

Crosscap states and duality of Ising field theory in two dimensions

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We propose two distinct crosscap states for the two-dimensional (2D) Ising field theory. These two crosscap states, identifying Ising spins or dual spins (domain walls) at antipodal points, are shown to be related via the Kramers-Wannier duality transformation. We derive their Majorana free field representations and extend bosonization techniques to calculate correlation functions of the 2D Ising conformal field theory (CFT) with different crosscap boundaries. Away from criticality, we develop a conformal perturbation theory to calculate the Klein bottle entropy (norm-square of the crosscap overlap) as a universal scaling function [Phys. Rev. Lett. **130**, 151602 (2023)]. For the Ising field theory, our analytical results support the conjectured monotonicity of the Klein bottle entropy under relevant perturbations. The formalism provides a general framework for studying perturbed 2D CFTs on nonorientable manifolds.

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Introduction. Although the two-dimensional (2D) classical Ising model was invented one century ago [1,2], studies on this model continue to deepen our understanding of critical phenomena at the present time. In the absence of external fields, Kramers-Wannier duality [3] and the exact solution [4–8] reveal a second-order phase transition at a critical temperature. At this critical point, scale invariance is promoted to conformal invariance [9–11], identifying the critical theory as the simplest conformal field theory (CFT) in two dimensions, known as the 2D Ising CFT [12]. In the vicinity of the critical point, the correlation length is much larger than the lattice spacing, so the long-wavelength physics can be captured using a continuous field theory description. Considering the two relevant perturbations (thermal and magnetic perturbations, driven by temperature and external field, respectively), the scaling limit of the 2D classical Ising model is described by the 2D Ising field theory [13,14], which has attracted considerable interest [15–20].

There have been significant advances in the study of 2D CFTs on nonorientable manifolds [21–28], such as the Klein bottle and the real projective plane ($\mathbb{RP}^2$), but the 2D classical Ising model is relatively less explored in this regard [29–32]. In general, crosscap boundary states (and the associated crosscap coefficients) are crucial information for

understanding the properties of 2D CFTs on nonorientable manifolds. For the 2D Ising CFT, the crosscap coefficients and certain two-point correlators on the $\mathbb{RP}^2$ manifold have been employed to conduct a nontrivial benchmark in the bootstrap program [33,34].

In this Letter, we demonstrate that the 2D Ising field theory supports two distinct crosscap states, related by Kramers-Wannier duality, which can be explicitly constructed both at and away from criticality. We begin with a lattice formulation (quantum Ising chain) and propose two physically motivated crosscap states on the lattice. These lattice crosscap states identify Ising spins and dual spins (domain walls) at antipodal points, respectively. Remarkably, the overlaps between the lattice crosscap states and the eigenstates of the critical Ising chain are universal (without finite-size corrections), enabling us to derive Majorana free field representations of the crosscap states in the continuum limit. Within the 2D Ising CFT, one of them is the standard Pradisi-Sagnotti-Stanev (PSS) crosscap state [22], while the other corresponds to an equal-weight superposition of the standard PSS crosscap state and a generalized PSS crosscap state labeled by a simple current [23,35,36]. Away from criticality, we develop a conformal perturbation theory to calculate the overlap of crosscap states with perturbed ground states, which we call *crosscap overlap*. This formalism is applicable to any 2D CFT perturbed by relevant operators, thus providing a systematic method to calculate the Klein bottle entropy [37–42] (norm square of the crosscap overlap) as a universal scaling function of dimensionless coupling constants [43]. For the A-series unitary minimal models, our conformal perturbation theory analysis provides a perturbative proof of the conjectured monotonicity of the Klein bottle entropy [43]. For the 2D Ising field theory,

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our leading-order analytical results further support this monotonicity, strengthening the conjecture.

Ising crosscap states. We start with the Hamiltonian formulation of the 2D Ising field theory [14]

$$H = H_0 - g_1 \int_0^L dx \varepsilon(x) - g_2 \int_0^L dx \sigma(x), \quad (1)$$

where H_0 is the Hamiltonian of the 2D Ising CFT with central charge $c = 1/2$, and ε and σ are primary fields of the Ising CFT with conformal weight $(1/2, 1/2)$ and $(1/16, 1/16)$, respectively. g_1 and g_2 are the couplings of the two relevant perturbations. The Hamiltonian (1) is defined on a circle of length L and can be viewed as the generator of the transfer matrix for the 2D classical Ising model in the scaling limit.

To reveal different crosscap states, we first focus on the critical point ($g_1 = g_2 = 0$) and consider the critical Ising chain as its faithful lattice realization,

$$H_{\text{latt}} = - \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x - \sum_{j=1}^N \sigma_j^z, \quad (2)$$

where σ_j^α ($\alpha = x, z$) are the Pauli operators at site j and N is the total number of sites. We consider even N and adopt a periodic boundary condition ($\sigma_{N+1}^\alpha = \sigma_1^\alpha$) throughout this work. The two relevant perturbations in the Ising field theory [Eq. (1)] are realized by adding $H'_{\text{latt}} = (1 - h_z) \sum_{j=1}^N \sigma_j^z - h_x \sum_{j=1}^N \sigma_j^x$, where the primary field $\varepsilon(x)$ [$\sigma(x)$] is identified as $-\sigma_j^z$ (σ_j^x) with coupling $g_1 \sim 1 - h_z$ ($g_2 \sim h_x$).

The Ising chain in Eq. (2) has a global $\mathbb{Z}_2$ symmetry, $[H_{\text{latt}}, Q] = 0$ with $Q = \prod_{j=1}^N \sigma_j^z$. The eigenvalues of Q define $\mathbb{Z}_2$ even ($Q = 1$) and $\mathbb{Z}_2$ odd ($Q = -1$) subspaces, which, following the CFT convention [12], are called Neveu-Schwarz (NS) and Ramond (R) sectors, respectively. If we restrict ourselves to the NS sector, the critical Ising chain (2) is also invariant (self-dual) under the Kramers-Wannier duality transformation. For our purpose, we define the Kramers-Wannier unitary operator as $U_{\text{KW}} = e^{i\frac{\pi}{4}N} \prod_{j=1}^{N-1} (e^{-i\frac{\pi}{4}\sigma_j^z} e^{-i\frac{\pi}{4}\sigma_j^x \sigma_{j+1}^x}) e^{-i\frac{\pi}{4}\sigma_N^z}$ [44–46], which acts on lattice operators as $U_{\text{KW}} \sigma_j^z U_{\text{KW}}^\dagger = \sigma_j^x \sigma_{j+1}^x$ and $U_{\text{KW}} \sigma_j^x U_{\text{KW}}^\dagger = \sigma_1^y \prod_{l=2}^j \sigma_l^z$ in the NS sector.

We propose one of the two lattice crosscap states as follows,

$$|\mathcal{C}_{\text{latt}}^+\rangle = \prod_{j=1}^{N/2} (1 + \sigma_j^x \sigma_{j+N/2}^x) |\uparrow\rangle, \quad (3)$$

where $|\uparrow\rangle \equiv |\uparrow_1 \uparrow_2 \cdots \uparrow_N\rangle$ is the fully polarized state in the σ^z basis. When placing $|\mathcal{C}_{\text{latt}}^+\rangle$ on the 1D circle, maximally entangled pairs $|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle$ identify Ising spins between two antipodal sites j and $j + N/2$ [47–52], as depicted in Fig. 1(a).

The other crosscap state is obtained by applying the Kramers-Wannier duality transformation to $|\mathcal{C}_{\text{latt}}^+\rangle$,

$$|\mathcal{C}_{\text{latt}}^-\rangle \equiv U_{\text{KW}} |\mathcal{C}_{\text{latt}}^+\rangle = \prod_{j=1}^{N/2} (1 + \mu_j \mu_{j+N/2}) \frac{1}{\sqrt{2}} (|\Rightarrow\rangle + |\Leftarrow\rangle), \quad (4)$$

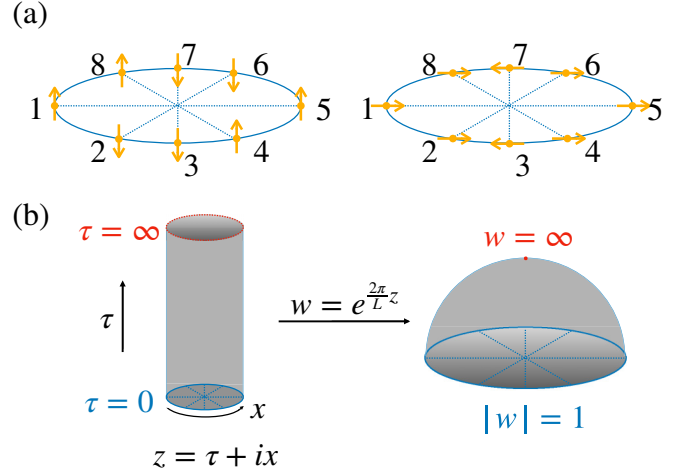


FIG. 1. Schematics of (a) typical configurations from lattice crosscap states $|\mathcal{C}_{\text{latt}}^+\rangle$ (left panel) and $|\mathcal{C}_{\text{latt}}^-\rangle$ (right panel) in the quantum Ising chain and (b) the conformal transformation from the semi-infinite cylinder with a crosscap boundary to the real projective plane ($\mathbb{RP}^2$).

where $\mu_j = \prod_{l=1}^j \sigma_l^z$ is the Ising disorder operator (dual spin). $|\Rightarrow\rangle \equiv |\rightarrow_1 \rightarrow_2 \cdots \rightarrow_N\rangle$ and $|\Leftarrow\rangle \equiv |\leftarrow_1 \leftarrow_2 \cdots \leftarrow_N\rangle$ are fully polarized states in the σ^x basis, i.e., $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$. As σ^z flips spins in the σ^x basis, $\mu_j \mu_{j+N/2}$ creates two *domain walls* at antipodal positions on top of $|\Rightarrow\rangle$ or $|\Leftarrow\rangle$, as illustrated in Fig. 1(a). Thus, the lattice crosscap state $|\mathcal{C}_{\text{latt}}^-\rangle$ identifies each pair of *dual spins* (domain walls) at the antipodal sites.

The two lattice crosscap states satisfy $U_{\text{KW}} |\mathcal{C}_{\text{latt}}^+\rangle = |\mathcal{C}_{\text{latt}}^-\rangle$, and therefore, are dual to each other under the Kramers-Wannier transformation. Remarkably, the dual state $|\mathcal{C}_{\text{latt}}^-\rangle$ can be expressed as [46]

$$|\mathcal{C}_{\text{latt}}^-\rangle = \frac{1}{\sqrt{2}} (|\mathcal{C}_{\text{latt}}^+\rangle + |\mathcal{C}'_{\text{latt}}\rangle), \quad (5)$$

which is an equal weight superposition of $|\mathcal{C}_{\text{latt}}^+\rangle$ and another modified crosscap state, defined by

$$|\mathcal{C}'_{\text{latt}}\rangle = \prod_{j=1}^{N/2} (1 - \sigma_j^x \sigma_{j+N/2}^x) |\uparrow\rangle. \quad (6)$$

The meaning of these lattice crosscap states will become more transparent in the continuum limit, as we shall demonstrate below.

To identify their field theory counterparts, we adopt the strategy of computing the overlaps of $|\mathcal{C}_{\text{latt}}^\pm\rangle$ with eigenstates of the critical Ising chain (2). As the eigenstates of the critical Ising chain can be easily identified with primary and descendant states of the 2D Ising CFT, the overlaps yield the expansion coefficients of the crosscap states in the Ising CFT basis.

To compute the overlaps, we solve the critical Ising chain (2) by using the Jordan-Wigner transformation [53], $\sigma_j^x = (c_j^\dagger + c_j) \prod_{l=1}^{j-1} e^{i\pi c_l^\dagger c_l}$ and $\sigma_j^z = 2c_j^\dagger c_j - 1$, and represent the eigenstates in the fermionic basis. As $Q|\mathcal{C}_{\text{latt}}^\pm\rangle = |\mathcal{C}_{\text{latt}}^\pm\rangle$, both crosscap states live in the NS sector, so it is sufficient to

consider the fermionized Hamiltonian of Eq. (2) in the NS sector. After the Fourier transform $c_j = \frac{1}{\sqrt{N}} \sum_{k \in \text{NS}} e^{ikj} c_k$ (lattice spacing set to one) and a Bogoliubov transformation, we arrive at

$$H_{\text{latt}}^{\text{NS}} = \sum_{k \in \text{NS}} 4 \cos \frac{k}{2} \left(d_k^\dagger d_k - \frac{1}{2} \right), \quad (7)$$

where $d_k = e^{i\pi/4} \sin \frac{k}{4} c_k + e^{-i\pi/4} \cos \frac{k}{4} c_{-k}^\dagger$ is the annihilation operator of the Bogoliubov mode and $k \in \text{NS}$ denotes allowed single-particle momenta in the NS sector: $k = \pm[\pi - \frac{2\pi}{N}(n_k - \frac{1}{2})]$ with $n_k = 1, 2, \dots, N/2$ ($n_k \in \mathbb{Z}^+$ in the continuum limit). The ground state of Eq. (7), annihilated by all d_k , can be written as

$$|0\rangle_d = \prod_{k>0} \left(\sin \frac{k}{4} + i \cos \frac{k}{4} c_k^\dagger c_{-k}^\dagger \right) |0\rangle_c, \quad (8)$$

where $|0\rangle_c$ is the vacuum of the Jordan-Wigner fermions, $c_j|0\rangle_c = 0 \forall j$ ($|0\rangle_c$ is just the fully polarized state $|\downarrow\rangle \equiv |\downarrow_1 \downarrow_2 \dots \downarrow_N\rangle$ in the spin basis). Excited states of Eq. (7) are obtained by applying an even number of $d_k^\dagger$'s (with distinct momenta) on top of the ground state $|0\rangle_d$.

A key observation enabling the overlap computation is that $|\mathcal{C}_{\text{latt}}^+\rangle$ can be expressed as a sum of two fermionic Gaussian states

$$\begin{aligned} |\mathcal{C}_{\text{latt}}^+\rangle &= \frac{1-i}{2} \prod_{j=1}^{N/2} (1 + i c_j^\dagger c_{j+N/2}^\dagger) |0\rangle_c \\ &\quad + \frac{1+i}{2} \prod_{j=1}^{N/2} (1 - i c_j^\dagger c_{j+N/2}^\dagger) |0\rangle_c. \end{aligned} \quad (9)$$

After transforming Eq. (9) into momentum space, its overlaps with the eigenstates of Eq. (7) can be calculated analytically. We find that only eigenstates of the form $|\psi_{k_1 \dots k_M}\rangle = \prod_{\alpha=1}^M (i d_{-k_\alpha}^\dagger d_{k_\alpha}^\dagger) |0\rangle_d$ ($0 < k_1 < \dots < k_M < \pi$) have nonvanishing overlaps with $|\mathcal{C}_{\text{latt}}^+\rangle$ [46]:

$$\langle \psi_{k_1 \dots k_M} | \mathcal{C}_{\text{latt}}^+ \rangle = \begin{cases} (-1)^{\sum_{\alpha=1}^M n_{k_\alpha}} \sqrt{\frac{2+\sqrt{2}}{2}}, & M \text{ even}, \\ i(-1)^{\sum_{\alpha=1}^M n_{k_\alpha}} \sqrt{\frac{2-\sqrt{2}}{2}}, & M \text{ odd}. \end{cases} \quad (10)$$

Most remarkably, the overlaps in Eq. (10) are free of any finite-size corrections and valid already for $N \geq 4$ [54].

For the other lattice crosscap state $|\mathcal{C}_{\text{latt}}^-\rangle$, the overlaps with the eigenstates of Eq. (7) can be computed with the help of the Kramers-Wannier duality. Using $U_{\text{KW}} d_k^\dagger U_{\text{KW}}^\dagger = i e^{-ik/2} d_k^\dagger$ and $U_{\text{KW}} |0\rangle_d = |0\rangle_d$ [46], we obtain $\langle \psi_{k_1 \dots k_M} | \mathcal{C}_{\text{latt}}^- \rangle = (-1)^M \langle \psi_{k_1 \dots k_M} | \mathcal{C}_{\text{latt}}^+ \rangle$.

In the continuum limit, the low-energy effective Hamiltonian for the lattice model (7) is just the Ising CFT Hamiltonian in the NS sector [12]

$$H_0^{\text{NS}} = \frac{2\pi}{L} \left[\sum_{n=1}^{\infty} \left(n - \frac{1}{2} \right) \left(b_{n-\frac{1}{2}}^\dagger b_{n-\frac{1}{2}} + \bar{b}_{n-\frac{1}{2}}^\dagger \bar{b}_{n-\frac{1}{2}} \right) - \frac{1}{24} \right], \quad (11)$$

with $b_{n-\frac{1}{2}}^\dagger$ ($b_{n-\frac{1}{2}}$) being creation (annihilation) operators of the left-moving Majorana fermion (right-moving ones

are similar). The connection of Eq. (11) with the lattice model is through the identification of low-energy modes in Eq. (7): $(d_k, d_k^\dagger) \Leftrightarrow (b_{n_k-\frac{1}{2}}, b_{n_k-\frac{1}{2}}^\dagger)$ and $(i d_{-k}, -i d_{-k}^\dagger) \Leftrightarrow (\bar{b}_{n_k-\frac{1}{2}}, \bar{b}_{n_k-\frac{1}{2}}^\dagger)$. This is valid for k close to $\pm\pi$, where the dispersion of the lattice model (7) can be linearized. The Kramers-Wannier duality of Eq. (11) is inherited from the lattice model: $U_{\text{KW}} b_{n-\frac{1}{2}} U_{\text{KW}}^\dagger = b_{n-\frac{1}{2}}$ and $U_{\text{KW}} \bar{b}_{n-\frac{1}{2}} U_{\text{KW}}^\dagger = -\bar{b}_{n-\frac{1}{2}}$.

In the continuum limit, the eigenstates $|\psi_{k_1 \dots k_M}\rangle$ which have nonvanishing overlaps with crosscap states [Eq. (10)] become $\prod_{\alpha=1}^M b_{n_{k_\alpha}-\frac{1}{2}}^\dagger \bar{b}_{n_{k_\alpha}-\frac{1}{2}}^\dagger |0\rangle_{\text{NS}}$ in the Ising CFT, where $|0\rangle_{\text{NS}}$ is the vacuum in the NS sector. By using Eq. (10), the continuum counterparts of the crosscap states are expressed in the Ising CFT basis as

$$\begin{aligned} |\mathcal{C}_\pm\rangle &= \frac{e^{i\pi/8}}{\sqrt{2}} \exp \left[\pm \sum_{n=1}^{\infty} (-1)^n b_{n-\frac{1}{2}}^\dagger \bar{b}_{n-\frac{1}{2}}^\dagger \right] |0\rangle_{\text{NS}} \\ &\quad + \frac{e^{-i\pi/8}}{\sqrt{2}} \exp \left[\mp \sum_{n=1}^{\infty} (-1)^n b_{n-\frac{1}{2}}^\dagger \bar{b}_{n-\frac{1}{2}}^\dagger \right] |0\rangle_{\text{NS}}. \end{aligned} \quad (12)$$

The crosscap Ishibashi states of the 2D Ising CFT are labeled by the primary fields as $|a\rangle_{\mathcal{C}}$ with $a = \mathbb{1}, \sigma, \varepsilon$ [21]. The crosscap states $|\mathcal{C}_\pm\rangle$ obtained in Eq. (12) are just linear combinations of two Ishibashi states [46]

$$|\mathcal{C}_\pm\rangle = \sqrt{\frac{2+\sqrt{2}}{2}} |\mathbb{1}\rangle_{\mathcal{C}} \pm \sqrt{\frac{2-\sqrt{2}}{2}} |\varepsilon\rangle_{\mathcal{C}}. \quad (13)$$

Comparing with the PSS crosscap states for the Ising CFT [23,35], $|\mathcal{C}_\mathbb{1}\rangle = \sqrt{2} \cos \frac{\pi}{8} |\mathbb{1}\rangle_{\mathcal{C}} + \sqrt{2} \sin \frac{\pi}{8} |\varepsilon\rangle_{\mathcal{C}}$ and $|\mathcal{C}_\varepsilon\rangle = \sqrt{2} \sin \frac{\pi}{8} |\mathbb{1}\rangle_{\mathcal{C}} - \sqrt{2} \cos \frac{\pi}{8} |\varepsilon\rangle_{\mathcal{C}}$, it becomes clear that $|\mathcal{C}_+\rangle = |\mathcal{C}_\mathbb{1}\rangle$ and $|\mathcal{C}_-\rangle = \frac{1}{\sqrt{2}} (|\mathcal{C}_\mathbb{1}\rangle + |\mathcal{C}_\varepsilon\rangle)$ [36]. Interestingly, Eqs. (5) and (6) provide a perfect lattice correspondence of the latter relation. It becomes evident once we notice that, in the continuum limit, $|\mathcal{C}_{\text{latt}}^+\rangle$ corresponds to $|\mathcal{C}_\varepsilon\rangle$ [46], by the same type of calculation that identifies $|\mathcal{C}_{\text{latt}}^+\rangle$ with $|\mathcal{C}_\mathbb{1}\rangle$.

Crosscap correlators. For the Ising CFT, the two crosscap states $|\mathcal{C}_\pm\rangle$ are indistinguishable from the partition function (with crosscap boundaries): $\langle \mathcal{C}_+ | e^{-\beta H_0} | \mathcal{C}_+ \rangle = \langle \mathcal{C}_- | e^{-\beta H_0} | \mathcal{C}_- \rangle$. To distinguish them, we calculate the crosscap correlators, which are defined as the conformal correlation functions on the semi-infinite cylinder with a crosscap state at the boundary, in the complex coordinate $z = \tau + ix$ [$\tau \in [0, \infty)$ and $x \in [0, L]$]. Geometrically, it is more convenient to perform a conformal transformation $w = e^{\frac{2\pi z}{L}}$ and interpret the crosscap correlators as the correlation functions on the real projective plane ($\mathbb{RP}^2$), as depicted in Fig. 1(b).

We extend bosonization techniques [12,55] to the Ising CFT with crosscap boundaries. In this approach, the Ising crosscap correlators can be expressed in terms of those of the $\mathbb{Z}_2$ -orbifold compactified boson CFT, enabling a systematic calculation. For instance, the crosscap correlators of two Ising spin fields are

$$\text{NS} \langle 0 | \sigma(w_1, \bar{w}_1) \sigma(w_2, \bar{w}_2) | \mathcal{C}_\pm \rangle = \sqrt{\frac{2+\sqrt{2}}{2}} \frac{G_\pm(\eta)}{|w_1 - w_2|^4}, \quad (14)$$

with

$$G_{\pm}(\eta) = \frac{\frac{\sqrt{2}}{2}\sqrt{1+\sqrt{1-\eta}} \pm \frac{2-\sqrt{2}}{2}\sqrt{1-\sqrt{1-\eta}}}{(1-\eta)^{\frac{1}{8}}}, \quad (15)$$

where w_1 and w_2 are the complex coordinates on $\mathbb{RP}^2$ and $\eta = \frac{|w_1-w_2|^2}{(1+|w_1|^2)(1+|w_2|^2)}$ is the crosscap cross ratio. The correlator with the boundary state $|\mathcal{C}_+\rangle$ is indeed consistent with that presented in Refs. [22,33,34]. Within the bosonization framework, we also obtain multipoint crosscap correlators for the 2D Ising CFT [46].

Universal scaling functions. While expressed in the Ising CFT basis, the validity of Ising crosscap states in Eq. (12) extends to the 2D Ising field theory when relevant perturbations are present. The resulting crosscap overlap, defined as the overlap of crosscap states with perturbed ground states, is a universal scaling function of the dimensionless couplings [43]. Despite its great potential for identifying critical theories in numerics, there was no systematic method to compute it analytically. Here, we take the first step by developing a conformal perturbation theory that allows for a perturbative expansion of the crosscap overlap in powers of the coupling constants, applicable to general 2D CFTs with perturbations.

We consider the Hamiltonian $H = H_0 + H_1$, where H_0 is the Hamiltonian of a unitary CFT defined on a circle of length L . The perturbation is given by $H_1 = -g \int_0^L dx \varphi(x)$, where φ is a primary operator of conformal dimension $(h, \bar{h})$ (we take $h = \bar{h}$ for simplicity and $h < 1$ so that the perturbation is relevant), normalized as $\lim_{x \rightarrow \infty} \lim_{L \rightarrow \infty} x^{4h} \langle \varphi(0) \varphi(x) \rangle = 1$ with the expectation value taken in the CFT vacuum. For notational simplicity we consider a single relevant perturbation; the extension to multiple perturbations is straightforward. The perturbed ground state, denoted as $|\psi_0(s)\rangle$, depends only on dimensionless coupling $s = gL^{2-2h}$ [56]. Denoting the crosscap state as $|\mathcal{C}\rangle$, our goal is to compute the crosscap overlap $\langle \psi_0(s) | \mathcal{C} \rangle$, which depends solely on s and is thus a universal scaling function [43].

To perform a perturbative analysis, we factorize the crosscap overlap into two parts,

$$\langle \psi_0(s) | \mathcal{C} \rangle = Z(s) \exp\left[\frac{1}{2}W(s)\right], \quad (16)$$

with $Z(s) = \langle \psi_0(s) | \mathcal{C} \rangle / \langle \psi_0(s) | \psi_0(0) \rangle$ and $\exp[\frac{1}{2}W(s)] = \langle \psi_0(s) | \psi_0(0) \rangle$. For practical calculations, $Z(s)$ and $W(s)$ are written as

$$Z(s) = \lim_{\beta \rightarrow \infty} \frac{\langle \psi_0(0) | \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} | \mathcal{C} \rangle}{\langle \psi_0(0) | \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} | \psi_0(0) \rangle},$$

$$W(s) = \lim_{\beta \rightarrow \infty} \left[\langle \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} \rangle_c - \frac{\pi\beta}{6L} \delta c(s) \right], \quad (17)$$

where $\mathcal{T}$ stands for time ordering and $H_1(\tau)$ is the perturbation term in the interaction picture, $\langle \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} \rangle_c$ denotes the *connected* contribution of $\langle \psi_0(0) | \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} | \psi_0(0) \rangle$, and $\delta c(s) = \lim_{\beta \rightarrow \infty} \frac{6L}{\pi\beta} \langle \mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)} \rangle_c$ is the change of the “running” central charge $c(s) \equiv \delta c(s) + c$. With the knowledge of the CFT correlators (on $\mathbb{RP}^2$ and the plane), the expansion of the time-ordered exponential $\mathcal{T} e^{-\int_0^\beta d\tau H_1(\tau)}$ (in

powers of the coupling g) allows us to calculate the crosscap overlap in a perturbative way.

We obtain a general expression for the first-order correction to the crosscap overlap [46],

$$\delta \langle \psi_0(s) | \mathcal{C} \rangle = \mathcal{A} \cdot \frac{{}_2F_1(h, 2h, h+1; -1)}{2(2\pi)^{1-2h} h} \cdot s + \dots, \quad (18)$$

where $\mathcal{A} = \lim_{w \rightarrow 0} \langle \psi_0(0) | \varphi(w) | \mathcal{C} \rangle$ is the crosscap coefficient in the one-point conformal crosscap correlator, and ${}_2F_1(a, b, c; x)$ is the hypergeometric function. Since ${}_2F_1(h, 2h, h+1; -1) > 0$ for $h < 1$, the sign of the first-order correction is solely determined by $\mathcal{A}$. As a concrete example, consider the A -series unitary minimal models [12], with central charge $c = 1 - \frac{6}{m(m+1)}$, $m \geq 3$. The primary fields are labeled by integer pairs (r, s) with $1 \leq r \leq m-1$ and $1 \leq s \leq m$, subject to $r+s = \text{even}$. The crosscap state associated with the identity field $(1,1)$ can be expressed in terms of the Ishibashi states as

$$|\mathcal{C}_{(1,1)}\rangle = \sum_{\substack{r+s=\text{even} \\ 1 \leq r \leq m-1, 1 \leq s \leq m}} \mathcal{A}_{(r,s)} |\mathcal{C}_{(r,s)}\rangle, \quad (19)$$

where $\mathcal{A}_{(r,s)}$ are the crosscap coefficients of the primary fields $\varphi_{(r,s)}$ [23,57],

$$\mathcal{A}_{(r,s)} = O_r O_s \sqrt{\frac{\sqrt{2} \cot\left(\frac{\pi r}{2m}\right) \cot\left(\frac{\pi s}{2(m+1)}\right)}{\sqrt{m(m+1)}}}, \quad (20)$$

with $O_n = \frac{1-(-1)^n}{2}$. Since $\mathcal{A}_{(r,s)} \geq 0$ for all (r, s) , Eq. (18) provides a perturbative proof of the monotonicity conjecture [43] for the crosscap overlap (and hence Klein bottle entropy) under the renormalization group flow in A -series unitary minimal models whenever the crosscap coefficient of the perturbing field is nonvanishing.

The conformal perturbation theory outlined above is directly applicable to the Ising field theory in Eq. (1), where $|\mathcal{C}_{(1,1)}\rangle$ in Eq. (19) reduces to $|\mathcal{C}_+\rangle$. The results are summarized as follows: (i) For the thermal perturbation, the crosscap overlap $\langle \psi_0(s_1) | \mathcal{C}_+ \rangle$, with $s_1 = g_1 L$, has been computed nonperturbatively in Ref. [43]. Our conformal perturbation theory calculations agree with the nonperturbative result up to second order [46]. (ii) For the magnetic perturbation, conformal perturbation theory shows that the crosscap overlap is an even function of the dimensionless coupling, $\langle \psi_0(s_2) | \mathcal{C}_+ \rangle = \sum_{n=0}^{\infty} \mathcal{C}_{2n} s_2^{2n}$, with $s_2 = g_2 L^{15/8}$, since crosscap correlators with an odd number of σ fields vanish. Utilizing the two-point crosscap correlator [Eq. (14)], we obtain the leading-order correction $\mathcal{C}_2 \approx -1.63528$ [46]. This shows that the crosscap overlap reaches its maximum at $s_2 = 0$, thereby supporting the monotonicity conjecture in Ref. [43]. An important future direction is to extend this perturbative proof to other perturbed CFTs in which the first-order correction of the crosscap overlap vanishes.

Summary and outlook. In conclusion, we have thoroughly investigated two distinct crosscap states in the 2D Ising field theory, connected by Kramers-Wannier duality. By developing a conformal perturbation theory for the crosscap overlap, we extended its definition beyond CFT and established a perturbative expansion in the coupling constant. This

framework not only deepens our understanding of field theories on nonorientable manifolds, but also enriches the numerical toolbox for identifying critical theories in lattice models [51,58–60]. The identification of lattice crosscap states and the bosonization techniques developed here also provide a natural route to the study of nonequilibrium dynamics involving crosscap states [61–66].

Beyond the Ising model, many field theories exhibit dualities (e.g., $\mathbb{Z}_N$ parafermion CFTs [67]), and it is natural to ask how crosscap states transform under such dualities. Exploring this interplay may reveal new structures of nonorientable CFTs. A second intriguing direction is to clarify the link between Klein bottle entropy and renormalization-group flow [43], in analogy with the c theorem [68] and g theorem [69,70]. Finally, crosscap overlaps may provide a direct route to extract crosscap coefficients in 3D CFTs [60], thereby

complementing bootstrap results where only ratios of such coefficients are accessible [33].

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Data availability. The data that support the findings of this article are openly available [71].

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- [1] E. Ising, Beitrag zur Theorie des Ferromagnetismus, *Z. Phys.* **31**, 253 (1925).
 - [2] B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, MA, 1973).
 - [3] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part I, *Phys. Rev.* **60**, 252 (1941).
 - [4] L. Onsager, Crystal statistics. I. A two-dimensional model with an order-disorder transition, *Phys. Rev.* **65**, 117 (1944).
 - [5] B. Kaufman, Crystal statistics. II. Partition function evaluated by spinor analysis, *Phys. Rev.* **76**, 1232 (1949).
 - [6] B. Kaufman and L. Onsager, Crystal statistics. III. Short-range order in a binary Ising lattice, *Phys. Rev.* **76**, 1244 (1949).
 - [7] C. N. Yang, The spontaneous magnetization of a two-dimensional Ising model, *Phys. Rev.* **85**, 808 (1952).
 - [8] T. D. Schultz, D. C. Mattis, and E. H. Lieb, Two-dimensional Ising model as a soluble problem of many fermions, *Rev. Mod. Phys.* **36**, 856 (1964).
 - [9] A. M. Polyakov, Conformal symmetry of critical fluctuations, *JETP Lett.* **12**, 381 (1970).
 - [10] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys. B* **241**, 333 (1984).
 - [11] S. Smirnov, Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model, *Ann. Math.* **172**, 1435 (2010).
 - [12] P. D. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, New York, 1997).
 - [13] A. B. Zamolodchikov, Integrals of motion and S-matrix of the (scaled) $T = T_c$ Ising model with magnetic field, *Int. J. Mod. Phys. A* **04**, 4235 (1989).
 - [14] P. Fonseca and A. B. Zamolodchikov, Ising field theory in a magnetic field: Analytic properties of the free energy, *J. Stat. Phys.* **110**, 527 (2003).
 - [15] G. Delfino, Integrable field theory and critical phenomena. The Ising model in a magnetic field, *J. Phys. A: Math. Gen.* **37**, R45 (2004).
 - [16] S. B. Rutkevich, Large- n excitations in the ferromagnetic Ising field theory in a weak magnetic field: Mass spectrum and decay widths, *Phys. Rev. Lett.* **95**, 250601 (2005).
 - [17] G. Delfino, P. Grinza, and G. Mussardo, Decay of particles above threshold in the Ising field theory with magnetic field, *Nucl. Phys. B* **737**, 291 (2006).
 - [18] A. Konechny, Rg boundaries and interfaces in Ising field theory, *J. Phys. A: Math. Theor.* **50**, 145403 (2017).
 - [19] B. Gabai and X. Yin, On the S-matrix of Ising field theory in two dimensions, *J. High Energy Phys.* **10** (2022) 168.
 - [20] H.-L. Xu, On the analyticity of the lightest particle mass of Ising field theory in a magnetic field, [arXiv:2405.09091](https://arxiv.org/abs/2405.09091).
 - [21] N. Ishibashi, The boundary and crosscap states in conformal field theories, *Mod. Phys. Lett. A* **04**, 251 (1989).
 - [22] D. Fioravanti, G. Pradisi, and A. Sagnotti, Sewing constraints and non-orientable open strings, *Phys. Lett. B* **321**, 349 (1994).
 - [23] G. Pradisi, A. Sagnotti, and Y. Stanev, Planar duality in $SU(2)$ WZW models, *Phys. Lett. B* **354**, 279 (1995).
 - [24] G. Pradisi, A. Sagnotti, and Y. S. Stanev, Completeness conditions for boundary operators in 2d conformal field theory, *Phys. Lett. B* **381**, 97 (1996).
 - [25] J. Fuchs, L. Huiszoon, A. Schellekens, C. Schweigert, and J. Walcher, Boundaries, crosscaps and simple currents, *Phys. Lett. B* **495**, 427 (2000).
 - [26] I. Brunner and K. Hori, Notes on orientifolds of rational conformal field theories, *J. High Energy Phys.* **07** (2004) 023.
 - [27] Z. Wei, Holographic dual of crosscap conformal field theory, *J. High Energy Phys.* **03** (2025) 086.
 - [28] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory* (Springer, Berlin, 2009).
 - [29] W. T. Lu and F. Y. Wu, Ising model on nonorientable surfaces: Exact solution for the Möbius strip and the Klein bottle, *Phys. Rev. E* **63**, 026107 (2001).
 - [30] C. H. O. Chui and P. A. Pearce, Finitized conformal spectra of the Ising model on the Klein bottle and Möbius strip, *J. Stat. Phys.* **107**, 1167 (2002).
 - [31] D. Cimasoni, The dimer and Ising models on Klein bottles, *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* **11**, 503 (2023).
 - [32] H. Shimizu and A. Ueda, Tensor network simulations for non-orientable surfaces, [arXiv:2402.15507](https://arxiv.org/abs/2402.15507).

- [33] Y. Nakayama, Bootstrapping critical Ising model on three dimensional real projective space, *Phys. Rev. Lett.* **116**, 141602 (2016).
- [34] Y. Nakayama and H. Ooguri, Bulk local states and crosscaps in holographic CFT, *J. High Energy Phys.* **10** (2016) 085.
- [35] L. Huiszoon, A. Schellekens, and N. Sousa, Klein bottles and simple currents, *Phys. Lett. B* **470**, 95 (1999).
- [36] W. Harada, J. Kaidi, Y. Kusuki, and Y. Liu, New crosscap states, [arXiv:2508.18357](https://arxiv.org/abs/2508.18357).
- [37] H.-H. Tu, Universal entropy of conformal critical theories on a Klein bottle, *Phys. Rev. Lett.* **119**, 261603 (2017).
- [38] W. Tang, L. Chen, W. Li, X. C. Xie, H.-H. Tu, and L. Wang, Universal boundary entropies in conformal field theory: A quantum Monte Carlo study, *Phys. Rev. B* **96**, 115136 (2017).
- [39] L. Chen, H.-X. Wang, L. Wang, and W. Li, Conformal thermal tensor network and universal entropy on topological manifolds, *Phys. Rev. B* **96**, 174429 (2017).
- [40] H.-X. Wang, L. Chen, H. Lin, and W. Li, Topological and geometric universal thermodynamics in conformal field theory, *Phys. Rev. B* **97**, 220407(R) (2018).
- [41] W. Tang, X. C. Xie, L. Wang, and H.-H. Tu, Klein bottle entropy of compactified boson conformal field theory, *Phys. Rev. B* **99**, 115105 (2019).
- [42] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, Topological aspects of the critical three-state Potts model, *J. Phys. A: Math. Theor.* **55**, 235002 (2022).
- [43] Y. Zhang, A. Hulsch, H.-C. Zhang, W. Tang, L. Wang, and H.-H. Tu, Universal scaling of Klein bottle entropy near conformal critical points, *Phys. Rev. Lett.* **130**, 151602 (2023).
- [44] B.-B. Chen, H.-H. Tu, Z. Y. Meng, and M. Cheng, Topological disorder parameter: A many-body invariant to characterize gapped quantum phases, *Phys. Rev. B* **106**, 094415 (2022).
- [45] N. Seiberg and S.-H. Shao, Majorana chain and Ising model - (non-invertible) translations, anomalies, and emanant symmetries, *SciPost Phys.* **16**, 064 (2024).
- [46] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/41qh-vws4> for further details, which include a brief review of the exact solution of the transverse-field Ising chain and its Kramers-Wannier duality; the construction and fermionic representation of lattice crosscap states and their overlaps with eigenstates of the transverse-field Ising chain; the identification of continuum crosscap states in the Ising conformal field theory; the bosonization of conformal crosscap states and the calculation of crosscap correlators; and the conformal perturbation theory for crosscap overlaps, with applications to the Ising field theory and the $\mathbb{Z}_3$ parafermion conformal field theory, which includes Refs. [72–76].
- [47] J. Caetano and S. Komatsu, Crosscap states in integrable field theories and spin chains, *J. Stat. Phys.* **187**, 30 (2022).
- [48] C. Ekman, Crosscap states in the XXX spin-1/2 spin chain, [arXiv:2207.12354](https://arxiv.org/abs/2207.12354).
- [49] T. Gombor, Integrable crosscap states in $gl(n)$ spin chains, *J. High Energy Phys.* **10** (2022) 096.
- [50] M. He and Y. Jiang, Integrable crosscap states: From spin chains to 1D Bose gas, *J. High Energy Phys.* **08** (2023) 079.
- [51] B.-Y. Tan, Y. Zhang, H.-C. Zhang, W. Tang, L. Wang, H.-H. Tu, and Y.-H. Wu, Extracting the Luttinger parameter from a single wave function, *Phys. Rev. Lett.* **134**, 076501 (2025).
- [52] Y. Yoneta, Thermal pure states for systems with antiunitary symmetries and their tensor network representations, *Phys. Rev. Res.* **6**, L042062 (2024).
- [53] P. Pfeuty, The one-dimensional Ising model with a transverse field, *Ann. Phys.* **57**, 79 (1970).
- [54] J. Kim and D.-H. Kim independently obtained the crosscap overlap for the ground state $|0\rangle_d$ and proved that it is free of finite-size corrections.
- [55] J. B. Zuber and C. Itzykson, Quantum field theory and the two-dimensional Ising model, *Phys. Rev. D* **15**, 2875 (1977).
- [56] H. Saleur and C. Itzykson, Two-dimensional field theories close to criticality, *J. Stat. Phys.* **48**, 449 (1987).
- [57] M. Bianchi, G. Pradisi, and A. Sagnotti, Planar duality in the discrete series, *Phys. Lett. B* **273**, 389 (1991).
- [58] Z.-Q. Li, L.-P. Yang, Z. Y. Xie, H.-H. Tu, H.-J. Liao, and T. Xiang, Critical properties of the two-dimensional q -state clock model, *Phys. Rev. E* **101**, 060105(R) (2020).
- [59] J. Kim, D. Kim, and D.-H. Kim, Neural-network quantum-state study of the long-range antiferromagnetic Ising chain, *Phys. Rev. E* **109**, 064123 (2024).
- [60] J.-M. Dong, Y. Zhang, K.-W. Huang, H.-H. Tu, and Y.-H. Wu, Numerical extraction of crosscap coefficients in microscopic models for $(2+1)$ D conformal field theory, *Phys. Rev. D* **112**, L121701 (2025).
- [61] K. Chalas, P. Calabrese, and C. Rylands, Quench dynamics of entanglement from crosscap states, *SciPost Phys.* **19**, 132 (2025).
- [62] Z. Wei and Y. Yoneta, Crosscap quenches and entanglement evolution, [arXiv:2412.18610](https://arxiv.org/abs/2412.18610).
- [63] H.-H. Chen, Exact quench dynamics from thermal pure quantum states, [arXiv:2510.05346](https://arxiv.org/abs/2510.05346).
- [64] C. Bai, M. T. Tan, B. Lapiere, and S. Ryu, Spatially structured entanglement from nonequilibrium thermal pure states, [arXiv:2510.25868](https://arxiv.org/abs/2510.25868).
- [65] R. Dulac and Z. Wei, No boundary density matrix in elliptic de Sitter $ds/\mathbb{Z}_2$, [arXiv:2512.00704](https://arxiv.org/abs/2512.00704).
- [66] K. Chalas, P. Calabrese, and C. Rylands, Entanglement evolution from entangled multipodal states, [arXiv:2512.03032](https://arxiv.org/abs/2512.03032).
- [67] A. B. Zamolodchikov and V. A. Fateev, Parafermionic currents in the two-dimensional conformal quantum field theory and selfdual critical points in $\mathbb{Z}(n)$ invariant statistical systems, *Sov. Phys. JETP* **62**, 215 (1985).
- [68] A. B. Zamolodchikov, “Irreversibility” of the flux of the renormalization group in a 2d field theory, *JETP Lett.* **43**, 730 (1986).
- [69] I. Affleck and A. W. W. Ludwig, Universal noninteger “ground-state degeneracy” in critical quantum systems, *Phys. Rev. Lett.* **67**, 161 (1991).
- [70] D. Friedan and A. Konechny, Boundary entropy of one-dimensional quantum systems at low temperature, *Phys. Rev. Lett.* **93**, 030402 (2004).
- [71] Y. Zhang, Y.-H. Wu, L. Wang, and H.-H. Tu, Crosscap states and duality of Ising field theory in two dimensions [dataset], Zenodo (2025), <https://doi.org/10.5281/zenodo.17225083>.

- [72] V. Petkova and J.-B. Zuber, Generalised twisted partition functions, *Phys. Lett. B* **504**, 157 (2001).
- [73] P. Bantay, The Frobenius-Schur indicator in conformal field theory, *Phys. Lett. B* **394**, 87 (1997).
- [74] T. Gannon, Integers in the open string, *Phys. Lett. B* **473**, 80 (2000).
- [75] J. von Delft and H. Schoeller, Bosonization for beginners—refermionization for experts, *Ann. Phys.* **510**, 225 (1998).
- [76] G. Albertini, S. Dasmahapatra, and B. M. McCoy, Spectrum doubling and the extended Brillouin zone in the excitations of the three state Potts spin chain, *Phys. Lett. A* **170**, 397 (1992).