

Geometric quantum drives and topological dynamical responses: Hyperbolically driven quantum systems and beyond

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We introduce a geometrical framework to construct a large class of time-dependent quantum systems in which the position of a classical particle moving autonomously on a smooth connected manifold is used to steer a quantum Hamiltonian over time. This results in quantum drives with structured temporal profiles and properties dependent on the local and global nature of the underlying choice of manifold. We show that our construction recovers the well-known classes of periodically driven and quasiperiodically driven quantum systems, but also unveils fundamentally new classes of quantum dynamics: by utilizing a compact two-dimensional hyperbolic Bolza surface and a nonorientable Klein-bottle surface, we demonstrate examples of a hyperbolically driven quantum system and a nonorientably driven quantum system respectively. Furthermore, we demonstrate that these driven systems exhibit unusual quantized dynamical responses reflecting their different underlying topologies, under the condition of being fully gapped and in the adiabatic limit, and which have interpretations as quantized crystalline electromagnetic responses in certain exotic effective tight-binding lattice models. We envision geometric quantum driving as a general framework to chart the landscape of time-dependent quantum systems and investigate the universal phase structures they exhibit, as well as a useful tool to enhance the capabilities of modern day quantum simulators.

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I. INTRODUCTION

Time-dependent controls are general and powerful tools in physics, routinely used to engineer new interactions or stabilize states not possible at equilibrium. A classic and well-known example is the Kapitza pendulum, where an originally unstable inverted equilibrium point can be made stable through rapid driving of the pivot [1]. In quantum systems, driving allows for the control of effective Hamiltonians for cold atoms in optical lattices and quantum materials [2–5]. In particular, periodic driving has been used to engineer artificial gauge fields and Floquet topological Bloch bands [6–8]. Novel nonequilibrium interacting phases of matter such as the discrete time crystal [9–12] or the anomalous Floquet insulator [13,14] have also been uncovered, which cannot intrinsically exist without explicit time-dependent modulation of the system.

Out of all possible driven quantum systems, time-periodic or Floquet systems [2–4,15,16] (and their generalization, time-quasiperiodic systems [17–20]), wherein system parameters are modulated with a single (few multiple) independent

fundamental frequency (frequencies), are perhaps the most well studied. This stems from the conceptual simplicity in understanding the effect of the drive: each frequency acts as an additional synthetic dimension [21], and thus the physical properties of a complex higher-dimensional system may be emulated in a lower-dimensional one. However, time-periodic or time-quasiperiodic driving constitute only a very limited subset of all possible time-dependent modulations of a system. This naturally raises a simple but important conceptual question: what are more general classes of quantum driving $H(t)$, generated by a time-dependent quantum Hamiltonian, which can lead to interesting responses and novel phase structure in dynamics? Crucially, the consideration of such generality is not merely academic but also relevant in practice: time-dependent controls in modern quantum simulator experiments are often implemented using *arbitrary waveform generators* (AWG) [22–24], which, in principle, allow for the creation of waveforms with arbitrary temporal profiles (within limitations such as finite bandwidths), thus according a multitude of ways to controllably take a system out of equilibrium.

A central challenge in tackling this question, though, is that the space of all possible drives $H(t)$ is very large. Moreover, it can be expected that the vast majority of drives will not lead to any interesting physical phenomena—for example, a random sequence of pulses applied results in the system responding chaotically and exhibiting a featureless state on average [25]. It is key that one has to impose *structure* on the temporal profiles of the drives to achieve nontrivial dynamics [9,26]. Previous work along this front have been scant, but a few

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notable ones include aperiodic drives generated by sequences of pulses following simple concatenation rules on basic characters of an alphabet (so-called "morphic" or "multipolar" sequences [27–29]). These have been predicted to give rise to interesting physical phenomena like time roudou crystals, a novel form of spatiotemporal ordering [30], and critically slow exploration of the Hilbert space [29].

In this work, we make headway on this question by introducing a general construction based on geometric considerations, which yields distinct classes of quantum drives $H(t)$ with structured temporal profiles. Our key idea is to use an autonomous classical particle moving on a choice of smooth connected manifold to steer a parent quantum Hamiltonian defined on the manifold; this results in a time-dependent quantum Hamiltonian whose behavior reflects the manifold's local geometry and global topology—motivating the naming of such classes of Hamiltonians as *geometric quantum drives*.

Employing the framework, we demonstrate two novel classes of quantum drives and predict interesting and exotic dynamical topological responses within them. Concretely, we first utilize a compact and hyperbolic space called the Bolza surface [31] to construct a *hyperbolically driven quantum system* (HDQS). We show a HDQS exhibits a quantized dynamical response set by the Chern number of its instantaneous bands in the fully gapped phases and adiabatic limit, which can be measured with a specially constructed observable. We then consider nonorientable surfaces like the Klein bottle and the real projective plane, which yield *nonorientably driven quantum systems* (NDQS). We show that a Klein bottle drive can exhibit a measurable quantized topological dynamical response associated with a dipolar Chern number characterizing the system's topology, and which has the physical interpretation of a crystalline response to an electromagnetic field in an effectivesolid-state lattice system with so-called nonsymmorphic momentum-space symmetry [32,33]. We envision geometric quantum driving as a powerful framework to chart the landscape of time-dependent quantum systems and investigate the universal phase structures they exhibit, and as well as a useful tool to extend the simulation capabilities of modern day quantum experiments.

II. GEOMETRIC QUANTUM DRIVING

We begin by introducing our theoretical framework of geometric quantum driving. We define a geometric quantum drive to be specified by a choice of triple $(M, H_Q(x), x_t)$. Here, M is a smooth manifold of dimension d , described by local coordinates $x = (x^1, \dots, x^d)$; $H_Q(x)$ is a parent quantum Hamiltonian parametrized by coordinates on M , which acts on a D -dimensional Hilbert space (assumed at least piecewise continuous and with bounded norm); and x_t is a trajectory on the manifold. We then construct a time-dependent quantum Hamiltonian as

$$H(t) := H_Q(x_t). \quad (1)$$

Interpreting x_t as the position of some classical particle moving on M , this time-dependent quantum Hamiltonian may be understood as arising from a particle "steering" the underlying parent quantum Hamiltonian (with no back action).

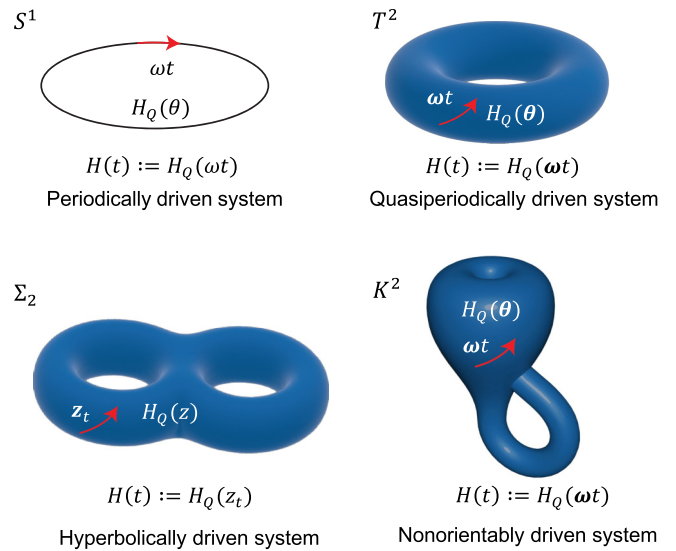


FIG. 1. Geometric quantum drives. Geodesic flows on a circle S^1 (top left), a torus T^2 (top right), a genus-2 torus Σ_2 (bottom left), and a Klein-bottle surface K^2 (bottom right), yield periodically, quasiperiodically, hyperbolically, and nonorientably driven quantum systems respectively. In this work, we present explicit constructions of the latter two systems and expound upon the novel physics they exhibit.

Stated this way, the construction is rather (perhaps overly) general—we have not specified the desired properties of the manifold M (like compactness or metricity); nor have we prescribed how the trajectory x_t is to be generated (e.g., through some vector flow, etc.), all of which can potentially affect the nature of the resulting driven quantum system $H(t)$. We now impose constraints on the triple $(M, H_Q(x), x_t)$ to make concrete predictions, but we stress that the framework is versatile and expect that different assumptions can lead to different classes of driven quantum systems. We note the related construction of Ref. [34], which also yields families of structured quantum drives with time dependence underpinned by classical dynamical flows. In our language, this is the choice that x_t is a classical dynamical flow on M , but our framework, in principle, need not require this.

In what follows we restrict ourselves to the case when the manifolds in consideration are closed, i.e., compact and boundaryless; and Riemannian, i.e., there exists a smooth positive-definite inner product $g_{ij}(x)$ defined on the tangent space at every point x . The existence of a Riemann metric allows for the consideration of a natural trajectory x_t on the manifold M —*geodesics* [35], which we also assume henceforth.

It is helpful at this stage to see our abstract discussion in action with these choices: they are powerful enough to already recover the well-known classes of periodically and quasiperiodically driven quantum systems (see Fig. 1). Take $M = S^1$, the circle, and use standard coordinates $\theta \in [0, 2\pi]$ (with end-points identified). Consider a bounded piecewise continuous parent quantum Hamiltonian $H_Q(\theta)$ on S^1 so it admits a Fourier series representation $H_Q(\theta) = \sum_{n \in \mathbb{Z}} H_n e^{in\theta}$. The metric on the circle is flat, $g(\theta) = 1$, yielding geodesics

$\theta_t = \omega t + \theta_0 \bmod 2\pi$, which describe a particle moving with constant speed or momentum $p_t = \omega$ around the circle. Using Eq. (1), the resulting time-dependent quantum Hamiltonian is then

$$H(t) := H_Q(\omega t + \theta_0) = \sum_{n \in \mathbb{Z}} H_n e^{in(\omega t + \theta_0)},$$

which is precisely a time-periodic quantum Hamiltonian, driven with frequency ω . Similarly, by choosing $M = T^d$, the d -torus for $d \geq 2$, described by local coordinates $\theta = (\theta^1, \dots, \theta^d) \in [0, 2\pi]^d$ with flat metric $g_{ij}(\theta) = \delta_{ij}$, geodesics are straight lines $\theta_t = \omega t + \theta_0 \bmod 2\pi$ where $\omega = (\omega^1, \dots, \omega^d)$ is a fixed momentum vector. Evaluation of Eq. (1) for a parent quantum Hamiltonian $H_Q(\theta)$ defined on the torus, assuming $\{\omega^i\}$ are rationally independent, gives

$$H(t) := H_Q(\omega t + \theta_0) = \sum_{n \in \mathbb{Z}^d} H_n e^{in \cdot (\omega t + \theta_0)},$$

precisely a quasiperiodically driven quantum Hamiltonian with frequency vector ω .

Next, we will go beyond these basic and familiar examples of the flat circle S^1 and flat torus T^d , and show how our geometric quantum driving framework can predict new and distinct classes of quantum drives with interesting physics. We consider two-dimensional manifolds with nontrivial topology, exemplified by a hyperbolic surface and a nonorientable surface.

III. HYPERBOLICALLY DRIVEN QUANTUM SYSTEMS

Let us consider a two-dimensional (2D) manifold with nontrivial negative curvature: a double torus Σ_2 , see Fig. 1. To be explicit, we take Σ_2 to be the Bolza surface, a particular compact, closed, hyperbolic Riemann surface of genus 2 [36], constructed as the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the hyperbolic plane and Γ is a Fuchsian group of translates; see [37] and the Supplemental Material (SM) for details [38]. In the Poincaré disk model $|z| < 1$ which has metric $g_{ij}(z) = g(z)\delta_{ij}$ with $g(z) = 4/(1 - |z|^2)^2$, the fundamental domain is an octagon wherein opposite sides (which are arcs of circles) are identified with the orientation shown in Fig. 2. We note the Bolza surface is not only a mathematical construct but has featured in various areas of physics: the Hadamard-Gutzwiller model [39–42] describes the motion of a free particle on the surface (an example of dynamical billiards), and is a minimal model of classical and quantum chaos; more recently, the Bolza surface has appeared in hyperbolic band theory [43] as a unit cell involved in a particular tiling of hyperbolic space. Our work adds to the growing list of interesting physical phenomena tied to hyperbolic geometry.

To understand the nature of geodesics on the Bolza surface, it is helpful to adopt the viewpoint of geodesics as arising from Hamiltonian dynamics of a free classical particle on the surface [35]. Precisely, let $H_{cl}(z, p) = \frac{1}{2}g^{ij}(z)p_i p_j$ be the classical Hamiltonian describing a unit-mass free classical particle on Σ_2 . This is a function on phase space described locally by position and momentum coordinates $(z, p) := (x^1 + ix^2, p_1 + ip_2) \in \mathbb{C}^2$; more precisely, the phase space is the cotangent bundle $T^*\Sigma_2$ of the Bolza. Here, $g^{ij}(z) = g(z)^{-1}\delta_{ij}$

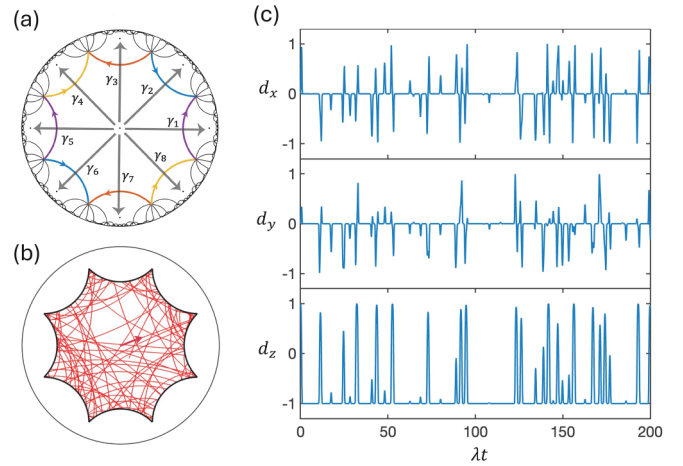


FIG. 2. (a) We take Σ_2 to be the Bolza surface, a compact Riemann surface with constant negative curvature. On the Poincaré disk, the fundamental domain is an octagon, with opposite sides identified in the orientation shown via Fuchsian translates γ_i . (b) Geodesic on the Bolza surface. (c) Temporal profile of the example hyperbolically driven qubit Hamiltonian considered in this work with $\epsilon = 0.5$.

is the inverse metric. Then, Hamiltonian's equations yield [38]

$$\dot{z} = \frac{1}{4}(1 - |z|^2)^2 p, \quad \dot{p} = \frac{1}{2}z(1 - |z|^2)|p|^2. \quad (2)$$

Upon specifying an initial condition (z_0, p_0) , there is a unique solution (z_t, p_t) ; the resulting z_t is the desired geodesic trajectory (note an appropriate Fuchsian translate has to be applied whenever the particle exits the fundamental region, see Fig. 2(b) and [38] for details). Note that given a geodesic solution (z_t, p_t) , we can always construct an infinite family of related geodesic solutions $(z_t^{(\lambda)}, p_t^{(\lambda)}) := (z_{\lambda t}, \lambda p_{\lambda t})$ for any $\lambda > 0$, each of which describes the same path on the manifold Σ_2 but differs by its conserved energy

$$H_{cl}(z_{\lambda t}, \lambda p_{\lambda t}) = H_{cl}(z_{\lambda 0}, \lambda p_{\lambda 0}) = \frac{1}{2}g(z_0)^{-1}|p_0|^2\lambda^2;$$

equivalently, they are geodesics of different speeds λ . We will henceforth assume that the normalization is such that $\lambda = 1$ corresponds to the unit-speed geodesic with energy $1/2$.

Lastly, given a choice of parent quantum Hamiltonian $H_Q(z)$ defined on the Bolza surface, we define a *hyperbolically driven quantum system* (HDQS) as

$$H_\lambda(t) := H_Q(z_t^{(\lambda)}), \quad (3)$$

where λ tunes the "speed" that the parent quantum Hamiltonian is being driven at, in analogy to the frequency ω in a periodically driven system. By virtue of our geometric construction, this is a class of quantum drives with temporal profiles clearly distinct from time-periodic or time-quasiperiodic drives. This distinction is especially transparent in their frequency content (i.e., upon decomposing the driving Hamiltonian in Fourier space): the Hamiltonians of hyperbolic drives generally have frequency spectra which are continuous, whereas periodic and quasiperiodic drives have spectra which are discrete, consisting of countably many fundamental frequencies. An explicit example of a HDQS based on the qubit model discussed below is shown in Fig. 2(c).

A. Topologically quantized response in the adiabatic limit

While our construction has given rise to a Hamiltonian with exotic temporal profile, we now explain that HDQS can exhibit novel physical phenomena. We will show how HDQS which have the property of being "fully gapped" (defined below), can be topologically characterized by quantized dynamical responses in the adiabatic limit, which are related to the underlying topology of the Bolza surface Σ_2 .

Let us call a D -level smooth parent quantum Hamiltonian $H_Q(z)$ "fully gapped" if its eigenstates $|\psi_n(z)\rangle$ have energies $E_n(z)$ obeying $E_n(z) < E_{n+1}(z)$ point-wise for every $z \in \Sigma_2$. Thus each $|\psi_n(z)\rangle$ ($n = 1, \dots, D$) forms a well-separated "band" over the manifold Σ_2 , the latter of which can be interpreted as a generalized Brillouin zone (BZ) in analogy to band theory of solid state physics (though our BZ crucially did *not* arise as the momentum space description of some lattice in either Euclidean or hyperbolic space). The topological properties of quantum systems parametrized by coordinates on a generic manifold M have been studied in great generality recently [44–48], but in our present case, standard arguments tell us that the topological classification of such bands is given by the obstruction to choosing a globally smooth gauge for $|\psi_n(z)\rangle$ over the surface [49]. This is captured by the $\mathbb{Z}$ -valued first Chern number

$$C_1^{(n)} = \frac{1}{2\pi} \iint_{\Sigma_2} dx^1 dx^2 \Omega_{x^1 x^2}^{(n)}(z), \quad (4)$$

the integral of the Berry curvature form $\Omega_{x^1 x^2}^{(n)}(z) = \partial_{x^2} A_{x^1}^{(n)}(z) - \partial_{x^1} A_{x^2}^{(n)}(z)$ over the fundamental region in the Poincaré disk defining the Bolza surface, where $A_{x^j}^{(n)}(z) = i \langle \psi_n(z) | \partial_{x^j} \psi_n(z) \rangle$ is the Berry connection (note there is no Poincaré metric under the integral). More sophisticatedly, the first Chern number can be understood as a topological classification of $U(1)$ fiber bundles on a genus 2 Riemannian surface (here, the Bolza surface), and takes values in the second cohomology group [50]

$$H^2(\Sigma_2; \mathbb{Z}) = \mathbb{Z}. \quad (5)$$

Intriguingly, we claim the (global) topological invariant $C_1^{(n)}$ characterizing the n th band may be extracted in the *dynamics* of the (local) D -level HDQS, as follows: fixing some choice of geodesic $(z_t^{(\lambda)}, p_t^{(\lambda)})$, initialize the quantum system in the n th instantaneous eigenstate $|\psi_n(z_0^{(\lambda)})\rangle$ at $t = 0$, and evolve the system with $H_\lambda(t) := H_Q(z_t^{(\lambda)})$ to obtain $|\psi(t)\rangle$. Then, measure the following judiciously constructed time-dependent observable of the D -level quantum system

$$O^{(\lambda)}(t) := 2g(z_t^{(\lambda)})^{-1} (p_{2,t}^{(\lambda)} \partial_{x^1} H_Q(z_t^{(\lambda)}) - 1 \leftrightarrow 2). \quad (6)$$

While the form of $O^{(\lambda)}(t)$ looks complicated, it is simply a observable constructed from the parent quantum Hamiltonian and thus has a time dependence explicitly determined by the choice of geodesic which can be easily written down. Finally, evaluate its (scaled) time-average expected value

$$w(T) := \frac{1}{\lambda^2 T} \int_0^T dt \langle \psi(t) | O^{(\lambda)}(t) | \psi(t) \rangle. \quad (7)$$

We argue that in the long-time and slow-driving limit (taken in that order), the response $w(T)$ is topologically

quantized to the first Chern number for almost all choices of geodesic flows [38]

$$\lim_{\lambda \rightarrow 0} \lim_{T \rightarrow \infty} w(T) = C_1^{(n)}. \quad (8)$$

Equation (8), the topologically quantized dynamical response of a HDQS, together with the identification of the nontrivial observable Eq. (6), constitute our main results in this section. In practice, a large T and small λ (but with both being finite) is sufficient to see good quantization, as we will see with the explicit model below [Fig. 3(c)]. We note that perfect quantization in a fully gapped HDQS can be achieved without the need for slow driving, with the use of counterdiabatic driving, see SM for details [38,51,52]. Note that the aforementioned observable extracts in dynamics the *extrinsic topology* of the manifold Σ_2 : specifically, the way that a $U(1)$ fiber bundle [given by the "band" $|\psi_n(z)\rangle$] wraps smoothly around it. It is also interesting to ask if there is an observable of the HDQS that can extract in dynamics the *intrinsic topology* to do with Σ_2 , such as Euler characteristic χ . In the SM [38], we indeed outline a possible general dynamical protocol that yields a metric tensor associated with the base manifold entering the construction of a geometric quantum drive, from which one can then extract such intrinsic geometrical and topological information.

B. Origin of topologically quantized response

We now briefly explain why one should expect a quantized topological response in $w(T)$. We argue that its quantization (taken within the appropriate limits) relies on two key properties of these HDQS which generalize to many other choices of M and many other choices of trajectories. First, under slow driving (more precisely, when the speed λ is much less than the energy gaps between the n th and $n \pm 1$ th "bands"—which can always hold due to the fully gapped condition), the system stays close to the n th instantaneous band by the quantum adiabatic theorem [53]. That is, $|\psi(t)\rangle \approx |\psi_n(z_t^{(\lambda)})\rangle + \dots$ (we omit the standard geometric and dynamical phases accumulated), where $\dots$ represent corrections in λ in an adiabatic expansion [54]. Thus, $w(T)$ may also be expanded in an adiabatic series. Second, our choice of path (geodesic) is *ergodic* on the Bolza surface Σ_2 . This nontrivial fact is guaranteed by a theorem of Hopf's [55,56], which states more generally that *the geodesic flow on a closed, negatively curved surface is ergodic*. For the Bolza surface, the implication (invoking Birkhoff's theorem [57]) is then as follows: consider an integrable function $f(z, p)$ defined on the cotangent bundle $T^*\Sigma_2$ of the Bolza surface, and its time-average along some geodesic $(z_t^{(\lambda)}, p_t^{(\lambda)})$. For almost all initial conditions $(z_0^{(\lambda)}, p_0^{(\lambda)})$, as $T \rightarrow \infty$,

$$\begin{aligned} \int_0^T \frac{dt}{T} f(z_t^{(\lambda)}, p_t^{(\lambda)}) &\rightarrow \iint_{\Sigma_2} dx^1 dx^2 \frac{g(z)}{4\pi} \\ &\times \int_0^{2\pi} \frac{d\theta}{2\pi} f(z, \lambda g(z)^{\frac{1}{2}} e^{i\theta}) \end{aligned}$$

(see Fig. S2 in the SM [38] for illustration). The meaning of the right-hand side (RHS) is as such: fix a point z and consider the possible momenta of the classical particle whenever it passes through that point. By energy conservation, these all

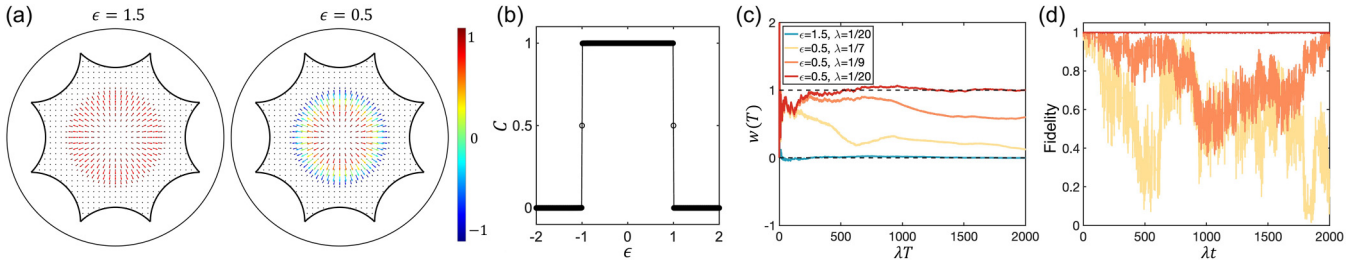


FIG. 3. (a) Spin texture $\mathbf{d}_\epsilon(z)$ of the parent qubit Hamiltonian $H_Q(z)$ on the Bolza surface for $\epsilon = 1.5, 0.5$. When $\epsilon = 1.5(0.5)$, the texture is ferromagnetic(skyrmionic), pointing mostly up(inverting from up to down). (b) The difference in textures can be quantitatively and sharply captured by the Chern number of the upper band of $H_Q(z)$, which changes from 0 (trivial) when $|\epsilon| > 1$, to 1 (topological) when $|\epsilon| < 1$. (c) Dynamical response $w(T)$ beginning from the instantaneous eigenstate of the upper band. For small λ , $w(T)$ converges to the Chern number of the band at late times, in both trivial and topological regimes. (d) Fidelity between time evolving state and instantaneous eigenstate for different λ s and $\epsilon = 0.5$.

have fixed length $\lambda g(z)^{\frac{1}{2}}$; so the only remaining degree of freedom is their direction $e^{i\theta}$. Ergodicity of the motion says that over long times, θ is sampled uniformly (over the circle). Further, it says this behavior occurs uniformly (with respect to the Poincaré metric) over the surface Σ_2 ; together this is nothing more than the familiar idea of ergodicity as time-averaging equaling space averaging (over the slice of $T^*\Sigma_2$ with constant energy $E = \lambda^2/2$). Utilizing this, we show in the SM [38] that the time-averaged observable $w(T)$ at large T can be approximated by a space integral over the Bolza surface; evaluating this the leading order term in λ vanishes, while the next order term is proportional to the Chern number $C_1^{(n)}$, leading to our claim Eq. (8). Importantly, it is clear that our result of quantized dynamical responses Eq. (8), in fact, constitute a topological *characterization* of fully gapped HDQS: their quantized responses are robust under perturbations of the system which preserve their fully gapped nature. We expect the responses are robust too under perturbations of the underlying classical path (e.g., noise) on Σ_2 , which determines the temporal profile of the Hamiltonian $H(t)$, as long as they are ergodicity-preserving—we leave these investigations for future work.

C. Qubit model and numerics

To substantiate our analytical claims, we explicitly construct an example qubit HDQS and numerically simulate its dynamical response. Consider the parent quantum Hamiltonian

$$H_Q(z) = \mathbf{d}_\epsilon(z) \cdot \boldsymbol{\sigma}, \quad (9)$$

where $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are standard Pauli matrices and $\mathbf{d}_\epsilon(z) = (d_{x,\epsilon}(z), d_{y,\epsilon}(z), d_{z,\epsilon}(z))$ is a real vector of field strengths varying over Σ_2 , tunable by a parameter $\epsilon \in \mathbb{R}$, defined as follows. We introduce a bump function $f(z)$, which changes smoothly and compactly from $\pi/2$ at $z = 0$ to $-\pi/2$ at $|z| = 0.6$ whose precise form is presented in the SM [38]. Then, $\mathbf{d}_\epsilon(z)$ is defined, for $|\epsilon| \neq 1$, as $\mathbf{d}_\epsilon(z) = \mathbf{d}'_\epsilon(z)/|\mathbf{d}'_\epsilon(z)|$ where

$$\mathbf{d}'_\epsilon(z) = \begin{pmatrix} \cos(f(z))x^1/|z| \\ \cos(f(z))x^2/|z| \\ \sin(f(z)) + \epsilon \end{pmatrix}; \quad (10)$$

see Fig. 3(a) for illustrative spin textures $\mathbf{d}_\epsilon(z)$. Since $|\mathbf{d}_\epsilon(z)| = 1$ for every z , $H_Q(z)$ is fully gapped. Furthermore, when $\epsilon > 1$ ($\epsilon < 1$), $\mathbf{d}_{z,\epsilon}$ is always positive (negative), describing a ferromagnetic spin texture, while when $|\epsilon| < 1$, the spin texture inverts moving from the origin to the boundary, describing a skyrmionic texture. We thus expect the bands of $H_Q(z)$ to be characterized by differing topological numbers in these two parameter regimes (respectively, trivial and topological), which is indeed seen in the Chern number plot, Fig. 3(b).

Turning to quantum dynamics, we consider a family of geodesics with varying speeds λ defined by initial conditions $z_0 = 0$ and $p_0 = \lambda e^{i\pi/9}$ [Fig. 2(b)], which defines the hyperbolically driven quantum Hamiltonian $H(t) := H_Q(z_t)$ [Fig. 2(c)]. We then measure $O^{(\lambda)}(t)$ and average it to yield $w(T)$. We see that when $\lambda \lesssim 1/20$, the motion is very close to adiabatic [Fig. 3(d)]. Moreover, $w(T)$ converges at late times to the Chern number 0 (1) in the trivial (topological) regimes of the model [Fig. 3(c)], demonstrating its nontrivial topological responses, Eq. (8), as predicted.

IV. NONORIENTABLY DRIVEN QUANTUM SYSTEMS

We next use our geometric quantum driving framework to construct yet another new and distinct class of driven quantum systems derived from the choice of *nonorientable* surfaces, i.e., 2D manifolds in which no globally consistent choice of normal direction can be defined, and unveil the physics it exhibits. In what follows we will focus on the paradigmatic example of the Klein bottle K^2 , see Fig. 1, but in the SM, we also provide another example given by the choice of real projective plane RP^2 . Collectively, these can be termed *nonorientably driven quantum systems* (NDQS).

Now, the Klein bottle can be realized as the quotient space $\mathbb{R}^2/\tau$, where $\mathbb{R}^2$ is the flat real plane and τ is the group generated by the following two transformations:

$$\tau_1 : (x, y) \mapsto (x + 2\pi, y), \quad \tau_2 : (x, y) \mapsto (2\pi - x, y + \pi). \quad (11)$$

As a result, the fundamental domain for K^2 can be taken to be $[-\pi, \pi] \times [-\pi, 0] \ni (\theta_x, \theta_y) = \boldsymbol{\theta}$, where vertical boundaries are identified with similar orientation while horizontal boundaries are identified with opposite orientation [gray region in Fig. 4(a)]. We note that two Klein bottles can be glued to

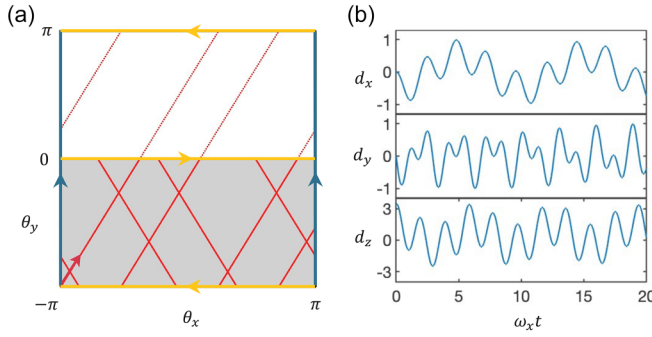


FIG. 4. (a) Geodesics on the Klein bottle (gray region) are depicted by red straight lines. Every time the particle leaves the gray region, it gets mapped back to a point within the fundamental region following the identification rules of the boundaries (yellow and blue arrows). Equivalently, one may consider two gluing copies of the Klein bottle as shown (white and gray regions) which yield a standard torus; a Klein bottle geodesic is then equivalent to a standard geodesic on the torus (dotted trajectory). (b) Temporal profile of the example Klein-bottle driven qubit Hamiltonian Eq. (18) utilized for numerical simulations, with $m = 0.5$.

form a standard torus as illustrated in Fig. 4(a) (white and gray regions); indeed, this stems from the fact that τ_1 and $\tau_2^2 : (x, y) \mapsto (x, y + 2\pi)$ generate the translation group of the torus.

As the metric on K^2 is flat, geodesics on it are generated by trajectories of a free particle, just like for the torus—with a key difference being how the particle traverses the boundaries, see Fig. 4(a) for an example. Let us specify the particle's initial velocity $\omega = (\omega_x, \omega_y)$ and initial position $\theta_0 = (\theta_{x,0}, \theta_{y,0})$. Then, the particle's position in the fundamental domain can be explicitly computed as follows:

$$\begin{aligned}\theta_x(t) &= (-1)^{n_y(t)}[\pi - (\omega_x t + \theta_{x,0} + \pi \bmod 2\pi)], \\ \theta_y(t) &= (\omega_y t + \theta_{y,0} \bmod \pi) - \pi,\end{aligned}\quad (12)$$

where $n_y(t) \equiv \lfloor (\omega_y t + \theta_{y,0})/\pi \rfloor$. These formulas simply encode a trajectory respecting the nontrivial gluing of the boundaries of the fundamental region to form K^2 .

Following our framework, we next consider a parent quantum Hamiltonian $H_Q(\theta)$ defined on the fundamental domain of K^2 . Combining everything, we finally construct a "Klein bottle drive" as $H(t) := H_Q(\theta_t)$. An example temporal profile is shown in Fig. 4(b), generated by a concrete qubit model used later in our numerical simulation. Since two copies of the Klein bottle can be combined to form a torus (as noted above), Klein bottle driving can be thought of as a special subclass of quasiperiodic driving, but with additional structure.

What sorts of physics do Klein bottle quantum drives exhibit? First we note that because the surface K^2 is nonorientable, a conventional first Chern number is always zero and thus does not characterize the topology of a band structure defined on it. Thus, one cannot hope to find an observable like Eq. (6) yielding a topologically quantized dynamical response like of Eq. (8) like in the HDQS case.

However, consider now that the parent quantum Hamiltonian $H_Q(\theta)$ obeys an additional y -mirror symmetry

$$M_y : H_Q(\theta_x, \theta_y) = U^\dagger H_Q(\theta_x, -\theta_y) U, \quad (13)$$

where U is a unitary operator satisfying $U^2 = \pm \mathbb{I}$. It was recently shown in studies of topological insulators in solid state physics that gapped energy bands $|\psi_n(\theta)\rangle$ of such $H_Q(\theta)$ are topologically classified by an invariant quantized to integer multiples of $\pi/2$ [32,33,58]:

$$D_y^{(n)} = \frac{1}{2\pi} \iint_{K^2} d\theta_x d\theta_y \Omega^{(n)}(\theta), \quad (14)$$

called the dipolar Chern number. Above, $\Omega^{(n)}(\theta) = \partial_{\theta_y} A_{\theta_x}^{(n)}(\theta) - \partial_{\theta_x} A_{\theta_y}^{(n)}(\theta)$ is the Berry curvature form, $A_{\theta_j}^{(n)}(\theta) = i\langle \psi_n(\theta) | \partial_{\theta_j} \psi_n(\theta) \rangle$ the Berry connection, and the integral is taken over the fundamental domain of the Klein bottle. Note in these solid state works [32,33,58,59], the Klein bottle K^2 (and also the real projective plane RP^2 [33]) arose as the Brillouin zone of lattice models with the imposition of certain unusual nonsymmorphic momentum-space symmetries. It was further shown that the dipolar Chern number Eq. (14) physically characterizes the (quantized) rate of a momentum current generated in response to an applied electric field, or equivalently, the (quantized) charge current induced by time-dependent strains on the lattice.

In our context, these exotic manifolds enter simply as *primitive* ingredients in our geometric quantum driving framework, which are simply and directly realized by time-dependent controls. Moreover, the topological invariant $D_y^{(n)}$ can be measured in our setup as follows: assuming a "fully gapped" condition on $H_Q(\theta)$ [again, this means all its energy levels $|\psi_n(\theta)\rangle$ form well-defined "bands" over K^2], first prepare an instantaneous eigenstate $|\psi_n(\theta_0)\rangle$ at $t = 0$ and evolve the system under $H(t) := H_Q(\theta_t)$, where θ_t describes a geodesic on the Klein bottle. Then measure the following time-dependent quantum operator

$$O^{(\omega)}(t) := \omega_y \theta_y(t) \frac{\partial H_Q(\theta_t)}{\partial \theta_x}. \quad (15)$$

Again, this operator is constructed from the parent quantum Hamiltonian $H_Q(\theta)$ and choice of geodesic θ_t and thus has a form which can be explicitly determined. Finally, we average its expectation value over time

$$v(T) = \frac{\pi}{\omega_y^2 T} \int_0^T dt \langle \psi(t) | O^{(\omega)}(t) | \psi(t) \rangle. \quad (16)$$

We claim that in the long-time and slow driving limit ($\|\omega\| \ll 1$, keeping the direction of ω fixed), $v(T)$ for generic driving parameters converges corresponding to the dipolar Chern number

$$\lim_{\|\omega\| \rightarrow 0} \lim_{T \rightarrow \infty} v(T) = D_y^{(n)}. \quad (17)$$

Equations (15) and (17) constitute our main claim in this section—a nontrivial topological response exhibited by NDQS whose derivation mirrors that of the HDQS (noting that geodesics on the Klein bottle are ergodic for generic driving frequencies, which allows us to replace the time integral with the space integral in the long-time limit). In practice, a clear quantization can already be observed by taking T large and $\|\omega\|$ small (but both finite), as illustrated in the explicit model of Fig. 5.

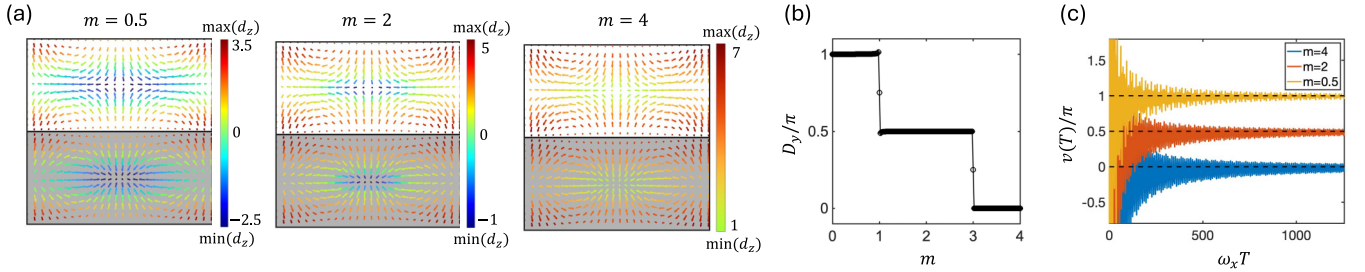


FIG. 5. (a) Spin texture $\mathbf{d}(\theta)$ defining the parent qubit Hamiltonian $H_Q(\theta)$ on the Klein bottle surface (gray regions) for $m = 0.5, 2, 4$, respectively. (b) The corresponding dipolar Chern number versus m is shown. For generic m it is quantized in units of $\pi/2$. (c) Dynamical response $v(T)$ starting from the instantaneous eigenstate of the upper band. For the small driving frequency ω stated in the main text, $v(T)$ approaches the dipolar Chern number of the band at late times.

Qubit model and numerics

To illustrate our predictions, we perform numerical simulations of a qubit Klein bottle-driven system, defined by the parent quantum Hamiltonian

$$H_Q(\theta) = \mathbf{d}(\theta) \cdot \boldsymbol{\sigma}, \quad (18)$$

and

$$\mathbf{d}(\theta) = \begin{pmatrix} \sin \theta_x \sin \theta_y \\ \cos \theta_x \sin 2\theta_y \\ m - \cos \theta_x + 2 \cos 2\theta_y \end{pmatrix}, \quad (19)$$

where m is a tunable parameter, motivated by Ref. [33]. This Hamiltonian possesses y -mirror symmetry Eq. (13) with the symmetry operator $U = \sigma^z$. Spin textures $\mathbf{d}(\theta)$ are illustrated in Fig. 5(a) for different m . It can be shown that the system is fully gapped for generic values of m ; in particular, it exhibits topologically nontrivial bands for $0 < m < 1$ with $|D_y| = \pi$ and for $1 < m < 3$ with $|D_y| = \pi/2$, while for $m > 3$, they are trivial with $|D_y| = 0$, see Fig. 5(b). For our simulations, we initialize the system at $t = 0$ in the instantaneous upper band at $(\theta_x, \theta_y) = (-\pi, -\pi)$, then drive the system with a geodesic with ω chosen as $\omega_x = 0.02$ and $\omega_y = \frac{1+\sqrt{5}}{2}\omega_x$, before measuring $O^{(\omega)}(t)$ and then computing $v(T)$. As illustrated in Fig. 5(c), $v(T)$ converges at late but finite times to the expected dipolar Chern number D_y across different regimes m , as predicted.

V. DISCUSSION AND CONCLUSION

In this work, we introduced a general geometric construction of driven quantum systems called *geometric quantum driving* whose key idea is to use a classical particle to steer a parent quantum Hamiltonian over a choice of trajectory and manifold. This results in a multitude of time-dependent systems with structured temporal profiles and physics dependent on the local geometry and global topology of the manifold. In particular, we demonstrated the construction of two new classes of quantum drives: HDQS and NDQS, with the choices of geodesic flows on the compact hyperbolic Bolza surface and nonorientable Klein bottle surface, respectively. We showed how these systems exhibit the physics of measurable, robust, topologically quantized dynamical responses under the conditions of slow driving and being fully gapped,

enabled by classical ergodicity of the geodesic trajectories over the manifolds underlying their construction.

We note these topological responses are reminiscent of a charge pump like the Thouless pump [60] or energy pump between drives in a quasiperiodically driven system [19,61–64], which can both be interpreted as a quantum Hall response in the extended frequency space picture [21] of Floquet or quasiperiodically driven systems, respectively. Indeed, for the Klein bottle quantum drive, we explained that the topological invariant extracted in dynamics has a direct physical interpretation as the crystalline and electromagnetic response in an effective topological insulator system with unusual nonsymmorphic momentum space symmetries, induced by artificial gauge or translational gauge fields [33,59]. For the HDQS, we explain in the SM [38] how their dynamics can similarly be mapped into a single-particle hopping problem defined on a nonstandard generalized "frequency" lattice governed by a quasienergy operator, formally called the "Koopman operator" [34,65,66], though the interpretation of their quantized topological response as a crystalline or electromagnetic response in this lattice is not immediately obvious. Nevertheless, what these examples show is how geometric quantum driving may be thought of as a useful tool to quantum simulate solid-state phenomena realized by nonstandard, exotic lattices.

More generally, our geometric construction is very versatile and one may consider other trajectories x_t (e.g., horocycle [67] or Asonov flows [68], etc.), and other manifolds M (e.g., Klein quartic [69]), giving rise to a plethora of different classes of driven quantum systems. In the adiabatic limit, and under the assumption of trajectories chosen to be ergodic, we expect by similar arguments in this work that topological invariants corresponding to surface integrals over the manifold M may be extracted through the dynamical responses of the driven system, using carefully constructed observables. Moving away from few-body topological physics, it would also be highly interesting to understand these systems' quantum ergodic properties studied via the newly introduced concept of *Hilbert space ergodicity* [25,29,70], as well as what novel dynamical phases of matter can be realized in *interacting* systems which are geometrically driven (not necessarily slowly). Geometric quantum driving thus serves as a powerful bridge linking the topics of quantum dynamics, differential geometry, ergodicity, and topology, and opens new possibilities in quantum simulation.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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