

Zeeman-activated Berry-curvature magnetotransport from the bulk of nonmagnetic metals with inversion symmetry

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The Berry curvature (BC)—a quantity encoding the geometry of electronic wavefunctions—governs various electronic transport effects in quantum materials. In magnetic systems, the BC is responsible for the intrinsic part of the anomalous Hall conductivity. Local concentrations of BC in noncentrosymmetric materials can lead instead to the quantum nonlinear Hall effect. Here, we argue that the bulk of nonmagnetic metals with inversion symmetry—systems where the BC is forced to vanish at any momentum—can be endowed with substantial concentrations of BC even with an infinitesimally small Zeeman coupling. This Zeeman-activated BC, independent of the magnetic field strength and instead related to the degree of nonparabolicity of the electronic bands, couples to the electronic orbital motion to generate a negative longitudinal magnetoresistance that scales with the relaxation time as the Drude resistivity. We show that the Zeeman-activated BC and the related intrinsic negative magnetoresistivity are generic: they appear in all centrosymmetric point groups and can occur both in topological and conventional conductors.

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I. INTRODUCTION

The electronic Bloch waves of a solid define a mapping from the Brillouin zone to a complex space equipped with a geometric structure that is encoded in a quantum geometric tensor [1]. The imaginary part of this tensor corresponds to an emergent field in momentum space called Berry curvature (BC). In metallic magnets, the flux of the BC through the Fermi surface is finite and regulates the intrinsic part of the anomalous Hall conductivity [2,3]. Anomalous Hall effect measurements thus provide an electronic transport footprint of the electrons quantum geometry. In nonmagnetic metals, the flux of the BC is forced to vanish by time-reversal symmetry. However, a segregation of positive and negative regions of BC is allowed in materials with noncentrosymmetric crystalline arrangements [4]. These local concentrations of BC can result in a BC dipole that governs the so-called nonlinear Hall effect with time-reversal symmetry [5–7]. A closely related disorder-induced nonlinear Hall effect [8], which has been shown to appear at the surfaces of nonmagnetic materials with trigonal symmetry, stems from a higher order moment of the Berry curvature dubbed BC triple [9,10].

In nonmagnetic metals with centrosymmetric crystalline arrangements the electronic BC is forced to vanish at any momentum. Nevertheless, externally applied electromagnetic fields can be used to break either inversion or time-reversal

symmetry and thus generate substantial BC concentrations. This typically occurs in two-dimensional materials with valley degrees of freedom where a protected level crossing is transmuted into an avoided level crossing by the external field, and BC hot-spots—regions in close proximity to near degeneracies where the Bloch waves are rapidly changing—appear. In bilayer graphene, for instance, an external electric field perpendicular to the layers generates a spectral gap reducing the point group symmetry from $\mathcal{D}_{3d}$ to $\mathcal{C}_{3v}$, and endows the materials with a BC leading to high-frequency rectification [11], and a BC dipole if an additional uniaxial strain field is present [12]. In single-valley bulk three-dimensional solids, a similar generation of substantial BC concentrations is not predicted, at least in the weak-field limit.

In this paper, we will demonstrate that contrary to these expectations, substantial BC concentrations arise in strongly spin-orbit coupled three-dimensional materials in the presence of an even infinitesimally small Zeeman coupling. This Zeeman-activated BC is independent of the magnetic field strength and Landé g -factor, with its strength that is only related to the degree of nonparabolicity of the electronic bands. Additionally, it appears in all centrosymmetric point groups both for topological and conventional electronic bands. We will further show that the Zeeman-activated BC couples via the anomalous velocity [13–15] to the orbital motion of Bloch electrons, and generate a weak-field negative longitudinal magnetoresistance (NLMR): a decrease in the electrical resistance when the driving electric field $\mathbf{E}$ is parallel to the externally applied magnetic field $\mathbf{B}$. Contrary to the NLMR observed in materials with strongly anisotropic dispersions [16] or due to extrinsic mechanisms, including anisotropic scattering [17–20] and inhomogeneous boundary

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effects [21,22], the NLMR we predict here violates the so-called Kohler relation [23,24], indicating that the details of scattering are irrelevant for this mechanism. Violation of the Kohler relation in the semiclassical regime has been discussed so far only in inversion symmetry broken systems or in inversion symmetric systems with well-formed Landau levels [24,25], including massive Dirac fermion systems [26,27]. We find in fact that the ratio between the finite field resistivity $\rho(\mathbf{B})$ and the Drude resistivity ρ_0 is independent of $\omega_c\tau$, with ω_c indicating the cyclotron frequency and τ the relaxation time. Put differently, the predicted negative magnetoresistivity $[\rho(B) - \rho(0)]/\rho(0)$ is an intrinsic quantity independent of disorder.

The weak-field NLMR due to the Zeeman-activated BC we put forward here, provides an explanation for the NLMR experimentally observed in Bi_2Se_3 thin films [28], and generally represents an analog for centrosymmetric materials of the weak-field NLMR appearing in a number of noncentrosymmetric systems, such as Dirac and Weyl semimetals [29–33], which possess nontrivial quantum geometric properties already in the complete absence of external fields [34–40]. We note that recent studies consider NLMR in Dirac semimetals [41] as well as in massive Dirac fermions systems with helical and chiral symmetry breaking [26,27] starting out from well-defined Landau levels. These studies differ qualitatively with the mechanism we will discuss below where Landau level formation is not required and the NLMR is given by an activated Berry curvature.

II. ZEEMAN-ACTIVATED BERRY CURVATURE

We first discuss the generic appearance of a Zeeman-activated BC that is independent of the magnetic field strength. To make things concrete, let us consider a crystalline system with the maximally symmetric point group equipped with inversion symmetry: the cubic $\mathcal{O}_h$ group. Additionally, we will consider that in our single valley system the low energy bands are centered around the Γ point of the Brillouin zone. Due to the concomitant presence of time-reversal symmetry Θ and inversion symmetry $\mathcal{I}$, all the electronic states are guaranteed to be twofold degenerate, with each pair of bands that are Kramers' related. The effective long wavelength Hamiltonian for a single Kramers' pair can be written as

$$\mathcal{H}(\mathbf{k}) = d_0(\mathbf{k})\sigma_0 + \vec{\mathbf{d}}(\mathbf{k}) \cdot \vec{\sigma}, \quad (1)$$

with the Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ acting in spin space and the crystalline momentum polynomials $\vec{\mathbf{d}}(\mathbf{k})$ constrained by point group symmetries. Inversion symmetry, for instance, requires that only terms quadratic in momentum $\mathbf{k}$ are allowed. Together with the fact that the Pauli matrices transform in the T_{1u} three-dimensional irreducible representation of the $\mathcal{O}_h$ group, we then obtain the effective Hamiltonian $\mathcal{H}_0(\mathbf{k}) = \hbar^2(k_x^2 + k_y^2 + k_z^2)\sigma_0/(2m^*)$ which implies that the two degenerate bands are perfectly parabolic and isotropic with effective mass m^* .

Let us now add a Zeeman coupling, which, without loss of generality, we take due to a magnetic field along a principal crystallographic direction, which we dub $\hat{y}$. This Zeeman coupling should be encoded in the usual term $\mathcal{H}_z = B_y\sigma_y$,

TABLE I. Mirror and $\mathcal{M}'$ symmetry constraints on the bilinear crystalline momentum terms appearing in the low-energy two-band Hamiltonian $\mathcal{H}(\mathbf{k}) = \vec{\mathbf{d}}(\mathbf{k}) \cdot \vec{\sigma}$ assuming a magnetic field in the $\hat{y}$ direction. The last column indicates the bilinears compatible with the three symmetries. Additional rotational symmetries impose constraints on coefficients. For instance the $\mathcal{C}_{4y}$ symmetries implies that $a = c$ and also $a_x = a_z$.

$d/\mathcal{M}$	$\mathcal{M}'_x = \mathcal{M}_x\Theta$	$\mathcal{M}_y$	$\mathcal{M}'_z = \mathcal{M}_z\Theta$	Allowed
d_x	$k_x k_y, k_x k_z$	$k_x k_y, k_z k_y$	$k_x k_y, k_i^2$	$ak_x k_y$
d_y	$b, k_i^2, k_y k_z$	$b, k_i^2, k_x k_z$	$b, k_i^2, k_x k_y$	$(b + a_i k_i^2)$
d_z	$a_0, k_i^2, k_y k_z$	$k_y k_x, k_y k_z$	$k_z k_x, k_z k_y$	$ck_y k_z$

which breaks the twofold degeneracy of the bands but does not yield band geometric properties. The crux of the story is that additional momentum-dependent terms can be activated by the Zeeman coupling. To show this, it is enough to note that with an external magnetic field along a principal crystallographic direction, the magnetic point group of the system is the centrosymmetric $4/m\bar{m}'m'$. This group is generated by a residual mirror symmetry $\mathcal{M}_y$, a fourfold rotation with the rotation axis along the applied field, and two combined antiunitary symmetries $\mathcal{M}_{x,z}\Theta$ squaring to one. Using the transformation properties of Pauli matrices and momentum bilinears under the mirror and the antiunitary $\mathcal{M}'$ symmetries listed in Table I, we find that the breaking of time-reversal and point-group symmetries due to the application of the external magnetic field yields a generalized Zeeman Hamiltonian with momentum dependent terms

$$\mathcal{H}_z = ak_x k_y \sigma_x + b\sigma_y + ak_y k_z \sigma_z, \quad (2)$$

where we have neglected the subleading terms (see Table I) of the form $k_{x,y,z}^2 \sigma_y$. The coupling constants a, b must be odd functions of the magnetic field strength, which implies that at weak fields they will grow linearly with B_y . In Eq. (9) we will derive the explicit form of these two parameters for the model we describe below.

Although in this work we focus on the magnetic point group $4/m\bar{m}'m'$, in this section we demonstrate how the p -wave Zeeman coupling generically arises in all centrosymmetric magnetic point groups compatible with ferromagnetism. To establish the emergence of a Berry curvature and to determine the structure of the generalized Zeeman coupling, we first note that inversion symmetry requires the coupling to be even in the crystalline momentum. In the absence of additional symmetry constraints, the most general Zeeman interaction up to second order in momentum can therefore be written as

$$\mathcal{H}_z = \mathbf{d}(\mathbf{k}) \cdot \vec{\sigma}, \quad (3)$$

where the vector $\mathbf{d}$ takes the general form

$$\begin{aligned} \mathbf{d} = & (a_0 + a_i k_i^2 + a_{ij} k_i k_j) \hat{x} + (b_0 + b_i k_i^2 + b_{ij} k_i k_j) \hat{y} \\ & + (c_0 + c_i k_i^2 + c_{ij} k_i k_j) \hat{z}, \end{aligned} \quad (4)$$

with $i \neq j$. Residual (magnetic) point-group symmetries impose strong constraints on the allowed coefficients of these momentum-dependent terms. To demonstrate the generic

TABLE II. Effective Zeeman coupling allowed in centrosymmetric magnetic groups compatible with ferromagnetism.

Magnetic group	d_x	d_y	d_z
$6/m\bar{m}'m'$	$a_{xy}k_xk_y$	$b_0 + b_x(k_x^2 + k_y^2) + b_yk_y^2$	$a_{xy}k_xk_z$
$6/m$	$a_{xy}k_xk_y$	$b_0 + b_x(k_x^2 + k_y^2) + b_yk_y^2$	$a_{xy}k_xk_z$
$-3m'$	$a_{xy}k_xk_y$	$b_0 + b_x(k_x^2 + k_y^2) + b_yk_y^2$	$a_{xy}k_xk_z$
-3	$a_{xz}k_xk_z - a_{yz}k_yk_z + a_x(k_x^2 - k_y^2) - 2a_{xy}k_xk_y$	$a_{xz}k_yk_z + a_{yz}k_xk_z - 2a_xk_xk_y - a_{xy}(k_x^2 - k_y^2)$	$c_0 + c_x(k_x^2 + k_y^2) + c_zk_z^2$
$4/m\bar{m}'m'$	ak_xk_y	$b_0 + b_x(k_x^2 + k_y^2) + b_yk_y^2$	ak_zk_y
$4/m$	$a_{xy}k_xk_y + a_{yz}k_yk_z$	$b_0 + b_x(k_x^2 + k_y^2) + b_yk_y^2$	$-a_{yz}k_xk_y + a_{xy}k_yk_z$
$m'm'm$	$a_{xy}k_xk_y$	$b_0 + b_ik_i^2$	$c_{xz}k_yk_z$
$2'/m'$	$a_0 + a_ik_i^2 + a_{zx}k_zk_x$	$b_{xy}k_xk_y + b_{zy}k_zk_y$	$c_0 + c_ik_i^2 + c_{zx}k_zk_x$
$2/m$	$a_{xy}k_xk_y + a_{zy}k_zk_y$	$b_0 + b_ik_i^2 + b_{xz}k_xk_z$	$c_{xy}k_xk_y + c_{zy}k_zk_y$
-1	$a_0 + a_ik_i^2 + a_{ij}k_ik_j$	$b_0 + b_ik_i^2 + b_{ij}k_ik_j$	$c_0 + c_ik_i^2 + c_{ij}k_ik_j$

appearance of a Zeeman-induced Berry curvature in centrosymmetric ferromagnets, we therefore systematically derive the form of the generalized Zeeman coupling for all magnetic point groups compatible with ferromagnetism, which in our case is stabilized by the presence of a homogeneous external magnetic field. The resulting symmetry-allowed couplings are summarized in Table II. The orientation of the magnetic field is identified by the nonvanishing momentum-independent term, corresponding to the conventional Zeeman coupling.

The generalized Zeeman Hamiltonian written above for magnetic point group $4/m\bar{m}'m'$ leads to an anisotropy in the energy spectrum that is directly proportional to the magnetic field strength. More importantly, the electronic states are now equipped with a d -wave BC [see Fig. 1], which can be explicitly written as

$$\bar{\Omega}^{\pm}(\mathbf{k}) = \mp \frac{a^2b}{2[b^2 + a^2k_y^2(k_x^2 + k_z^2)]^{3/2}} \{k_xk_y, -k_y^2, k_yk_z\}. \quad (5)$$

In the equation above, the $\pm$ sign distinguishes the two spin-split bands forming the Kramers' pair. A few remarks are in order here. First, the BC written above scales as $B_y/|B_y| = \text{sign}(B_y)$ in the weak-field limit. The BC of a single band displays a discontinuity as $B_y \rightarrow 0$, which guarantees that precisely at $B_y \equiv 0$, i.e., in presence of time-reversal and

inversion symmetries, the BC vanishes. Second, the fact that the BC of the two bands is opposite implies that any net anomalous Hall contribution, if existing, can only come from the difference in the shape of the Fermi surfaces that instead scales linearly with B_y . Third, in spin-orbit free systems, any BC is precluded: the $SU(2)$ spin symmetry at $B_y \equiv 0$ implies that the generalized Zeeman coupling of Eq. (2) can be expressed using a single Pauli matrix even if momentum dependent terms are allowed. Finally, the momentum dependent terms in the generalized Zeeman coupling also yield a finite orbital magnetic moment with the simple form $\bar{\mathbf{m}}^{\pm} \propto \pm |\bar{\mathbf{d}}(\mathbf{k})| \bar{\Omega}^{\pm}(\mathbf{k})$. A direct consequence is that the orbital magnetic moment grows linearly with both the magnetic field strength B_y and the Landé g -factor. This also implies that the contributions to transport mediated by the orbital magnetic moment are expected to be smaller than the BC contributions, at least in materials with small g -factors.

Before discussing the consequences that Eq. (5) entails in magnetotransport, we show how the generalized Zeeman coupling of Eq. (2) can arise in a cubic system. Consider a time-reversal symmetric four-band model for two Kramers doublets of bands (one conduction and one valence) of opposite parity—for instance deriving from an atomic s orbital and a $p_{1/2}$ atomic orbital. The criterion derived by Fu and Kane [42] asserts that the level ordering between the two pairs of bands distinguishes a normal from a nontrivial topological class. Crucially, the occurrence of the Zeeman-activated BC and its consequences on magnetotransport are independent of whether the two pairs undergo level crossing and hence on the topology of the electronic states. The effective $\mathbf{k} \cdot \mathbf{p}$ Hamiltonian away from the Γ point can be obtained in a straightforward manner. Since all the states have definite parity eigenvalues and the momentum $\mathbf{p}$ has odd parity, only matrix elements between states of opposite parity are nonvanishing. This also implies that the effective Hamiltonian to the linear order in $\mathbf{k}$ corresponds to a three-dimensional Dirac theory similar to the one [43,44] governing the electronic states in the bulk of Bi_2Se_3 . By considering additional quadratic terms in $\mathbf{k}$, we obtain an effective Dirac Hamiltonian that can be written as

$$\mathcal{H}_{\text{eff}} = \epsilon_0(\mathbf{k})\tau_0\sigma_0 + \mathcal{M}(\mathbf{k})\tau_z \otimes \sigma_0 - \frac{P}{\sqrt{3}}\tau_x \otimes \boldsymbol{\sigma} \cdot \mathbf{k}, \quad (6)$$

where the $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ matrices act in spin and orbital space, respectively, $\epsilon_0(\mathbf{k}) = \bar{M}k^2$ is a particle-hole symmetry breaking

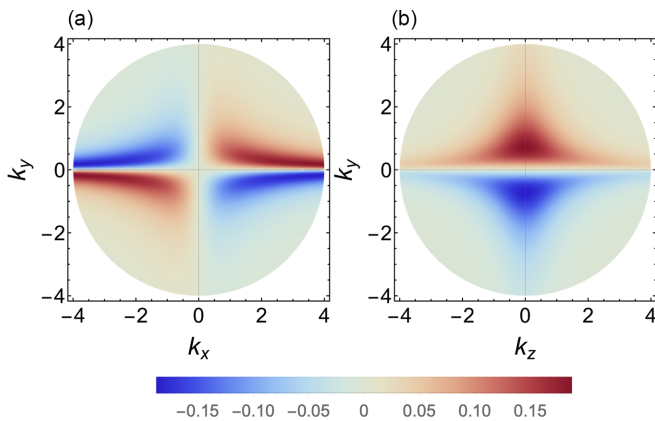


FIG. 1. Density plots of the d -wave Berry curvature $\Omega_x^+(\mathbf{k})$ induced by the generalized Zeeman coupling in the k_x, k_y and k_y, k_z planes. In both cases we have chosen $b = a = 1$.

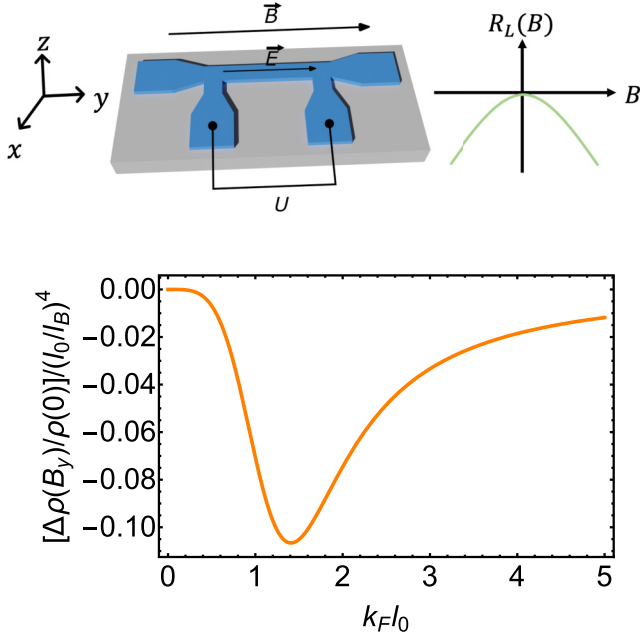


FIG. 2. Top panel: Sketch of a device for longitudinal magnetoresistance measurement with the direction of the driving electric field and the applied in-plane magnetic field. The right panel shows an ensuing negative longitudinal magnetoresistance quadratic in the magnetic field. Bottom panel: Behavior of the negative magnetoresistance $\Delta\rho(B_y)/\rho(0)$ obtained using the generalized Zeeman coupling Eq. (2) as a function of the Fermi wavevector k_F measured in units of the inverse of the characteristic material dependent length scale $l_0 = \sqrt{|a|/|b|}$. The magnetoconductivity is rescaled by the factor $(l_0/l_B)^4$ with $l_B = \sqrt{\hbar/(eB_y)}$ the Landau magnetic length.

term, and $\mathcal{M}(\mathbf{k}) = M_0 + M_1 k^2$. The relative sign between M_0 and M_1 governs the band ordering [44] and thus distinguishes a normal insulator from a strong three-dimensional topological insulator. The parameter P instead regulates the degree of nonparabolicity of the twofold degenerate electronic bands

$$\epsilon(\mathbf{k}) = \epsilon_0(\mathbf{k}) \pm [\mathcal{M}(\mathbf{k})^2 + P^2 k^2/3]^{1/2}. \quad (7)$$

We emphasize that in the normal band ordering, the model written above describes the Γ_6 s bands and the spin-orbit split-off Γ_7 bands in cubic semiconductors. We next add a Zeeman coupling with orbital dependent Landé g -factors [44] due to a B_y magnetic field, which can be thus cast as

$$\Delta\mathcal{H}_{\text{eff}} = \bar{g} \mu_B B_y \tau_0 \otimes \sigma_y + \delta g \mu_B B_y \tau_z \otimes \sigma_y, \quad (8)$$

where $\bar{g}$ and δg are the average and the difference between the two orbital Landé g -factor, respectively, and μ_B is the Bohr magneton. By performing a Löwdin partitioning [45] we find (see the Supplemental Material [46]) that in, for instance, the conventional band ordering and for the pair of conduction bands, the projected two-band Hamiltonian has a generalized Zeeman coupling precisely of the Eq. (2) form with the two couplings

$$\begin{aligned} a &= (\bar{g} - \delta g) \mu_B P^2 B_y / (6M_0^2), \\ b &= (\bar{g} + \delta g) \mu_B B_y. \end{aligned} \quad (9)$$

The linear B_y dependence of these couplings guarantees that the ensuing BC is independent of the magnetic field strength. Furthermore, the BC is only dependent on the ratio between the two orbital dependent Landé g -factors and not on their absolute values. We have also computed the BC of the four-band model Hamiltonian $\mathcal{H}_{\text{eff}} + \Delta\mathcal{H}_{\text{eff}}$ using the method outlined in Ref. [47]. In perfect agreement with our foregoing analysis, we find (see the Supplemental Material [46]) the existence of weak field contributions, opposite for the two bands, scaling as $B_y/|B_y|$ and with the same crystalline momentum dependence of Eq. (5), with one notable exception. The $\Omega_y(\mathbf{k})$ component displays a momentum-independent contribution that is completely allowed, but it is not predicted by our symmetry-based theory. We will now show that remarkably the weak-field $\Omega_y(\mathbf{k})$ BC does not contribute to the NLMR: the transport properties derived by the symmetry-based and the model Hamiltonian approaches are equivalent.

III. SEMICLASSICAL TRANSPORT THEORY

The semiclassical equation of motion of charge carriers incorporating the BC of Bloch electrons [4,13,48] are given by

$$\dot{\mathbf{r}} = (1/\hbar) \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}} - \dot{\mathbf{k}} \times \boldsymbol{\Omega}, \quad (10)$$

$$\dot{\mathbf{k}} = (-e/\hbar)(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}). \quad (11)$$

These coupled equations can be solved for $\dot{\mathbf{r}}$ and $\dot{\mathbf{k}}$ to obtain

$$\dot{\mathbf{r}} = D(\mathbf{k}) \left[\frac{1}{\hbar} \frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} + \frac{e}{\hbar} \mathbf{E} \times \boldsymbol{\Omega} + \frac{e}{\hbar} \left(\frac{1}{\hbar} \frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} \cdot \boldsymbol{\Omega} \right) \mathbf{B} \right], \quad (12)$$

$$\dot{\mathbf{k}} = D(\mathbf{k}) \left[-\frac{e}{\hbar} \mathbf{E} - \frac{e}{\hbar^2} \frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} \times \mathbf{B} - \frac{e^2}{\hbar^2} (\mathbf{B} \cdot \mathbf{E}) \boldsymbol{\Omega} \right], \quad (13)$$

with the factor $D(\mathbf{k}) = (1 + \frac{e}{\hbar} \mathbf{B} \cdot \boldsymbol{\Omega})^{-1}$. The first and second terms in the r.h.s. of Eq. (13) are the classical electric and Lorentz force, respectively. The last term instead is caused by the BC of Bloch electrons and represents a force that is nonvanishing even for collinear $\mathbf{E}$ and $\mathbf{B}$.

To proceed further, we use that the electronic distribution function $f_{\mathbf{k}}$ obeys the Boltzmann equation

$$\partial_t f_{\mathbf{k}} + \dot{\mathbf{r}} \cdot \nabla_{\mathbf{r}} f_{\mathbf{k}} + \dot{\mathbf{k}} \cdot \nabla_{\mathbf{k}} f_{\mathbf{k}} = \mathcal{I}_{\text{coll}}[f_{\mathbf{k}}], \quad (14)$$

which we solve (see the Supplemental Material [46]) in the relaxation time approximation where the collision integral $\mathcal{I}_{\text{coll}}[f] = -g_{\mathbf{k}}/\tau$, and $g_{\mathbf{k}}$ is the deviation of the electronic distribution function from the equilibrium f_{eq} Fermi-Dirac distribution. The out of equilibrium distribution takes the simple form

$$g_{\mathbf{k}} = -\frac{e\tau}{\hbar} D\Psi \cdot \left(\frac{\partial f_0}{\partial \mathbf{k}} \right), \quad (15)$$

where $\Psi = \Phi + \Upsilon$ is the driving field. The first term $\Phi = \frac{e}{\hbar} M_{\Omega} \mathbf{E}$ with $M_{\Omega} = \boldsymbol{\Omega} \otimes \mathbf{B}$, while the field Υ reads

$$\Upsilon = \left(I - \tau \frac{e}{\hbar} M_L \right)^{-1} \left[\mathbf{E} + \tau \frac{e^2}{\hbar^2} (\mathbf{B} \cdot \mathbf{E}) \mathbf{B} \times \mathbf{C} \right] \quad (16)$$

with $\mathbf{C} = \nabla_{\mathbf{k}}(\mathbf{\Omega} \cdot \mathbf{D}\mathbf{v})$ and

$$M_L = \begin{pmatrix} B_y \frac{\partial Dv_x}{\partial k_z} - B_z \frac{\partial Dv_x}{\partial k_y} & B_y \frac{\partial Dv_y}{\partial k_z} - B_z \frac{\partial Dv_y}{\partial k_y} & B_y \frac{\partial Dv_z}{\partial k_z} - B_z \frac{\partial Dv_z}{\partial k_y} \\ B_z \frac{\partial Dv_x}{\partial k_x} - B_x \frac{\partial Dv_x}{\partial k_z} & B_z \frac{\partial Dv_y}{\partial k_x} - B_x \frac{\partial Dv_y}{\partial k_z} & B_z \frac{\partial Dv_z}{\partial k_x} - B_x \frac{\partial Dv_z}{\partial k_z} \\ B_x \frac{\partial Dv_x}{\partial k_y} - B_y \frac{\partial Dv_x}{\partial k_x} & B_x \frac{\partial Dv_y}{\partial k_y} - B_y \frac{\partial Dv_y}{\partial k_x} & B_x \frac{\partial Dv_z}{\partial k_y} - B_y \frac{\partial Dv_z}{\partial k_x} \end{pmatrix}. \quad (17)$$

In the present work, it is enough to take $\mathbf{Y} = \mathbf{E}$, since the other terms give transversal contribution related to Lorentz force.

By computing the electric current $\mathbf{j} = -e \text{tr}(\dot{\mathbf{r}}\mathbf{f}_{\mathbf{k}})$, we subsequently obtain (see the Supplemental Material [46]) the conductivity $\sigma_{ii} = \partial j_i / \partial E_i$. This can be written in a classical Drude form $\sigma_{ii}(\mathbf{B}) = e^2 \tau \sum_{\mathbf{k}} \dot{\mathbf{r}}_i \dot{\mathbf{r}}_i (-\partial f_{eq} / \partial \epsilon_{\mathbf{k}})$ with a BC-induced renormalization of the density of states [49] $\sum_{\mathbf{k}} = \int (d^d \mathbf{k} / (2\pi)^d) (1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega})$, and the velocity components given by Eq. (12) apart from the anomalous velocity term governing the anomalous Hall conductivity [2,3,50–56]. Explicitly, we have

$$\sigma_{ii}(\mathbf{B}) = e^2 \tau \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \frac{[v_i + \frac{e}{\hbar} (\mathbf{v} \cdot \mathbf{\Omega}) B_i]^2}{(1 + \frac{e}{\hbar} \mathbf{B} \cdot \mathbf{\Omega})} \left(-\frac{\partial f_{eq}}{\partial \epsilon_{\mathbf{k}}} \right), \quad (18)$$

where we introduced the group velocity $\mathbf{v} = \hbar^{-1} \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}}$. In the zero temperature limit $-\partial f_{eq} / \partial \epsilon_{\mathbf{k}} \approx \delta(\epsilon_{\mathbf{k}} - \epsilon_{0\mathbf{k}})$, the zero-field conductivity corresponds to the usual Drude term $\sigma_{ii}(0) = e^2 \tau k_F^3 / (3m^* \pi^2)$. At finite $\mathbf{B}$, there are two field-induced corrections to the conductivity that must be taken into account. The first correction is due to the Zeeman-induced correction to the carrier group velocity whereas the second one is due to Zeeman-activated BC. Crucially, while the former is proportional to the Zeeman coupling strength (via the Bohr magneton μ_B) the latter can only depend on the ratio

between orbital-dependent effective Landé g -factors, and will thus yield the largest correction to the conductivity. Additionally, all corrections linear in magnetic fields vanish in agreement with the Onsager's relation $\sigma(\mathbf{B}) = \sigma(-\mathbf{B})$. The leading correction to the Drude conductivity (see the Supplemental Material [46]) can be finally written as

$$\Delta \sigma_{yy}(B_y) = \frac{2e^4 \tau}{\hbar^2} B_y^2 \int \frac{d^3 \mathbf{k}}{(2\pi)^3} (v_{x0} \Omega_x + v_{z0} \Omega_z)^2 \left(-\frac{\partial f_{eq}}{\partial \epsilon_{0\mathbf{k}}} \right), \quad (19)$$

where $v_{0x/0z}$ and $\epsilon_{0\mathbf{k}}$ indicate the bare group velocities and the band energy in the absence of the generalized Zeeman coupling. In deriving the equation above, we also used that the contributions of the two conduction bands sum up due to the fact that only bilinears of the (opposite in sign) BC components appear. Equation (19) is the second main result of this study: it provides the expression for a magnetic field-induced increase of conductivity in a spin-orbit coupled material with a centrosymmetric cubic crystalline arrangement. Explicitly, the longitudinal conductivity reads

$$\frac{\Delta \sigma_{yy,2}(B_y)}{\sigma_{yy}(0)} = \left(\frac{l_0}{l_B} \right)^4 \frac{3}{16\pi} (k_F l_0)^4 I(k_F l_0), \quad (20)$$

where $\sigma_{yy}(0)$ is the Drude conductivity defined above and the integral is

$$I(k_F l_0) = \int_0^{2\pi} d\phi \int_0^\pi \sin(\theta) d\theta \frac{\sin^2(\theta) \sin^2(\phi) (\sin^2(\theta) \cos^2(\phi) + \cos^2(\theta))^2}{[1 + (k_F l_0)^4 \sin^2(\theta) \sin^2(\phi) (\sin^2(\theta) \cos^2(\phi) + \cos^2(\theta))]^3}. \quad (21)$$

The ensuing NLMR $\Delta \rho_{yy}(B_y) / \rho(0) \simeq -\Delta \sigma_{yy}(B_y) / \sigma(0)$ grows quadratically with the externally applied magnetic field strength [see Fig. 2] while being independent of the relaxation time and thus intrinsic.

Discussion

A direct computation of Eq. (19) for the generalized Zeeman coupling in Eq. (2) leads to the NLMR shown in Fig. 2. It scales as $(l_0 / l_B)^4 f(k_F l_0)$ (see the Supplemental Material [46]) where $l_B = \sqrt{\hbar / e B_y}$ is the magnetic length while $l_0 = \sqrt{|a|/|b|}$ is a material-dependent characteristic length. As discussed before, the latter can be also expressed in terms of the linear in $\mathbf{k}$ coupling P between Kramers' pairs of opposite parity, directly controlling the nonparabolicity of the bands, and the direct gap between conduction and valence bands M_0 . Specifically, $l_0 = [P^2 |\bar{g} - \delta g| / (6|\bar{g} + \delta g| M_0^2)]^{1/2}$. The Fermi momentum function $f(k_F l_0)$ is instead independent of specific material properties and hence universal. The ensuing NLMR

displays a nonmonotonous behavior that is a characteristic fingerprint of BC-mediated transport effects, with a maximum value $\Delta \rho_{yy}(B_y) / \rho(0) \simeq 0.1 (l_0 / l_B)^4$.

For materials such as Bi_2Se_3 with a linear coupling $P \simeq 3.5 \text{ eV}\text{\AA}$ [44] and a gap $M_0 \simeq 0.1 \text{ eV}$ [57], we can estimate $l_0 \simeq 2 \text{ nm}$ assuming the g -factors of electrons and holes are comparable [57], i.e., $\delta g = 0$. This leads to a characteristic NLMR of the order of 0.5% at a few Teslas which is close to the values experimentally observed [28]. We note that this NLMR value is independent of the g -factor, in marked contrast with a previous theoretical estimate [58] where a large Landé g -factor of 30 has been used, and the appearance of the NLMR was strictly tied to the Zeeman splitting between the conduction bands. The NLMR we predict is completely independent of this splitting since it only relies on the presence of time-reversal symmetry breaking. As a result, it remains sizable even in materials with very small Landé g -factors. Additionally, the magnetoresistance in the direction

orthogonal to the magnetization takes a different form given by

$$\frac{\sigma_{xx,2}(B_y)}{\sigma_{xx}(0)} = \left(\frac{l_0}{l_B}\right)^4 \frac{3}{16\pi} (k_F l_0)^4 \delta I_{xx}(k_F l_0) \quad (22)$$

with the angular integral

$$\delta I_{xx}(k_F l_0) = \int d\phi \int \sin\theta d\theta \left(\frac{\sin^6(\theta) \sin^4(\phi) \cos^2(\phi)}{[1 + k_F^4 l_0^4 \sin^2(\theta) \sin^2(\phi) (\sin^2(\theta) \cos^2(\phi) + \cos^2(\theta))]^3} \right). \quad (23)$$

This anisotropy of the magnetoresistance implies the occurrence of a complementary π -periodic planar Hall effect whose angular dependence will have a $\pi/4$ shift with respect to the magnetoresistance angular dependence, similar to the situation encountered in, for instance, Weyl semimetals.

IV. CONCLUSIONS

Thus far, NLMR in centrosymmetric and time-reversal invariant conductors has been shown to occur either in the $\omega_c \tau \gg 1$ regime with well-formed Landau levels [59,60] or considering very large Landé g -factors as a result of the orbital magnetic moment of Zeeman-split electronic bands [58]. We have instead identified a previously unknown mechanism of NLMR at weak fields in centrosymmetric and time-reversal invariant materials. We have shown that the nonparabolicity of spin-orbit coupled bands leads to a generalized Zeeman coupling in the projected doublet of bands participating in transport. This generalized Zeeman coupling contains momentum dependent terms that lead to a d -wave Berry curvature independent of the magnetic field strength, which directly couples to the orbital motion of Bloch electrons. In contrast to the mechanism based on Zeeman-split bands [58], our theory is independent of the Landé g -factor and the orbital magnetic moment. The NLMR mechanism individuated in this study can appear in all centrosymmetric point groups (see [46]). Additionally, it is independent of the topology of the electronic bands and can appear both in

doped topological insulators and in conventional conductors. We therefore expect the weak-field NLMR to act full force in a vast class of nonmagnetic spin-orbit coupled materials, including, for instance, $\text{Bi}_{1-x}\text{Sb}_x$. In this compound, the electronic pockets at the L point of the Brillouin zone have a small direct energy gap and a strongly nonparabolic, nearly linear dispersion [61] for small x . This can lead to strongly enhanced values of the characteristic length l_0 and consequently of its intrinsic weak-field NLMR. The intrinsic weak-field NLMR can be parsed using the explicit scattering time dependence of other contributions due to, e.g., orbital magnetoresistance effects, sample inhomogeneities, and anisotropies, which obey Kohler's rule [23,24]. Magnetoresistance experiments at varying temperature can be therefore used to clearly identify the BC-induced NLMR unveiled in our study.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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