

Instability of Laughlin fractional quantum Hall liquids into gapless power-law correlated states with continuous exponents in ideal Chern bands: Rigorous results from plasma mapping

Saranyo Moitra^{*} and Inti Sodemann Villadiego[†]

Institut für Theoretische Physik, Universität Leipzig, 04103 Leipzig, Germany



(Received 7 October 2025; revised 27 February 2026; accepted 9 March 2026; published 1 April 2026)

We investigate the fate of Laughlin's wave function in ideal Chern bands which can be mapped to generalized zero Landau levels in spatially dependent magnetic fields. By exploiting its exact mapping onto a classical Coulomb gas and leveraging previous results of one-component plasmas in nonuniform neutralizing backgrounds, we demonstrate that the ideal Laughlin wave function undergoes a phase transition from its well-known fully gapped topologically ordered plasma state into a power-law correlated dielectric state even for the fixed filling of $1/3$, as the magnetic field becomes increasingly more inhomogeneous. This dielectric state is gapless even though it does not spontaneously break translational symmetry. Remarkably, for a fixed filling $\nu = 1/m$, the exponent governing density correlations in this state changes continuously as a function of the degree of spatial inhomogeneity of the magnetic field, and can range from 4 near a Berezinskii-Kosterlitz-Thouless transition to the plasma state, up to $2m$ in the limit of fields generated by point solenoids.

DOI: [10.1103/3y87-96dw](https://doi.org/10.1103/3y87-96dw)

Introduction. A curious property of 2D Dirac fermions [1] and parabolic fermions with g factor $g = 2$ [2], is that their zero-energy states in a magnetic field remain exactly degenerate at zero energy even when the field is spatially nonuniform. These wave functions span a generalized zero Landau level that can be viewed as an ideal Chern band [3–13]. Such ideal Chern bands (ICBs) have been found in the chiral limit of twisted bilayer graphene [6,7], argued to approximate the bands of twisted MoTe_2 [13–16], be relevant to rhombohedral multilayer graphene [17–19], where fractional quantum Hall effect at zero magnetic field has been observed [20–24], and proposed in Dirac materials placed on type-II superconductors [25].

These ICBs, also referred to as Aharonov-Casher (AC) bands [14], admit an explicit writing of a generalized Laughlin wave function [7]. However, unlike the standard zero Landau level where the Laughlin wave functions are fixed, the ICBs realize a continuum *family* of Laughlin wave functions, which can be viewed as parametrized by a spatially dependent magnetic field $B(\mathbf{r})$ [26]. The central question of our study is what is the nature of the states realized by these generalized Laughlin wave functions? As we will see, in ICBs the Laughlin wave function can realize not only the well-known topologically ordered state with fractionalized quasiparticles, but also a variety of other complex states. We will demonstrate this by

exploiting the exact mapping of the probability density of the Laughlin wave function onto a classical Coulomb gas [27], which has been recently used to investigate optical properties of these states [28]. In this plasma mapping, the electrons act as pointlike -1 charges fluctuating at temperature $T = 1/2m$ (at filling $\nu = 1/m$), and the magnetic field as a fixed neutralizing background with charge density $\rho_b(\mathbf{r}) = B(\mathbf{r})/(m\Phi_0)$.

In this setting, the standard topologically ordered phase of the Laughlin wave function is a classical plasma where correlations decay exponentially due to its perfect screening properties. However, as we will see, in ICBs the spatial variations of the neutralizing background tend to localize the electrons, transforming the Laughlin plasma into a dielectric [29–31]. In the case of the ICBs relevant to moiré systems, the unit cell contains one effective flux quantum, and, therefore, the neutralizing background has fractional charge $1/m$ per unit cell. In these situations, the dielectric localization of electrons is expected to be accompanied by spontaneous breaking of discrete translations, i.e., charge-density-wave (CDW) formation with m -fold enlargement of the unit cell.

However, as we will demonstrate, these dielectrics realized by the Laughlin wave function are not ordinary CDWs, but rather complex states with density correlations that decay as power laws with continuously changing exponents. Such decay of correlations implies that these states are gapless, but this gaplessness does not seem to fit within the Goldstone mode paradigm because the continuous translational symmetry has been broken explicitly by the spatially dependent field. To be able to tread more carefully through these complexities, we will focus on a simpler model that deviates from the situation in moiré materials and has m fluxes per unit cell, so that each cell contains $+1$ background charge and the dielectric tends to localize one electron per unit cell. A state with one electron strongly localized per lattice site, e.g., by a deep

^{*}Contact author: saranyo.moitra@uni-leipzig.de

[†]Contact author: sodemann@uni-leipzig.de

single-particle potential, is expected to be adiabatically connected to a trivial insulator described by a Slater determinant. However, as we will see, the dielectric Laughlin state with one electron per unit cell generically remains in a complex gapless and power-law correlated state with continuously varying exponents.

Model and preliminaries. A simple ICB is realized by massless two-dimensional Dirac fermions placed in an inhomogeneous magnetic field $\mathbf{B} = -B(\mathbf{r})\hat{\mathbf{z}}$, which have the following zero-energy wave functions [1]:

$$\psi_n(\mathbf{r}) = z^n e^{-\phi(\mathbf{r})}, \quad n = 0, 1, 2, \dots, \quad (1)$$

where we use the Coulomb gauge $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$, $z = x + iy$ is the complex position, and the exponent is determined from $\nabla^2 \phi(\mathbf{r}) = eB(\mathbf{r})/\hbar$ (electron charge is $-e$, $e > 0$). The above wave functions are normalizable when $B(\mathbf{r})$ has a positive global spatial average, but we will focus on cases when $B(\mathbf{r})$ is spatially periodic [32]. The generalized Laughlin state of N electrons is given by

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i < j} (z_i - z_j)^m \prod_{i=1}^N e^{-\phi(\mathbf{r}_i)}. \quad (2)$$

For a uniform field, $B(\mathbf{r}) = B_0$, $\phi(\mathbf{r}) = |\mathbf{r}|^2/4\ell_B^2$, $\ell_B = \sqrt{\hbar/eB_0}$, Eq. (2) reduces to the usual Laughlin state. As in the usual zero Landau level, the above wave function corresponds to the highest-degree holomorphic polynomial that is a zero-energy eigenstate of a short-ranged Hamiltonian [33,34], whose projection onto the ICB we view as the ideal Hamiltonian defining our problem. For general ICBs (including those relevant for moiré materials) an effective $B(\mathbf{r})$ can be defined, by demanding that the probability density of the Laughlin wave function is identical to that of Eq. (2) [7,12,13,16]. Remarkably, the probability density of these generalized Laughlin states is identical to the finite temperature probability density of a gas of classical particles repelling via 2D Coulomb forces and with a neutralizing background charge density, namely $|\Psi|^2 = \exp[-\beta\mathcal{U}]$, with

$$\mathcal{U} = - \sum_{i < j}^N \ln |\mathbf{r}_i - \mathbf{r}_j| + \sum_{i=1}^N \frac{\phi(\mathbf{r}_i)}{m}, \quad (3)$$

where we have chosen the electrons to have Coulomb charge -1 and $\beta = 2m$. The term $\phi(\mathbf{r}_i)/m$ is the potential energy experienced by an electron at $\mathbf{r}_i$ generated by a background with charge density $\rho_b(\mathbf{r}) = B(\mathbf{r})/(m\Phi_0)$, where $\Phi_0 = h/e$. Thus, we see that any region, R , containing a single flux quantum, $\int_R d^2r B(\mathbf{r}) = \Phi_0$, acts as a background source of charge $+1/m$.

In usual zero Landau levels, the above is the map to a one-component Coulomb gas with a uniform neutralizing background $\rho_b(\mathbf{r}) = 1/(2m\pi\ell_B^2) \equiv B_0/(m\Phi_0)$, introduced by Laughlin [27]. This one-component gas is in a plasma phase at high temperatures (the quantum topologically ordered state) and undergoes a phase transition into a crystal at a very large exponent $m \gtrsim 70$ [35,36]. However, from numerical [29,31] and analytical studies [30] of the one-component Coulomb gas in nonuniform neutralizing backgrounds, we expect that even the most common Laughlin state with $m = 3$ undergoes a transition from its plasma state into a dielectric state where

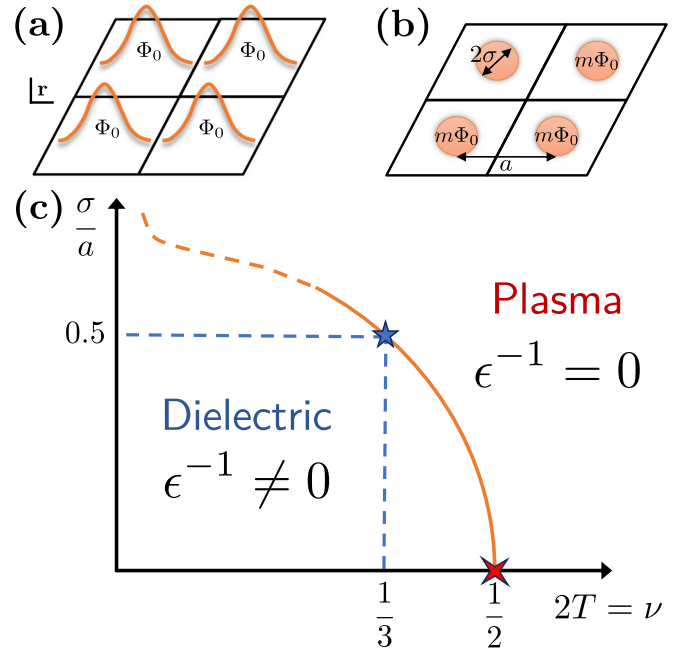


FIG. 1. (a) Depiction of effective magnetic field in moiré Chern bands with one flux per unit cell, and (b) in ICB of solenoids with m fluxes per unit cell. (c) Phase diagram of the Coulomb gas with neutralizing background of the ions from panel (b), adapted from Ref. [31]. The BKT transition is expected to occur for small radius and at $T\epsilon = 1/4$.

the electrons largely freeze their positions, when the spatial variations of the background charge density [i.e., of $B(\mathbf{r})$] are sufficiently large.

When the magnetic field $B(\mathbf{r})$ contains one flux quantum per unit cell, as realized in Chern bands of some moiré materials (see, e.g., [37] for twisted MoTe_2), there will be an average of $1/m$ electrons per unit cell, and the Laughlin dielectric state would tend to break spontaneously the discrete magnetic translational symmetries defined by $B(\mathbf{r})$. However, we will see that such a dielectric state is not a trivial CDW state (i.e., one approximated by a Slater determinant), but rather a complex power-law correlated state that remains gapless even in the absence of spontaneous symmetry breaking.

To argue more clearly for this, we adopt a setting where it is manifest that such a dielectric Laughlin state does not need to spontaneously break translational symmetry. To do so, we take $B(\mathbf{r})$ to be a periodic function with m fluxes per unit cell. In this case, the neutrality of the Coulomb gas implies that we must have an average of one electron per unit cell, and therefore the dielectric can form without spontaneously breaking translations. For concreteness, we take $B(\mathbf{r})$ to be uniform inside a collection of solenoids of radius σ and zero outside the solenoids. The solenoid positions, $\mathbf{X}_i$, are taken to form a triangular lattice [i.e., with separation $a = (4\pi m/\sqrt{3})^{1/2}\ell_B$, see Fig. 1(b)]. None of these details are crucial for our main conclusions [38], but this geometry has the advantage that it has been previously investigated numerically [31] and analytically [30]. Since each solenoid carries flux $m\Phi_0$, it behaves as a fixed disk with positive charge $+1$, and, therefore, this Coulomb gas is a 2D version of a classical

crystal of “Thomson plum pudding” of hydrogen atoms. The Coulomb potential in Eq. (3) originating from these positive background ions is a sum of the single ion potentials [denoted by $\chi(\mathbf{r})$]:

$$\frac{\phi(\mathbf{r})}{m} = \sum_i \chi(\mathbf{r} - \mathbf{X}_i), \quad \chi(\mathbf{r}) = \begin{cases} \frac{|\mathbf{r}|^2 - \sigma^2}{2\sigma^2} & r < \sigma \\ \ln \frac{|\mathbf{r}|}{\sigma} & r > \sigma \end{cases}. \quad (4)$$

This model has two dimensionless parameters: the dimensionless solenoid radius, σ/a , and the filling (or temperature) $\nu = 1/m = 2T$, and its phase diagram, adapted from Ref. [31], is depicted in Fig. 1(c). Several arguments [30] support that in the “dilute” limit of pointlike solenoids ($\sigma/a \ll 1$), the high-temperature plasma state with ionized electrons (i.e., the quantum topologically ordered phase) undergoes a Berezinskii-Kosterlitz-Thouless (BKT) transition into a dielectric state in which each background positive ion binds one electron, at the temperature that corresponds to the filling of the bosonic Laughlin state at $\nu = 1/2$ [red star in Fig. 1(c)]. For the fermionic Laughlin state at $\nu = 1/3$, from the results of Ref. [31], we estimate that the plasma-dielectric transition occurs at $\sigma/a \sim 0.5$ [blue star in Fig. 1(c)], i.e., when the solenoids are barely touching.

Dielectric properties and quasiholes. The dielectric constant ϵ plays a central role in the properties of the Coulomb gas. In the plasma state $\epsilon^{-1} = 0$, while in the dielectric ϵ is finite and changes continuously as a function of the solenoid radius, σ/a , and temperature $T = 1/(2m)$. At the BKT transition ϵ^{-1} experiences a universal jump from $\epsilon^{-1} = 4T = 2/m$ to $\epsilon^{-1} = 0$. More generally, the screening of external potentials is controlled by the dielectric response function, $\epsilon^{-1}(\mathbf{r}, \mathbf{r}')$, which is related to the density correlator $S(\mathbf{r}, \mathbf{r}')$ [30,39] (see [38] for details), as follows:

$$\epsilon^{-1}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') + 2m \int d\mathbf{r}'' \ln \frac{|\mathbf{r} - \mathbf{r}''|}{\sigma} S(\mathbf{r}'', \mathbf{r}'), \quad (5)$$

where $S(\mathbf{r}, \mathbf{r}') := \langle \rho(\mathbf{r})\rho(\mathbf{r}') \rangle - \langle \rho(\mathbf{r}) \rangle \langle \rho(\mathbf{r}') \rangle$, and $\rho(\mathbf{r}) = \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)$ is the electron density. In our setting, these correlation functions are only invariant under the discrete lattice translations of the periodic magnetic field, and thus, to facilitate the analysis of their long-distance behaviors, it is convenient to define their *cell-averaged* counterparts as follows:

$$\epsilon_{\text{ca}}^{-1}(\mathbf{r}) := \frac{1}{A_{\text{uc}}} \int_{\mathbf{x} \in \text{uc}} d^2\mathbf{x} \epsilon^{-1}(\mathbf{r} + \mathbf{x}, \mathbf{x}), \quad (6)$$

where A_{uc} is the area of a unit cell and we define similarly $S_{\text{ca}}(\mathbf{r})$. The Fourier transform of the cell-averaged Eq. (5) is $\epsilon_{\text{ca}}^{-1}(\mathbf{q}) = 1 - 4\pi m S_{\text{ca}}(\mathbf{q})/|\mathbf{q}|^2$, and the dielectric constant ϵ is obtained from its limit $\mathbf{q} \rightarrow 0$ [30,39]:

$$\epsilon^{-1} = 1 - \lim_{\mathbf{q} \rightarrow 0} \frac{4\pi m S_{\text{ca}}(\mathbf{q})}{|\mathbf{q}|^2} = 1 + \pi m \int d^2\mathbf{r} |\mathbf{r}|^2 S_{\text{ca}}(\mathbf{r}), \quad (7)$$

The above equations assume that the system has at least C_3 rotational invariance, and imply that $S_{\text{ca}}(\mathbf{r})$ must decay faster than $\sim 1/|\mathbf{r}|^4$ at long distances ($1/|\mathbf{r}|^4$ is the decay at the BKT line). The above reduces to the Stillinger-Lovett perfect screening condition [39–41] in the plasma state. From it one can also relate the dielectric constant to the variance of particle

positions [42] as follows ($\mathbf{r}_{\text{CM}} \equiv (\sum_i \mathbf{r}_i)/N$):

$$\epsilon^{-1} = 1 - \frac{2m\pi}{A_{\text{uc}}} N (\langle \mathbf{r}_{\text{CM}}^2 \rangle - \langle \mathbf{r}_{\text{CM}} \rangle^2). \quad (8)$$

Now, the probabilities of our Coulomb gas with an additional pointlike test particle located at the origin and charge $-1/m$, are identical to the probability density of the Laughlin quasihole wave function [27] generalized to ICB, i.e., $\Psi_{\text{qh}} = (\prod_{i=1}^N \mathbf{z}_i) \Psi$, with Ψ the wave function from Eq. (2). This quasihole is also an exact zero-energy eigenstate of the same pseudopotential Hamiltonian projected into the ICB, for which Ψ is a zero-energy eigenstate. At long distances, the Coulomb electric field emanating from this test charge would be reduced by the dielectric constant ϵ . In the plasma phase there is perfect screening ($\epsilon^{-1} = 0$), and this test charge is surrounded by an exponentially localized screening cloud which exactly cancels its bare charge, leading to the classic result that quasihole carries a fractional physical electric charge $q_{\text{qh}} = +e/m$ (i.e., a depletion of electrons). In the dielectric state ϵ is finite, and the screening is imperfect, and the electric field far away from the test particle asymptotes to the value $-\hat{\mathbf{r}}/(\epsilon m |\mathbf{r}|)$. From this it follows that the quasihole in the dielectric state carries a physical electric charge:

$$q_{\text{qh}} = (1 - \epsilon^{-1}) \frac{e}{m}. \quad (9)$$

The above is remarkable because in the dielectric state ϵ changes continuously with the solenoid radius, even for fixed m , and therefore the quasihole charge changes continuously. This quasihole is a nonlocal object because the decay of its screened electric field with distance implies that electrons have a displacement away from the solenoid centers that decays as $\sim 1/|\mathbf{r}|$ away from the location of the quasihole. These features indicate that the dielectric Laughlin state cannot be a fully gapped phase of matter, since in 2D these are expected to have quasiparticles with physical charges that are fixed rational values of the electron charge [43]. We will provide a more rigorous argument for this based on the behavior of correlation functions in the coming section.

Correlations and cluster expansion. In this section we will demonstrate one of the most remarkable properties of the dielectric Laughlin states in ICBs, namely, that their density-density correlations have the following power-law decay at long distances:

$$S(\mathbf{r}, \mathbf{r}') \sim -\frac{C}{|\mathbf{r} - \mathbf{r}'|^{2m\epsilon^{-1}}}. \quad (10)$$

Notably, the exponent $2m\epsilon^{-1}$ changes continuously in the dielectric state, even for a fixed m , as a function of the solenoid radius σ . Since gapped ground states of local Hamiltonians necessarily have exponentially decaying correlations [44], the above implies that the dielectric Laughlin states must have gapless excitations, even if they do not spontaneously break translations. The above decay of density correlations has been widely discussed for the dielectric state of the more commonly studied two-component Coulomb gas [45–50], but also occurs in the one-component Coulomb gas with nonuniform

background [30]. Here, we will reproduce these results and expand on the details of the cluster expansion method proposed in Ref. [30] for our one-component Coulomb gas with nonuniform background.

Deep in the dielectric phase ($\sigma/a \ll 1, m^{-1} \ll 1$) each positive ion strongly binds one electron. This motivates a cluster expansion controlled by these small parameters [30]. We start by an exact rewriting of the Boltzmann weight from Eq. (3) as follows:

$$e^{-2m\mathcal{U}} = \prod_{i=1}^N p(\mathbf{r}_i - \mathbf{X}_i) \prod_{j < k} (1 + f_{jk}), \quad (11)$$

where $p(\mathbf{r}) = e^{-2m\chi(\mathbf{r})}/w$, $w = \int d^2\mathbf{r} e^{-2m\chi(\mathbf{r})} \simeq \mathcal{O}(\sigma^2)$ is the probability density of a single electron and a single ion (i.e., as if no other ions or electrons were present), and the Mayer factors are $f_{jk} := \exp[-2m\delta\mathcal{U}_{jk}] - 1$, with the residual potential defined as $\delta\mathcal{U}_{jk} = -\ln(|\mathbf{r}_j - \mathbf{r}_k|/\sigma) - \chi(\mathbf{X}_j - \mathbf{X}_k) + \chi(\mathbf{r}_j - \mathbf{X}_k) + \chi(\mathbf{r}_k - \mathbf{X}_j)$. We have introduced a few innocuous constants in the definition of $\mathcal{U}$ to facilitate its normalization [38]. The cluster expansion is a perturbation series in f_{jk} , justified from the smallness of the residual potential, $\delta\mathcal{U}_{jk}$, when particles and ions are tightly bound. To leading order ($f_{jk} \approx 0$) one gets $S(\mathbf{r}, \mathbf{r}') \approx \sum_i p(\mathbf{r} - \mathbf{X}_i)(\delta(\mathbf{r} - \mathbf{r}') - p(\mathbf{r}' - \mathbf{X}_i))$, whose cell average decays at long distances, as follows [38]:

$$S_{\text{ca}}(\mathbf{r}) \approx -\frac{2}{A_{\text{uc}}} p(\mathbf{r}) = -\frac{2}{A_{\text{uc}}w} \frac{\sigma^{2m}}{|\mathbf{r}|^{2m}}, \quad |\mathbf{r}| \gg \sigma. \quad (12)$$

The above proves Eq. (10) in the dilute limit ($\sigma/a \rightarrow 1$) where one expects $\epsilon^{-1} \rightarrow 1$, and is consistent with a BKT transition into the plasma state at $\beta = 2m = 4$ [30]. Within this leading order, one finds $N(\mathbf{r}_{\text{CM}}^2) = \int d^2\mathbf{r} r^2 p(\mathbf{r})$, and from Eq. (8), ϵ is estimated to be

$$\epsilon^{-1} = 1 - \frac{(m-1)(m-2 + (m+2)e^{-m})}{(m-2)(m-1 + e^{-m})} \frac{2\pi\sigma^2}{A_{\text{uc}}}. \quad (13)$$

Notice that ϵ^{-1} remains finite as the temperature approaches zero ($m \rightarrow \infty$) and one recovers the polarizability of a Thomson atom from Ref. [31] (Ref. [30] considered the ions to be impenetrable, which leads to a low temperature instability as $m \rightarrow \infty$ [38]). Interestingly, to order σ^2/A_{uc} , ϵ^{-1} depends only on the detailed magnetic field distribution within a single solenoid, but it is independent of the global arrangement of solenoids and thus is the same regardless of the type of solenoid lattice (i.e., triangular vs square, or even a disordered arrangement). An interesting implication of this is that in the dilute limit ($\sigma \rightarrow 0$), the BKT transition is expected to occur at $\nu = \frac{1}{2}$ regardless of how the solenoids are arranged in space [see Fig. 1(c)]. The first order in f_{jk} corrections to $S_{\text{ca}}(\mathbf{r})$ leads to the following modifications at large distances [38]:

$$S_{\text{ca}}(\mathbf{r}) \sim \frac{\sigma^{2m}}{|\mathbf{r}|^{2m}} \left[1 + 2m(1 - \epsilon^{-1}) \ln \frac{|\mathbf{r}|}{\sigma} \right], \quad (14)$$

where ϵ^{-1} is given by Eq. (13). By noting that $1 - \epsilon^{-1}$ is of order σ^2/a^2 , Eq. (14) can be re-exponentiated in the small σ/a limit to read $S_{\text{ca}}(\mathbf{r}) \simeq (\sigma/|\mathbf{r}|)^{2m\epsilon^{-1}}$ [30]. This decay of density-density correlations also implies that the density of the screening cloud surrounding the quasihole decays as $\delta n(\mathbf{r}) \sim A/|\mathbf{r}|^{2(m\epsilon^{-1}-1)}$, and therefore this is sufficiently fast to make the quasihole charge from Eq. (9) well defined in the dielectric state since $m\epsilon^{-1} > 2$ (except at the BKT line where $m\epsilon^{-1} \rightarrow 2$).

Discussion. We have used a Coulomb gas mapping to investigate the nature of the Laughlin wave function in ideal Chern bands which can be viewed as generalized zero Landau levels with nonuniform magnetic fields. By taking a model of a field generated by a lattice of solenoids carrying m fluxes, we have demonstrated that the Laughlin wave function is not always in its celebrated plasma state with topological order, but can transition into a new dielectric state where electrons tend to localize and bind to these solenoids. This dielectric Laughlin state displays power law decaying correlations with continuous exponents, which implies that the quantum state must be gapless, but this gaplessness appears to be beyond the standard Goldstone mode paradigm, since there is no continuous translational symmetry in the model.

Quantum ground states with continuously varying exponents in two space dimensions are quite rare. However, similar behavior has been reported in a supersymmetric XY model [51], and in quantum-dimer and related models [52–56], which have been argued to be critical states with continuously varying exponents captured by a 2D quantum Lifshitz field theory [52,54,55,57]. Continuously changing exponents were also argued to be realized at a multicritical line of the Toric-code model with Zeeman fields [58]. Thus, it is possible that the dielectric Laughlin states in ICBs might form a similar type of critical states. We hope future studies can clarify this interesting possibility, the connection to the criticality driven by periodic potentials [59], and the mechanisms underlying their gaplessness which appears to be beyond the Goldstone mode paradigm.

An important open question concerns ICBs relevant to moiré materials, where $B(r)$ carries one flux quantum per unit cell rather than the $m\Phi_0$ flux considered in our model. In this case, the corresponding wave functions map onto a classical crystal with a fractional background charge per unit cell, a regime that, to our knowledge, has not been studied previously. One possibility is that a CDW-like Laughlin dielectric state with true spontaneous symmetry breaking of discrete translations occurs in these settings. This possibility in ICBs and its implications for realistic moiré materials are important open problems for future investigations.

Acknowledgments. We would like to thank Qi Hu, Markus Kohler, Anish Koley, Akihiro Ozawa, Nemin Wei, Luca Delacrétaz, Dam T. Son, Paul Wiegmann, Nicolás Morales Durán, Gil Young Cho, Jennifer Cano, Lei Chen, Trithep Devakul, Cenke Xu, Daniel Arovass, Ying Ran, Ashvin Vishwanath, Debanjan Chowdhury, Ravindra N. Bhatt, Ganpathy Murthy, Daniel Barci, Jörg Schmalian, Patrick Ledwith, Julien Vidal, and Liang Fu, for stimulating discussions and

pointing out useful references. We thank Chuan Chen for a thorough reading of the manuscript and for pointing out an issue with an earlier version of Eq. (12). I.S.V. is also thankful for the hospitality at the Aspen Center for Physics, supported by National Science Foundation (NSF) Grant No. PHY-2210452, where some of these discussions

took place. We are also thankful for the support from the Deutsche Forschungsgemeinschaft (DFG) through research Grants Projects No. 542614019, No. 518372354, and No. 555335098.

Data availability. No data were created or analyzed in this study.

-
- [1] Y. Aharonov and A. Casher, Ground state of a spin- $\frac{1}{2}$ charged particle in a two-dimensional magnetic field, *Phys. Rev. A* **19**, 2461 (1979).
 - [2] B. A. Dubrovin and S. P. Novikov, Ground states of a two-dimensional electron in a periodic magnetic field, *Sov. Phys. JETP* **52**, 511 (1980).
 - [3] R. Roy, Band geometry of fractional topological insulators, *Phys. Rev. B* **90**, 165139 (2014).
 - [4] T. S. Jackson, G. Möller, and R. Roy, Geometric stability of topological lattice phases, *Nat. Commun.* **6**, 8629 (2015).
 - [5] M. Claassen, C. H. Lee, R. Thomale, X.-L. Qi, and T. P. Devereaux, Position-momentum duality and fractional quantum Hall effect in Chern insulators, *Phys. Rev. Lett.* **114**, 236802 (2015).
 - [6] G. Tarnopolsky, A. J. Kruchkov, and A. Vishwanath, Origin of magic angles in twisted bilayer graphene, *Phys. Rev. Lett.* **122**, 106405 (2019).
 - [7] P. J. Ledwith, G. Tarnopolsky, E. Khalaf, and A. Vishwanath, Fractional Chern insulator states in twisted bilayer graphene: An analytical approach, *Phys. Rev. Res.* **2**, 023237 (2020).
 - [8] J. Wang, J. Cano, A. J. Millis, Z. Liu, and B. Yang, Exact Landau level description of geometry and interaction in a flatband, *Phys. Rev. Lett.* **127**, 246403 (2021).
 - [9] T. Ozawa and B. Mera, Relations between topology and the quantum metric for Chern insulators, *Phys. Rev. B* **104**, 045103 (2021).
 - [10] B. Mera and T. Ozawa, Kähler geometry and Chern insulators: Relations between topology and the quantum metric, *Phys. Rev. B* **104**, 045104 (2021).
 - [11] P. J. Ledwith, A. Vishwanath, and D. E. Parker, Vortexability: A unifying criterion for ideal fractional Chern insulators, *Phys. Rev. B* **108**, 205144 (2023).
 - [12] B. Estienne, N. Regnault, and V. Crépel, Ideal Chern bands as Landau levels in curved space, *Phys. Rev. Res.* **5**, L032048 (2023).
 - [13] J. Dong, J. Wang, P. J. Ledwith, A. Vishwanath, and D. E. Parker, Composite Fermi liquid at zero magnetic field in twisted MoTe₂, *Phys. Rev. Lett.* **131**, 136502 (2023).
 - [14] N. Morales-Durán, N. Wei, J. Shi, and A. H. MacDonald, Magic angles and fractional Chern insulators in twisted homobilayer transition metal dichalcogenides, *Phys. Rev. Lett.* **132**, 096602 (2024).
 - [15] J. Shi, N. Morales-Durán, E. Khalaf, and A. H. MacDonald, Adiabatic approximation and Aharonov-Casher bands in twisted homobilayer transition metal dichalcogenides, *Phys. Rev. B* **110**, 035130 (2024).
 - [16] B. Li and F. Wu, Variational mapping of Chern bands to Landau levels: Application to fractional Chern insulators in twisted MoTe₂, *Phys. Rev. B* **111**, 125122 (2025).
 - [17] T. Tan and T. Devakul, Parent Berry curvature and the ideal anomalous Hall crystal, *Phys. Rev. X* **14**, 041040 (2024).
 - [18] J. Dong, T. Wang, T. Wang, T. Soejima, M. P. Zaletel, A. Vishwanath, and D. E. Parker, Anomalous Hall crystals in rhombohedral multilayer graphene. I. Interaction-driven Chern bands and fractional quantum Hall states at zero magnetic field, *Phys. Rev. Lett.* **133**, 206503 (2024).
 - [19] T. Tan, J. May-Mann, and T. Devakul, Variational wave-function analysis of the fractional anomalous Hall crystal, *Phys. Rev. Lett.* **135**, 036604 (2025).
 - [20] J. Cai, E. Anderson, C. Wang, X. Zhang, X. Liu, W. Holtzmann, Y. Zhang, F. Fan, T. Taniguchi, K. Watanabe, Y. Ran, T. Cao, L. Fu, D. Xiao, W. Yao, and X. Xu, Signatures of fractional quantum anomalous Hall states in twisted MoTe₂, *Nature (London)* **622**, 63 (2023).
 - [21] Y. Zeng, Z. Xia, K. Kang, J. Zhu, P. Knüppel, C. Vaswani, K. Watanabe, T. Taniguchi, K. F. Mak, and J. Shan, Thermodynamic evidence of fractional Chern insulator in moiré MoTe₂, *Nature (London)* **622**, 69 (2023).
 - [22] H. Park, J. Cai, E. Anderson, Y. Zhang, J. Zhu, X. Liu, C. Wang, W. Holtzmann, C. Hu, Z. Liu, T. Taniguchi, K. Watanabe, J.-H. Chu, T. Cao, L. Fu, W. Yao, C.-Z. Chang, D. Cobden, D. Xiao, and X. Xu, Observation of fractionally quantized anomalous Hall effect, *Nature (London)* **622**, 74 (2023).
 - [23] F. Xu, Z. Sun, T. Jia, C. Liu, C. Xu, C. Li, Y. Gu, K. Watanabe, T. Taniguchi, B. Tong, J. Jia, Z. Shi, S. Jiang, Y. Zhang, X. Liu, and T. Li, Observation of integer and fractional quantum anomalous Hall effects in twisted bilayer MoTe₂, *Phys. Rev. X* **13**, 031037 (2023).
 - [24] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, Fractional quantum anomalous Hall effect in multilayer graphene, *Nature (London)* **626**, 759 (2024).
 - [25] J. Dong, J. Wang, and L. Fu, Dirac electron under periodic magnetic field: Platform for fractional Chern insulator and generalized Wigner crystal, *arXiv:2208.10516*.
 - [26] In general this field is a function that parametrizes single-particle wave functions and not necessarily a physical field.
 - [27] R. B. Laughlin, Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations, *Phys. Rev. Lett.* **50**, 1395 (1983).
 - [28] T. Wolf, Y.-C. Chao, A. H. MacDonald, and J.-J. Su, Intra-band collective excitations and spatial correlations in fractional Chern insulators, *Phys. Rev. Lett.* **134**, 116501 (2025).
 - [29] J. Clerouin, J.-P. Hansen, and B. Piller, Two-dimensional classical electron gas in a periodic field: Delocalization and dielectric-plasma transition, *Phys. Rev. A* **36**, 2793 (1987).
 - [30] A. Alastuey, F. Cornu, and B. Jancovici, Comment on “Two-dimensional classical electron gas in a periodic field:

- Delocalization and dielectric-plasma transition”, *Phys. Rev. A* **38**, 4916 (1988).
- [31] P. Choquard, B. Piller, R. Rentsch, J. Cl  rouin, and J.-P. Hansen, Ionization and phase diagram of classical Thomson atoms on a triangular lattice, *Phys. Rev. A* **40**, 931 (1989).
- [32] $B(\mathbf{r})$ is allowed to change sign as long as it does not average to zero. For examples, see [7,13,16].
- [33] F. D. M. Haldane, Fractional quantization of the Hall effect: A hierarchy of incompressible quantum fluid states, *Phys. Rev. Lett.* **51**, 605 (1983).
- [34] S. A. Trugman and S. Kivelson, Exact results for the fractional quantum Hall effect with general interactions, *Phys. Rev. B* **31**, 5280 (1985).
- [35] J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen, A Monte Carlo study of the classical two-dimensional one-component plasma, *J. Stat. Phys.* **28**, 325 (1982).
- [36] E. V. Herland, E. Babaev, P. Bonderson, V. Gurarie, C. Nayak, L. Radzihovsky, and A. Sudb  , Freezing of an unconventional two-dimensional plasma, *Phys. Rev. B* **87**, 075117 (2013).
- [37] F. Wu, T. Lovorn, E. Tutuc, I. Martin, and A. H. MacDonald, Topological insulators in twisted transition metal dichalcogenide homobilayers, *Phys. Rev. Lett.* **122**, 086402 (2019).
- [38] See Supplemental Material at <http://link.aps.org/supplemental/10.1103/3y87-96dw> for (1) linear response relation between the dielectric function and charge correlation function, (2) sum rule for the dielectric constant and its relation to particle position fluctuations, (3) extension to more general inhomogeneous magnetic fields, and (4) asymptotics of the cell averaged correlator at leading order and next-to-leading order of cluster expansion.
- [39] P. A. Martin and Ch. Gruber, A new proof of the Stillinger-Lovett complete shielding condition, *J. Stat. Phys.* **31**, 691 (1983).
- [40] D. J. Mitchell, D. A. McQuarrie, A. Szabo, and J. Groeneveld, On the second-moment condition of Stillinger and Lovett, *J. Stat. Phys.* **17**, 15 (1977).
- [41] P. A. Martin, Sum rules in charged fluids, *Rev. Mod. Phys.* **60**, 1075 (1988).
- [42] ϵ^{-1} is related to fluctuations of electric polarization [36,60–62], which reduce to the variance of particle positions for a one component Coulomb gas with charges $q = -1$ [38].
- [43] X.-G. Wen, *Quantum Field Theory of Many-Body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2007).
- [44] M. B. Hastings and T. Koma, Spectral gap and exponential decay of correlations, *Commun. Math. Phys.* **265**, 781 (2006).
- [45] V. L. Berezinski  , Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group I. Classical systems, *Sov. Phys. JETP* **32**, 493 (1971).
- [46] V. L. Berezinski  , Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum systems, *Sov. Phys. JETP* **34**, 610 (1972).
- [47] J. M. Kosterlitz and D. J. Thouless, Ordering, metastability and phase transitions in two-dimensional systems, *J. Phys. C: Solid State Phys.* **6**, 1181 (1973).
- [48] A. Alastuey and F. Cornu, Correlations in the Kosterlitz-Thouless phase of the two-dimensional Coulomb gas, *J. Stat. Phys.* **66**, 165 (1992).
- [49] A. Alastuey and P. J. Forrester, Correlations in two-component log-gas systems, *J. Stat. Phys.* **81**, 579 (1995).
- [50] P. Minnhagen, The two-dimensional Coulomb gas, vortex unbinding, and superfluid-superconducting films, *Rev. Mod. Phys.* **59**, 1001 (1987).
- [51] A. E. Rana and S. M. Girvin, Soluble supersymmetric quantum XY model, *Phys. Rev. B* **48**, 360 (1993).
- [52] E. Ardonne, P. Fendley, and E. Fradkin, Topological order and conformal quantum critical points, *Ann. Phys.* **310**, 493 (2004).
- [53] E. Fradkin, D. A. Huse, R. Moessner, V. Oganesyan, and S. L. Sondhi, Bipartite Rokhsar–Kivelson points and Cantor deconfinement, *Phys. Rev. B* **69**, 224415 (2004).
- [54] P. Ghaemi, A. Vishwanath, and T. Senthil, Finite-temperature properties of quantum Lifshitz transitions between valence-bond solid phases: An example of local quantum criticality, *Phys. Rev. B* **72**, 024420 (2005).
- [55] A. Vishwanath, L. Balents, and T. Senthil, Quantum criticality and deconfinement in phase transitions between valence bond solids, *Phys. Rev. B* **69**, 224416 (2004).
- [56] S. V. Isakov, P. Fendley, A. W. W. Ludwig, S. Trebst, and M. Troyer, Dynamics at and near conformal quantum critical points, *Phys. Rev. B* **83**, 125114 (2011).
- [57] B. Hsu and E. Fradkin, Dynamical stability of the quantum Lifshitz theory in 2+1 dimensions, *Phys. Rev. B* **87**, 085102 (2013).
- [58] S. Dusuel, M. Kamfor, R. Or  s, K. P. Schmidt, and J. Vidal, Robustness of a perturbed topological phase, *Phys. Rev. Lett.* **106**, 107203 (2011).
- [59] X.-G. Wen and Y.-S. Wu, Transitions between the quantum Hall states and insulators induced by periodic potentials, *Phys. Rev. Lett.* **70**, 1501 (1993).
- [60] A. Vallat and H. Beck, Coulomb-gas representation of the two-dimensional XY model on a torus, *Phys. Rev. B* **50**, 4015 (1994).
- [61] P. Olsson, Monte Carlo analysis of the two-dimensional XY model. I. Self-consistent boundary conditions, *Phys. Rev. B* **52**, 4511 (1995).
- [62] E. V. Herland, E. Babaev, P. Bonderson, V. Gurarie, C. Nayak, and A. Sudb  , Screening properties and phase transitions in unconventional plasmas for Ising-type quantum Hall states, *Phys. Rev. B* **85**, 024520 (2012).