

Spectral decimation of quantum many-body Hamiltonians

Feng He , Arthur Hutsalyuk , Giuseppe Mussardo , and Andrea Stampiggi ^{*}
International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy
and INFN Sezione di Trieste, via Valerio 2, 34127 Trieste, Italy



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We develop a systematic theory of spectral decimation for quantum many-body Hamiltonians and show that it provides a quantitative probe of emergent symmetries in statistically mixed spectra. Building on an analytical description of statistical mixtures, we derive an explicit expression for the size of a characteristic symmetry sector (CSS), defined as the largest subsequence of levels exhibiting non-Poissonian correlations. The CSS dimension is shown to be the size-biased average of the underlying symmetry sectors, establishing a direct link between spectral statistics and Hilbert-space structure. We apply this framework to two paradigmatic settings: Hilbert-space fragmentation and disorder-induced many-body localization. In fragmented systems, the CSS reproduces the mixture prediction and isolates correlated subsectors even when the full spectrum appears nearly Poissonian. In the disordered Heisenberg chain, spectral decimation reveals the gradual emergence of integrability through a shrinking CSS, whose statistics exhibit signatures consistent with local integrals of motion. We introduce a characteristic symmetry entropy as a finite-size scaling observable and extract, within accessible system sizes, the crossover exponents. Our results establish spectral decimation as a controlled, unbiased, and computationally inexpensive diagnostic of hidden structure in many-body spectra, capable of distinguishing between chaotic dynamics, statistical mixtures, and emergent integrability.

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I. INTRODUCTION

One of the central limitations in the study of quantum many-body systems is the exponential growth of the Hilbert space with system size. In the absence of exact analytical techniques—such as the Jordan-Wigner transformation or the Bethe Ansatz—numerical methods become indispensable. Over the years, a wide array of numerical approaches has been developed to provide nonperturbative insights, including density-matrix renormalization group algorithms [1–5], tensor-network techniques [6–9], machine-learning-based methods [10–18]—see also Ref. [19] for the first-ever application of neural networks to quantum disordered systems—and more recently spectral decimation algorithms [20].

Among the basic features of a quantum many-body Hamiltonian is its energy spectrum. Yet, despite its apparent simplicity, the spectrum contains a remarkably rich amount of information, and overall, it determines the signatures between integrable, nonintegrable and noninteracting systems. Even purely numerical spectra—devoid of analytic input—can reveal essential structural properties of the underlying Hamiltonian. As early as the 1950s, Wigner, Dyson, and

Porter [21–24], and later Berry and Tabor [25] introduced the modern viewpoint that the statistics of gaps, i.e., the differences between consecutive energy levels, provide a powerful diagnostic of quantum dynamics. Indeed, through gap statistics, one can probe and quantify the statistical signatures of integrability, chaos, and intermediate regimes. Nowadays, it is well understood that generic, nonintegrable quantum systems exhibit Wigner-Dyson level statistics, whereas integrable models, endowed with a nontrivial extensive set of commuting conserved operators, feature independent energy levels—in other words, Poisson spacings. It is worth recalling that, in the modern theory of out-of-equilibrium systems, the asymptotic regime of nonintegrable dynamics can be addressed through the eigenstate thermalization hypothesis (ETH) [26–31], whereas the asymptotic stationary states of integrable models are captured by the framework of the generalized Gibbs ensemble (GGE) (see Ref. [32] and references therein).

Spectral analysis has recently become a key diagnostic tool in the study of phenomena related to the emergence of symmetry. Such interesting phenomenology can stem from constraints on the Hilbert space or from the disorder preventing thermalization. The first case is commonly known as Hilbert-space fragmentation (HSF) [33], which has been studied extensively as a form of ETH or weak-ETH violation. Conventional literature often distinguishes between weak and strong HSF. Overall, fragmentation is diagnosed as the emergence of many disconnected emergent symmetry sectors, ranging from one-dimensional states to exponentially large subblocks. The phenomenology of models featuring HSF is vast, ranging from dipole-conserving systems [31,33–37] to

^{*}Contact author: astampig@sisssa.it

more exotic pair-flip Hamiltonians [38–42], and we defer the reader to Ref. [43] for a detailed review on this topic.

As for the second case, the presence or absence of thermalization in certain disordered quantum many-body systems is a topic of wide research interest. Several groups have focused on the study of many-body localization from different perspectives, for example, by studying the thermalization of observables, e.g., quantum quench dynamics in the many-body localized (MBL) phase [44,45], entanglement growth and scaling properties of eigenstates [45–51], renormalization-group approaches to the MBL transition [52–54], and spectral statistics and related diagnostics of ergodic–localized crossovers [55,56]. For a more extensive list of references and contributions to this active field, we refer the reader to broader reviews on many-body localization and thermalization [57–59].

When considering spectral analysis, it is, however, crucial to recognize that serious obstructions may arise from the presence of emergent “hidden” symmetries or, more generally, nontrivial constraints on the Hilbert space. Such constraints, as well as ordinary symmetries, split the Hilbert space into multiple dynamical and disconnected sectors. The global spectrum, being the juxtaposition of the individual sectors, is that of a “statistical mixture.” Even if each individual sector features level repulsion, i.e., an analogous distribution to the Wigner-Dyson (WD) one appearing in the study of the Gaussian orthogonal ensemble (GOE) in random matrix theory [60], the gap probability of the mixture is nonvanishing for arbitrarily small gaps. Consequently, a naive inspection of gap statistics may falsely indicate integrability or suggest the emergence of a nonergodic phase.

This obstacle was already recognized in Porter’s early work [21], where higher-order gap statistics were introduced to expose deviations from true Poissonian behavior. Recently, new approaches have been developed to address hidden symmetries and to extract reliable information from many-body spectra—see Refs. [20,61] and references therein. Fundamentally, the central question can be phrased as follows: given a spectrum whose level gaps appear Poissonian, can one determine whether energy levels are independent, or whether they arise from the superposition of many independent spectral blocks in a statistical mixture?

In Ref. [20], a new idea based on exact numerical sampling techniques, precursors of Monte Carlo sampling, was introduced. It was shown that it is possible to discern, from a top-down approach, if a many-body spectrum pertains to a mixture or to a set of independent energy levels—thereby signaling strong evidence of integrability. Through the application of consecutive removals of Poisson gaps, the spectrum can be decimated to a smaller set of values, whose gap statistics may be GOE-like, as in the case of mixtures, or Poissonian.

In this paper, we provide a more systematic theory of the spectral decimation of Ref. [20], and perform an in-depth analysis of statistical mixtures, with applications to constrained and disordered quantum many-body Hamiltonians. Starting from the known analytical result for statistical mixtures [22,60,62], we derive an approximation for the gap statistics to quantify the fraction of uncorrelated Poisson spacings and the set of correlated ones. We call the latter the

“characteristic symmetry sector” (CSS), and derive a formula for its size: it is the size-biased average of the individual component dimensions, $d_{\text{out}} = \sum_i d_i^2/d$. Each subblock contributes to the CSS proportionally to its density of states d_i/d , reflecting the probability that two randomly selected levels in the mixture belong to the same block. In this way, spectral decimation dynamically isolates the statistically dominant symmetry sector even when the full spectrum appears Poissonian.

We also present several concrete and illustrative applications of this method; in particular, our focus is on two phenomena:

(1) Hilbert-space fragmentation [33,43,63,64]. HSF occurs when the Hilbert space dynamically decomposes into disjoint sectors due to the presence of constraints, understood as an emergent commutant algebra [38]. Importantly, HSF provides an extreme stress-test for gap diagnostics, as it can involve exponentially many sectors, difficult to deduce from a naive study of the Hamiltonian since it generates a large Poisson contribution to the mixture. HSF thus offers a natural arena in which spectral decimation can distinguish genuinely nonintegrable dynamics from a deceptively integrable-looking spectrum.

(2) Many-body localization [56,65–69]. MBL is a phenomenon in which strongly disordered interacting systems undergo a transition to a phase¹ with emergent integrability diagnosed through the presence of independent energy levels, typically through the statistics of the r ratio [56]. Statistical decimation provides a more refined diagnostic, imputing the origin of emergent integrability not to genuine independent levels, but to those of a disorder-dependent statistical mixture. This phenomenon is well-captured by the dimension of the CSS and quantified by the “characteristic symmetry entropy” (CSE) as a function of both the disorder strength and system size.

The paper is organized as follows. Section II defines statistical mixtures, recalls the exact gap statistics and derives the small-gap approximation, from which the dimension of the CSS is obtained. Section III provides a high-level description of the spectral decimation algorithm. Section IV covers the application of the spectral decimation to HSF and MBL systems. Our conclusions are gathered in Sec. V.

This paper presents three Appendixes aimed at complementing the results of the main text. Appendix A discusses the unfolding procedure utilized in this work, which explicitly preserves the positivity of the energy density. Appendix B discusses the technical implementation of the spectral decimation, and Appendix C contains the statistical analysis of the decimation applied to statistical mixtures of random matrices.

II. STATISTICAL MIXTURES

The simplest model of a statistical mixture is that of a Hamiltonian system, where each energy level belongs to one

¹Whether this is a true phase or merely a finite-size MBL regime, with ETH restored in the thermodynamic limit [55,70–77], depends on the specific model. Here, we focus on cases where the existence of a bona fide MBL phase is established.

of N disjoint “sectors.” In principle, these sectors may be generated by a symmetry with N irreducible representations on the physical Hilbert space $\mathcal{H}$ or by some constraints. Each sector $\mathcal{H}_i$ is of dimension d_i , and there are as many energy levels $E_j^{(i)}$, distributed according to the density $\rho_i(E)$, which can be found, for instance, through a best-fit approximation, as outlined in Appendix A. In practice, one needs the cumulative density of energies $c_i(E) = \int_{-\infty}^E dx \rho_i(x)$, from which the “unfolded energies” are simply $\epsilon_j^{(i)} = c_i(E_j^{(i)})$ and the j th consecutive (unfolded) gap is $s_j^{(i)} = \epsilon_{j+1}^{(i)} - \epsilon_j^{(i)}$.

For each subblock, we denote the statistics of the gaps as $p_i(s)$. To compare with physical spectra, one requires $p_i(s)$ to be normalizable and to have a finite average $\mu_i = \int_0^\infty ds s p_i(s) = 1/\bar{\rho}_i$, with $\bar{\rho}_i$ being the average density. Customarily, $p_i(s)$ is compared to a universal distribution of mean and average one: to this aim, it is always possible to rescale the $s_i \rightarrow s'_i = s_i \bar{\rho}_i$, such that $\mu_i = 1$.

As it will be clear later, the Poisson component of a mixture can come from consecutive energy levels of two distinct blocks, or if $p_i(s)$ is Poissonian, from consecutive energies of block i . We restrict ourselves here to the assumption that Poisson gaps come only from levels of distinct blocks; thus, we assume each $p_i(s)$ to exhibit level repulsion, modeled through a Brody-like small gap behavior: $p_i(s) \sim s^{v_i}$ as $s \rightarrow 0$, for some $v_i > 0$.

A statistical mixture of the N spectra is the union of the energy levels of each subblock: $\{E\}_{(N)} = \bigcup_j \{E\}_j$. We are interested in $p_N(s)$, the probability density function (PDF) of the consecutive gaps of the mixture, which in general has a very different statistics compared to the one of the individual components, since energy levels belonging to different spectra may fall arbitrarily close, contributing to a Poisson component in $p_N(s)$. Analytical formulas were derived by Porter and Rosenzweig [22], Berry and Robnik [62], and Mehta [60]. While exact, they involve first and second integrals of the $p_i(s)$'s, which may be difficult to interpret and manipulate. We shall derive an approximate formula holding in the region $s \ll 1$, which takes a particularly simple form. Before doing that, let us recall the main results by Refs. [22,60,62].

Denote by $d = \sum_i d_i$ the size of the statistical mixture, and $\rho_i = d_i/d$ the density of levels of type i , i.e., the probability that two consecutive energies belong to the same subblock i . Let S be the total gap before unfolding, so that the average gap is $\bar{S} = 1/\rho$, with $\rho = \sum_i \rho_i$. Let

$$P_i(S) = \int_S^\infty d\sigma p_i(\sigma), \quad E_i(S) = \int_S^\infty d\sigma P_i(\sigma) \quad (1)$$

be respectively the probability of a gap larger than S (coming from two consecutive levels of the i th spectrum) and the probability that any two consecutive levels have a gap larger than S . When evaluated in $S = 0$, $P_i(0) = 0$ is the overall normalization and $E_i(S) = 1/\rho_i$ is the average spacing. The PDF of the mixture is then given by

$$p_N(S) = \frac{d^2}{dS^2} E(S), \quad E(S) = \frac{1}{\rho} \prod_{i=1}^N \rho_i E_i(S). \quad (2)$$

The unfolded gap is $s = \rho S$, and $p_N(s) = p_N(S)/\rho$.

We seek an approximate form when $s \ll 1$ and the constituents follow a Brody-like distribution [78]

$$p_B(S, \rho', \nu) = \rho' (1 + \nu) a s^\nu \exp(-a(S\rho')^{1+\nu}),$$

$$a = \left[\Gamma\left(\frac{2+\nu}{1+\nu}\right) \right]^{1+\nu}, \quad (3)$$

which interpolates between the GOE ($\nu = 1$) and Poisson ($\nu = 0$) gap distributions. The Brody-like distribution is properly normalized and of mean $1/\mu'$. For these distributions, one can compute the integrals explicitly

$$P_B(S, \rho', \nu) = e^{-(S\rho')^{1+\nu} a},$$

$$E_B(S, \rho', \nu) = \frac{\Gamma\left(\frac{1}{1+\nu}, (S\rho')^{1+\nu} a\right)}{\rho \Gamma\left(\frac{1}{1+\nu}\right)}, \quad (4)$$

where $\Gamma(\alpha, z) = \int_z^\infty dt e^{-t} t^{\alpha-1}$ denotes the incomplete gamma function. The small- S expansions

$$P_B(S, \rho, \nu) \sim 1 - (S\rho\Gamma\left(\frac{2+\nu}{1+\nu}\right))^{1+\nu},$$

$$E_B(S, \rho, \nu) \sim 1 - s + \frac{(S\rho)^{2+\nu}}{2+\nu} \Gamma\left(\frac{2+\nu}{1+\nu}\right)^{1+\nu} \quad (5)$$

can be inserted in Eq. (2) to obtain the approximation

$$p_N(s) \sim \sum_{i=1}^N \left(\frac{\rho_i}{\rho}\right)^2 p_i\left(\frac{\rho_i}{\rho} s\right) e^{-s(1-\rho_i/\rho)} + M e^{-s},$$

$$M = 1 - \sum_{i=1}^N \left(\frac{\rho_i}{\rho}\right)^2, \quad (6)$$

where here $p_i(s)$ is the standardized distribution with average 1. In the presence of one spectrum, this formula reduces to $p_1(s)$, when $\rho = 1$, and in the limit $N \rightarrow \infty$, as $\rho_i \sim 1/N$, only the Poisson component survives, irrespective of the p_i 's. Concerning the validity of the approximation, even when N is small, it qualitatively describes the exact solution [22,60,62] in the interval $s \lesssim 0.5$, see, for example, the plots in Appendix C.

Generally, this Poisson component of the mixture is a fraction M of it, which also corresponds to the value of the PDF near the origin. What remains, of size $(1 - M)$ is the correlated component of the mixture, the CSS. The spectral decimation, applied to a mixture, separates the Poisson component from the CSS. Its output, a spectrum of size d_{out} , will then be distributed according to the correlated component of Eq. (6). The size of the CSS extracted through the spectral decimation is a Gaussian random variable with an expectation value

$$\mathbb{E}[d_{\text{out}}] = d(1 - M) = \sum_i \frac{d_i^2}{d}. \quad (7)$$

This is the central result of this paper: d_{out} is the size of CSS where all energies are statistically dependent. If all spectra have the same dimensions $d_i = d/N$, then it is easy to see that also $\mathbb{E}[d_{\text{out}}] = d/N$. However, if the densities are not equal, then d_{out} is dominated by the largest sectors. Finally, if the system has uncorrelated energy levels—which is the case of

strongly interacting integrable systems and free ones—then $\mathbb{E}[d_{\text{out}}] = 1$, since $d_i = 1$. Information about the variance of d_{out} can be found in Appendix B. In other words, in the absence of explicit symmetry generators that allow us to sift the energies of the N blocks, the spectrum of the mixture only retains information about a CSS with a “size-biased” dimension, or, in other words, by the frequency that two consecutive energies belong to the same subblock. The CSS is a natural probe of the two-level spectral correlations. However it also influences dynamical quantities which probe long-range correlations, like the spectral form factor. In Appendix C this latter dynamical quantity is studied varying the size of the CSS.

III. SPECTRAL DECIMATION

From a statistical perspective, Poissonian spacings signal the absence of level repulsion, either because the system is integrable or because levels from different symmetry sectors cross freely. Spectral decimation exploits this distinction by iteratively removing those spacings that are statistically indistinguishable from a Poisson process. Since level repulsion is preserved under this procedure, correlations originating within the same dynamical sector survive the decimation, while accidental inter-sector crossings are progressively eliminated. Repeating this process isolates a characteristic subset of correlated levels, whose size reflects the dominant symmetry structure of the spectrum.

Thus, the purpose of the algorithm is to extract, via consecutive extractions, the Poisson component of a set of initial d unfolded gaps, obtained from the spectrum of a quantum many-body Hamiltonian—either the full spectrum or a fraction of it. By construction, it removes the largest fraction of Poisson gaps, leaving a set of d_{out} remainder gaps. Moreover, it is a self-averaging algorithm: If the input are spectra obtained by n different realizations of the same Hamiltonian (for example, one featuring disorder), then it is possible to not only decimate individual samples, but also to collect the gaps together, across all samples. The sample obtained by combining the gaps in this way is called a “pooled sample.” The d_{out} obtained by averaging the size of the decimated samples is equivalent to the decimated size of the pooled sample divided by n . While the decimation of one sample is more intuitive and sometimes the only option, when possible it is always best to pool many realizations together, as larger samples reduce the statistical fluctuations.

Aside from the initial set of gaps d , two other inputs are necessary to the algorithm: the first is an “extraction fraction” f , i.e., the fraction of Poisson gaps which are removed from the current sample. While f is, in general, arbitrary, too small a fraction can lead to systematic underestimates of d_{out} , while, if too large, it can lead to the exact opposite. Appendix B discusses how to properly choose f to ensure the algorithm stability. The second parameter of the algorithm is the halting size d_{halt} , which is associated to a confidence level $\alpha = 1 - d_{\text{halt}}/d$. When the sample has been decimated to a size d_{halt} , at least a fraction α of the initial set of gaps is Poisson distributed. d_{halt} prevents the algorithm from running indefinitely. When pooling n realizations together, the halting number can be set to $d_{\text{halt}} = n$, implying that if the pooled

sample has been decimated to $d_{\text{out}} \leq n$, then the individual samples will have subblocks of dimension one, on average.

When the spectrum is that of a mixture, the algorithm ends before meeting the halting threshold: $d_{\text{out}} > d_{\text{halt}}$. In this case, Eq. (7) provides the expectation of d_{out} . However, if the algorithm ends after meeting the threshold d_{halt} , then one concludes that with confidence α the spectrum is composed of uncorrelated energy levels. If the remainder gap statistics is compatible with a Poissonian, then this provides a strong hint for the integrability of the quantum many-body system. It may also happen that the remainder gap statistics features distinct peaks: in this case, it is likely that the spectrum is free or weakly interacting. These outcomes are reported in Fig. 1.

Below, give here a short description of the spectral decimation developed in Ref. [20], referring the reader to Appendix B for implementation details and more details on its statistical properties.

Structure of the algorithm

Given a sample of size d , an iteration m of the decimation algorithm, with extraction fraction f and halting threshold d_{halt} , proceeds as follows:

(1) the empirical PDF π_j of the available “remainder” gaps, $\mathcal{R}_m$ of $d_R(m) = (1 - f)^m d$ elements, is computed with bin width

$$\delta(m) = \left(\frac{12}{d_R(m)} \right)^{1/3}. \quad (8)$$

At the first iteration, $m = 0$, the set of remainder gaps coincides with the original set $\mathcal{R}_0$ of d gaps. The bin width δ is chosen according to the Freedman-Diaconis rule for the Poisson PDF [79], which minimizes the integrated mean-squared error between the empirical and true PDF—see Appendix B 2 for a derivation of the formula.

(2) To the set $\mathcal{R}_m$, one applies rejection-sampling (RS)—see Appendix B—with target probability $\tilde{q}(s) = e^{-s}$ and acceptance ratio

$$M(m) = \min \{\pi_0, 1\}. \quad (9)$$

Here $M(m)$ is the fraction of Poisson gaps, up to order $O(\delta)$, at iteration m , estimated from the zero-bin empirical PDF π_0 . One therefore obtains a set $\mathcal{A}_m$ of accepted gaps which follow the Poisson distribution.

(3) A set of $d_E(m) = f d_R(m)$ Poisson gaps is extracted from $\mathcal{A}_m$ uniformly. This yields a subset of $\mathcal{E}_m$ of d_E Poisson gaps at each iteration which is removed from the set of available ones

$$\mathcal{R}_{m+1} = \mathcal{R}_m - \mathcal{E}_m. \quad (10)$$

(4) The set $\mathcal{R}_{m+1}$ of $d_R(m+1) = (1 - f)^{m+1} d$ remainder gaps can be utilized as the input for the iteration $(m+1)$.

1. Halting conditions

H1. If after (1) $d_R(m) \leq d_{\text{halt}}$, then the algorithm ends and the global spectrum is Poissonian with confidence α .

H2. If after (3) and $d_R(m) > d_{\text{halt}}$, then the number of accepted gaps, whose expected value is $\mathbb{E}[d_A(m)] = M(m)d_R(m)$, is less than $d_E(m) = f d_R(m)$, the algorithm halts because the requested fraction has not been found. In this case,

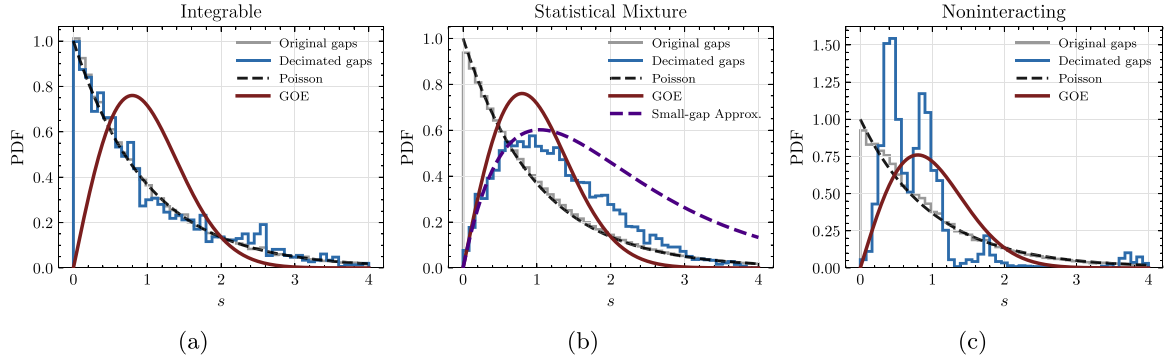


FIG. 1. Outcome of the decimation in the case the parent Hamiltonian is (a) integrable, (b) statistical mixture of chaotic spectra, (c) noninteracting. In panel (a) the periodic spin-1/2 XXX model is diagonalized for $L = 22$ sites in the $SU(2)$ irreducible representation. The remainder gaps are Poissonian and the decimation succeeds for any d_{halt} . (b) The parent Hamiltonian is a mixture of 24 random GOE (Gaussian orthogonal ensemble) matrices of same size $d = 10^4$. Here the decimation ends at the size of one GOE block and the gap statistics clearly shows signs of level repulsion. The small-gap expansion found in Sec. II closely follows the decimated gap statistics for $s \lesssim 1$. (c) Decimated spectrum of a $L = 10$ free fermionic chain with random on-site energies. In particular, they are i.i.d. uniformly in $[0,1]$. The gap statistics of the decimated gaps presents an excess in the zero-gap probability compared to the pure Poisson case, due to i.i.d. nature of the frequencies.

the algorithm ends and the global spectrum is not Poissonian with confidence α .

2. Generalizations

The spectral decimation algorithm can be generalized to sample different statistics rather than the one of consecutive gaps. Another choice is the r ratio

$$r_j = \frac{\min\{E_{j+1} - E_j, E_{j+2} - E_{j+1}\}}{\max\{E_{j+1} - E_j, E_{j+2} - E_{j+1}\}}. \quad (11)$$

While the decimation procedure can be applied to different spectral observables (e.g., gaps or r ratios), the statistical mixture approximation leading to Eq. (7) is strictly derived for independent spacings. Since r ratios involve correlated triplets of levels [61], their mixture structure does not generically obey the same size-biased law. Therefore, the CSS extracted from r ratios needs not coincide quantitatively with the one extracted from gaps, and its derivation is an interesting problem which lies beyond the scope of this paper. We noticed that the CSS size in fragmented Hamiltonians (Sec. IV A) and in toy-models of statistical mixtures (Appendix C) differs from Eq. (7) when computed with the r ratio. However, for disordered quantum Hamiltonians (Sec. IV B), the CSS size found through the r ratio and gap statistics present the same qualitative behavior, while still featuring quantitative deviations, especially at large disorder couplings. Nevertheless, the r ratio remains a valuable numerical tool as it is inherently independent of the local density of states, bypassing the need for spectral unfolding.

IV. APPLICATIONS TO QUANTUM MANY-BODY HAMILTONIANS

In this section, we consider realistic quantum many-body Hamiltonians where the Hilbert space presents disconnected sectors, such as in the case of Hilbert-space fragmentation, or disorder, which is the case of systems featuring many-body localization. Emergent symmetries, due either to the presence of many Krylov subspaces or local integrals of motion, can cause the global statistics to appear Poissonian. In this section we

investigate if the CSS presents the same Poisson statistics, or if it underlies the presence of a statistical mixture, and if its size can inform us about the underlying emergent symmetry.

A. Hilbert-Space fragmentation

Hilbert-space fragmentation has emerged as a distinct and conceptually rich form of ergodicity breaking in quantum many-body systems, one that is neither rooted in integrability nor driven by quenched disorder [42,43,63,64,80–82] (see also Refs. [33,83–100] for an extensive list of models). In fragmented models, the physically natural (often product-state) basis reveals that the Hamiltonian dynamics does not explore the full Hilbert space but instead decomposes it into an exponential number of dynamically disconnected Krylov subspaces. Unlike conventional symmetry sectors—whose number grows at most polynomially with system size—these Krylov subspaces are generated by microscopic kinetic constraints or conserved dipole-like quantities and can span a broad spectrum of dimensionalities, from finite or logarithmically growing manifolds to subspaces that remain exponentially large. As emphasized in the recent review of fragmentation phenomena [43], this structural decomposition leads to a rich variety of dynamical behaviours: certain subspaces support effectively integrable dynamics with Poissonian spectral statistics, while others exhibit nonintegrable, quantum-chaotic behavior consistent with GOE statistics. Consequently, Hilbert-space fragmentation provides a controlled setting in which integrable and chaotic sectors coexist within a single Hamiltonian, and the global spectral statistics becomes an unreliable indicator of the underlying dynamical properties.

A salient implication of this phenomenon, particularly relevant for our present study, is that conventional diagnostics, such as level-spacing ratios or eigenstate entanglement scaling, can be profoundly obscured by the superposition of spectra originating from multiple dynamically isolated sectors. In weakly fragmented systems, only a measure-zero subset of eigenstates violates ETH, and the majority of sectors behave thermally, often yielding GOE statistics once

symmetries are resolved. In strongly fragmented systems, by contrast, the proliferation of small-dimensional Krylov subspaces produces extensive degeneracies and unconventional spectral statistics, typically mimicking Poisson behavior even when the dynamics within individual large sectors remain nonintegrable. This mixing problem underscores the need for refined spectroscopic tools capable of disentangling the true dynamical character of each sector. The spectral decimation protocol is precisely designed to meet this challenge: by progressively thinning the spectrum and isolating statistically consistent subsequences, it enables us to diagnose hidden nonintegrable sectors in weakly fragmented models and to characterize the emergent integrability structure in strongly fragmented ones. In this sense, Hilbert-space fragmentation provides an ideal testbed for demonstrating the power of spectral decimation in resolving complex ergodicity-breaking mechanisms.

As an example of HSF, we consider the spin-1/2 pair-flip (PF) model introduced in Refs. [38,39] of the Hamiltonian

$$H_{\text{PF}} = \sum_{i=1}^L \sum_{\alpha,\beta=1}^2 [g_{i,j}^{\alpha,\beta} |\alpha_i \alpha_{i+1}\rangle \langle \beta_i \beta_{i+1}| + \text{H.c.}] + \sum_{i=1}^L \sum_{\alpha=1}^2 h_i^\alpha |\alpha_i\rangle \langle \alpha_i|, \quad (12)$$

where $g_{i,j}^{\alpha,\beta}$ and h_i^α are arbitrary constants.

This model features HSF for any $g_{i,j}^{\alpha,\beta}$ and h_i^α in the computational basis, i.e., the total Hilbert space is decomposed into $(L+1)$ disconnected Krylov spaces, each associated with a $U(1)$ charge. As pointed out in Ref. [39], each Krylov subspace can be labeled by an “irreducible string”: for a given state, it is obtained by the deletion of all possible next-nearest-neighbor pairs, going from left to right and repeating if, after a deletion, new pairs arise. For example, if we denote down and up spin states as 1 and 2, then the irreducible string for the state $|1221\rangle$ is $\emptyset$, and it is 2 for the state $|22211\rangle$. Each dynamical disconnected Krylov subspace is uniquely characterized by an irreducible string and is spanned by the basis connected by all allowed pair-flip moves. For example, for the irreducible string $\emptyset$ with $L=4$, one can start from the initial state $|1221\rangle$ and perform all possible pair-flips to generate the basis of the corresponding Krylov subspace, reproducing the action of the Hamiltonian. One can easily notice that there is no pair flip that allows transitions between states belonging to subspaces with different irreducible strings; therefore, the Hilbert space is fragmented. Alternatively, it is also possible to start from the 2^L states of the computational basis, and identify its irreducible strings with a “pushdown automaton” [39].

The Hilbert space of this PF model can be expressed as a direct sum of dynamically connected subspaces labeled by an irreducible string x , i.e.,

$$\mathcal{H} = \bigoplus \mathcal{H}_x, \quad \mathcal{H}_x = \text{span}\{|\omega\rangle : \text{irr}(\omega) = x\}, \quad (13)$$

where $|\omega\rangle$ represents the quantum state in Hilbert space, and x denotes the corresponding irreducible string of the state $|\omega\rangle$. In the spin-1/2 case, if we denote the total number of 1 and 2 in the irreducible strings as $k = 2M$, then it follows that all the possible irreducible strings for a chain with even length L

are

$$\emptyset, 12, 21, 1212, 2121, 121212, 212121, \dots \quad (14)$$

Each irreducible string x of length k carries a $U(1)$ conserved charge $Q(x) = \sum_{i=1}^{L-1} (-1)^i (N_i^{(1)} - N_i^{(2)}) = \pm k$, where $N_i^{(1)}$ and $N_i^{(2)}$ count the number of down and up spins in site i . If we define an effective magnetization as $M = -Q(x)/2$, then the dimension of the Krylov subspace corresponding to H_x is

$$D_{L,M} = \binom{L}{\frac{L}{2} - |M|}. \quad (15)$$

Therefore, the entire Hilbert space of the PF model is decomposed into $(L+1)$ Krylov subspaces. The largest block is of dimension $D_{L,>}$, which is equal to $D_{L,0}$ for even L and $D_{L,1/2}$ for odd L , respectively. The average block size is $\langle D_L \rangle = 2^L/(L+1)$.

As the pair-flip model is, to all effects, a statistical mixture, the size of the CSS when all blocks are nonintegrable is, according to Eq. (7),

$$\mathbb{E}[d_{\text{out}}^{\text{PF}}] = 2^{-L} \sum_M D_{L,M}^2 = 2^{-L} \binom{2L}{L} \sim \frac{2^L}{\sqrt{\pi L}}. \quad (16)$$

As a particular realization of the PF Hamiltonian, we choose the binary random couplings $|g_{i,j}^{\alpha,\beta}| = J(1 + \delta_{\alpha,\beta})/2$ and h_i^α to be uniformly i.i.d. in $[-w, w]$ for some disorder strength such that $w/J \ll 1$. With this choice of coefficients, the Krylov subsectors exhibit almost perfect GOE-like level spacing and gap ratio statistics, and therefore serve as an ideal testbed for spectral decimation.

For each L , we pool several realizations of the PF Hamiltonian together and perform the gap decimation. Figure 2 shows remarkable agreement of the spectral decimation algorithm with Eq. (7). This pooling approach enlarges linearly the input of the spectral decimation algorithm, and at the same time, it removes the sample-by-sample fluctuations. Therefore, we have demonstrated that the size of the CSS in a generic mixture, Eq. (7), can be probed through the spectral decimation algorithm, even when HSF occurs.

B. Many-body localization

MBL [56–59,66–69,101] has emerged as a paradigmatic mechanism of ergodicity breaking in disordered quantum systems, providing a robust alternative to thermalization in highly excited states [102–127]. Unlike conventional isolated many-body systems, whose long-time behavior is governed by the ETH, MBL systems fail to act as their own thermal reservoirs: they retain a persistent memory of their initial conditions and never approach a Gibbs ensemble. This phenomenology is reflected directly in the structure of the many-body eigenstates: instead of exhibiting volume-law entanglement as predicted by ETH, MBL eigenstates obey area-law scaling even at finite energy density, signaling the emergence of an extensive set of quasi-local integrals of motion (LIOMs or ℓ -bits) [46,128–134] (see also Ref. [135]). This emergent integrability underpins all aspects of the MBL phase, and it distinguishes MBL sharply from both noninteracting Anderson insulators and integrable models described by generalized Gibbs ensembles.

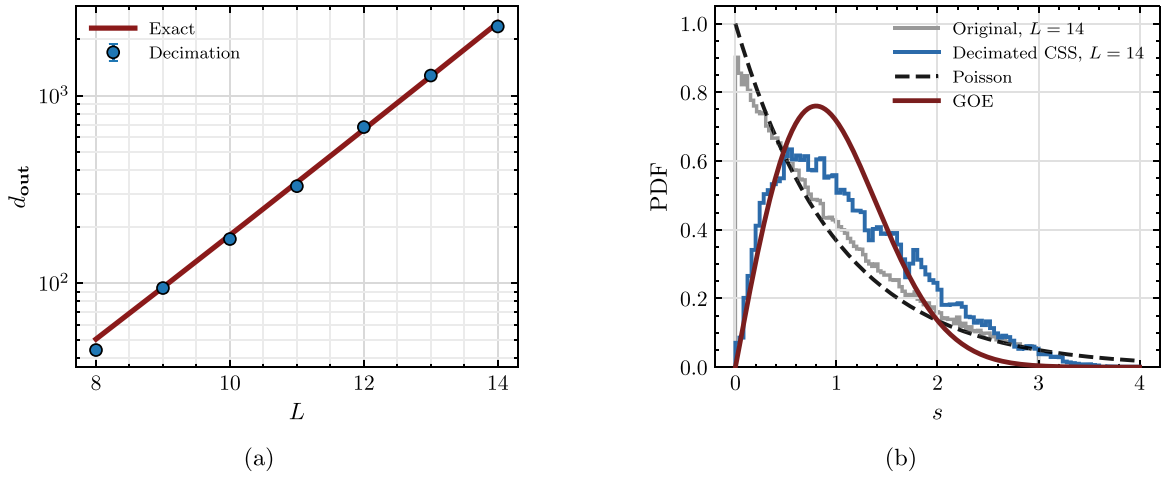


FIG. 2. Application of spectral decimation to the pair-flip model of Eq. (12). (a) The decimation output, d_{out} , follows closely the theoretical prediction Eq. (7). (b) Despite the global distribution of the gaps follows closely the Poisson statistics, the distribution of the remainder gaps, the one surviving the decimation, feature clear sign of level repulsion and therefore the CSS is nonintegrable.

A central challenge in contemporary studies of MBL lies in characterizing the transition between thermal and localized phases—a problem that remains subtle due to finite-size effects, sample-to-sample fluctuations, and the coexistence of thermal and localized regions (avalanches). Conventional diagnostics such as the level-spacing ratio, entanglement entropy, or transport coefficients capture some aspects of this transition, but they are often obscured by Griffiths effects and by the slow approach to the thermodynamic limit. Here, statistical spectroscopy offers a powerful complementary perspective. Because MBL is intrinsically a form of emergent integrability, its onset leaves sharp fingerprints in the spectral statistics: GOE-like distributions characteristic of chaotic dynamics give way to Poisson statistics as LIOMs proliferate. Yet the precise manner in which this crossover occurs—particularly in finite systems where mixed or “rare-region-dominated” spectra are common—requires more refined tools than simple gap-ratio analysis. The spectral-decimation protocol developed in this work is tailored precisely for this purpose: by extracting statistically consistent subsequences of the spectrum, it exposes the latent integrable structure hidden beneath disorder-induced spectral mixing, thereby providing a clean, unbiased probe of the emergent integrability that defines the MBL phase.

As a paradigmatic case of MBL, we analyze the spin-1/2 disordered XXZ model [55,69,136,137] of Hamiltonian

$$H = J \sum_{i=1}^L S_i \cdot S_{i+1} + J(\Delta - 1) \sum_{i=1}^L S_i^Z S_{i+1}^Z + \sum_{i=1}^L h_i S_i^Z. \quad (17)$$

We set $J = 1$ and $\Delta = 1$. The h_i 's are random i.i.d. variables uniformly distributed in $[-W, W]$, with W the strength of disorder. The model features $U(1)$ symmetry with a generator $\hat{M} = \sum_i \sigma_i^Z$. Throughout this paper, we focus on the $M = 0$ sector for even L and $M = 1/2$ for odd L . The dimension of the Hilbert space is given by Eq. (15).

We now summarize the results of the spectral decimation applied to Eq. (17), offering both qualitative and quantitative analyses. For each value of $L = \{9, \dots, 16\}$ we considered

at least $N = 1000$ samples for each (W, L) . Then, for each (W, L) we pool all samples together and perform the decimation on this larger set, resampled $r = 25$ times.

Qualitative features of the CSS

Overall, the CSS remains close to the Hilbert-space dimension for $W \lesssim 2$ and then decreases exponentially with an (W, L) -dependent slope. Crucially, even at moderate volumes, we observe in the strong disorder region, $W = 10$, that the CSS approaches values on the order of a few percent of $d(L)$, up to 1%. In that region, energy levels tend to form small CSS, implying that global spectral correlations are weakening. The operators enabling these emergent symmetries are the LIOMs and the CSS may be interpreted as an estimator of the effective symmetry sectors generated by them. Thus, spectral decimation provides an independent spectral diagnostic consistent with the presence of LIOMs in the disordered XXZ Hamiltonian.

We emphasize that the extracted CSS size is robust under moderate variations of the halting threshold d_{halt} , provided that the extraction fraction f is chosen according to Appendix B so that it does not introduce systematic over- and under-estimates of the decimation output. As explained in Sec. III, the self-averaging pooling procedure suppresses finite-sample fluctuations, rendering the CSS stable against such variations. We also compared the CSS size obtained using the r ratios instead of the gaps, as an internal test of the consistency of the spectral decimation. Qualitatively, the CSS size presents similar L and W dependence, but their quantitative values differ, as outlined in Sec. III.

Figure 3 shows the underlying distribution of the CSS gaps and r ratios, at weak and strong disorder. We observe that, at weak disorder, the gap and that of the r ratio statistics are GOE-like. At very large disorder, the leading term of the Hamiltonian is $H_Z = \sum_{i=1}^L h_i S_i^Z$, whose spectrum is that of a sum of independent, uniformly i.i.d., L on-site energies. In the limit $L \rightarrow \infty$, the statistics of the “truncated oscillator” approaches the Poissonian, similarly to the $N \rightarrow \infty$ limit of the mixture (see Sec. II). As already pointed out in Ref. [25], the

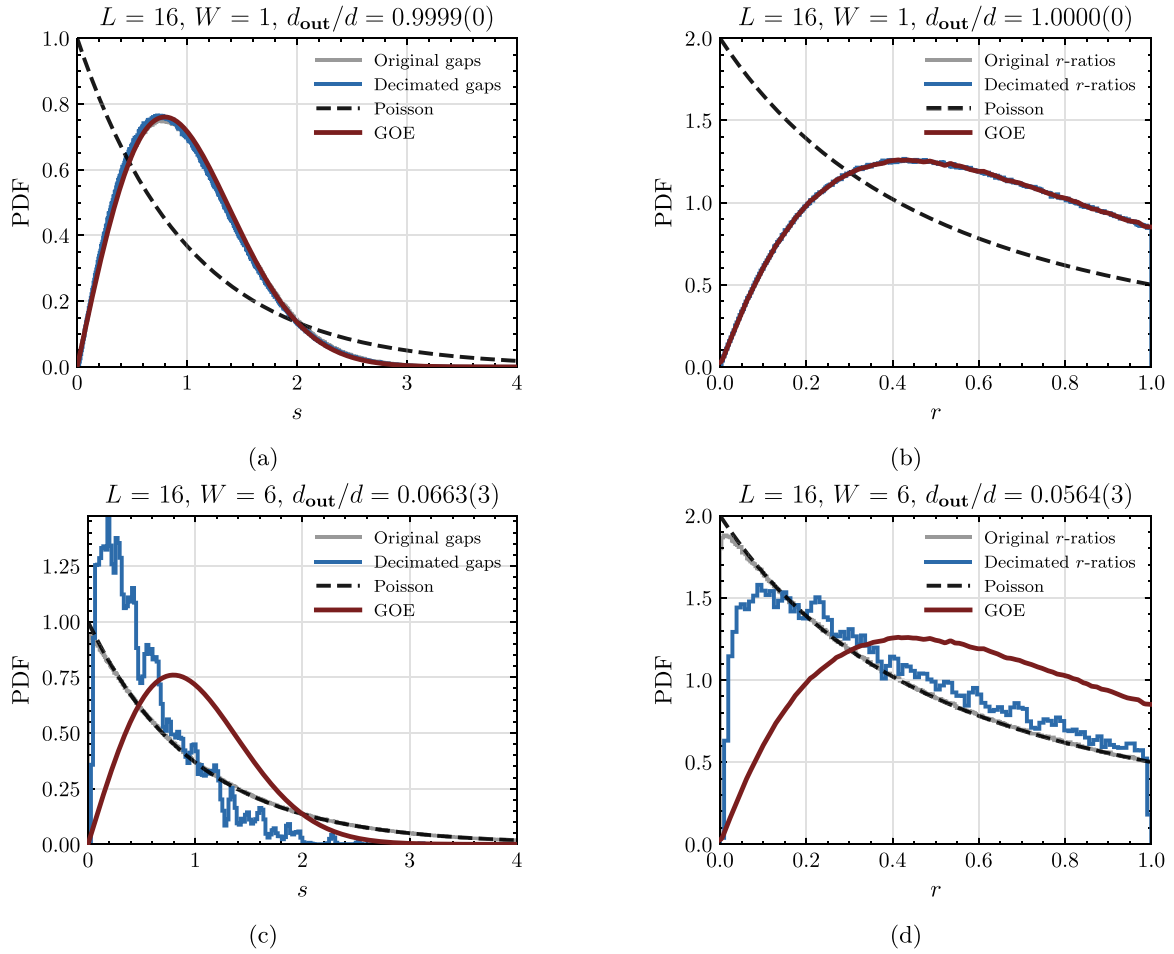


FIG. 3. Histograms of gaps and r ratios of the one-dimensional disordered Heisenberg chain for $L = 16$ at weak ($W = 0.5$) and strong ($W = 6.5$) disorders, in a energy window of $\Delta E = 90\%$ of the full spectrum. At weak disorder, one observe excellent agreement of the distribution of gaps (a) and r ratios (b) with the Wigner-Dyson distribution of GOE random matrices. The value of d_{out}/d is approximately 1 in both cases, the one obtained from the r ratios subject to smaller statistical fluctuations. At strong disorder, however, while the r -ratio statistics (d) seems to be compatible with a semi-Poissonian (which eventually will be restored as $W \rightarrow \infty$), the gap statistics (c) clearly deviates from the Poisson one and accumulates toward $s \rightarrow 0$. Sharp peaks are characteristic of effectively free or weakly interacting spectra, where spectra are regular and energy differences expressible as differences of linear combinations of one-particle energies. This distinction between free and strongly interacting integrable system statistics is not captured by the r -ratio distribution, which is the same. We also observe that the CSS size extracted from r -ratio decimation deviates quantitatively from that of gap decimation, as discussed in Sec. III.

level statistics of the truncated oscillator with few frequencies, corresponding in our case to finite L , features characteristic peaks due to equidistant energy levels. Moreover, these peaks can not be explained, for example, by a Brody parameter, which only interpolates between GOE and Poisson statistics. The CSS features the same phenomenology at strong disorder. While this is evident from the gap statistics [especially if compared to the free-fermion statistics Fig. 1(c)], we notice that this feature is absent in the one of the r ratio, which mildly deviates from $p(r) = 2/(1+r)^2$.

The analysis of the CSS statistics provides new insight into the "emergent integrability" of the one-dimensional disordered XXZ Hamiltonian. It should not be understood as the emergence of generic integrable dynamics, like the one of the same model in absence of disorder, the strongly interacting XXX chain, but of an asymptotically free one, like that of weakly interacting fermions.

C. Characteristic symmetry entropy

For any (W, L) , the CSS size $d_{\text{out}}(W, L)$ obtained by running multiple times (resampling) the gap decimation of the pooled samples varies exponentially with W and L . Therefore, the CSS size cannot be used as a sensible observable for a finite-size scaling (FSS) analysis.

We start by noticing that, as the asymptotics of the Hilbert-space dimension is

$$\ln d(L) \sim (\ln 2)L - \frac{1}{2} \ln L + \frac{1}{2} \ln \frac{2}{\pi}, \quad (18)$$

one can consider $\frac{1}{L} \ln d(L)$ as an intensive quantity, up to $O((\ln L)/L)$ corrections. Therefore, we study the ratio of the original size and the one of the CSS dimension. This quantity expresses the effective number of blocks of dimension of the CSS—assuming them uniformly. The logarithm of this ratio

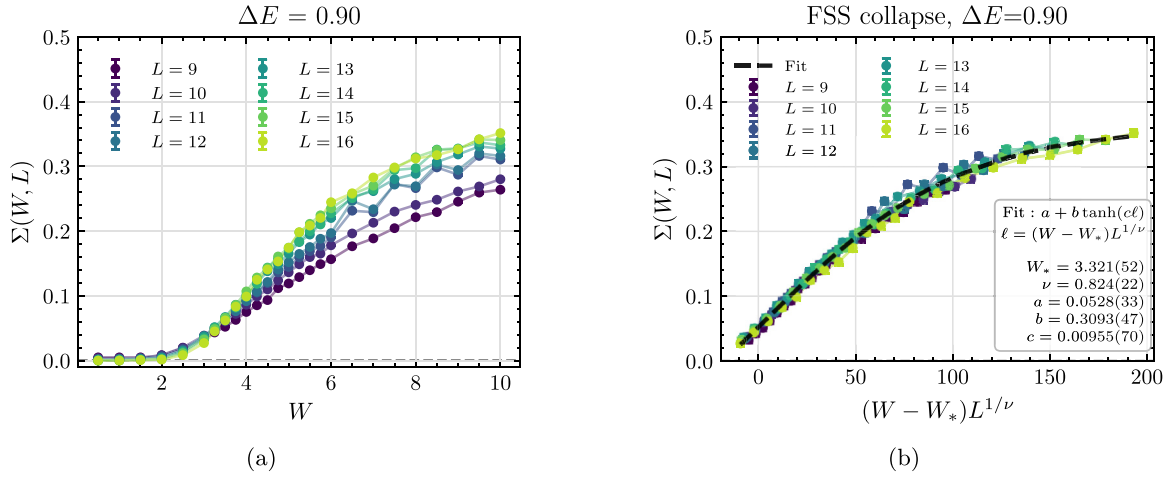


FIG. 4. (a) Plot of the CSE over a large window of the spectrum ($\Delta E = 0.9$), varying the disorder strength W , for different volumes L . For $W \lesssim 3$, the CSE remains close to 0, indicating that the system is close to GOE, while it increases for larger W 's. For the disorder strengths and volumes studied, the CSE does not approach unity, even though the CSS may be as low as 1% of the Hilbert-space dimension. As such, the CSE is a sensible probe of the fine spectral structure of disordered quantum Hamiltonians. (b) FSS collapse of the various $\Sigma(W, L)$ curves. Fitted parameters, as well as the fitting form, are reported in the box on the bottom right.

can be rescaled to define the intensive CSE:

$$\Sigma(W, L) = \frac{1}{L \ln 2} (\ln d(L) - \ln d_{\text{out}}(W, L)), \quad (19)$$

where $d_{\text{out}}(W, L)$ is the average of the output of the spectral decimation. Vanishing $\Sigma(W, L)$ corresponds to fully chaotic (GOE-like) behavior, while larger values indicate increasing effective fragmentation.

We notice that decimations performed with gaps or with r ratios produce qualitatively similar CSE, for the disordered XXZ Hamiltonian. We also observe that it never saturates to unity for the system sizes and disorder parameters considered, reaching a maximum value of approximately 0.4, see Fig. 4(a). This immediately puts into question if the system sizes under consideration represent a legitimate thermodynamic regime, where true MBL scaling dominates. This issue can not be resolved by exact diagonalization alone, and it can be clarified by a renormalization-group study of the CSS, a task which we will not pursue in this current work.

Finite-size scaling across multiple energy windows

Below, we present a FSS analysis of the CSE. Its purpose is not to draw conclusions on the physics of MBL, rather to show that the CSE is an observable which has FSS collapse. Indeed, the validity of this analysis is based on a phenomenological ansatz and not on a renormalization-group scaling form. Moreover, it is likely that the system sizes utilized are not representative of a true thermodynamic phase, but of a crossover regime. For these reasons, the following analysis will only be indicative rather than conclusive.

While conventional studies of MBL focus on thermal states in a small energy window ΔE , to compensate for the limited system sizes at our disposal, we will also consider wider windows, ranging from the bulk to the full spectrum, where the spectral decimation becomes more statistically accurate. As customary, we introduce a scaling variable

$$\ell = L^{1/\nu} (W - W_*), \quad (20)$$

where conventionally W_* is the critical disorder parameter and ν is the dynamical exponent characterizing the divergence of the correlation length: $\xi \sim |W - W_*|^{-\nu}$. It is also worth noting that this ansatz may, in principle, be wrong, as renormalization-group studies [52, 53, 138] argue that the transition may be of the Kosterlitz-Thouless (KT) type, where $\nu = \infty$. Therefore, the scaling parameters should be interpreted as effective crossover quantities rather than true thermodynamical exponents. We tried fitting with both a KT law, $\ln \xi \sim (-\sqrt{W - W_*})$, and a power-law one, but found that the KT FSS ansatz does not provide a collapse for the system sizes under study.

Overall, the CSE presents three qualitatively distinct regions: for weak disorder, $W \lesssim 3$, it is qualitatively vanishing, then it undergoes a crossover, $3 \lesssim W \lesssim 6$, with an L -dependent slope toward a saturation region, $W \gtrsim 10$. This suggests that the FSS form may be a sigmoid or an hyperbolic tangent. Indeed, the last option offers a collapse of the Σ versus W at different L 's:

$$\Sigma(W, L) = a + b \tanh(c\ell), \quad (21)$$

where a , b , and c are numerical constants. The FSS ansatz is robust in the disorder window $3 \leq W \leq 10$, offering a collapse of the CSE; see Fig. 4(b). Varying the energy window, we observe that the estimated critical point lies in the range $3 \lesssim W_* \lesssim 4$, and that the critical exponent $\nu < 1$; see Fig. 5.

While the value of W_* for small ΔE agrees with other estimates from exact diagonalization studies [55], the value of ν violates the Harris bound ($\nu > 2/D$ where D is the spatial dimension). Nevertheless, among the ansatzes tested, the power-law form provides the best collapse of the available data, and we report in Fig. 5 its parameters.

Nonetheless, the CSE can be used as an observable easily accessible from numerical data. Moreover, it can be computed either from the gaps, which require unfolding, or the r ratios, which gauge three-level correlations but do not require fitting of the energy density. Thus, it can be considered as a complementary probe of eigenvalue correlations, like the average r

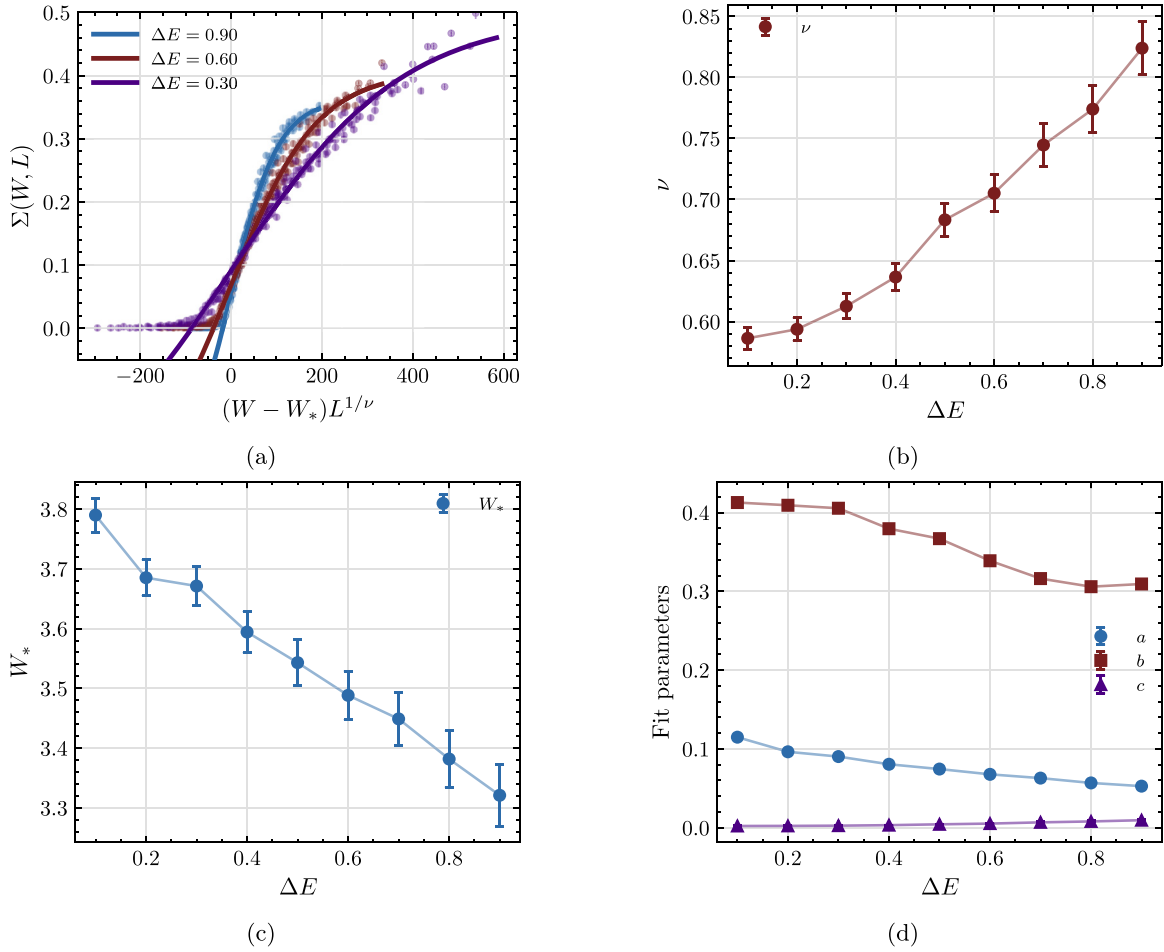


FIG. 5. Results of finite-size scaling (FSS) of the characteristic symmetry entropy (CSE). (a) the FSS of the CSE is reported for few energy windows ΔE , showing that as ΔE increases, the steepness of the curves, dictated by the critical exponent ν diminishes. (b) The same critical exponent ν is plotted against ΔE for the energy windows studied. (c) Similar plot for W_* . Finally, the fit constants are varying monotonically with the energy window (d).

ratio. The CSE is, moreover, consistent with a power-law FSS ansatz, which brings it to a volume-independent collapse.

V. CONCLUSIONS

In this paper, we applied the spectral decimation procedure [20] to one-dimensional quantum many-body Hamiltonians featuring nontrivial forms of emergent symmetries. We argued that the underlying framework in which they can be understood is that of statistical mixtures, for which we derived the size of a CSS. The CSS is the largest sector with non-Poissonian statistics, characterized either by level repulsion or by isolated peaks, and its size is a size-biased average of the symmetry sectors present in the Hamiltonian.

For systems featuring Hilbert-space fragmentation, in particular, the pair-flip model, we showed that the CSS follows the relation predicted by the statistical mixture approximation and demonstrated that, while the overall global statistics may be close to Poissonian, the statistics of the CSS clearly features level repulsion.

For systems undergoing many-body localization, we focused on the one-dimensional disordered Heisenberg chain and showed that the CSS can be used as a probe of the

emergent symmetries generated by the local integrals of motion. We clarified the nature of the emergent integrability by looking at the statistics of the CSS at large disorder, which shows clear peaks reminiscent of those of weakly interacting or almost-free systems. The size of the CSS can be transformed into a novel order parameter, the CSE, which is computationally inexpensive since it is derived from the Hamiltonian spectrum. Alike other observables, such as the r ratio [56,139,140] or the zeros of the renormalization group β function [52,53,141–146], it features universal finite-size scaling collapse. An interesting future direction is to derive the CSE finite-size scaling form from an explicit renormalization-group description of LIOM formation, and clarify the nature of the transition, if of the Kosterlitz-Thouless type or governed by a different infinite-randomness scenario.

In conclusion, we demonstrated that the CSS and the CSE can be used both as qualitative and quantitative probes of the statistical properties of many-body spectra, able to offer guidance in discriminating between strongly interacting integrable systems, free ones and statistical mixtures of nonintegrable components. Moreover, as spectral decimation operates directly on spectral statistics without relying on entanglement or dynamical probes, it provides a computationally

inexpensive and broadly applicable diagnostic for emergent integrability.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [147].

APPENDIX A: UNFOLDING PROCEDURE WITH BERNSTEIN POLYNOMIALS

In this Appendix, we describe the unfolding procedure used to obtain consecutive spacings from the energy spectrum, defined as

$$s_i = e_i - e_{i-1}, \quad e_i = \int_{-\infty}^{E_i} dE \rho(E), \quad (\text{A1})$$

where ρ is the density of energy. Fitting directly ρ is impractical here, first because it is subject to fluctuations, especially for very small system sizes. Second, we recognize that e_i is the cumulative energy density (CDE), which is a much better quantity to fit, as it is monotonic.

Thus, we seek to fit the CDE, whose empirical values are

$$\tilde{c}_i = \sum_{j=1}^d \theta(E_i - E_j), \quad (\text{A2})$$

i.e., it counts the number of energies less than or equal to E_i . We notice that, to obtain s_i , we can rescale the energies in the region $[0, 1]$:

$$\tilde{E}_i = \frac{E_i - E_1}{E_d - E_1}. \quad (\text{A3})$$

Then, the problem is reduced to fit the data $(i/d, \tilde{c}_i)$ in the interval $[0, 1]$.

The most important property that the fit must preserve is the monotonicity of the $\tilde{c}$'s, which guaranties that ρ will always be positive. A linear polynomial fit with basis functions x^n does not provide generic monotonic $\tilde{c}$'s. In $[0, 1]$ there exists a basis of polynomials, called Bernstein polynomials $b_{v,n}(x)$:

$$b_{v,n}(x) = \binom{n}{v} x^v (1-x)^{n-v}, \quad v \in \{0, 1, \dots, n\}, \quad (\text{A4})$$

which is of order n .

Bernstein polynomials satisfy

$$\frac{d}{dx} b_{v,n}(x) = n[b_{v-1,n-1}(x) - b_{v,n-1}(x)]. \quad (\text{A5})$$

If then one considers a generic polynomial approximation of degree n of a monotonic function $f(x)$, i.e.,

$$f_n(x) = \sum_{v=0}^n \gamma_v b_{v,n}(x), \quad (\text{A6})$$

then it is easy to show that

$$\gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_n \rightarrow \frac{d}{dx} f_n(x) \geq 0. \quad (\text{A7})$$

Introducing auxiliary constants $\beta_i \geq 0$ such that $\gamma_k = \sum_{i=0}^k \beta_i$, one can easily perform a constrained linear least-squares fit to obtain the β_i 's and therefore obtain the best monotonic approximation of a monotonic $f(x)$ through a polynomial of degree n . This is exactly the strategy we use to obtain the CDE and, as a byproduct, consecutive gaps ensuring the fitted density is positive.

APPENDIX B: DETAILS OF THE DECIMATION ALGORITHM

This Appendix details certain technical aspects of spectral decimation, which complement the discussion in the main text. Here, we discuss in detail the RS algorithm, the optimal bin width, the choice of the extraction fraction f and sources of systematic errors. Finally, we offer a theoretical prediction of the variance of the decimation algorithm, as it is repeated on the same spectrum many times (resample).

1. Rejection sampling

The RS algorithm is an exact simulation technique to construct a prescribed target distribution $\tilde{q}(s)$ from samples $\{s\}$ drawn according to an original distribution $q(s)$. The construction proceeds by generating an empirical distribution $\pi(s)$, which converges to $\tilde{q}(s)$ in the limit of infinitely many samples. Such a reconstruction is feasible only when the target distribution is fully enveloped within the original one. In particular, if $\tilde{q}(s)$ is defined over $\tilde{d}$ elements and $q(s)$ over $d > \tilde{d}$, then $\tilde{q}(s)$ can be reproduced exactly, without any distortion, provided that $\tilde{q}(s)\tilde{d} \leq q(s)d$, for all $s \geq 0$.

From the envelope principle, it follows that only a ratio

$$M = \min_s \frac{\tilde{d}\tilde{q}(s)}{dq(s)} \quad (\text{B1})$$

of the original d gaps can reproduce exactly the distribution $\tilde{q}(s)$.

The set of accepted values $\mathcal{A}$ is that for which

$$\mathcal{A} = \{s_j \in \{s_1, \dots, s_d\}, \text{ such that } u_j q(s_j) \leq M \tilde{q}(s_j), \\ u_j \sim \text{Unif}(0, 1)\}. \quad (\text{B2})$$

In other words, for each s_j , a uniform i.i.d. number in $[0, 1]$, u_j , is extracted and the gap s_j is conditionally accepted according to the inequality $u_j \leq \alpha(s_j)$, where $\alpha(s) = \min\{M\tilde{q}(s)/q(s), 1\}$ is the acceptance probability.

It follows that M becomes the “acceptance rate” of the RS. Indeed, for each s_j , the probability of acceptance is $\alpha(s_j)$ and therefore the total number of accepted values is

$$\mathbb{E}[d_A] = d \int_0^\infty ds q(s) \alpha(s) = Md. \quad (\text{B3})$$

Thus, in one realization of the RS, the number of accepted values d_A is the sum of d Bernoulli variables, accepted with probability M . The number of accepted counts, d_A , has therefore mean $\mathbb{E}[d_A]$ and variance $\sigma_A^2 = dM(1 - M)$. Compactly, we write: $d_A \sim \text{Bern}(d, M)$.

2. Optimal bin width

A central problem in numerical estimation of continuous PDFs is the estimation of the proper histogram bin width. If the bin width is too large, then the histogram fails to capture the intrinsic variations of the PDF, while if it is too small, then bin counts fluctuations fail to reproduce a smooth curve. For these reasons, choosing an appropriate bin width is not only of qualitative, but also of quantitative importance.

Let us consider a sample of d i.i.d. elements drawn from the PDF $p(s)$. For its discrete representation, we homogeneously partition the interval $s \in [0, \infty)$ in bins of width δ , such that bin j comprises values $s \in [\delta j, \delta(j + 1)]$. Let d_j be the number of samples falling in bin j . The empirical histogram PDF is

$$\tilde{\pi}_j = \frac{d_j}{\delta d}, \quad (\text{B4})$$

where $\tilde{\pi}_j$ is in principle a random variable, because d_j is the sum of Bernoulli counts. Indeed, the expectation of $\tilde{\pi}_j$ is the mean of the continuous PDF over the bin:

$$\mathbb{E}[\tilde{\pi}_j] = \pi_j = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} ds p(s). \quad (\text{B5})$$

Its variance, however, is, for $d_j \gg 1$,

$$\text{var}[\tilde{\pi}_j] = \frac{\text{var}[d_j]}{(\delta d)^2} = \frac{d_j}{(\delta d)^2}. \quad (\text{B6})$$

The optimal bin width is that which minimizes the fluctuations between $\tilde{\pi}_j$, π_j , and $p(s)$ in each bin, over the whole histogram. This quantity is called the integrated mean-squared error (IMSE), defined as

$$\text{IMSE} = \sum_j \delta (\tilde{\pi}_j - \pi_j)^2 + \sum_j \int_{j\delta}^{(j+1)\delta} ds (p(s) - \pi_j)^2. \quad (\text{B7})$$

The reason for the second term as follows. Denoting as $s_j = \delta(j + 1/2)$ the midpoint value, π_j approximates $p(s_j)$ to an order δ^2 , which is subleading in the IMSE.

We evaluate the two terms separately. The first is quantified by the variance of $\tilde{\pi}_j$, which comes from the Bernoulli distribution of d_j . When summed over bins, it gives

$$\sigma_{\text{hist}}^2 = \sum_j \delta \frac{d_j}{(\delta d)^2} = \frac{1}{\delta d}. \quad (\text{B8})$$

However, the variance of the integral mean is computed by expanding $q(s)$ around the midpoint δ_j :

$$\sigma_{\text{mean}}^2 = \frac{\mathcal{I}_{1,2}[p]}{12} \delta^2, \quad \mathcal{I}_{n,m}[p] = \int_0^\infty d\sigma \left(\frac{d^n p(\sigma)}{d\sigma^n} \right)^m. \quad (\text{B9})$$

The minimum of their sum yields the Freedman-Diaconis rule [79]:

$$\delta = \left(\frac{6}{\mathcal{I}_{1,2}[p]d} \right)^{1/3}. \quad (\text{B10})$$

For the Poisson distribution, $\mathcal{I}_{1,2} = 1/2$.

3. Choice of the extraction fraction

The extraction fraction f is a free parameter of the spectral decimation algorithm. However, if f is chosen too small, then gaps can be deemed Poissonian even if the underlying spectrum is that of a random matrix, like a GOE. Therefore, for a spectrum of d elements, one fixes f by looking at the first-bin value of a GOE.

For a GOE matrix, the bulk gap PDF is well-approximated by the Wigner-Dyson surmise $p_{\text{GOE}}(s) = \frac{\pi}{2} s e^{-s^2 \pi/4}$. However, it is exact only for 2×2 GOE matrices, but not for arbitrary size, see Ref. [60] for a derivation of lower and upper bounds on the GOE CDF. This discrepancy is noticeable at small s , where the true GOE PDF behaves as $\pi s/6$; hence, the value of the zero-bin PDF is

$$\pi_{0,\text{GOE}}(\delta) = \frac{\pi^2}{12} \delta, \quad (\text{B11})$$

and its variance is

$$\sigma_{\text{GOE},0}^2 = \frac{\pi_{0,\text{GOE}}(1 - \delta \pi_{0,\text{GOE}})}{d\delta}. \quad (\text{B12})$$

In practice, one chooses $f = \pi_{0,\text{GOE}}(\delta) + z_q \sigma_{\text{GOE},0}$, such that if, at a given iteration, a small-gap probability is compatible with a GOE, then the algorithm halts. Assuming d is sufficiently large, z_q is the probit of a normal distribution $\mathcal{N}(0, 1)$ of a certain quantile q . If at a certain iteration the algorithm ends, then the underlying remainder distribution is GOE-like with a zero-bin value compatible with $\pi_{0,\text{GOE}}$ within confidence q . For example, when $q = 0.95$, $z_q = 1.96$. This choice of f guaranties that any GOE gap distribution will result in the decimation algorithm halting, thereby returning its actual size.

In the case of systems with randomness, we utilized $f = \pi_{0,\text{GOE}}(\delta) + 1.96 \sigma_{\text{GOE},0}$ for sample-by-sample decimations and $f = \pi_{0,\text{GOE}}(\delta)$ when samples are pooled together, as fluctuations are averaged in each bin.

4. Sources of systematic error

The choices of a finite sample d and resolution δ introduce numerical artifacts, independent of f . This is especially relevant when the initial gaps are i.i.d. Poisson ones, as this case is a benchmark of the stability of the spectral decimation algorithm.

Indeed, in this case, the RS coincides theoretically with uniform sampling, as $\alpha(s) = p(0) = 1$. This implies that the ratio of accepted values, in each bin, α_j^{RS} is a constant. However, this does not occur in practice, because of the inherent variance of the Poisson sample. This phenomenon is more evident when either small initial samples are considered or at later stages of the decimation.

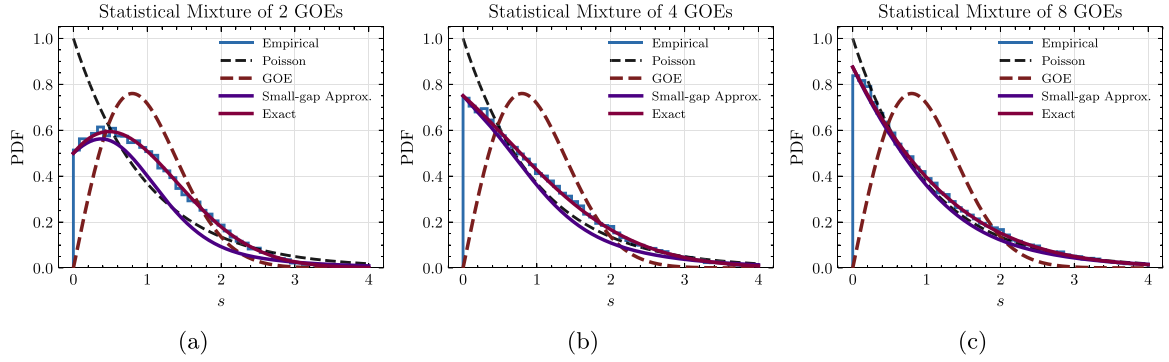


FIG. 6. Statistical Mixture of N GOEs of the same size $d_{\text{GOE}} = 10^4$, respectively, $N = 2$ (a), $N = 4$ (b), and $N = 8$ (c). At moderately small $s \lesssim 0.5$, the small s approximation, see Sec. II, qualitatively describes the gap distribution, whose exact PDF [22,60,62] is shown. In particular, it predicts the correct value of the PDF near the origin. As N increasing, the Poisson component of the mixture becomes dominant with respect to the non-Poissonian one, featuring level repulsion, and the approximation becomes better in describing the mixture for larger values of s .

For bin j of width δ , the mean Poisson probability is

$$\pi_j^P = \frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} ds e^{-s} = e^{-j\delta} \frac{1 - e^{-\delta}}{\delta}, \quad (\text{B13})$$

and the expected number of counts is $\mathbb{E}[d_j] = d_j^P = d\delta\pi_j^P$, with variance $\text{var}[d_j] = d_j^P$. The actual number of counts in bin j follows a Bernoulli distribution: $d_j \sim \text{Bern}(d, \delta\pi_j^P)$.

Therefore, the empirical acceptance ratio $\alpha_j = \mathbb{E}[d_j]/d_j$ has variance $\text{var}[\alpha_j] = 1/d_j$, implying that the standard deviation behaves as

$$\sigma_\alpha(s) = \frac{e^{s/2}}{\sqrt{d\delta}}. \quad (\text{B14})$$

As $\alpha(s)$ is a probability and cannot take values larger than 1, statistical noise becomes stronger as s increases. In particular, if one chooses a threshold $\sigma_\alpha(s) = 1 - \epsilon$, then one obtains noisy acceptance probabilities starting from

$$s_{\text{noise}}(\epsilon) = \ln(d\delta) - 2\ln(1 - \epsilon). \quad (\text{B15})$$

At the value $s_{\text{noise}}(1) = \ln(d\delta)$, the acceptance probability has the wildest oscillations, and it is possible that true Poisson gaps are refused only because of statistical noise.

A practical solution to this problem is to perform a normality test between the empirical counts and the expected Poisson counts. As this should only stabilize true Poisson samples, it is independent with the following logic:

(1) One checks if the first bin counts are compatible with the Poisson estimate;

(1) if the condition is true, one proceeds with the checks for the other bins. If the bin, say j , has counts compatible with the Poisson expectation, then the acceptance ratio is set to $\alpha_j \equiv 1$.

(2) if the condition is false, implying that the counts are not compatible with the Poisson estimate, the acceptance ratio for that bin is left untouched.

5. Variance of the output after resamplings

Let us consider the application of multiple decimations on the same spectrum of size d . Let the expectation value of the output be $d_{\text{out}} = D$, implying that the Poisson fraction is $M =$

$1 - D/d$. As the decimation algorithm is probabilistic, which is the variance of d_{out} obtained after resampling the spectrum R times?

When the spectrum has been decimated to the size D , the number of extracted Poisson gaps in the first bin has expectation $\mathbb{E}[d_0] = Md\delta_D$, where δ_D is the optimal bin width according to the Freedman-Diaconis rule.

The same quantity d_0 can be computed by tracking the number of Poisson gaps extracted at each iteration. At iteration m_* , the spectrum has been decimated to D , thus $D = d(1 - f)^{m_*}$ and

$$m_* = \frac{\ln(D/d)}{\ln(1 - f)}. \quad (\text{B16})$$

Henceforth, d_0 will be the sum of the Poisson gaps extracted at every iteration up to m_* , in average:

$$\begin{aligned} \mathbb{E}[d_0] &= \sum_{m=0}^{m_*} d_E(m)\delta(m) \\ &= 12^{1/3} \frac{f}{1 - (1 - f)^{2/3}} (d^{2/3} - D^{2/3}) \\ &\underset{f \rightarrow 0}{\sim} \frac{3}{2} 12^{1/3} (d^{2/3} - D^{2/3}). \end{aligned} \quad (\text{B17})$$

It is easily checked that

$$\text{var}[d_0] = \left(\frac{d\mathbb{E}[d_0]}{dD} \right)^2 \text{var}[D] = \delta_D^2 \text{var}[D]. \quad (\text{B18})$$

As d_0 is the sum of Bernoulli variables in the first bin, its variance is $\text{var}[d_0] = \mathbb{E}[d_0]$. This equality leads to

$$\text{var}[D] = \frac{Md}{\delta_D} = \left(1 - \frac{D}{d} \right) \frac{dD^{1/3}}{12^{1/3}}. \quad (\text{B19})$$

APPENDIX C: DECIMATION OF STATISTICAL MIXTURES

This Appendix provides a check of the stability of the decimation algorithm. The setup is studying a generic statistical mixture with N GOE spectra, each of them of size $d_i = 10^4$. Gap statistics for $N = 2, 4$ and 8 are reported in Fig. 6 and compared with the small- s approximation of Sec. II.

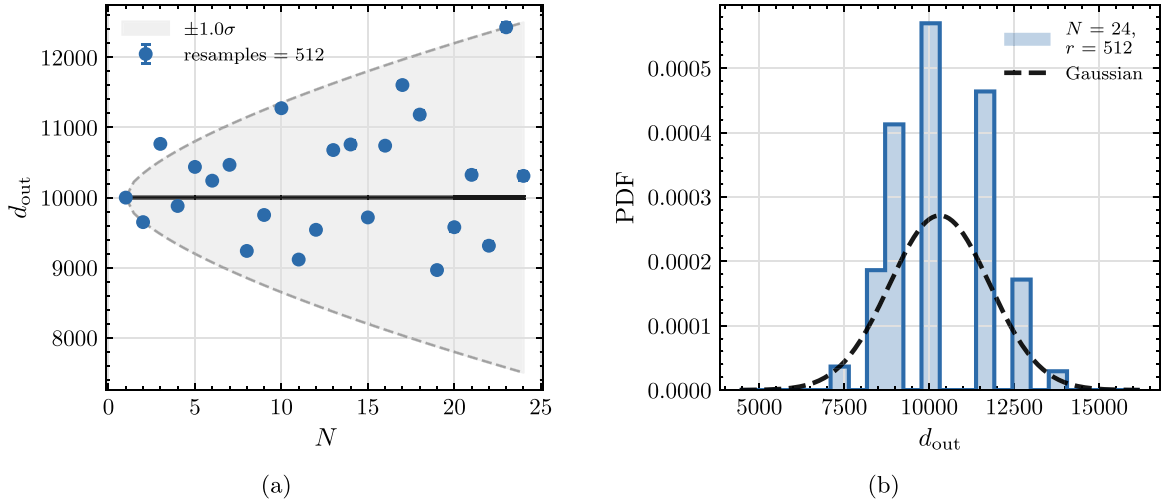


FIG. 7. Decimation of a single realization of a statistical mixture, with $d_{\text{GOE}} = 1$. (a) After 512 resamples for each $N \leq 24$, d_{out} is plotted against the expected value d_{GOE} . Almost all data points fall within one standard deviation from the mean. (b) Distribution of d_{out} for $N = 24$, resampled $r = 512$ times. Overall, the distribution follows a Gaussian with variance predicted from the spectral decimation, Eq. (B15). Because of the discrete extraction fraction values, d_{out} can only assume discrete values, thus the peaks in the PDF.

1. One-sample analysis

For a mixture of N GOE spectra of the same dimension d_{GOE} , Eq. (7) predicts that the spectral decimation algorithm output, $d_{\text{out},N}^{(1)}$, to be compatible with d_{GOE} . However, $d_{\text{out},N}^{(1)}$ is a random variable, as can be easily checked by repeating the decimation on the same spectrum multiple times (resample).

For one sample, the Poisson fraction, whose expectation is $\mathbb{E}[M] = 1 - 1/N$, is itself a random variable, quantified by the number of counts in the first bin: $d_0 = M\delta d$. As $\text{var}[d_0] = d_0$, one obtains that $\text{var}[M] = \frac{M}{\delta d}$. Hence,

$$\text{var}[d_{\text{out},N}^{(1)}] = \left(1 - \frac{1}{N}\right) \frac{d}{\delta} = 12^{-1/3} \left(1 - \frac{1}{N}\right) (Nd_{\text{GOE}})^{4/3}. \quad (\text{C1})$$

For large enough N , the standard deviation grows as $N^{1/3}$, implying that $d_{\text{out},N}^{(1)}$ can naively differ very much from d_{GOE} , while still being compatible.

This prediction can be easily checked with a numerical experiment, see Fig. 7. The knowledge of one sample, even if resampled many times, is not sufficient to estimate d_{GOE} with arbitrary accuracy. To this aim, the analysis of multiple samples is needed.

2. Analysis of multiple samples

The analysis of t samples of mixtures with the same N can be performed by generating (tN) GOE spectra of dimension d_{GOE} . It is also possible to diagonalize a sufficiently large number of GOE matrices—in our case 512,—and construct

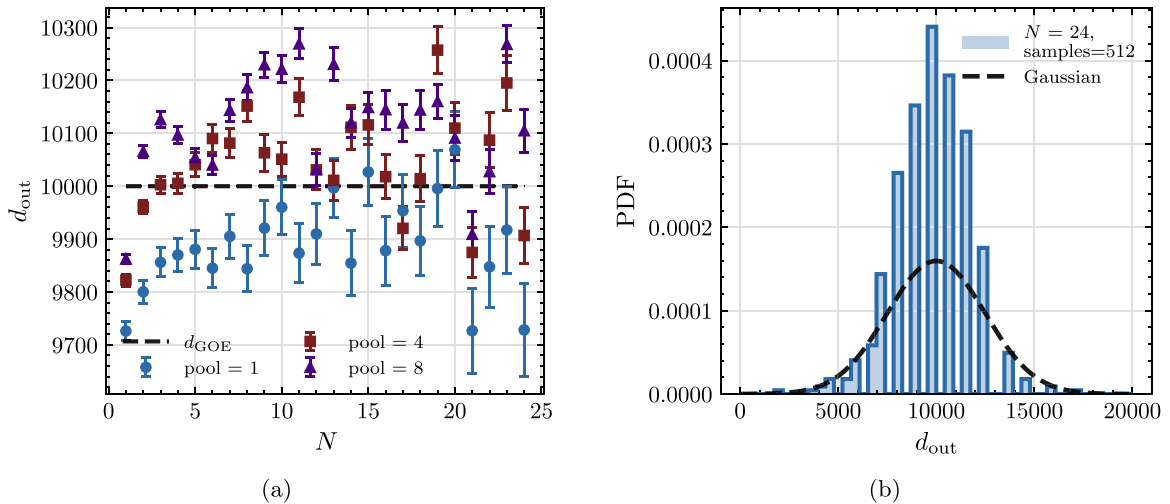


FIG. 8. Result of the decimation procedure on statistical mixtures generated by N identical GOEs of size $d_{\text{GOE}} = 10^4$. (a) Decimation algorithm output, d_{out} as function of the number N . Each data point is obtained by decimating 512 mixtures, individually or by pooling them in groups of p . For each pooling p , data are contained within a window of 3% of the true value d_{GOE} . (b) Focusing on $N = 24$ and $p = 1$, we check the distribution of the decimation outputs and observe that they are distributed according to a Gaussian with mean d_{GOE} and variance Eq. (C1). As both the initial size and extraction fraction are fixed, the possible d_{out} are discrete, hence the reason for the peaks.

the mixture by superposing N randomly chosen GOE spectra from the original pool. Because of computational efficiency, we resort to the last approach to generate the t samples.

The goal of this analysis is to assess whether the decimation can predict d_{GOE} accurately, up to the finite-size artifacts which are intrinsic to the statistics. These errors can be significantly reduced by pooling the t samples together since we are interested only in the average d_{out} .

Figure 8 shows that d_{GOE} can be accurately estimated from the decimation procedure and that the pooling technique is equivalent to averaging the d'_{out} s of individual samples. For each N , the outcomes of the decimation follow a Gaussian distribution with a mean of d_{GOE} and variance given by Eq. (C1).

3. Dynamical signatures of the CSS

We conclude this Appendix by considering the behavior of dynamical probes of quantum ergodicity, such as the spectral form factor (SFF), defined as

$$K(t) = \left\langle \sum_{m,n} e^{it(E_m - E_n)} \right\rangle. \quad (\text{C2})$$

The SFF is a measure of spectral rigidity, also known as two-level form factor [60], which depends not only on short-range correlations, and also on additional properties of the Hamiltonian, including long-range spectral correlations and global density-of-states effects.

We consider Hamiltonians H_N whose spectra are obtained as statistical mixtures of N independent GOE spectra, each of size 10^4 . Since all ensembles share the same semicircular density of states, we focused on $m = 10^3$ consecutive unfolded levels in the bulk and computed the corresponding SFF for different values of N . The results were further averaged over

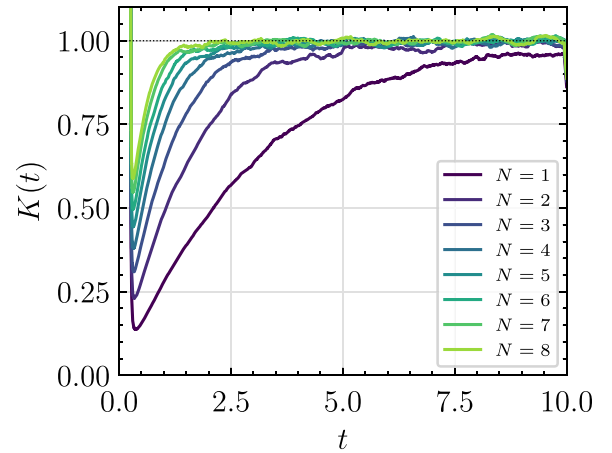


FIG. 9. Normalized spectral form factor (SFF) of the Hamiltonians H_N , obtained as mixtures of N GOE's of size 10^4 . For each line, $m = 10^3$ eigenvalues are sampled in the bulk of the density of states – Wigner's semicircle. We observe that, increasing N (equivalently decreasing the size of the CSS $d(N) = n/M$), the SFF interpolates between the GOE curve ($m = 1$) and the Poisson one ($m \rightarrow \infty$).

disorder realizations and locally smoothed in time to resolve the ramp structure. Its plot, for $N \leq 8$, is given in Fig. 9.

The effective size of the CSS in this construction is $d(N) = m/N$. Figure 9 shows a clear and systematic dependence of the SFF on the CSS size. As $d(N)$ decreases from its maximal value at $N = 1$ —corresponding to a pure GOE spectrum—two characteristic effects emerge:

- (1) the ramp of the SFF becomes progressively steeper;
- (2) the depth of the correlation hole is strongly suppressed.

Together, these effects continuously drive the SFF from the GOE behavior toward the Poisson limit $K_{\text{Poisson}}(t) = 1$, which is recovered as $N \rightarrow \infty$, namely when the spectrum becomes completely uncorrelated.

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