

Quadrature for the generalized hydrodynamics equation and absence of shocks in the Lieb-Liniger model

Friedrich Hübner[✉] and Benjamin Doyon[✉]

Department of Mathematics, *King's College London*, Strand, London WC2R 2LS, United Kingdom



(Received 7 November 2024; accepted 26 September 2025; published 1 December 2025)

In conventional fluids, it is well known that Euler-scale equations are plagued by ambiguities and instabilities. In one dimension smooth initial conditions may develop shocks, and weak solutions, such as for domain wall initial conditions (the paradigmatic Riemann problem of hydrodynamics), are not unique (and must be supplemented by physically motivated conditions such as non-negative entropy production). The absence of shock formation experimentally observed in quasi-one-dimensional cold-atomic gases, which are described by the Lieb-Liniger model, provides perhaps the strongest pointer to a modification of the hydrodynamic equation due to integrability. Generalized hydrodynamics (GHD) is the required hydrodynamic theory, taking into account the infinite number of conserved quantities afforded by integrability. We provide a new quadrature for the GHD equation—a solution in terms of a Banach fixed-point problem where time has been explicitly integrated. The quadrature is an efficient numerical solution tool; and it allows us, in the Lieb-Liniger model, to rigorously show that no shock may appear at all times, and, when combined with recent hydrodynamic fluctuation theories, to obtain new expressions for correlations in nonstationary states, establishing for the first time the presence of discontinuities that are fundamental to the nonequilibrium physics of integrable systems.

DOI: [10.1103/5w4p-bxnt](https://doi.org/10.1103/5w4p-bxnt)

Introduction. In recent years there has been an increased interest in studying one of the oldest and most powerful frameworks for many-body systems out of equilibrium: that of hydrodynamics. It captures the large space-time behavior of local observables, and is built out of the local conservation laws admitted by the system, and the thermodynamic equations of state. While in most physical systems there are only a few local conservation laws (such as particle number, momentum, and energy), one-dimensional *integrable systems* admit an infinite number. A new universality class for their hydrodynamic equations has been uncovered, referred to as generalized hydrodynamics (GHD) [1–3]. Most notably, GHD provides the relevant equation for the large-scale dynamics of quasi-one-dimensional cold atomic gases realized in experiments [4–7] and described by the Lieb-Liniger model [8], as reviewed in [9]. It is also the hydrodynamic theory underlying kinetic equations of soliton gases observed in optic fibers and water tanks [10]. GHD applies to almost every known many-body integrable system (spin chains, field theories, classical and quantum gases, etc.) [11–14].

The Euler-scale dynamics of integrable models is described by a “quasiparticle” density in phase space, $\rho(t, x, \lambda)$, which encodes the densities of all conservation laws, typically including the particle, momentum, and energy densities. The variable λ parametrizes a *continuum* of hydrodynamic modes. The quasiparticle density has a simple physical definition:

in the gas, take out a mesoscopic-length fluid cell element $[x, x + dx]$, put it in the vacuum, and let it expand for a long time. Then the quantity $\rho(t, x, \lambda)dx d\lambda$ is the number of particles (or more generally asymptotic objects, such as solitons, waves, etc.) observed with asymptotic momenta lying in $[\lambda, \lambda + d\lambda]$. This process is experimentally realizable [7].

Finding an efficient expression for $\rho(t, x, \lambda)$ that solves the GHD equations without the need for finite-element analysis is however still an open problem. This is important both for numerical purposes, and in order to establish fundamental properties of GHD, including the behavior of $\rho(t, x, \lambda)$ itself, and of its correlations via the fluctuation transport principles of Ballistic Macroscopic Fluctuation Theory [15,16]. In particular, a drastic phenomenological difference between solutions to conventional hydrodynamic (CHD) equations for nonintegrable systems, and GHD, is the appearance of shocks. Early attempts to explain experiments on quasi-one-dimensional cold atomic gases by using CHD failed dramatically because in CHD, typically, shocks develop in finite time. Yet experimentally, the gas appears to be spatially smooth at all times. In GHD, it was indeed observed that shocks do not develop [17]. In fact, finite-component reductions of GHD have the property of “linearly degeneracy” [18,19], known to prevent the appearance of shocks [20–22], and a continuum analog of this property holds for GHD [11]. Shocks encode physics beyond the Euler scale, and break uniqueness of solutions, requiring physically motivated entropy-production conditions [23]. This is clearest in the Riemann problem of hydrodynamics: a domain-wall initial condition (“partitioning protocol”) is not a well-defined initial-value problem in CHD. By contrast GHD solutions appear to be well defined, composed of a continuum of contact

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

singularities [1], which are entropy-preserving. The appearance of shocks and entropy production at the Euler scale is a fundamental aspect of the large-scale physics of conventional systems, and their observed absence, the starkest signature of integrability. Given that, it is important to show, beyond any doubt, that GHD indeed does not develop shocks and is well-posed even for the Riemann problem.

In this paper, we introduce a new solution method for the GHD equation, a “quadrature” presented as integral equations where the space-time variables have been explicitly integrated, which significantly improves on earlier quadratures [24,25]. The new quadrature has both practical and conceptual importance. First, it gives a new, numerically efficient *solution algorithm* for GHD, which, we believe, surpasses previous algorithms. Second, we explain how it gives rise to a *rigorous proof* (carried out in our companion paper [26]), in the Lieb-Liniger model, that smooth initial conditions stay smooth at all times—hence no shocks develop—and that weak solutions are unique and entropy-preserving—hence contact singularities give the unique solution to the Riemann problem. Thus, we have established the phenomenology of the GHD equation for cold atomic gases. Third, it also allows us to obtain a new *explicit formula for large-scale correlation functions*. Going beyond previous studies [15,27,28] we establish, from this, that the equal-time two-point correlation functions instantaneously develops a PT-breaking jump at equal points, by contrast to equilibrium correlations. This is a fundamental element of the nonequilibrium dynamics of integrable systems, at the basis of its diffusive-scale corrections [29].

GHD in normal coordinates. The GHD equation [1–3] for single particle type (the framework for many particle types is presented in [11,27], see the SM [30]) is

$$\partial_t \rho(t, x, \lambda) + \partial_x (v^{\text{eff}}(t, x, \lambda) \rho(t, x, \lambda)) = 0, \quad (1)$$

where the so-called effective velocity $v^{\text{eff}}(t, x, \lambda)$ of the quasiparticles satisfies the following self-consistency equation:

$$v^{\text{eff}}(t, x, \lambda) = v(\lambda) + \int d\mu \varphi(\lambda, \mu) \rho(t, x, \mu) \times (v^{\text{eff}}(t, x, \mu) - v^{\text{eff}}(t, x, \lambda)). \quad (2)$$

The intuition behind this equation is illustrated in Fig. 1. Each quasiparticle evolves with its own bare (or asymptotic) velocity $v(\lambda)$. During the scattering of a particle with momentum λ with a particle with momentum μ , effectively, the particle trajectory is displaced by a semiclassical spatial shift $\varphi(\lambda, \mu)$, similar to two-soliton scattering of integrable PDEs. The effective velocity is the bare velocity plus the contribution from the scattering with other particles, which is proportional to the scattering shift $\varphi(\lambda, \mu)$, the density $\rho(t, x, \mu)$, and the differences of effective velocities (this is because faster particles scatter with more particles in the same unit time). In the Lieb-Liniger model, with quantum Hamiltonian $H = \sum_i p_i^2/2 + (c/2) \sum_{i \neq j} \delta(x_i - x_j)$, one finds $v(\lambda) = \lambda$ and $\varphi(\lambda, \mu) = 2c/(c^2 + (\lambda - \mu)^2)$ [8].

Equation (1) may seem rather unwieldy: it is highly nonlinear, and nonlocal in the momentum λ variable, involving the solution to an integral equation. Finite-element numerics is possible (for state of the art solvers see the IFluid project [33]),

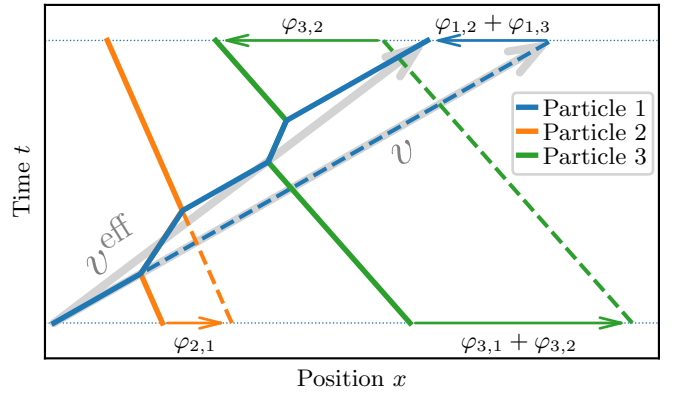


FIG. 1. Illustration of the intuition behind GHD [18,19,31] (here using the so-called semiclassical Bethe models [32]): The quasiparticles (solid lines) obtain a Wigner-time delay during scattering, leading to an effective position shift $\varphi_{ij} = \varphi(\lambda_i, \lambda_j)$. Therefore, the particles effectively move with a different velocity $v^{\text{eff}} \neq v$. The normal coordinate $\hat{x}$ (dashed lines) is obtained by adding the scattering shifts of all particles to their left to their position x , which in the continuum limit becomes (3). In these new coordinates the evolution is noninteracting with velocity $v(\lambda_i)$.

but more efficient solution schemes are required to obtain a better understanding.

A fundamental property of GHD was established in [24], at the basis of its physics. Consider the rapidity- and state-dependent coordinate change $x \mapsto \hat{x} = \hat{X}(t, x, \lambda)$,

$$\hat{X}(t, x, \lambda) = x + \int_{-\infty}^x dy \int d\mu \varphi(\lambda, \mu) \rho(t, y, \mu). \quad (3)$$

Then the quasiparticle density in the new coordinates,

$$\hat{\rho}(t, \hat{X}(t, x, \lambda), \lambda) d\hat{x} = \rho(t, x, \lambda) dx, \quad (4)$$

satisfies the *free-particle kinetic equation*

$$\partial_t \hat{\rho}(t, \hat{x}, \lambda) + v(\lambda) \partial_{\hat{x}} \hat{\rho}(t, \hat{x}, \lambda) = 0. \quad (5)$$

Physically, the “normal coordinate” $\hat{x}$ is a position that takes into account the “density of available space” (see Fig. 1). Indeed, the quantity

$$2\pi \rho_s(t, x, \lambda) = \frac{d\hat{X}(t, x, \lambda)}{dx} = 1 + \int d\mu \varphi(\lambda, \mu) \rho(t, x, \mu) \quad (6)$$

has the interpretation as a (non-negative) density of the space available for particles to travel through freely [24] (the factor 2π is conventional). This is similar to the metric in general relativity, which changes the physical space based on the amount of mass present. In the $\hat{x}$ space trajectories of particles are linear. The map (3) is a generalization of the “excluded volume” technique used in the hard-rod gas [34] and extended and used in other contexts [32,35–38], and is fundamentally related to the factorized scattering property of integrable systems [11].

Quadratures. In [24] this was used to find a quadrature for the GHD equations, where time t is essentially integrated. Indeed, Eq. (5) is solved trivially, $\hat{\rho}(t, \hat{x}, \lambda) = \hat{\rho}(0, \hat{x} - v(\lambda)t, \lambda)$; and $\hat{\rho}(t = 0, \hat{x}, \lambda)$ can be computed directly from $\rho(t = 0, x, \lambda)$. Thus, $\hat{\rho}(t, \hat{x}, \lambda)$ is known at any

time t and one only need solve for $\rho(t, x, \lambda)$ in (3) and (4). An algorithm for this was proposed and illustrated in [24,27,28].

Such quadratures are powerful techniques: in order to compute the solution at time t one does not need to compute the solution at any other time, and thus, by contrast to finite-element analysis, numerical inaccuracies introduced by finite-time elements $dt \rightarrow \Delta t$ disappear, and the computational effort to obtain the solution remains constant as time increases. Simulating long time dynamics, which is crucial for hydrodynamic phenomenology, becomes feasible, and one may then gain analytical control over general properties of solutions such as stability, as well as efficient expressions for correlations by linear response or fluctuation theories [15,16]. However, the algorithm proposed in [24] is time consuming and numerically delicate (see [28]), involving whole-space numerical integrations, and its mathematical structure is opaque.

The new quadrature. In this work we present a powerful new quadrature. It allows us to solve both the conceptual and practical problems discussed above: to find the solution directly at any fixed space-time point (t, x) , giving an efficient solution algorithm; to show, rigorously, the absence of shocks and the uniqueness of solutions at all times; and to uncover fundamental properties of correlations out of equilibrium hitherto unobserved. The idea is to introduce height-fields (for now we assume for simplicity that ρ and $\hat{\rho}$ decay sufficiently fast as $x, \hat{x} \rightarrow -\infty$)

$$\begin{aligned}\Psi(t, x, \lambda) &= \int_{-\infty}^x dy \rho(t, y, \lambda) \\ \hat{\Psi}(t, \hat{x}, \lambda) &= \int_{-\infty}^{\hat{x}} d\hat{y} \hat{\rho}(t, \hat{y}, \lambda),\end{aligned}\quad (7)$$

which satisfy the equations

$$\partial_t \Psi(t, x, \lambda) + v^{\text{eff}}(t, x, \lambda) \partial_x \Psi(t, x, \lambda) = 0 \quad (8)$$

$$\partial_t \hat{\Psi}(t, \hat{x}, \lambda) + v(\lambda) \partial_{\hat{x}} \hat{\Psi}(t, \hat{x}, \lambda) = 0. \quad (9)$$

Note that (8) in the usual hydrodynamic terminology means that $\Psi(t, x, \lambda)$ is a Riemann invariant (there are many more such Riemann invariants, see SM [30]). From (9) one can again solve explicitly $\hat{\Psi}(t, \hat{x}, \lambda) = \hat{\Psi}(0, \hat{x} - v(\lambda)t, \lambda)$. Furthermore, one can show from (4) that $\hat{\Psi}(t, \hat{X}(t, x, \lambda), \lambda) = \Psi(t, x, \lambda)$. Using this and (3) we find

$$\Psi(t, x, \lambda) = \hat{\Psi}\left(0, x - v(\lambda)t + \int d\mu \varphi(\lambda, \mu) \Psi(t, x, \mu), \lambda\right). \quad (10)$$

Alternatively we can plug this into (3) and find

$$\hat{X}(t, x, \lambda) = x + \int d\mu \varphi(\lambda, \mu) \hat{\Psi}(0, \hat{X}(t, x, \mu) - v(\mu)t, \mu). \quad (11)$$

(10) and (11) are “self-consistent” fixed-point equations for $\Psi(t, x, \lambda)$ and $\hat{X}(t, x, \lambda)$, respectively. Observe that they fully decouple for different t, x . Solutions to such equations can be obtained for instance by fixed-point iteration, for which strong results on convergence are known (see below). Once a solution is found, one can either compute the solution from (10) via $\rho(t, x, \lambda) = \partial_x \Psi(t, x, \lambda)$ or from (11) by first computing

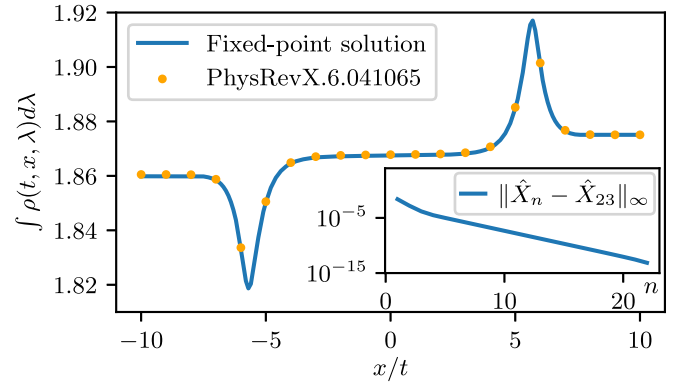


FIG. 2. Momentum-integrated particle density profile of a partitioning protocol in the Lieb-Liniger model. The initial state and model parameters correspond to Fig. 8 in [1]. The result obtained via fixed-point iteration (solid line) of (11) agrees with the solution given in [1] (dots). The inset shows the error of $\hat{X}(t=1, x=0, \lambda)$ after the n -th iteration compared to the final result approximated by taking $n=23$ steps; we see the exponential convergence.

the so-called occupation function

$$n(t, x, \lambda) = 2\pi \hat{\rho}(0, \hat{X}(t, x, \lambda) - v(\lambda)t, \lambda). \quad (12)$$

By taking a derivative of (10) we find

$$\rho(t, x, \lambda) = n(t, x, \lambda) \left(1 + \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu) \rho(t, x, \mu)\right). \quad (13)$$

This linear equation can either be solved again by fixed-point iteration, or by using a linear solver. Therefore, we conclude the following:

Result 1. By solving first (11) and then (13) one is able to find the solution to the GHD equation directly at an arbitrary space-time point t, x . It is only required to solve self-consistency equations in the remaining degree of freedom λ , and these equations have clear convergence properties.

We demonstrate that this new efficient algorithm works in practice by reproducing a simulation of a previous publication [1], see Fig. 2.

Absence of shocks, existence, and uniqueness. It turns out that our new fixed-point approach is extremely useful beyond providing a new efficient algorithm. Using the example of the repulsive Lieb-Liniger model, one can show that the GHD equation (1) has a unique solution, which does not develop shocks.

Recall that the quasiparticles of the Lieb-Liniger model have fermionic statistics, so that $0 \leq n(t, x, \lambda) \leq 1$ (the occupation function can be interpreted as the average occupation of quantum numbers). For a simpler expression of the results, we exclude $n(t, x, \lambda) = 1$ here; we discuss it in the appendix. Using (12) it follows $\hat{\rho}(t, x, \lambda) < 1/2\pi$, from which we find $|\hat{\Psi}(0, \hat{x}, \lambda) - \hat{\Psi}(0, \hat{y}, \lambda)| < |\hat{x} - \hat{y}|/(2\pi)$. Furthermore, one can explicitly compute $\int d\mu \varphi(\lambda, \mu) = 2\pi$.

Using these facts we can obtain the following result:

Result 2. In the repulsive Lieb-Liniger model the solution to the GHD equation exists and is unique for all times t (see appendix for detailed statements).

To see this, denote the fixed-point map associated to (11) by $C_{t,x}[f] = x + \int d\mu \varphi(\lambda, \mu) \hat{\Psi}(0, f(\mu) - v(\mu)t, \mu)$. Now

observe that in the supremum norm $\|f\|_\infty = \sup_\lambda |f(\lambda)|$ for any two functions $f_1(\lambda)$ and $f_2(\lambda)$:

$$\begin{aligned} & \|C_{t,x}[f_1] - C_{t,x}[f_2]\|_\infty \\ & \leq \int d\mu \varphi(\lambda, \mu) |\hat{\Psi}(0, f_1(\mu) - v(\mu)t, \mu) \\ & \quad - \hat{\Psi}(0, f_2(\mu) - v(\mu)t, \mu)| \end{aligned} \quad (14)$$

$$< \frac{1}{2\pi} \left(\int d\mu \varphi(\lambda, \mu) \right) \|f_1(\mu) - f_2(\mu)\|_\infty \quad (15)$$

$$= \|f_1(\mu) - f_2(\mu)\|_\infty. \quad (16)$$

This is the definition of a contracting fixed-point map. The Banach fixed-point theorem states that a contracting fixed-point map has a unique fixed-point, from which we conclude that $C_{t,x}[f]$ has only one fixed-point $f(\lambda) = \hat{X}(t, x, \lambda)$. Because $\hat{X}(t, x, \lambda)$ is unique, $\Psi(t, x, \lambda)$ is as well and thus $\rho(t, x, \lambda) = \partial_x \Psi(t, x, \lambda)$ is the unique solution to the GHD equation. By solving (13) one can then establish that $\rho(t, x, \lambda)$ exists as a bounded function. But now note the following: Since $\rho(t, x, \lambda)$ is bounded, its height-field $\Psi(t, x, \lambda)$ is differentiable. Assuming that initially $\hat{\rho}(0, x, \lambda)$ is smooth, we find that $n(t, x, \lambda)$ is differentiable and thus $\rho(t, x, \lambda)$ is also differentiable. This then implies that $\Psi(t, x, \lambda)$ is twice differentiable. By iterating this argument we see that $\rho(t, x, \lambda)$ will be a smooth function. We thus have established the following:

Result 3. In the repulsive Lieb-Liniger model, if the initial state $\rho(0, x, \lambda)$ is smooth, then it will remain smooth for all times. Hence, shocks or any other type of gradient singularities are absent (see appendix section for *precise statements*).

The Banach fixed-point theorem also includes another, more practical result: It states that the usual fixed-point iteration algorithm (which was for instance used in Fig. 2) converges exponentially fast.

Result 4. In the repulsive Lieb-Liniger model, the solution to (10) or (11) can be obtained by fixed point iteration from any initial $\Psi(t, x, \lambda)$ or $\hat{X}(t, x, \lambda)$ respectively, which converges exponentially fast.

The arguments can straightforwardly be extended to other models as well. They will also give existence, uniqueness, absence of shock formation, and exponentially fast-fixed point convergence if the initial state satisfies $\hat{\rho}(0, \hat{x}, \lambda) < C/(\sup_\lambda \int d\mu |\varphi(\lambda, \mu)|)$, with $C = 2\pi$ if $\varphi(\lambda) \geq 0$ and $C = \pi$ else. This applies to a large family of initial states that are physically relevant. The idea is more general than this specific construction. Indeed, there are models where this precise approach does not apply, for instance with infinitely-many particle species (e.g., the “strings” in the XXZ chain [39]), where the total phase-shift (generalized to multiple particle species, see SM [30]) diverges $\sum_j \int d\mu |\varphi_{ij}(\lambda, \mu)| = \infty$. But workarounds to get converging fixed-point equations exist following similar strategies, see [40].

Application: Large-scale correlations. A crucial application of exact solutions is to understand the dynamics of correlations. Suppose the initial state $\rho(0, x, \lambda)$ is perturbed $\rho(0, x, \lambda) \rightarrow \rho(0, x, \lambda) + \delta\rho(0, x, \lambda)$. The corresponding perturbation of the fixed-point problems (10) and (11) gives the perturbed solution $\rho(t, x, \lambda) \rightarrow \rho(t, x, \lambda) +$

$\delta\rho(t, x, \lambda)$ which satisfies (see the SM [30])

$$\begin{aligned} \delta\rho(t, x, \lambda) &= \hat{\mathbf{G}}_t \delta\rho(0, x, \lambda) \\ &:= \int dy d\mu G_t(x, \lambda|y, \mu) \delta\rho(0, y, \mu) \end{aligned} \quad (17)$$

with the evolution kernel

$$\begin{aligned} G_t(x, \lambda|y, \mu) &= \frac{1}{2} \partial_x \left[\text{sgn}(x - Y(t, y, \lambda)) \right. \\ &\quad \times \left. \left(\delta(\lambda - \mu) - \frac{1}{2\pi} n(t, x, \lambda) \varphi(\lambda, \mu) \right) \right]^{\text{drT}}. \end{aligned} \quad (18)$$

Here, in the standard terminology of GHD $Y(t, x, \lambda) = \hat{X}^{-1}(t, \hat{X}(0, x, \lambda) - v(\lambda)t)$ is the GHD characteristic, i.e., the trajectory of a quasiparticle starting at x with momentum λ . The operation f^{drT} gives the solution to the so-called transposed dressing equation

$$f^{\text{drT}}(\lambda) = f(\lambda) + n(t, x, \lambda) \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu) f^{\text{drT}}(\mu). \quad (19)$$

The evolution kernel, which is interpreted as the response of the system to a perturbation $\delta\rho(0, x, \lambda) = \delta(x - y)\delta(\lambda - \mu)$ had been studied previously [[27], Eq. (3.12)] and computed numerically in [28], but (18) is a much simpler and more compact formula. We provide a plot of $G_t(x, \lambda|y, \mu)$ in Fig. 3 and more details about its structure in the appendix *Further details of the evolution of perturbations*. As correlations $\langle \rho(t, x, \lambda) \rho(s, y, \mu) \rangle^c$ can be viewed as perturbations spreading on top of the GHD evolution, one then gets [15,16,27]

$$\langle \rho(t, x, \lambda) \rho(s, y, \mu) \rangle^c = \langle (\hat{\mathbf{G}}_t \rho(0, x, \lambda)) (\hat{\mathbf{G}}_s \rho(0, y, \mu)) \rangle. \quad (20)$$

The initial state $\langle . . . \rangle$ may or may not be at local equilibrium. In a local equilibrium state, $\langle \rho(0, x, \lambda) \rho(0, y, \mu) \rangle^c = \delta(x - y) \mathbf{C}(x, \lambda, \mu)$ where $\mathbf{C}(x, \lambda, \mu)$ is the local covariance [41]. Using (18), we find that the evolved equal-time correlations have the form (see SM [30], Eq. (41) for the explicit formula)

$$\begin{aligned} & \langle \rho(t, x, \lambda) \rho(t, y, \mu) \rangle^c|_{x \approx y} \\ &= \delta(x - y) \mathbf{C}(x, \lambda, \mu) + \text{sgn}(x - y) \\ &\quad \times (\tilde{\mathbf{C}}(x, \lambda, \mu) \partial_x \rho(t, x, \lambda)) + (\text{continuous}). \end{aligned} \quad (21)$$

Crucially, the so-called long range correlations [15] have a jump at $x = y$ proportional to $\partial_x \rho(t, x, \lambda)$, stemming from the sign function, representing a *local spontaneous breaking of PT symmetry*. Interestingly, this jump develops “instantaneously” (from Euler time $t = 0^+$) [42].

Result 5. The equal time two-point correlation function locally is given by a $\delta(x - y)$ peak representing the local GGE plus a finite part which shows a jump proportional to $\partial_x \rho(t, x, \lambda)$ at $x = y$.

This plays a fundamental role on diffusive effects, as discussed in a separate manuscript [29].

Conclusion. Using “height fields”, we have constructed new Riemann invariants for the GHD equation, the universal equation of Euler-scale hydrodynamics in many-body

integrable systems. From these, we have obtained a new quadrature—integral equations that solve the GHD equation for fixed position and time. First, this provides a powerful numerical solution technique to the GHD equation. Second, it provides deeper insights into the dynamics of integrable models: the fixed-point problem Eq. (11) has a unique solution in the case of the Lieb-Liniger model for all physically relevant states, and solutions to the GHD equation exist and are unique, without shocks, at all times, in stark contrast to conventional Euler hydrodynamics (the proofs are given in [26]). Establishing such properties in Euler hydrodynamics is in general an extremely difficult task, and this is the first result of this kind for the GHD equation. Finally, it allows us to extract a fundamental phenomenological property of correlations out of equilibrium, which is crucial for the understanding of diffusive effects: even if the system is initially in a local equilibrium state, its correlations quickly differ from the equilibrium correlations. We believe the techniques developed here may be used to establish such results in many more models, where one may need other spaces in which (11) (or similar quadratures) have unique solution; investigating soliton gases [10] would be particularly useful, as strong mathematical methods are available. It would be interesting to see whether GHD may indeed fail to have unique solutions, such as in models with infinitely-many particle species; and to generalize the results to the presence of external forces [43] and diffusive [29,44,45] and dispersive [46] corrections. One could also extend our results to higher-order correlations functions, and use it to simplify the framework of ballistic macroscopic fluctuation theory [16] in integrable models. Perhaps Banach-space techniques such as those used here could be employed to study the *emergence* of the GHD equation from microscopic dynamics.

Acknowledgments. F.H. acknowledges funding from the faculty of Natural, Mathematical & Engineering Sciences at King's College London. B.D. was supported by the Engineering and Physical Sciences Research Council (EPSRC) under Grant No. EP/W010194/1. Numerical computations were partially done in Julia [47], in particular using the amazing ApproxFun.jl library [48], and ran on the CREATE cluster [49].

Appendix: Other quadrature formulation. It is instructive to also derive a related quadrature directly from the GHD equation. The GHD equation for $n(t, x, \lambda)$ is given by [1]

$$\partial_t n(t, x, \lambda) + v^{\text{eff}}(t, x, \lambda) \partial_x n(t, x, \lambda) = 0. \quad (\text{A1})$$

An important concept is the dressing equation $f^{\text{dr}}(\lambda)$ [1]:

$$f^{\text{dr}}(t, x, \lambda) = f(\lambda) + \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu) n(t, x, \lambda) f^{\text{dr}}(t, x, \mu). \quad (\text{A2})$$

It can be shown that [1]

$$\begin{aligned} 2\pi \rho_s(t, x, \lambda) &= 1^{\text{dr}}(t, x, \lambda) \\ v^{\text{eff}}(t, x, \lambda) &= v^{\text{dr}}(t, x, \lambda) / 1^{\text{dr}}(t, x, \lambda), \end{aligned} \quad (\text{A3})$$

and furthermore

$$\partial_t 1^{\text{dr}}(t, x, \lambda) + \partial_x v^{\text{dr}}(t, x, \lambda) = 0. \quad (\text{A4})$$

From this we can follow that there exists a function $K(t, x, \lambda)$ s.t.

$$\partial_x K(t, x, \lambda) = 1^{\text{dr}}(t, x, \lambda) \quad \partial_t K(t, x, \lambda) = -v^{\text{dr}}(t, x, \lambda). \quad (\text{A5})$$

Using (A5) and (A4) in (A1) we infer $\partial_x K(t, x, \lambda) \partial_t n(t, x, \lambda) = \partial_t K(t, x, \lambda) \partial_x n(t, x, \lambda)$, which is only possible if a function $\tilde{n}(k, \lambda)$ exists s.t.

$$n(t, x, \lambda) = \tilde{n}(K(t, x, \lambda), \lambda). \quad (\text{A6})$$

Similar to $\hat{\rho}(t, \hat{x}, \lambda)$, $\tilde{n}(k, \lambda)$ can be interpreted as expressing the solution to the GHD equation in ‘spatial coordinate’ $K(t, x, \lambda)$ where the GHD equation trivialities $\partial_t \tilde{n}(k, \lambda) = 0$. In fact, if we choose $K(0, 0, \lambda) = 0$ we can express $K(t, x, \lambda) = \hat{X}(t, x, \lambda) - v(\lambda)t - \hat{X}(0, 0, \lambda)$. Defining $\tilde{N}(k, \lambda) = \int_0^k dk \tilde{n}(k, \lambda)$ and using (A5) we have

$$\begin{aligned} K(t, x, \lambda) &= \int_{(0,0)}^{(t,x)} 1^{\text{dr}} dx - v^{\text{dr}} dt \\ &= x - v(\lambda)t + \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu) \\ &\quad \times \int_{(0,0)}^{(t,x)} n(t, x, \mu) (1^{\text{dr}} dx - v^{\text{dr}} dt) \\ &= x - v(\lambda)t + \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu) \tilde{N}(K(t, x, \mu), \mu) \\ &=: G_{t,x}[K]. \end{aligned} \quad (\text{A7})$$

This is again a fixed-point equation, which can be studied similar to (11). Once $K(t, x, \lambda)$ has been found the solution $n(t, x, \lambda)$ can be obtained via (A6). This formulation has the advantage that it also applies to cases like the partitioning protocol, where $\rho(t, x, \lambda)$ does not vanish as $x \rightarrow -\infty$. Furthermore, we would also like to present the following observation. Under a perturbation of the fixed-point map $G \rightarrow G + \delta G$ the fixed point changes as $K \rightarrow K + \delta K$ with $\delta K = (\delta G)^{\text{dr}}$. This means that the dressing equation is a local version of the fixed-point equation or alternatively the fixed-point equation is a global version of the dressing equation.

Precise statements of existence and uniqueness results. The following results are proven in a separate paper [26], but we would like to present them here for completeness.

We assume the following property of the initial state:

$$\sup_{\lambda} \int d\mu \varphi(\lambda, \mu) \sup_x n(0, x, \mu) < C, \quad (\text{A8})$$

where $C = 2\pi$ if $\varphi(\lambda, \mu) \geq 0$ or $C = \pi$ else. This is slightly weaker than $n(0, x, \mu) < 1/(\sup_{\lambda} \int \frac{d\mu}{2\pi} \varphi(\lambda, \mu))$. In the Lieb-Liniger model it allows us to include all physically relevant states studied until now, zero-entropy states [17]; in particular, $n(0, x, \mu)$ can reach its maximal value 1.

Note that we also allow initial states with discontinuities (this is for instance relevant for the important case of the partitioning protocol). In this case the solution $\rho(t, x, \lambda)$ will not be differentiable and thus (1) does not make sense. It only makes sense in a weak form; for this purpose, we use the

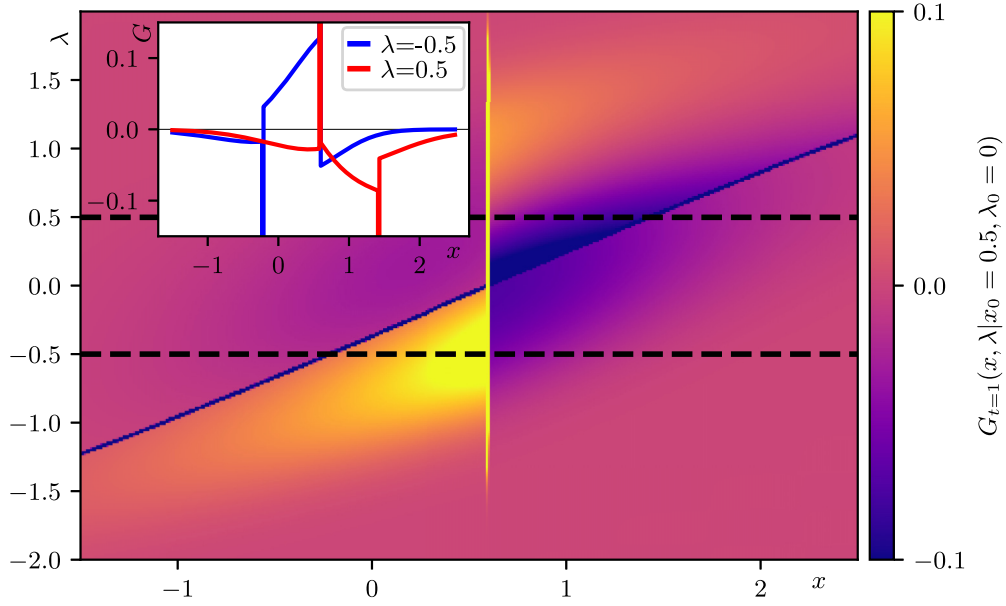


FIG. 3. Solution kernel $G_{t=1}(x, \lambda | x_0 = 0.5, \lambda_0 = 0)$ to the linearized GHD equation of the (repulsive) Lieb-Liniger model $c = 1$ starting from a delta peak at $x_0 = 1, \lambda_0 = 0$ at $t = 0$. The background density at initial time is given by $\rho(t = 0, x, \lambda) = e^{-x^2/2 - \lambda^2/2}/4$. The inset shows cuts at $\lambda = \pm 1/2$ (indicated by dashed lines). The solution kernel has two delta functions in x , a positive one at the position of the GHD characteristic starting at (x_0, λ_0) , which is independent of λ and a negative one at the position of the GHD characteristic starting at (x_0, λ) . At the locations of the δ peaks the nonsingular part of $G(t = 1, x, \lambda | x_0 = 0.5, \lambda_0 = 0)$ has a jump given by (A10).

global form of the conservation law:

$$\begin{aligned} & \int_{x_1}^{x_2} dx (\rho(t_2, x, \lambda) - \rho(t_1, x, \lambda)) \\ &= \int_{t_1}^{t_2} dt (v^{\text{eff}}(t, x_1, \lambda) \rho(t, x_1, \lambda) \\ & \quad - v^{\text{eff}}(t, x_2, \lambda) \rho(t, x_2, \lambda)). \end{aligned} \quad (\text{A9})$$

Then we obtain the following statements:

(1) Existence and uniqueness of weak solutions: If (A8) and $\sup_{x, \lambda} |v(\lambda)n(0, x, \lambda)| < \infty$, then the solution to

(A9) exists and is unique, and satisfies $\sup_{t, x, \lambda} n(t, x, \lambda) = \sup_{x, \lambda} n(0, x, \lambda)$.

(2) Absence of shocks: If further $n(0, x, \lambda)$ is smooth in x and $\sup_{x, \lambda} (1 + |\lambda|^{r+1}) |\partial_x^r n(0, x, \lambda)| < \infty$ for every r , then the (strong) solution to (1) exists, is unique, and is smooth as a function of (t, x) .

As part of this, we have also shown that the solution $v^{\text{eff}}(t, x, \lambda)$ to (2) exists and is unique for all x, t , and that the dressing operation is well defined.

Further details on the evolution of perturbations. It is instructive to extract the singular part from the evolution kernel $G(t, x, \lambda | y, \mu)$:

$$\begin{aligned} G(t, x, \lambda | y, \mu) &= \delta(x - Y(t, y, \mu)) [\delta(\lambda - \mu)]^{\text{drT}} - \delta(x - Y(t, y, \lambda)) \frac{1}{2\pi} n(t, x, \lambda) \varphi(\lambda, \mu) - \frac{1}{2} \text{sgn}(x - Y(t, y, \lambda)) \\ & \quad \times \frac{1}{2\pi} \partial_x n(t, x, \lambda) \varphi(\lambda, \mu) + \frac{1}{2} \text{sgn}(x - Y(t, y, \mu)) [\partial_x n(t, x, \lambda) \hat{\mathbf{T}} [\delta(\lambda - \mu)]^{\text{drT}}]^{\text{drT}} + (\text{continuous}). \end{aligned} \quad (\text{A10})$$

As we can read off, the solution kernel consists of two δ functions with different coefficients that have opposite signs. The positive one is independent of λ and located at the GHD characteristic $Y(t, y, \mu)$ corresponding to μ . In addition there is also a negative δ peak at the characteristic $Y(t, y, \lambda)$ of

a particle with rapidity λ starting at y . In addition there is a background density, which jumps at the location of the δ peaks (also see Fig. 3 where we give a plot of the solution kernel for a specific example). The jump of this nonsingular part is precisely what causes the jump in the evolution of the correlation functions as well.

[1] O. A. Castro-Alvaredo, B. Doyon, and T. Yoshimura, Emergent hydrodynamics in integrable quantum systems out of equilibrium, *Phys. Rev. X* **6**, 041065 (2016).

[2] B. Bertini, M. Collura, J. De Nardis, and M. Fagotti, Transport in out-of-equilibrium XXZ chains: Exact profiles of charges and currents, *Phys. Rev. Lett.* **117**, 207201 (2016).

- [3] G. El, The thermodynamic limit of the Whitham equations, *Phys. Lett. A* **311**, 374 (2003).
- [4] M. Schemmer, I. Bouchoule, B. Doyon, and J. Dubail, Generalized hydrodynamics on an atom chip, *Phys. Rev. Lett.* **122**, 090601 (2019).
- [5] J.-S. Caux, B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Hydrodynamics of the interacting Bose gas in the Quantum Newton Cradle setup, *SciPost Phys.* **6**, 070 (2019).
- [6] F. Møller, C. Li, I. Mazets, H.-P. Stimming, T. Zhou, Z. Zhu, X. Chen, and J. Schmiedmayer, Extension of the generalized hydrodynamics to the dimensional crossover regime, *Phys. Rev. Lett.* **126**, 090602 (2021).
- [7] N. Malvania, Y. Zhang, Y. Le, J. Dubail, M. Rigol, and D. S. Weiss, Generalized hydrodynamics in strongly interacting 1D Bose gases, *Science* **373**, 1129 (2021).
- [8] E. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, *Phys. Rev.* **130**, 1605 (1963).
- [9] I. Bouchoule and J. Dubail, Generalized hydrodynamics in the one-dimensional Bose gas: theory and experiments, *J. Stat. Mech.* (2022) 014003.
- [10] P. Suret, S. Randoux, A. Gelash, D. Agafontsev, B. Doyon, and G. El, Soliton gas: Theory, numerics, and experiments, *Phys. Rev. E* **109**, 061001 (2024).
- [11] B. Doyon, Lecture notes on Generalised Hydrodynamics, *SciPost Phys. Lect. Notes* **18** (2020).
- [12] A. Bastianello, B. Bertini, B. Doyon, and R. Vasseur, Introduction to the special issue on emergent hydrodynamics in integrable many-body systems, *J. Stat. Mech.* (2022) 014001.
- [13] F. H. Essler, A short introduction to Generalized Hydrodynamics, *Physica A* **631**, 127572 (2023), Lecture Notes of the 15th International Summer School of Fundamental Problems in Statistical Physics.
- [14] H. Spohn, *Hydrodynamic Scales of Integrable Many-Body Systems* (World Scientific, Singapore, 2024).
- [15] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, Emergence of hydrodynamic spatial long-range correlations in nonequilibrium many-body systems, *Phys. Rev. Lett.* **131**, 027101 (2023).
- [16] B. Doyon, G. Perfetto, T. Sasamoto, and T. Yoshimura, Ballistic macroscopic fluctuation theory, *SciPost Phys.* **15**, 136 (2023).
- [17] B. Doyon, J. Dubail, R. Konik, and T. Yoshimura, Large-scale description of interacting one-dimensional Bose gases: Generalized hydrodynamics supersedes conventional hydrodynamics, *Phys. Rev. Lett.* **119**, 195301 (2017).
- [18] G. El, A. Kamchatnov, M. V. Pavlov, and S. Zykov, Kinetic equation for a soliton gas and its hydrodynamic reductions, *J. Nonlinear Sci.* **21**, 151 (2011).
- [19] M. V. Pavlov, V. B. Taranov, and G. A. El, Generalized hydrodynamic reductions of the kinetic equation for a soliton gas, *Theor. Math. Phys.* **171**, 675 (2012).
- [20] E. Ferapontov, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, *Phys. Lett. A* **158**, 112 (1991).
- [21] B. L. Rozdestvenskii and A. D. Sidorenko, On the impossibility of ‘gradient catastrophe’ for weakly nonlinear systems, *Comput. Math. & Math. Phys.* **7**, 1176 (1967).
- [22] T. P. Liu, On the impossibility of ‘gradient catastrophe’ for weakly nonlinear systems, *J. Diff. Equ.* **33**, 92 (1979).
- [23] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem*, Vol. 20 (Oxford University Press on Demand, Madrid, 2000).
- [24] B. Doyon, H. Spohn, and T. Yoshimura, A geometric viewpoint on generalized hydrodynamics, *Nucl. Phys. B* **926**, 570 (2018).
- [25] V. B. Bulchandani, R. Vasseur, C. Karrasch, and J. E. Moore, Solvable hydrodynamics of quantum integrable systems, *Phys. Rev. Lett.* **119**, 220604 (2017).
- [26] F. Hübner and B. Doyon, Existence and uniqueness of solutions to the generalized hydrodynamics equation, [arXiv:2411.04922](https://arxiv.org/abs/2411.04922) [math-ph].
- [27] B. Doyon, Exact large-scale correlations in integrable systems out of equilibrium, *SciPost Phys.* **5**, 054 (2018).
- [28] F. S. Møller, G. Perfetto, B. Doyon, and J. Schmiedmayer, Euler-scale dynamical correlations in integrable systems with fluid motion, *SciPost Phys. Core* **3**, 016 (2020).
- [29] F. Hübner, L. Biagetti, J. De Nardis, and B. Doyon, Diffusive hydrodynamics from long-range correlations, *Phys. Rev. Lett.* **134**, 187101 (2025).
- [30] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/5w4p-bxnt> for a discussion of (1) our results with other momentum parametrisations, (2) an application to partitioning protocols, (3) expressions for further Riemann-invariants, (4) equivalence of two fixed-point problems, and (5) a derivation of conservation of total charge and entropy, which includes Refs. [50–54].
- [31] B. Doyon, T. Yoshimura, and J.-S. Caux, Soliton gases and generalized hydrodynamics, *Phys. Rev. Lett.* **120**, 045301 (2018).
- [32] B. Doyon, F. Hübner, and T. Yoshimura, New classical integrable systems from generalized $T\bar{T}$ -deformations, *Phys. Rev. Lett.* **132**, 251602 (2024).
- [33] F. S. Møller and J. Schmiedmayer, Introducing iFluid: a numerical framework for solving hydrodynamical equations in integrable models, *SciPost Phys.* **8**, 041 (2020).
- [34] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer, Berlin, Heidelberg, 1991).
- [35] B. Pozsgay, T. Gombor, A. Hutsalyuk, Y. Jiang, L. Pristýák, and E. Vernier, Integrable spin chain with Hilbert space fragmentation and solvable real-time dynamics, *Phys. Rev. E* **104**, 044106 (2021).
- [36] B. Pozsgay, T. Gombor, and A. Hutsalyuk, Integrable hardrod deformation of the Heisenberg spin chains, *Phys. Rev. E* **104**, 064124 (2021).
- [37] B. Doyon, F. Hübner, and T. Yoshimura, Generalised $T\bar{T}$ -deformations of classical free particles, [arXiv:2312.14855](https://arxiv.org/abs/2312.14855) [cond-mat.stat-mech].
- [38] D. A. Croydon and M. Sasada, Generalized hydrodynamic limit for the BoxBall system, *Commun. Math. Phys.* **383**, 427 (2021).
- [39] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, 1999).
- [40] F. Hübner, E. Vernier, and L. Piroli, Generalized hydrodynamics of integrable quantum circuits, *SciPost Phys.* **18**, 135 (2025).
- [41] B. Doyon and H. Spohn, Drude weight for the Lieb-Liniger Bose gas, *SciPost Phys.* **3**, 039 (2017).
- [42] This means the jump develops faster than the hydrodynamic time scale.
- [43] B. Doyon and T. Yoshimura, A note on generalized hydrodynamics: inhomogeneous fields and other concepts, *SciPost Phys.* **2**, 014 (2017).

- [44] J. De Nardis, D. Bernard, and B. Doyon, Hydrodynamic diffusion in integrable systems, *Phys. Rev. Lett.* **121**, 160603 (2018).
- [45] J. De Nardis, D. Bernard, and B. Doyon, Diffusion in generalized hydrodynamics and quasiparticle scattering, *SciPost Phys.* **6**, 049 (2019).
- [46] J. De Nardis and B. Doyon, Hydrodynamic gauge fixing and higher order hydrodynamic expansion, *J. Phys. A: Math. Theor.* **56**, 245001 (2023).
- [47] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, Julia: A fresh approach to numerical computing, *SIAM Rev.* **59**, 65 (2017).
- [48] S. Olver and A. Townsend, A practical framework for infinitedimensional linear algebra, in *Proceedings of the 1st Workshop for High Performance Technical Computing in Dynamic Languages – HPTCDL '14* (IEEE, New Orleans, 2014).
- [49] King's College London. (2022). King's computational research, engineering and technology environment (CREATE). Retrieved June 30, 2023, from <https://doi.org/10.18742/rnvf-m076>.
- [50] T. Bonnemain, B. Doyon, and G. El, Generalized hydrodynamics of the KdV soliton gas, *J. Phys. A: Math. Theor.* **55**, 374004 (2022).
- [51] C. N. Yang and C. P. Yang, Thermodynamics of a one dimensional system of bosons with repulsive delta-function interaction, *J. Stat. Phys.* **10**, 1115 (1969).
- [52] A. Zamolodchikov, Thermodynamic Bethe ansatz in relativistic models: Scaling 3-state potts and Lee-Yang models, *Nucl. Phys. B* **342**, 695 (1990).
- [53] P. Fendley and H. Saleur, Nonequilibrium dc noise in a Luttinger liquid with an impurity, *Phys. Rev. B* **54**, 10845 (1996).
- [54] B. Doyon and H. Spohn, Dynamics of hard rods with initial domain wall state, *J. Stat. Mech.* (2017) 073210.