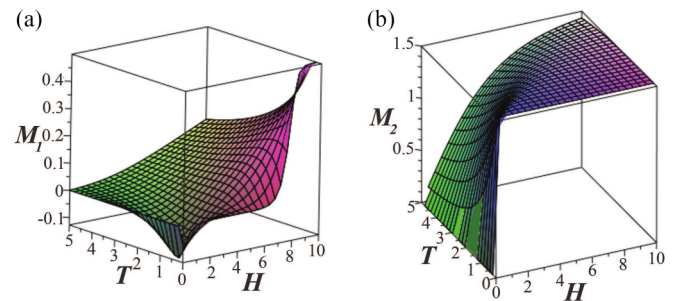


FIG. 2. (a) Temperature and magnetic field dependence of the magnetic moment of odd spins M_1 of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the magnetic moment of even spins M_2 of that chain.

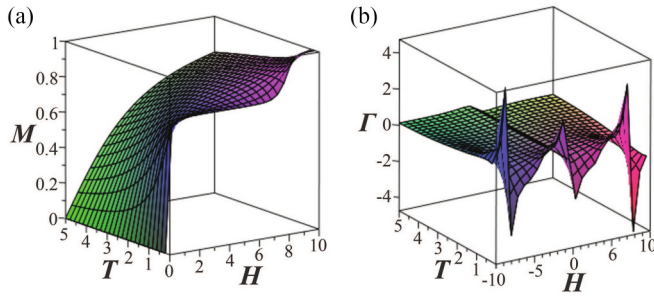


FIG. 3. (a) Temperature and magnetic field dependence of the total magnetic moment $M = (M_1 + M_2)/2$ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the magnetic Grüneisen parameter Γ of that chain.

One can see from Fig. 1 the remnant values of the entropy at $H = 0$ and at the (quantum) critical value of the magnetic field. The remnant entropy at those values of the magnetic field is caused by the macroscopic degeneracy of eigenstates of the model. The specific heat does not show narrow maxima. Such a behavior is characteristic for spin ices [43–45].

Figure 2 manifests the behavior of magnetic moments of odd and even spins of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth chain. One can see that magnetic moments of even spins show the behavior similar to the one of the single noninteracting spin 1 with the easy-axis magnetic anisotropy. The behavior of magnetic moments of odd spins is more complicated. The decrease of the value of M_1 with the growth of the magnetic field for low values of H is characteristic for one-dimensional ferrimagnets.

Finally, in Fig. 3 we show the temperature and magnetic field dependence of the total magnetic moment $M = (M_1 + M_2)/2$ and the magnetic Grüneisen parameter Γ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. It can be seen that the magnetic moment shows the jump at $H = 0$, and plateaux like behavior for larger values of H , characteristic for the magnetic system with the geometric frustration, e.g., spin ices, see [43–45].

Correlation functions decay exponentially with the distance for that case.

B. Antiferromagnetic NN and ferromagnetic NNN bilinear couplings with the easy-axis single-ion anisotropy: Quasiordering

In this subsection, we show the characteristic results for the antiferromagnetic NN and ferromagnetic NNN bilinear couplings of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain. Again, the figures are obtained using the exact solution of the model. We use the values of the NN bilinear coupling $J_1 = -1$, the NNN bilinear coupling $J = 10$ (for the case of large J , the effect of quasiordering is more pronounced), the parameters of the single-ion anisotropy $D_1 = 1$, $D_2 = 2$, i.e., the easy-axis anisotropy, the constants of the biquadratic NN

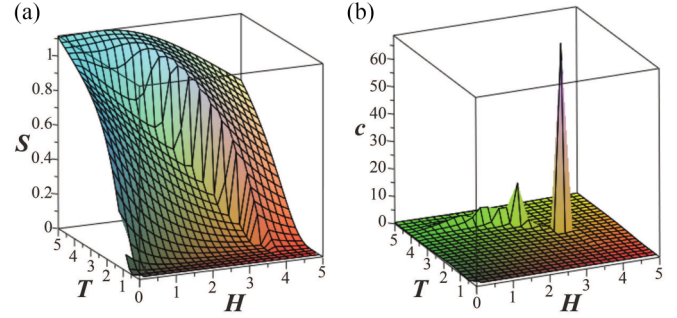


FIG. 4. (a) Temperature and magnetic field dependence of the entropy S of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the specific heat c of that chain.

exchange $K_1 = -0.5$, and large positive NNN one $K = 12$. The effective g factors are $g_1 = 0.5$ and $g_2 = 1.5$.

One can see from Fig. 4 that the entropy manifests a jump-like behavior, and the specific heat shows a narrow maximum at the quasicritical temperature T_q dependent on the value of the magnetic field. Notice that for the magnetic field close to the critical one $H = 4$ at $T \neq 0$, the maximum of the specific heat becomes very narrow, and not seen in the figure. We will discuss the onset of those features in the next subsections using analytic results.

Figure 5 manifests the behavior of magnetic moments of odd and even spins of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth chain. One can see that the magnetic moments of even spins show, in general, a behavior similar to the one of a noninteracting spin-1 system with a large easy-axis magnetic anisotropy. However, at temperature of the quasiordering, we see the change of the behavior for large values of the magnetic field, comparing to the single noninteracting spin. The behavior of odd spins manifests the large decrease of the value of M_1 with the growth of the magnetic field and the plateaux with the negative value of the magnetic moment for low values of H , characteristic for

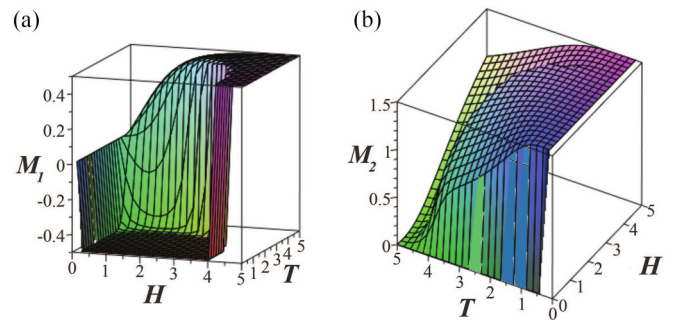


FIG. 5. (a) Temperature and magnetic field dependence of the magnetic moment of odd spins M_1 of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the magnetic moment of even spins M_2 of that chain.

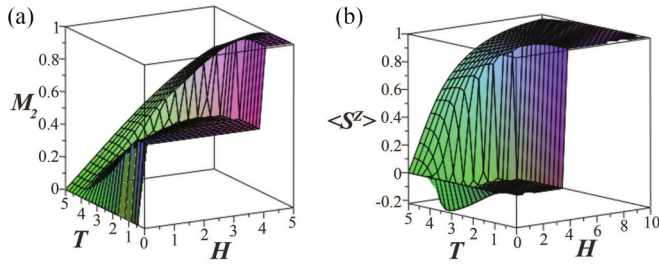


FIG. 6. (a) Temperature and magnetic field dependence of the total magnetic moment $M = (M_1 + M_2)/2$ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy. (b) Temperature and magnetic field dependence of the expectation value of the z -projection of the total spin $\langle S^z \rangle = \langle S_{1,n}^z + S_{2,n}^z \rangle/2$ of that chain.

ferrimagnets with the quasiordering. The most dramatic changes are seen at T below the quasiordering temperature.

In Fig. 6, the temperature and magnetic field dependence of the total magnetic moment $M = (M_1 + M_2)/2$ and the expectation value of the z projection of the total spin $\langle S_{1,n}^z + S_{2,n}^z \rangle/2$ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion anisotropy. It can be seen that the magnetic moment shows the most dramatic changes for temperatures below the quasiordering temperature T_q . In contrast to the one for the magnetic moment M , the expectation value of the projection of the total spin can be negative for positive values of the magnetic field H at nonzero temperatures.

Figure 7 shows the behavior of the magnetic Grüneisen parameter. It manifests the largest values of the magnetocaloric effect for temperatures of order of the quasiordering temperature T_q , unlike Fig. 3 for the antiferromagnetic NN and NNN interactions, where the main features of the magneto-caloric effect are seen at $T = 0$.

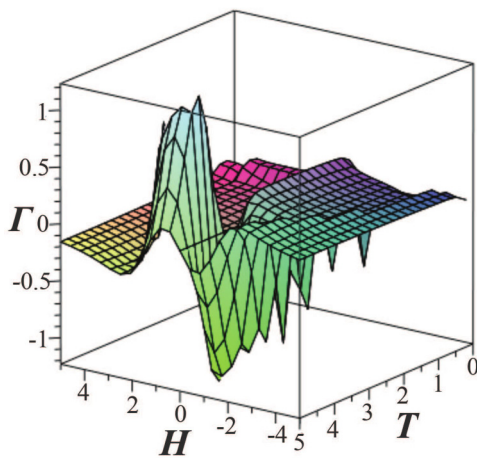


FIG. 7. Magnetic field and temperature dependencies of the magnetic Grüneisen parameter Γ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain with antiferromagnetic NN couplings, ferromagnetic NNN bilinear coupling, and NNN biquadratic interaction; the easy-axis single-ion magnetic anisotropy.

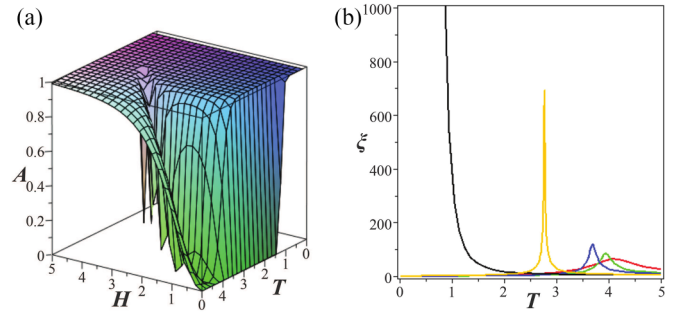


FIG. 8. (a) Temperature and magnetic field dependence of the correlation function of odd spins $A = \langle S_{1,n}^z S_{1,n+x}^z \rangle$ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy for $x = 10^6$. (b) The temperature dependence of the correlation length ξ of that chain for various values of the magnetic field. Red curve: $H = 0.5$; green curve: $H = 2.5$; blue curve: $H = 3$; orange curve: $H = 3.5$; black curve: $H = 4$.

Finally, Fig. 8 presents the results for the temperature and magnetic field dependence of the correlation function of odd spins $A = \langle S_{1,n}^z S_{1,n+x}^z \rangle$ of the Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions and the easy-axis single-ion magnetic anisotropy for $x = 10^6$, and the related correlation length as a function of temperature for several values of the magnetic field. One can see that the correlation length at the quasiordering temperature and for the magnetic field close to the critical value, cf. Figs. 4–7 and 8(a), becomes very large (larger than the interspin distance; it is not shown in the figure to have the possibility to see other curves), and the correlation function almost does not decay for temperatures below the quasiordering temperature T_q , and decays for larger values of the temperature.

C. Easy-plane single-ion anisotropy for the same bilinear spin-spin interactions: Spin ice behavior

Let us change the signs of the single-ion magnetic anisotropy D_1 and D_2 and the NNN biquadratic spin-spin coupling K . Figures 9 and 10 show the typical behavior of thermodynamic characteristics of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain with the same parameters as in Figs. 4–8 but with the different sign of $D_1 = -5$, $D_2 = -4$, i.e., the easy-plane magnetic anisotropy, and the biquadratic NNN interaction $K = -12$.

Figure 9 presents the magnetic field and temperature behavior of the entropy and the total magnetic moment of odd and even spins of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth chain with the easy-plane single-ion anisotropy. It can be seen that the behavior is reminiscent of the one shown in Figs. 1 and 3 for the antiferromagnetic NN and NNN interactions: there exists the remnant entropy at $H = 0$ and at quantum critical points. The remnant entropy is related to the high level of degeneracy at those points. Such a behavior is characteristic for spin ices. The behavior of magnetic moments of even and odd spins, as well as the

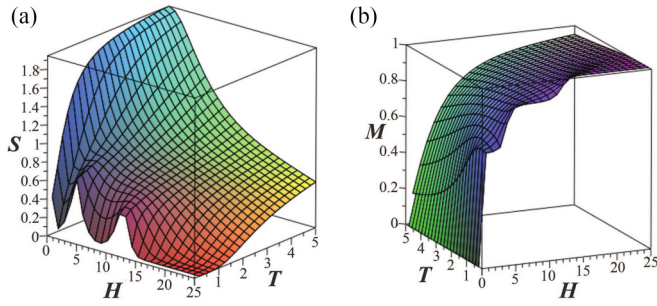


FIG. 9. (a) Temperature and magnetic field dependence of the entropy of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain for the antiferromagnetic NN and ferromagnetic NNN bilinear spin-spin interactions, the biquadratic large negative NNN coupling $K < 0$, and the easy-plane single-ion magnetic anisotropy. (b) The temperature and magnetic field dependence of the total magnetic moment M of that chain.

behavior of the magnetic Grüneisen parameter, is qualitatively similar to the ones shown in Figs. 2 and 3.

Finally, Fig. 10 shows the behavior of the correlation length ξ .

One can see that the correlation length is small in that case. Correlation function decays exponentially with the distance between odd spins.

D. Conditions for the remnant entropy and the quasiordering

Analyzing the temperature behavior of the entropy, we get the following results after variation of parameters of the Hamiltonian.

First, for the absence of biquadratic couplings, quasiordering exists for the antiferromagnetic NN bilinear spin-spin interaction J_1 for the ferromagnetic NNN bilinear coupling for any sign of the (relatively small for the easy-plane situation) single-ion anisotropy. In the case of quasiordering correlation

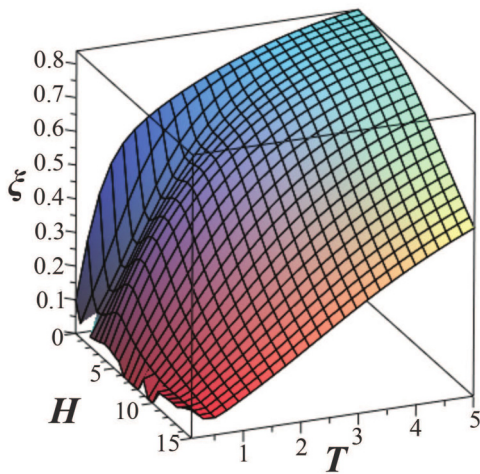


FIG. 10. Magnetic field and temperature dependencies of the correlation length ξ of the ferrimagnetic Blume-Emery-Griffiths sawtooth chain with antiferromagnetic NN couplings, ferromagnetic NNN bilinear coupling, and antiferromagnetic large NNN biquadratic interaction for the easy-plane single-ion magnetic anisotropy.

function of odd spins weakly decays for $T < T_q$ and strongly decays for $T > T_q$. Magnetic field behavior of magnetic moments manifests nonmonotonic behavior for $T < T_q$. The magnetocaloric effect is maximal in the vicinity of T_q . Second, nonzero positive (ferromagnetic) values of the biquadratic couplings K_1 and K do not produce principal changes into that picture, while large antiferromagnetic (negative) biquadratic couplings can reduce the existence of the quasiordering.

To study the spin ice behavior, we can write the eigenvalues of each configuration of spins of each triangle with $S_{1,n}^z$, $S_{2,n}^z$, and $S_{1,n+1}^z$ of the sawtooth chain for $H = 0$. Let us denote eigenstates of each configuration of spins of that triangle as $(s_{n,1}, s_{2,n}, s_{1,n+1})$. There exist 27 configurations. The one with $(0,0,0)$ has zero energy per spin, and it is nondegenerate. Two configurations with $(0, \pm 1, 0)$ are doubly degenerate with the energies per spin $-D_2/3$. Four configurations with $(\pm 1, 0, 0)$, $(0, 0, \pm 1)$ have the energy per spin $-(K_1/3) - (D_1/6)$ (they are fourfold degenerate). Four configurations with $(\pm 1, \pm 1, 0)$, $(0, \pm 1, \pm 1)$ have the energy per spin $-(K_1/3) - (D_1/6) - (D_2/3) - (J_1/3)$ (they are also fourfold degenerate). Four configurations with $(\pm 1, \mp 1, 0)$, $(0, \mp 1, \pm 1)$ have the energy per spin $-(K_1/3) - (D_1/6) - (D_2/3) + (J_1/3)$ and they are fourfold degenerate. Two configurations $(\pm 1, 0, \pm 1)$ have the energy per spin $-(2K_1/3) - (K/3) - (J/3) - (D_1/3)$ are twofold degenerate. Two configurations $(\pm 1, 0, \mp 1)$ have the energy per spin $-(2K_1/3) - (K/2) + (J/2) - (D_1/2)$ are twofold degenerate. Two configurations $(\pm 1, \pm 1, \pm 1)$ have the energy per spin $-(2K_1/3) - (K/3) - (J/3) - (D_1/3) - (2J_1/3)$ and are also twofold degenerate. Two configurations $(\pm 1, \mp 1, \pm 1)$ have the energy per spin $-(2K_1/3) - (K/3) - (J/3) - (D_1/3) + (2J_1/3)$ and are twofold degenerate too. Finally, four configurations with $(\pm 1, \pm 1, \mp 1)$ have the energy per spin $-(2K_1/3) - (K/3) + (J/3) - (D_1/3) - (D_2/3)$ and are fourfold degenerate. Since $S_{1,n}^z$, $S_{1,n+1}^z$, and $S_{2,n}^z$ commute for any n , the (macroscopic) degeneracies of most configurations of the sawtooth chain at $H = 0$ are thermodynamically large. We see that the degeneracy of the ground state configuration at $H = 0$ depends on the relative values of the parameters of the Hamiltonian (bilinear and biquadratic couplings and the single-ion anisotropy). Only one ground state, with z projections of all spins of the chain being zeros, is not macroscopically degenerate at $H = 0$. It is related to the very large easy-plane value of the single-ion anisotropy. The remnant entropy can be related to the degeneracy Ω of the ground state as $S(H = 0)|_{T \rightarrow +0} = k_B \ln \Omega$. It can be written from the general expression for the entropy in the limit $T \rightarrow 0$. The easy-axis single ion magnetic anisotropy makes the energies of configurations (which depend on D_1 and D_2) lower, while the easy-plane anisotropy makes the energies higher. One can also see that the ferromagnetic (positive) biquadratic coupling makes energies of configurations with $s_{1,2,n,n+1} = \pm 1$ lower, and the antiferromagnetic (negative) makes them higher comparing to the case of the Blume-Capel sawtooth chain, cf. [51]. It is interesting that only several (degenerate) ground state configurations depend on the NN and NNN bilinear couplings J and J_1 . However, which configuration is realized in the ground state depends on the relative values of $D_{1,2}$, J , J_1 , K , and K_1 . Analyzing the temperature behavior of the entropy we see that the remnant entropy at $H = 0$, characteristic for spin

ices, takes place for NNN bilinear spin-spin coupling $J > 0$ for the easy-plane single-ion anisotropy, and for $J < 0$ for the easy-axis anisotropy.

Similar analysis can be performed for the critical values of the magnetic field in the ground state, at which quantum phase transitions take place. For those values of H , the ground state is also macroscopically degenerate.

It is important that for some ranges of parameters, the sawtooth ferrimagnetic Blume-Emery-Griffiths chain can manifest the quasiordering together with the spin-ice behavior. For example, it happens for antiferromagnetic NN and ferromagnetic NNN bilinear coupling for a weak easy-plane single-ion anisotropy in the absence of biquadratic couplings or for positive biquadratic couplings.

E. Analytic results

We have obtained some analytic results in the limiting cases, which show why such a behavior of the ferrimagnetic Blume-Emery-Griffiths model on the sawtooth chain takes place.

The case of the large easy-axis magnetic anisotropy of odd spins or large positive biquadratic spin-spin couplings K_1 or K permits some analytic results. It is possible to show analytically that in this situation quasiordering can take place for the ferromagnetic NNN coupling $J > 0$, with the jumplike behavior of the entropy at the quasicritical temperature T_q . It is possible to show analytically that quasiordering corresponds to almost nondecaying spin-spin correlation function for odd spins at temperatures below the quasiordering one, while for larger temperatures the correlation function decays exponentially with distance between spins. Also, the behavior of even and odd spins in the chain manifests some interesting features for temperatures below the quasicritical one.

For large values of the easy-axis magnetic anisotropy constant $D_1 > 0$ of odd spins or large biquadratic positive exchange parameters K_1 or K for the fixed temperature, the transfer matrix of the system can be reduced (we used the smallness of some elements of the original transfer matrix for that case) to the effective 2×2 matrix $\hat{\tau}_a$ with the elements

$$\begin{aligned}\tau_{a11} &= e^{(D_1/T)+(2K_1/T)+(K/T)+(J/T)+(g_1H/T)} [1 \\ &\quad + 2e^{D_2/T} \cosh[(2J_1 + g_2H)/T]], \\ \tau_{a12} &= \tau_{a21} = e^{(D_1/T)+(2K_1/T)+(K/T)-(J/T)} [1 \\ &\quad + 2e^{(D_2/T)} \cosh(g_2H/T)], \\ \tau_{a22} &= e^{(D_1/T)+(2K_1/T)+(K/T)+(J/T)-(g_1H/T)} [1 \\ &\quad + 2e^{D_2/T} \cosh[(2J_1 - g_2H)/T]].\end{aligned}\quad (10)$$

The reduced matrix can be written as

$$\hat{\tau}_a = \frac{\tau_{a11} + \tau_{a22}}{2} \times I + \frac{\tau_{a11} - \tau_{a22}}{2} \times \sigma^z + \tau_{a12} \times \sigma^x, \quad (11)$$

where I is the unity matrix and $\sigma^{x,z}$ are Pauli matrices. Here we used the property of the studied systems $\tau_{a12} = \tau_{a21}$. The diagonalization of the expression (11) is trivial using the rotation in the $\sigma^{x,z}$ space around the y axis. We get for the eigenvalues of the transfer matrix

$$\tau_{a1,2} = \frac{\tau_{a11} + \tau_{a22}}{2} \pm \sqrt{\left(\frac{\tau_{a11} - \tau_{a22}}{2}\right)^2 + \tau_{a12}^2}. \quad (12)$$

The effective Helmholtz free energy of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth chain per spin for that case can be approximately written as $f \approx -T \ln(\tau_{a1})$. The parameters D_1 , K_1 , and K trivially change the ground state energy. All thermodynamic characteristics of the chain in this limit can be obtained by the differentiation of the effective free energy. For the mentioned large parameters using the above mentioned rotation in the σ space in the thermodynamic limit, we can write

$$\begin{aligned}\langle S_{1,n}^z S_{1,n+x}^z \rangle &\approx \frac{1}{\left(\frac{\tau_{a11} - \tau_{a22}}{2}\right)^2 + \tau_{a12}^2} \\ &\times \left[\left(\frac{\tau_{a11} - \tau_{a22}}{2}\right)^2 + \left(\frac{\tau_{a2}}{\tau_{a1}}\right)^x \tau_{a12}^2 \right].\end{aligned}\quad (13)$$

Obviously, the effective correlation length for that correlation function is

$$\xi \approx -\frac{1}{\ln(\tau_{a2}/\tau_{a1})}. \quad (14)$$

Let us assume antiferromagnetic bilinear NN coupling $J_1 < 0$ and $g_2 > g_1$. For the ferromagnetic NNN coupling $J > 0$, for very large values of J at the fixed temperature, one can neglect the term τ_{a12}^2 comparing to $(\tau_{a11} - \tau_{a22})^2/4$. Then, in this case, the partition function has a feature at $\tau_{a11} = \tau_{a22}$. The latter defines the quasicritical value of the temperature T_q :

$$\begin{aligned}e^{(2g_1H/T_q)} [1 + 2e^{(D_2/T_q)} \cosh[(2J_1 + g_2H)/T_q]] \\ = [1 + 2e^{(D_2/T_q)} \cosh[(2J_1 - g_2H)/T_q]].\end{aligned}\quad (15)$$

Note that the onset of the quasicritical behavior is not determined by the single-ion magnetic anisotropy of even spins (see below, though, the case of large negative D_2). In the vicinity of the quasicritical temperature, the entropy of the system manifests the strong increase. That increase implies the maximum of the specific heat in that vicinity of T_q . Note that at the value $T_q \neq 0$, the entropy of the model does not manifest the real jump (as in the case of the real phase transition). The change of the entropy is smooth, however, it is maximal in the region $T_q - \delta T, T_q + \delta T$. This is why the specific heat is not diverging at $T = T_q$. Thus, the behavior of thermodynamic characteristics of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth chain is only reminiscent of ones at the phase transition and may be called the quasicritical one. It becomes critical in the limit $J \rightarrow \infty$.

The width $2\delta T$ of the temperature crossover region around the quasicritical temperature ($T_q - \delta T, T_q + \delta T$) is determined from the condition $|\tau_{a11} - \tau_{a22}| = 2\tau_{a12}$ at $T_q \pm \delta T$. For the fixed value of T_q , the width δT decreases with the growth of the NNN bilinear ferromagnetic coupling J as $\exp(-2J/T_q)$.

For the large easy-axis magnetic anisotropy of even spins D_2 , one can neglect unities in the expressions (10) and (15). It yields the analytical expression for the quasicritical value of the field $H_q = -2J_1/g_2$, and the quasicritical temperature is determined from

$$e^{-(2g_2H/T_q)} = -\frac{\sinh[(g_1H + 2J_1)/T_q]}{\sinh[(g_1H - 2J_1)/T_q]} \quad (16)$$

or

$$e^{(4J_1/T_q)} = \frac{\sinh[(g_2 - g_1)H/T_q]}{\sinh[(g_1 + g_1)H/T_q]}. \quad (17)$$

For small values of the magnetic field, it produces $T_q \approx 4J_1 / \ln[(g_2 - g_1)/(g_2 + g_1)]$. On the other hand, for the large easy-plane magnetic anisotropy of even spins $D_2 < 0$, one has $T_q \rightarrow \infty$ at the finite values of the magnetic field. The width of the region of the quasiordering can be written as

$$\begin{aligned} \delta T &\approx - \frac{T_q^2 e^{-(2J_{\text{eff}}/T_q)}}{g_1 \cosh(H_{\text{eff}}/2T_q) [H_{\text{eff}} - T \frac{\partial H_{\text{eff}}}{\partial T}]} \Big|_{T=T_q} \\ &\sim \frac{T_q e^{-2J_{\text{eff}}/T_q}}{g_1} \left[\frac{\partial H_{\text{eff}}}{\partial T} \right]^{-1} \Big|_{T=T_q}, \end{aligned} \quad (18)$$

where

$$\begin{aligned} J_{\text{eff}} &= J + \frac{T}{4} \ln(\tau_{a11} \tau_{a22} / \tau_{a12}^2), \\ H_{\text{eff}} &= H + \frac{T}{2g_1} \ln(\tau_{a11} / \tau_{a22}). \end{aligned} \quad (19)$$

One can see that the quasiordering temperature is determined mostly by the NN bilinear spin-spin interaction, while the narrow width of the temperature region, in which the quasiordering takes place, is defined mostly by the NNN bilinear coupling.

Our results correlate with recent ones for the analysis of quasycritical temperatures associated with nearly reducible transfer matrices [52,53].

For $\tau_{a12}^2 \ll (\tau_{a11} - \tau_{a22})^2/4$, the correlation function of odd spins almost does not decay with the distance between spins, and it decays exponentially with x for other values of T , see Eq. (13). This is why the quasiorder parameter is the value of the z component of the odd spin moment.

Some analytical results can be also obtained for the case of the large easy-plane magnetic anisotropy $D_1 < 0$ of odd spins, or large negative be-quadratic spin-spin couplings $K, K_1 < 0$ too. For the large negative values of those parameters with the temperature fixed, the transfer matrix of the considered ferrimagnetic Blume-Emery-Griffiths sawtooth model can be also reduced to the effective ones. One of the eigenvalues of the reduced 3×3 transfer matrix becomes zero. Other two values are

$$\tau_{p1,2} = \frac{\tau_{p22}}{2} \pm \left(\frac{\tau_{p22}^2}{4} + \tau_{p12}^2 + \tau_{p23}^2 \right)^{1/2}, \quad (20)$$

with nonzero components of the reduced transfer matrix

$$\begin{aligned} \tau_{p12} &= \tau_{p21} = e^{(K_1/T) + (D_1/2T) + (g_1 H/2T)} [1 \\ &\quad + 2e^{D_2/T} \cosh([J_1 + g_2 H]/T)], \\ \tau_{p22} &= 1 + 2e^{D_2/T} \cosh(g_2 H/T), \\ \tau_{p23} &= \tau_{p32} = e^{(K_1/T) + (D_1/2T) - (g_1 H/2T)} [1 \\ &\quad + 2e^{D_2/T} \cosh([J_1 - g_2 H]/T)]. \end{aligned} \quad (21)$$

In the main approximation of large negative D_1 , or K , or K_1 , one has $\tau_{p1} \approx \tau_{p22}$ and $\tau_{p2} = 0$. That approximation describes

the noninteracting even spins 1. The next correction yields the effective free energy of the considered ferrimagnetic Blume-Emery-Griffiths model as $f \approx -T \ln(\tau_{p1})$. One can see that in the limit of the large easy-plane magnetic anisotropy of odd spins or large negative biquadratic couplings, the effective free energy does not depend on the NNN spin-spin bilinear J and biquadratic K couplings.

Physically, one expects that the strong easy-axis single ion anisotropy D_1 of odd spins produces lowest energy levels with z projections of those spins ± 1 much below the ones with 0. This is why such a case is reminiscent of the one for the Ising spins 1/2, and the situation is reduced to the case studied in [17]. However, the effect of the large positive biquadratic spin-spin couplings is totally many-body one in nature.

IV. SUMMARY

In summary, in this work the exactly solvable ferrimagnetic model—the Blume-Emery-Griffiths model on the sawtooth chain with nearest neighbor bilinear and biquadratic couplings, and the next-nearest neighbor bilinear and biquadratic interaction between odd spins, and the single-ion magnetic anisotropy has been studied. The conditions for the spin ice behavior of the remnant entropy caused by the geometrical frustration have been determined. On the other hand, we have shown that for the antiferromagnetic nearest neighbor and ferromagnetic next-nearest-neighbor bilinear spin-spin coupling the model manifests quasiordering of odd spins. That quasiordering reveals itself in the almost distance-independent correlation functions, the jumplike temperature behavior of the entropy, the quasispecularity in the temperature behavior of the specific heat at the quasycritical temperature. It has also been shown that the temperature behaviors of the magnetic moments of even and odd spins, the total magnetization, and the magnetic Grüneisen ratio drastically change their properties at the quasycritical temperature. Obtained results can be used for the theoretical description of the magnetic field and temperature behavior of some rare-earth based olivines [40]. Also, various spin models with Ising-like behavior were used for explanation of some effects in optical lattices [54], electronic circuits [55], and photonic networks [56]. Since quasiordering implies strong features in various characteristics and the position and the width of those features are regulated by different parameters, the application of one-dimensional ferrimagnets, e.g., as sensors in those fields of physics is possible. The magnetocaloric effect is maximal at the quasycritical temperature (if the latter exists), and at $T = 0$ in other cases, therefore one can apply considered ferrimagnetic systems for cooling at the necessary temperature range.

ACKNOWLEDGMENTS

A.A.Z. acknowledges support from the Deutsche Forschungsgemeinschaft via the SFB 1143. The research of A.A.Z. was performed within the program “Ensembles of random matrices, models of quantum multi-level and multi-particle systems, nonlinear integrable equations and abstract evolution equations” of the NAS of Ukraine, the registration number 0126U001858. The research of

G.A.Z. was performed within the program “Conductive, superconducting, magnetic and sensory properties of novel functional materials” of the NAS of Ukraine, the registration number 0122U001501.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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