

Engineering complex dispersion relations

R. G. Edge¹, S. A. R. Horsley¹, T. A. Starkey¹, and G. J. Chaplain¹

*Centre for Metamaterial Research and Innovation, Department of Physics and Astronomy,
University of Exeter, Exeter EX4 4QL, United Kingdom*



(Received 4 April 2025; revised 4 June 2025; accepted 11 June 2025; published 7 July 2025)

Motivated by recent theoretical and experimental interest in metamaterials comprising nonlocal coupling terms, we present an analytic framework to realize materials with arbitrary complex dispersion relations. Building on the inverse design method proposed by Kazemi *et al.* [*Phys. Rev. Lett.* **131**, 176101 (2023)], which demonstrated arbitrary real dispersion relations with real nonlocal coupling terms, we show that using nonconservative, nonlocal couplings not only can band structures be drawn, but arbitrary attenuation of the supported modes can be achieved by tuning their trajectory in the complex frequency plane. This is demonstrated using a canonical mass-spring lattice with higher-order spatial connections; we consider the energy velocity in such systems and show that competing loss and gain of the complex interactions can produce exotic wave dispersion.

DOI: [10.1103/3nt5-j5mp](https://doi.org/10.1103/PhysRevB.112.014304)

I. INTRODUCTION

Wave propagation, in structured materials or otherwise, is fundamentally governed by the dispersive properties of the media in which propagation occurs. There has been a concerted research effort to influence, and even engineer wave dispersion across almost all regimes [1–3]. By design of a composite, underlying structure (typically subwavelength, periodic, and resonant), so-called metamaterials arise with unique, exotic, and even bespoke wave dispersion that can be utilized to control wave propagation for a variety of far-reaching applications, such as the slowing and trapping of light [4–6], directional beam steering and lensing behaviors [7–10], enhanced acoustic absorption and noise reduction [11–13], and energy harvesting [14,15], and seismic isolation [16,17], to name a few.

Lattices, in particular infinitely periodic structures, play a crucial role in the study of wave dynamics; their elegant analytical framework is often traced back to the seminal work of Brillouin [18]. Recently there has been a renaissance in the ideas of Brillouin, incorporating nonlocal couplings that manifest as beyond-nearest-neighbor interactions through spatially extending beyond the unit cell [19–25]. These couplings offer routes to engineering features in dispersion curves, for instance, Dirac points [26] or evanescent Bloch states [27], and enable the control of multiple (potentially arbitrarily many) regions of zero-group-velocity in the dispersion curves (band structures) of such crystals with applications in noise control [28], energy harvesting [14,29], and mode trapping [30,31]. It has been recently shown, via inverse design, that indeed any arbitrary dispersion curve on the real

frequency-wavenumber (ω - k) plane can be “drawn” [32] in a lattice of masses and springs by leveraging the Fourier series representation of the periodic band structure. The result being, with every additional Fourier component, additional nonlocal elements are required; for arbitrary curves this leads to a large number of beyond-nearest-neighbors, with possibly negative spring constants.

Often, advances in such systems have been made using conventional, conservative mass-spring models, i.e., dissipation is neglected. Important implications of this, particularly in the realm of elasticity, have been highlighted [33–36], where limits on harvesting applications have been derived in the context of “meta-damping”, with extensions to generalized Brillouin zone theory considered [37]. Loss is an intrinsic property of materials, shaping the dynamical response of structures, and is often not possible, nor desirable, to be neglected [38–40]. Considered even less in such systems, is the inverse case; gain is seldom accounted for in simplistic mass-spring models, although with the advent of time-varying metamaterials [41–45], it is often a requirement and appears naturally in non-Hermitian systems.

In non-Hermitian systems gain and loss can give rise to unique wave-matter interactions [46], with solutions typically presenting as complex eigenvalues, due to instabilities and unconserved probabilities. Exceptions do exist to this, particularly in parity-time ($\mathcal{PT}$) symmetric systems which can stabilise and exhibit non-Hermitian behaviors [47]. This demonstrates that, at critical points, the spontaneous breaking of $\mathcal{PT}$ symmetry can transition between stable systems with purely real eigenvalues and ones possessing complex eigenvalues [48]. Despite this, encompassing the principles of non-Hermitian behaviors can facilitate the emergence of new effects not previously possible in conventional systems, leading to unique wave-matter interactions and enhanced performance in a range of applications [49].

In this paper, we combine recent advances in beyond-nearest-neighbor (BNN) metamaterials with inverse design,

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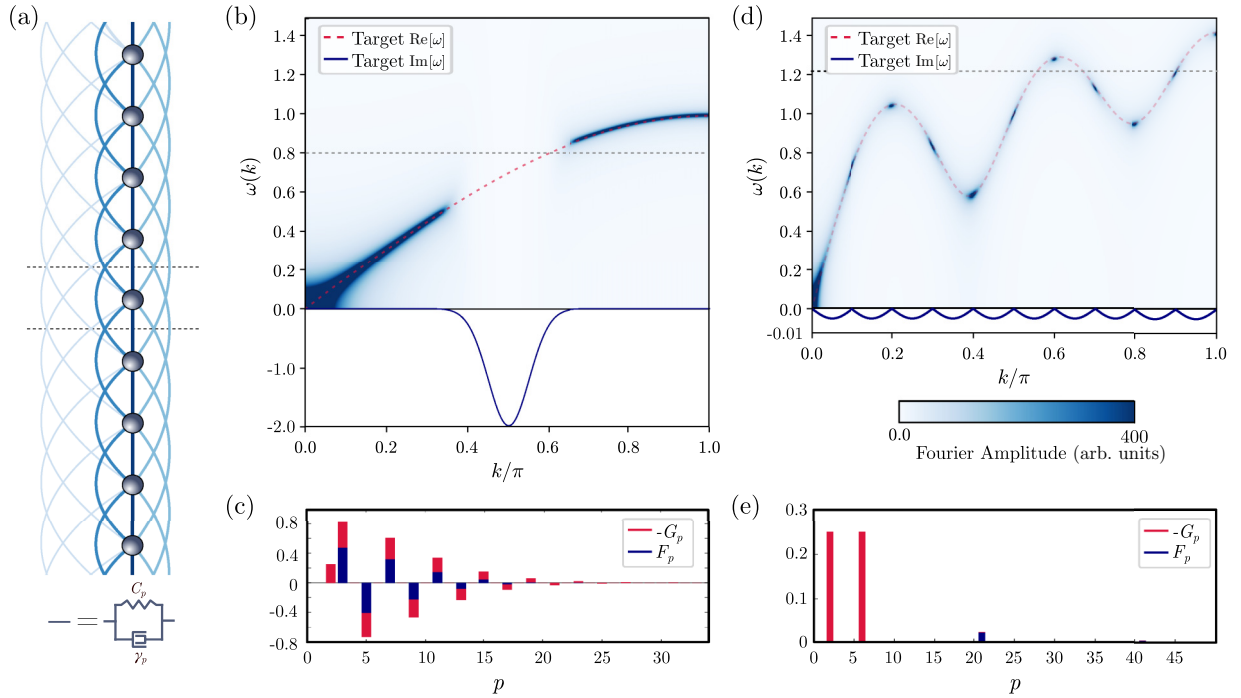


FIG. 1. (a) Schematic of a BNN mass-spring chain, where inter-mass connections are modelled by parallel spring (elastic restoring forces) and dashpot elements (dissipation or gain). (b) Fourier amplitude representation of an engineered dispersion curve, exhibiting a k -vector band gap. The target real and imaginary functions generating this curve are indicated by dashed red and solid blue lines, respectively. (c) Stiffness (G_p) and dissipative (F_p) Fourier coefficients derived from the inverse design protocol for the targets in (b). Panel (d) depicts the Fourier amplitude representation of an exotic passive dispersion relation, designed via the positive definite convolution identity (see the Supplemental Material Part I [50]), displaying a “dashed dispersion”. Panel (e) shows the associated Fourier coefficients. The gray-horizontal dashed lines in (b) and (d) highlight the excitation frequencies of the Green’s functions solutions presented in Fig. 2.

achieving arbitrary *complex* dispersion relations through combining periodic mass-spring lattices with nonlocal interactions that include both gain and loss. Each spatial order of lattice connection is characterized by uniquely defined spring constants and damping terms, a schematic is illustrated in Fig. 1(a). We then consider classes of sensible functions possessing only positive-definite Fourier coefficients, thus circumventing the requirement for gain or negative spring constants.

The result is a robust method that allows for a more nuanced control over the system’s dispersion relation. We provide multiple examples that confirm the freedom of the design protocol and highlight the unique physics that can be achieved in such systems, for example, topical wavevector band gaps [51,52]. We present numerical solutions of the governing eigenproblem and of the system’s Green’s function, whilst providing commentary on the energy and group velocities in such systems and motivate physical realization of such lattices.

II. THEORY

Our theory begins by considering damped oscillators in the canonical 1D periodic mass-spring chain [37]. We begin with the case of monatomic masses connected via springs that may have arbitrary nonconservative spring constants, i.e., they may contribute gain or loss to the system, and then expand this notation by incorporating p beyond-nearest-neighbor (BNN)

spatial connections, as visualized in Fig. 1(a). From Newton’s second law of motion, we obtain a real-space equation of motion for the displacement (U_n) of the n th point mass

$$\sum_{p=1}^{\infty} [\gamma_p (\ddot{U}_{n+p} + \ddot{U}_{n-p} - 2\ddot{U}_n) + C_p (U_{n+p} + U_{n-p} - 2U_n)], \quad (1)$$

where m denotes the mass, C_p denotes the spring constant for each BNN order p (i.e., connecting to the p th nearest neighbor), and γ_p likewise represents the damping of the connection to the p th nearest neighbor. Note that we assume the physical size of the unit cell is unity ($a = 1$), and for harmonic excitations assume the convention $e^{-i\omega t}$, e.g., displacement oscillations.

Because of lattice periodicity, we restrict our analysis to a single unit cell and employ Bloch’s theorem [18], assuming the displacement in cell n is related to that in the fundamental cell by $U_n = U_0 e^{ik \cdot n}$, with k the wavenumber along the chain. Thereupon, we obtain the reciprocal space equation of motion

$$\underbrace{(\omega^2 - i\omega F(k) - G(k))}_R U = 0, \quad (2)$$

where

$$F(k) = \sum_{p=1}^{\infty} 2\beta_p \alpha_p, \quad G(k) = - \sum_{p=1}^{\infty} \alpha_p \omega_{0,p}^2 \quad (3)$$

with $\beta_p = \frac{\gamma_p}{2m}$, $\omega_{0,p}^2 = \frac{C_p}{m}$, and $\alpha_p = 2(\cos(kp) - 1)$. We refer to the bracketed term R in (2) as the solvability condition, such that $R = 0$ yields the lattice's dispersion relation.

The generalized form of (2), permits the arbitrary exploration of the forward design problem, configuring lattices with unique combinations of nearest and beyond-nearest-neighbor spatial connections. Motivation is drawn from Kazemi *et al.* [32], to investigate the *inverse* design problem of general complex dispersion relations.

To address the inverse design problem, we first note that the dispersion curve is necessarily a repeating function, and so we can express any functional dependence of $\omega(k)$ as a Fourier series in k . Here, the real functions $F(k)$ and $G(k)$ represent the dissipative and reactive behaviors of the lattice respectively, noting that $G(k)$ is a positive real function. The goal is to now find the BNN couplings C_p and γ_p that set desired real and imaginary parts of a given dispersion relation $\omega(k)$. Clearly, $\text{Re}[\omega(k)]$ prescribes the conventional band structure, whilst $\text{Im}[\omega(k)]$ characterizes the global nonconservative behaviors; a negative (positive) $\text{Im}[\omega(k)]$ signifying damping (gain) within the system.

This inversion of the dispersion relation can be achieved through completing the square in (2),

$$\left(\omega(k) - \frac{iF(k)}{2}\right)^2 = G(k) - \frac{|F(k)|^2}{4}, \quad (4)$$

enabling the identification of the two functions $F(k)$ and $G(k)$ in terms of the dispersion relation $\omega(k)$,

$$\begin{aligned} F(k) &= 2\text{Im}[\omega(k)], \\ G(k) &= \text{Re}[\omega(k)]^2 + \text{Im}[\omega(k)]^2. \end{aligned} \quad (5)$$

Here, we note that $\text{Im}[\omega(k)]$ and $\text{Re}[\omega(k)]$ characterize the target dispersion functions, where k is taken as a real parameter. Equally, the complex wavenumber/real frequency convention could be chosen.

The associated damping and stiffness Fourier coefficients appearing in (2) for each specified dispersion relation $\omega(k)$ are then found via a Fourier transform

$$\begin{aligned} F_p &= \int_{-\pi}^{\pi} F(k) e^{-ikpa} dk = \frac{\gamma_p}{m}, \\ G_p &= \int_{-\pi}^{\pi} G(k) e^{-ikpa} dk = -\frac{C_p}{m}, \end{aligned} \quad (6)$$

such that the damping and spring constants take real values.

The above Fourier expansion (6) combined with the inverse design formulas (5) enables us to find combinations of reactive and dissipative coupling constants for any complex dispersion relation $\omega(k)$. Whilst it is valuable to explore these theoretical limits, it is equally important to consider the practical implications and the physical realization of the structures under examination. Note that, if gain is required (negative Fourier coefficients F_p), a physical realization necessitates active control. Similarly, negative spring constants ($C_p < 0$) require more exotic inertial-resonant elements and are typically narrowband [36]. Conversely, dispersion curves achievable without gain are theoretically obtainable through passive structures. Therefore, based on the sign of the Fourier

coefficients defining the target dispersion relations, functions can be classified as either exotic passive (no gain) or exotic active (with gain). Motivated by the design of an exotic passive dispersion relation, we present a condition for designing target dispersion relations that ensures the inverse design protocol only yields real, positive-definite Fourier coefficients (see the Supplemental Material Part I [50]).

III. RESULTS AND DISCUSSION

The precise engineering of $\text{Im}[\omega(k)]$ governs the attenuation characteristics in the designed dispersion profiles. When $\text{Im}[\omega(k)]$ is tailored to exhibit a magnitude comparable to $\text{Re}[\omega(k)]$, apparent band gap phenomena can be introduced into the complex dispersion relations, replicating behaviors observed in Hermitian systems. To highlight the power of this method, Fig. 1(b) presents an engineered system that exhibits a k -vector band gap in a *real* fundamental phononic dispersion relation. Here, $\text{Re}[\omega(k)]$ is defined as the canonical nearest-neighbor monatomic mass-spring dispersion relation ($\text{Re}[\omega(k)] \propto |\sin(ka)|$ [18]), and $\text{Im}[\omega(k)]$ is chosen as a Gaussian centered on $|k| = 0.5\pi$ (given by [53]). To showcase the designed attenuation, our analysis is extended into the inhomogeneous regime, numerically solving the analytic Green's function ($\mathcal{F}^{-1}(R^{-1} \cdot f)$) for some applied point forcing f , with $\mathcal{F}^{-1}$ denoting the inverse Fourier transform. Figure 2(a) shows the Green's function at $\omega = 0.8$, within the apparent k band gap, illustrating evanescent wave behavior. We perform the Green's function simulations, cycling through excitation frequency, and plot the absolute Fourier magnitude by the color-scales in Figs. 1(b) and 1(d), where the apparent k -gaps are evident. These gaps are "apparent" in the sense that the propagating wave components are effectively attenuated; closely mimicking the behavior of traditional band gaps where propagation is forbidden.

To generate the dispersion relation depicted in Fig. 1(b), a combination of both positive and negative Fourier coefficients F_p are necessary, as shown in Fig. 1(c). This implies that some of the lattice connections exhibit gain, i.e., by active control such as digital meta-atom approaches [41]. Although interesting, practical considerations usually make gain difficult to implement. This raises the question: Can we restrict our inverse design formulas (5) to yield only passive coupling parameters?

The answer to this question is yes; provided that careful consideration is given to the target function $\omega(k)$. To eliminate the gain requirement, we introduce an additional design condition; we restrict all orders of Fourier coefficients F_p to positive definite values. We express $F(k)$ as the convolution of two k -dependent functions $[h(k), s(k)]$, as outlined in the Supplemental Material Part I [50]. Imposing the condition $s(k) = h(-k)$, ensures real coefficients and taking the Fourier transform of $F(k)$ as detailed in (6), yields $F_n = |H|^2$, which is both real and positive. Here, H is the Fourier transform of $h(k)$. As an example, the function $|\sin(\mu k)|$ satisfies this condition (proof in the Supplemental Material Part I [50]), and we use this in the application of our design method shown in Fig. 1(d).

Figure 1(d) shows that the designed dispersion relation now incorporates a finite number of real, positive-definite

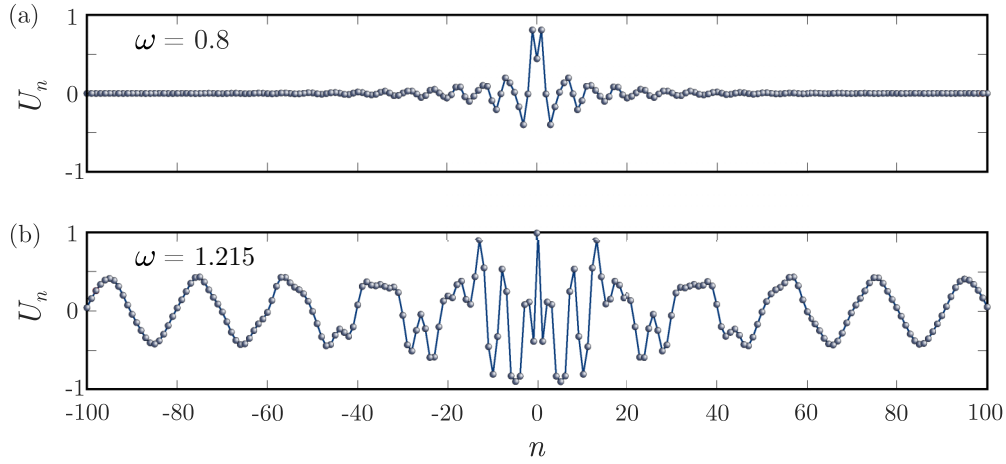


FIG. 2. Green's function solutions (real point mass displacements) at select frequencies in the engineered dispersion curves presented in Fig. 1. (a) Solution driven at $\omega = 0.8$ in dispersion curve in Fig. 1(b), in the apparent k -vector band gap. (b) Solution driven at $\omega = 1.215$, in dispersion curve in Fig. 1(d), which intersects the dispersion curve at three distinct k , two of which observe increased attenuation.

coefficients for both the dissipative F_n , and reactive G_n response [see Fig. 1(e)]. In this example, the real component $\text{Re}[\omega(k)]$ replicates the behavior of a mass-spring lattice with first- and fifth-order spatial connections, somewhat more complex than the recent “Roton-like” dispersion curves [19], while the imaginary component $\text{Im}[\omega(k)] = -|\sin(10k)|$ fulfills the aforementioned convolution identity. As shown in Fig. 1(d), increased attenuation is observed in line with the maximal points of $\text{Im}[\omega(k)]$, creating a dashed-like appearance [Fourier amplitude in Fig. 1(d)]. This is further demonstrated in Fig. 2(b), which displays the Green's function at a driving frequency ($\omega = 1.215$) that intersects three k solutions, two of which exhibit increased attenuation. The decay envelopes of the decay modes are localized near $n = 0$, while a singular driven mode dominates outside this region.

We now investigate the energy transport within the structure corresponding to the dispersion shown in Fig. 1(d). One approach to this would be to consider the elastic Poynting vector and group velocity; however, in this non-Hermitian regime we must be careful in applying conventional wave analysis techniques. Dissipative effects engender a complex reshaping of a wave packet during propagation, meaning that we must be careful in inferring information from the group velocity, something known well in both the electromagnetic and elastic literature [54,55]. Here, we explore energy transport throughout the lattice via the energy flux and the closely related energy velocity, which characterizes the rate at which energy propagates within a lattice. Specifically, we consider the rate of loss of kinetic energy in a given unit cell, finding the contributions owing to propagation and dissipation. Supplemental Material Part II contains the derivation of energy flux and velocity following the approach of Brillouin [50].

We delineate between the two constituent contributions to the total energy flux, namely, the dissipated energy flux (Φ_D), representing energy lost within the system, and the transmitted energy flux (Φ_T), which quantifies the energy transfer between adjacent unit cells such that the total energy flux (rate of loss of kinetic energy) is $\Phi = \Phi_T + \Phi_D$. We show the derivation (generalized from Brillouin [18] to include dissipation

and BNNs) in the Supplemental Material Part II [50],

$$\Phi = \sum_p^P \left(\frac{U_0^2 C_p \text{Re}(\omega) p}{2} \sin(kp) + p \left(\frac{U_0^2 C_p \text{Im}(\omega)}{2} + \frac{U_0^2 \gamma_p |\omega|^2}{2} \right) (1 - \cos(kp)) \right), \quad (7)$$

where the first line gives the contribution of Φ_T with the second giving Φ_D , respectively. Graphically shown in Fig. 3(a), Φ_D converges to zero in alignment with the tailored dissipation profile [Fig. 3(c)], reflecting the critical role of $\text{Im}[\omega(k)]$. Moreover, the dissipated energy flux remains strictly positive ($\Phi_D > 0 \forall k \geq 0$), indicating the absence of gain mechanisms within the system, reinforcing the exotic passive condition arising from the convolution identity. Complementary to this, the transmitted energy flux exhibits features akin to those of the equivalent undamped system, revealing localized regions where $\Phi_T = 0$; these regions correspond to turnover points in $\text{Re}[\omega(k)]$. Equally significant is the development of negative regions in the transmitted energy flux, a consequence of the competition between power channels, which drives a system response exhibiting apparent retrograde energy flow [25].

Figure 3(b) plots the closely related energy velocity, defined as the energy flux divided by the energy density [Fig. 3(a), inset]—this is required to be analyzed as the notion of group velocity loses significant meaning in nonconservative systems [55]. For completeness we show the group velocity of $\text{Re}[\omega(k)]$, not shown is $v_g(k = 0) = 0$ —this discrepancy arises from a discontinuity in v_g at $k = 0$, which separates the positive ($k \geq 0$) and negative ($k < 0$) regions of the first Brillouin zone. The distinction between E_v and v_g elucidates the expected challenges in characterizing energy propagation via the group velocity in dissipative media. Yet, there is an expected commonality when dissipation becomes negligible ($\text{Im}[\omega(k)] \rightarrow 0$), with $E_v \rightarrow v_g$, revealing their analogous behavior in conservative systems.

Before concluding, we note here that with this powerful theoretical freedom there are some challenges to highlight; high gradients in target dispersion relations are sensitive to

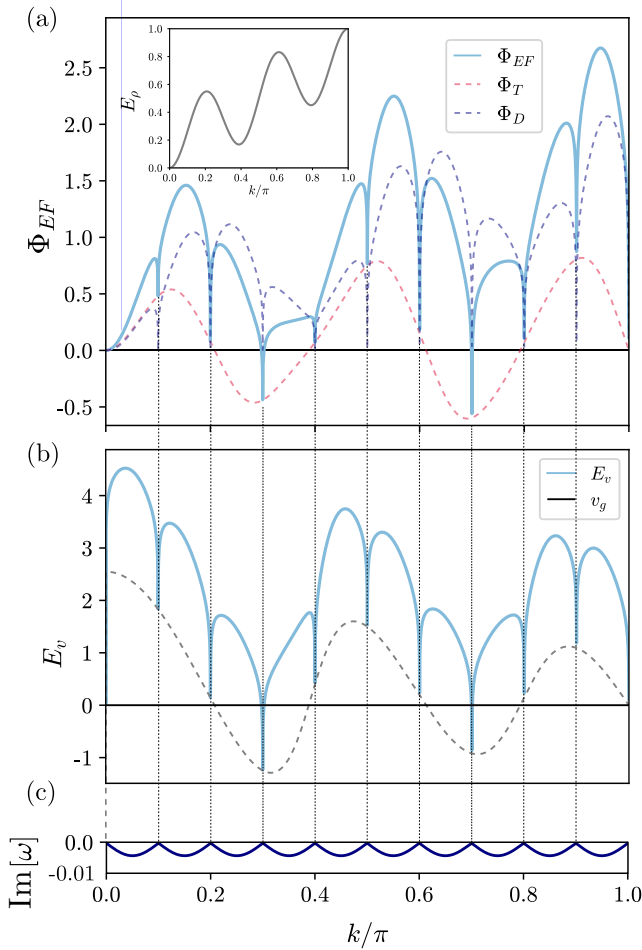


FIG. 3. (a) Net-energy flux arising from the kinetic energy of the mass for the systems presented in Fig. 1(d). Φ_T and Φ_D , represent the transmitted and dissipated flux channels contributing to net power flow (Φ), while the inset illustrates the energy density, the total kinetic and potential energies. Panel (b) is the closely related energy velocity (E_v), which is the flux divided by the energy density, characterizing the rate of energy flow. For comparison, the group velocity (v_g) of the equivalent undamped system ($\text{Re}[\omega(k)]$ only) is presented. Panel (c) is $\text{Im}[\omega(k)]$, outlining the designed attenuation.

Gibbs phenomena [56], prompting the exploration of continuous, smooth functions to mitigate this. Furthermore, the designed dispersion relations are constrained by the condition, $\omega(k=0) = 0$, arising from the recurring α_p term in the previously defined equation of motion (2), which aligns with the physical motivation presented in [32]. However, it is important to recognize that this condition is not universally applicable, as demonstrated in systems with zero-frequency band gaps, where the behavior at $k=0$ can be fundamentally different [57,58].

IV. CONCLUSIONS

Dispersion engineering offers a compelling avenue for advancing technologies across a diverse range of fields such as noise reduction, seismic isolation, sensing, and even energy harvesting [3,24,59]. We have presented a versatile inverse design protocol where both the real and imaginary parts of any dispersion relation can be related to a fixed set of BNN coupling constants. We have applied this method to explore a set of engineered dispersion relations, which includes the formation of apparent k -vector band gaps, and “dashed” dispersions, where only narrow ranges of k values exhibit lossless propagation. Extending the inverse design of lattice couplings to encompass non-Hermitian behavior thus provides a powerful framework for tailoring wave propagation characteristics.

ACKNOWLEDGMENTS

R.G.E. and T.A.S. acknowledge the financial support of Defence Science and Technology Laboratory (Dstl) through Grants No. DSTL0000022047, No. DSTLXR1000154754, and No. AGR 0117701. S.A.R.H. and G.J.C. acknowledge financial support from the EPSRC via the META4D Programme Grant (No. EP/Y015673/1).

DATA AVAILABILITY

The data supporting this study’s findings are available within the article.

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