

# Classification of mass terms in kagome semimetals

Simone Ciceri<sup>1,2,3,\*</sup>, Matteo Massaro<sup>4,1</sup> and Lars Fritz<sup>1,†</sup>

<sup>1</sup>*Institute for Theoretical Physics and Center for Extreme Matter and Emergent Phenomena, Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands*

<sup>2</sup>*Einstein Center for Neurosciences Berlin, Charité Universitätsmedizin Berlin, 10117 Berlin, Germany*

<sup>3</sup>*Modelling of Cognitive Processes, Technical University of Berlin, 10587 Berlin, Germany*

<sup>4</sup>*Department of Physics and Materials Science, University of Luxembourg, L-1511 Luxembourg, Luxembourg*



(Received 19 November 2024; revised 22 June 2025; accepted 8 July 2025; published 21 July 2025)

In the last years, kagome materials have received massive attention by virtue of being candidate hosts for a large variety of quantum phases: spin liquids, unconventional superconductivity, and topological phases of matter, to name the more exotic. One of the most interesting features is tunability: changing the filling, the noninteracting band structure can be tuned from flat bands to conventional metallic phases as well as to semimetals. In this paper, we concentrate on the latter. At specific lattice filling, the electronic bands have a semimetallic structure, hosting Dirac, massless quasiparticles, such as in graphene or other layered two dimensional materials. Specifically, we determine what terms can be added to the nearest-neighbor hopping that opens at the gap at the said Dirac point. These terms can, in principle, arise through external perturbations, interactions, or collective instabilities. We classify the 16 possible gap-opening terms according to the broken symmetries. Furthermore, we identify concrete microscopic realizations, allowing for an interpretation of these phases.

DOI: [10.1103/2gzg-vnf9](https://doi.org/10.1103/2gzg-vnf9)

## I. INTRODUCTION

Kagome metals have recently become a focal point for research into two-dimensional electronic materials. Due to the nontrivial geometry of the underlying lattice, their noninteracting band structure hosts a variety of interesting configurations, depending on the electronic filling. It hosts conventional metals and a completely flat band, as well as a semimetal-like graphene. The flat bands as well as van Hove singularities make these systems an interesting playground for the study of strong correlation effects [1]. In addition, they have been proposed as hosts for quantum spin liquids [2,3], unconventional superconductivity [4], and topological phases of matters [5], as well as other exotic quantum phases [6,7].

In this article, we concentrate on the least studied case of the system, which is when the filling is tuned to the semimetallic limit. There, an effective Dirac theory emerges described by massless Dirac fermions [8–11]. We study and classify perturbations to the noninteracting nearest neighbor (NN) hopping model that open a gap at said Dirac points, rendering the quasiparticles massive and effectively turning the system insulating. This classification was carried out in graphene a decade ago. It is organized according to symmetries [12] and it analyzes the properties of the underlying insulator [13–18]. In

this work, we carry out a related analysis for kagome and give both a physical interpretation of the mass terms as well as a recipe on how to trigger the respective phases (at least in some cases). For some mass terms, the best candidate realization is a collective instability. Graphene is usually considered a system that is on the verge of being strongly interacting. However, Mazin *et al.* [11] showed that Sc-Herbertsmithite is strongly interacting compared to graphene, with a 3 times larger fine-structure constant.<sup>1</sup> This suggests an increased likelihood to undergo spontaneous symmetry breaking to gapped phases.

Our paper is structured as follows. We initially review the theory of noninteracting, spinless electrons hopping on the kagome lattice. We then discuss lattice symmetries as well as fundamental symmetries of the underlying quantum theory. We then classify possible mass terms according to the broken symmetries, allowing reverse-engineer microscopic lattice realizations of perturbations that have the said effect. In parallel, we investigate the relationship with graphene, which shares the same low-energy theory and the underlying lattice symmetries, albeit with a simpler unit cell.

## II. MODEL

In this section, we investigate the band structure of a spinless, noninteracting nearest-neighbor hopping model. From

\*Contact author: [simone.ciceri@tu-berlin.de](mailto:simone.ciceri@tu-berlin.de)

†Contact author: [l.fritz@uu.nl](mailto:l.fritz@uu.nl)

<sup>1</sup>The fine-structure constant, defined as  $\alpha = \frac{e^2}{4\pi\epsilon_0\epsilon_r} \frac{1}{\hbar v_F}$ , controls the strength of the Coulomb interaction:  $\epsilon_r$  is the relative permittivity of the medium,  $e$  the elementary charge, and  $v_F$  the Fermi velocity of the electrons.

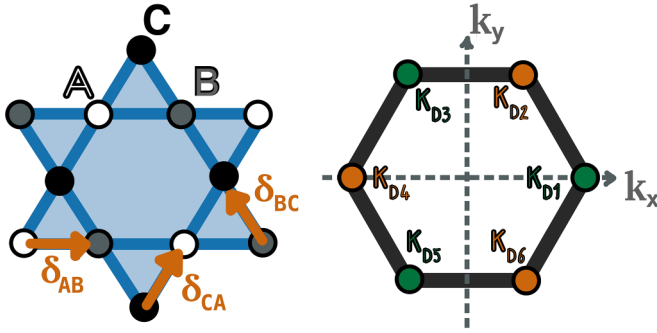


FIG. 1. Kagome lattice (left) and its Brillouin zone (right). Sublattices  $A, B, C$ , sublattice vectors  $\delta_{ij}$ , and the Dirac points  $K_{Dj}$  are indicated.

there, we proceed to a derivation of the low-energy theory and a discussion of all its symmetries.

### A. Tight-binding model on the kagome lattice

As a minimal model for the kagome metal, we introduce the tight-binding Hamiltonian for fermions hopping on the kagome lattice. Initially, we disregard spin-orbit coupling (SOC) and start from spinless particles:

$$H = -t \sum_{\langle i, j \rangle} \psi_i^\dagger \psi_j. \quad (1)$$

The summation extends over nearest-neighbors  $\langle i, j \rangle$ . The lattice has three sublattices, henceforth called  $A, B, C$ , and lattice vectors  $\delta_{AB} = a(1, 0)^T$ ,  $\delta_{BC} = \frac{a}{2}(1, \sqrt{3})^T$ ,  $\delta_{CD} = \frac{a}{2}(-1, \sqrt{3})^T$ , where  $a$  is the lattice constant; see Fig. 1. The Hamiltonian can be block-diagonalized in momentum space with a Fourier transformation thanks to the discrete translational symmetry of the lattice:

$$H = \sum_{\mathbf{k} \in \mathcal{B}} \Psi^\dagger(\mathbf{k}) h(\mathbf{k}) \Psi(\mathbf{k}). \quad (2)$$

We defined a vector of field operators according to  $\Psi(\mathbf{k}) = (\hat{A}(\mathbf{k}), \hat{B}(\mathbf{k}), \hat{C}(\mathbf{k}))^T$ , where the components refer to the field operators on the respective sublattice. The Bloch Hamiltonian reads

$$h(\mathbf{k}) = -2t \begin{pmatrix} 0 & \cos(\mathbf{k} \cdot \delta_{AB}) & \cos(\mathbf{k} \cdot \delta_{AC}) \\ \cos(\mathbf{k} \cdot \delta_{AB}) & 0 & \cos(\mathbf{k} \cdot \delta_{BC}) \\ \cos(\mathbf{k} \cdot \delta_{AC}) & \cos(\mathbf{k} \cdot \delta_{BC}) & 0 \end{pmatrix}. \quad (3)$$

$\mathbf{k} = (k_x, k_y)^T$  is a momentum in the first Brillouin zone shown in Fig. 1. A diagonalization of the Bloch Hamiltonian reveals three bands, shown in Fig. 2:

$$E(\mathbf{k}) = \begin{cases} -t(1 \pm \sqrt{3 + 2 \sum_{i < j} \cos(2\mathbf{k} \cdot \delta_{ij})}) \\ 2t. \end{cases} \quad (4)$$

The two lower bands touch at six Dirac points, among which only two are independent. We refer to them as valleys. A possible choice is  $\mathbf{K}_{D1} = (\frac{2\pi}{3a}, 0)^T$  and  $\mathbf{K}_{D4} = (-\frac{2\pi}{3a}, 0)^T$ , pointed out in Fig. 1. Different choices of Dirac points result in different low-energy Hamiltonians that are related by gauge transformations (see discussion in Appendix A).

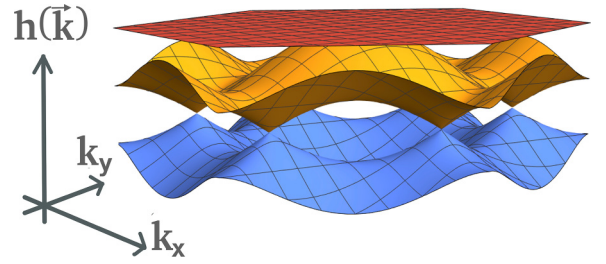


FIG. 2. Electronic bands of the tight-binding model on the kagome lattice [Eq. (4)]: A flat band (red), and two lower bands (orange, blue) that touch at six Dirac points where the density of states vanishes. The focus of this work is on the low-energy theory that emerges when the filling is tuned at these Dirac points (for instance, when only the lower, blue band is completely filled).

### 1. Low-energy expansion

First, we want to investigate the mass terms in the low-energy theory. To that end, we will expand the Bloch Hamiltonian to linear order in  $\mathbf{k}$  and perform a unitary transformation. This reveals the two-dimensional effective Dirac theory. The diagonalization matrix is given by

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, & \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \end{bmatrix}. \quad (5)$$

We consider the expansion of the transformed Hamiltonian  $\tilde{h}(\mathbf{k}) = U h(\mathbf{K}_{Di} + \mathbf{k}) U^{-1}$  up to linear order in  $\mathbf{k}$ , centered at the two Dirac points  $\mathbf{K}_{D1}$  and  $\mathbf{K}_{D4}$ . In terms of two new momenta  $k_1 = \frac{1}{2}(k_x - \sqrt{3}k_y)$  and  $k_2 = \frac{1}{2}(\sqrt{3}k_x + k_y)$ , the transformed, linearized Hamiltonian reads

$$\tilde{h}(\mathbf{K}_{D1} + \mathbf{k}) \simeq \begin{pmatrix} -t - v_f k_1 & -v_f k_2 \\ -v_f k_2 & -t + v_f k_1 \\ \frac{v_f}{\sqrt{2}} k_2 & \frac{v_f}{\sqrt{2}} k_1 \\ \frac{v_f}{\sqrt{2}} k_1 & \frac{v_f}{\sqrt{2}} k_2 \end{pmatrix} \begin{pmatrix} \frac{v_f}{\sqrt{2}} k_2 \\ \frac{v_f}{\sqrt{2}} k_1 \\ 2t \end{pmatrix} \quad (6)$$

$$\tilde{h}(\mathbf{K}_{D4} + \mathbf{k}) \simeq \tilde{h}(\mathbf{K}_{D1} - \mathbf{k}), \quad (7)$$

where  $v_f = \sqrt{3}at$  can be seen as the effective Fermi velocity. Note that the Jacobian of this transformation is one. The highlighted  $2 \times 2$  block in the left-upper corner describes the emerging Dirac theory, while the other terms couple it to the flat band with  $E = 2t$ . Integrating out this flat band contributes terms of order  $\mathcal{O}(k^2)$  to the Dirac theory and can therefore be neglected. For a more detailed description of these calculations, we refer to a previous work [19], where this is shown in more detail.

In the two-dimensional subspace of the Dirac theory, the Hamiltonian reads

$$\tilde{h}(\mathbf{K}_{D1} + \mathbf{k}) = -t\mathbb{1} - v_f(k_1 \tau_3 + k_2 \tau_1), \quad (8)$$

$$\tilde{h}(\mathbf{K}_{D4} + \mathbf{k}) = \tilde{h}(\mathbf{K}_{D1} - \mathbf{k}), \quad (9)$$

where  $\tau_i$  is the  $i$ th Pauli matrix. Note that the corresponding field operators, defined through Eq. (5), are linear combinations of the three sublattices fields  $\hat{A}, \hat{B}, \hat{C}$ . We later refer to this degree of freedom as a *sublattice*, in analogy with the sublattices of graphene. This is abuse of language due to the presence of three sublattices and the wave function

being distributed over all of them. At  $1/3$  lattice filling, the term proportional to the identity is absorbed in the chemical potential and we can neglect it. A more general form of the Hamiltonian can be written, including the spin and valley degrees of freedom, according to

$$\tilde{\mathcal{H}}(\mathbf{k}) = -v_f (k_1 s_0 \otimes \sigma_3 \otimes \tau_3 + k_2 s_0 \otimes \sigma_3 \otimes \tau_1), \quad (10)$$

where  $s_i$  and  $\sigma_i$  are the  $i$ th Pauli matrices acting on the spin and valley subspace, respectively. The valley dependence is derived from Eqs. (6) and (7). The identity in the spin space reflects the absence of spin-orbit coupling, which is studied in the next chapter in the context of perturbations. The field operators in this picture are

$$\hat{\Psi}(\mathbf{k}) = (\hat{\varphi}_{\mathbf{k}\uparrow+}, \hat{\eta}_{\mathbf{k}\uparrow+}, \hat{\varphi}_{\mathbf{k}\uparrow-}, \hat{\eta}_{\mathbf{k}\uparrow-}, \hat{\varphi}_{\mathbf{k}\downarrow+}, \hat{\eta}_{\mathbf{k}\downarrow+}, \hat{\varphi}_{\mathbf{k}\downarrow-}, \hat{\eta}_{\mathbf{k}\downarrow-})^T, \quad (11)$$

where  $\hat{\varphi}$ ,  $\hat{\eta}$  represent the two *sublattice* field operators with defined valley ( $\pm$ ) and spin quantum numbers ( $\uparrow\downarrow$ ). The respective low-energy effective Hilbert space is given by

$$\mathbb{H}_{\text{eff}} \equiv \underbrace{\mathbb{C}^2}_{\text{spin space}} \otimes \underbrace{\mathbb{C}^2}_{\text{valleys sp.}} \otimes \underbrace{\mathbb{C}^2}_{\text{sublatt. sp.}}. \quad (12)$$

For later reference, when we reverse engineer couplings in the microscopic model where the three sublattices are explicitly represented, we refer to the Hilbert space

$$\mathbb{H}_{\text{full}} \equiv \underbrace{\mathbb{C}^2}_{\text{spin space}} \otimes \underbrace{\mathbb{C}^2}_{\text{valleys sp.}} \otimes \underbrace{\mathbb{C}^3}_{\text{sublatt. sp.}}. \quad (13)$$

The Dirac structure of the Hamiltonian Eq. (10) is similarly found in graphene at half filling, albeit with different form in terms of Pauli matrices (Appendix C).

### B. Symmetries of the model

The lattice symmetries (and, equivalently, those of the honeycomb lattice) are encoded in the wallpaper group  $p6m$ : it consists of six reflection axes, one rotation center of order six, two of order three, and three of order two [20]. Reflections and the six rotations with the same center are shown in Fig. 3. Their operator form, which commutes with the tight-binding Hamiltonian, is derived explicitly in Appendix A. The main results are summarized in Table III. Note that rotations and reflections interchange Dirac points.  $R_1$ , for instance, the anticlockwise rotation of  $\pi/3$ , transforms  $K_{D1}$  and  $K_{D4}$  into  $K_{D2}$  and  $K_{D5}$ , respectively. A linear expansion of the Hamiltonian around two specific Dirac points is consequently transformed into the linear expansion around two corresponding ones.

Besides lattice symmetries and discrete translational symmetry (accounted for with the Fourier transform), we consider three additional symmetries: time reversal, chirality, and charge conjugation. In the low-energy effective Hilbert space  $\mathbb{H}_{\text{eff}}$ , time reversal assumes the form  $\mathcal{T} = U_T \circ \mathcal{K} = is_2 \otimes \sigma_1 \otimes \tau_0 \circ \mathcal{K}$ , where  $\mathcal{K}$  represents complex conjugation and  $U_T$  is unitary. The Hamiltonian is  $\mathcal{T}$  invariant since  $U_T \tilde{\mathcal{H}}^*(\mathbf{k}) U_T^{-1} = \tilde{\mathcal{H}}(-\mathbf{k})$ . The breaking of time-reversal symmetry has been investigated in a study by Feng *et al.* [7], with results that are in part related to the present work. Charge conjugation is an antiunitary symmetry that relates a particle excitation to a hole excitation of opposite charge

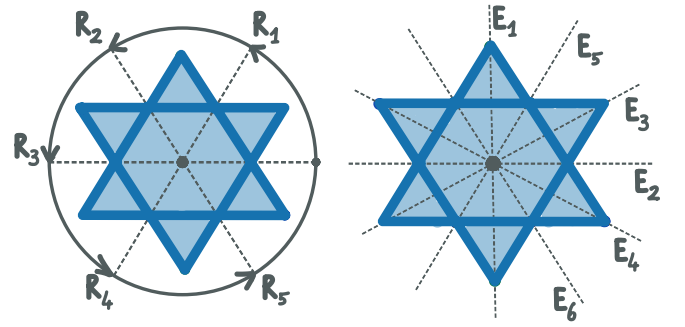


FIG. 3. Rotation symmetries  $R_i$  (left figure) and reflection symmetries  $E_i$  (right figure) of the kagome lattice. The tight-binding Hamiltonians Eqs. (1) and (3) are invariant under these transformations. However, Dirac points are interchanged by rotations and reflections. A linear expansion of the Hamiltonian around two specific Dirac points is accordingly transformed into a linear expansion around two different ones. Explicit examples are shown in Appendix A.

and opposite energy. Chirality (also known as sublattice symmetry) is defined as the composition of time-reversal and charge symmetry, and it is unitary. In contrast to graphene, the kagome tight-binding Hamiltonian enjoys neither chiral symmetry nor charge symmetry. In simple terms, it is due to the flat band, which makes the spectrum asymmetric under a sign flip  $t \rightarrow -t$ . However, the low-energy theory Eq. (10) obtained by integrating out the flat band is charge and chiral symmetric. The chirality operator  $\mathcal{S} = s_0 \otimes \sigma_3 \otimes \tau_2$  anticommutes with the effective Dirac Hamiltonian:  $\{\tilde{\mathcal{H}}(\mathbf{k}), \mathcal{S}\} = 0$ . Charge conjugation  $\mathcal{C}$  is obtained by the combination of time reversal and the inverse of sublattice symmetry:  $\mathcal{C} = \mathcal{T} \circ \mathcal{S}^{-1}$ . The simplest form is obtained in the low-energy subspace  $\mathbb{H}_{\text{eff}}$ , where  $\mathcal{C} = U_C \circ \mathcal{K} = s_2 \otimes \sigma_2 \otimes \tau_2 \circ \mathcal{K}$ . The effective Hamiltonian is charge symmetric since  $U_C \tilde{\mathcal{H}}^*(-\mathbf{k}) U_C^{-1} = -\tilde{\mathcal{H}}(\mathbf{k})$ . Symmetry relations for  $\mathcal{C}$  and  $\mathcal{S}$  only hold if the flat band is neglected. This approximation is justified when the theory is expanded around the Dirac cones and only low-energy effects are retained, as illustrated in the previous section. To summarize, while the tight-binding theory of graphene is charge and chiral symmetric, this holds in kagome only when the flat band is ignored and the filling is tuned to the Dirac point.

### III. MASS TERMS

Many different bilinear terms could be added to the band structure introduced above. We concentrate on bilinear terms that open a gap at the Dirac points: these terms convert the semimetal to an insulator. The most general local bilinear term involves spin and valley degrees and freedom, and within the low-energy subspace  $\mathbb{H}_{\text{eff}}$  it takes the form  $m \mathcal{M}_{ijk} = m s_i \otimes \sigma_j \otimes \tau_k$ . The scalar  $m$  encodes the amplitude and  $s_i$ ,  $\sigma_j$ , and  $\tau_k$  are Pauli matrices acting on the spin, valley, and sublattice subspaces, respectively. If  $\mathcal{M}_{ijk}$  anticommutes with the Dirac Hamiltonian, i.e.,  $\{\tilde{\mathcal{H}}, \mathcal{M}\} = \tilde{\mathcal{H}}\mathcal{M} + \mathcal{M}\tilde{\mathcal{H}} = 0$ , this results in a gap in the excitation spectrum. The resulting dispersion relation reads  $\epsilon_{\mathbf{k}} = \pm\sqrt{\mathbf{k}^2 + m^2}$ . From now on, we refer to all anticommuting terms as gap terms or mass terms. According to our classification, these operators do not involve couplings

TABLE I. Kagome mass terms and their microscopic realization. Indices  $i, j, k$  in the first column label each term according to  $m\mathcal{M}_{ijk} = m s_i \otimes \sigma_j \otimes \tau_k$ . The quantum symmetries of the massless Hamiltonian (parity, time reversal, charge, and chiral symmetry) are either preserved (✓) or broken (✗) by each term. A similar result for graphene is reported in Table IV for comparison.

| Kagome mass terms $\mathcal{M}_{ijk}$ |        |       |        |        |                                        |         |
|---------------------------------------|--------|-------|--------|--------|----------------------------------------|---------|
| Idx $\{ijk\}$                         | Parity | T-rev | Charge | Chiral | Lattice realization                    | Section |
| $\{0,0,2\}$                           | ✓      | ✗     | ✓      | ✗      | Haldane's model (spin chiral)          | IV A    |
| $\{0,1,0\}$                           | ✓      | ✓     | ✓      | ✓      | Plaquette phase (rings)                | IV C    |
| $\{0,2,0\}$                           | ✗      | ✓     | ✓      | ✓      | Plaquette phase (rings)                | IV C    |
| $\{0,3,2\}$                           | ✗      | ✓     | ✗      | ✗      | Lattice dimerization and trimerization | IV D    |
| $\{1,1,0\}$                           | ✓      | ✗     | ✗      | ✓      | Resonating plaquette phase: $x$        | IV C    |
| $\{2,1,0\}$                           | ✓      | ✗     | ✗      | ✓      | Resonating plaquette phase: $y$        |         |
| $\{3,1,0\}$                           | ✓      | ✗     | ✗      | ✓      | Resonating plaquette phase: $z$        |         |
| $\{1,2,0\}$                           | ✗      | ✗     | ✗      | ✓      | Resonating plaquette phase: $x$        |         |
| $\{2,2,0\}$                           | ✗      | ✗     | ✗      | ✓      | Resonating plaquette phase: $y$        |         |
| $\{3,2,0\}$                           | ✗      | ✗     | ✗      | ✓      | Resonating plaquette phase: $z$        |         |
| $\{1,0,2\}$                           | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: $x$               | IV B    |
| $\{2,0,2\}$                           | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: $y$               |         |
| $\{3,0,2\}$                           | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: $z$               |         |
| $\{1,3,2\}$                           | ✗      | ✗     | ✓      | ✗      | SOC in trimerized lattice $x$          | IV E    |
| $\{2,3,2\}$                           | ✗      | ✗     | ✓      | ✗      | SOC in trimerized lattice $y$          |         |
| $\{3,3,2\}$                           | ✗      | ✗     | ✓      | ✗      | SOC in trimerized lattice $z$          |         |

to the flat band. We embed the gap terms in the full space according to

$$\tilde{M}_{ijk} = s_i \otimes \sigma_j \otimes \begin{pmatrix} \tau_k & 0 \\ 0 & 0 \end{pmatrix}. \quad (14)$$

While there are 64 possible bilinear terms, only 16, reported in Table I, anticommute with the Hamiltonian and represent mass terms. Note that  $\tilde{M}_{ijk}$  is written in the decoupled representations, in which the Hamiltonian is diagonal at the Dirac points. The mass terms in the original representation of the full Hilbert-space [same as Eq. (3)] are obtained using  $M_{ijk} = U^{-1}\tilde{M}_{ijk}U$ . We classify the mass terms according to the broken and preserved symmetries (Appendix A), relevant for identifying microscopic realization and possible experimental signatures.

The described gap-opening mechanism is equivalently present in graphene at half filling. The graphene Dirac Hamiltonian takes, however, a different form in terms of the Pauli matrices. Therefore, it gives rise to a distinct set of mass terms and different microscopic realizations. One classification with regard to the broken symmetries has been presented in Refs. [12,21] and is reviewed in Appendix C (Table IV).

#### IV. MICROSCOPIC REALIZATION OF MASS TERMS

The main result of our study and focus of this section is to present lattice realizations for each of the possible mass terms. We start every paragraph with a table that lists all the symmetries. To distinguish their presence or absence, we note preserved with a green check mark (✓) while we denote broken with a red cross (✗). Time-reversal symmetry is denoted by  $\mathcal{T}$ , charge conjugation by  $\mathcal{C}$ , and chirality by  $\mathcal{S}$ , while reflections are written as  $E_i$  and rotations as  $R_i$ .

Table I summarizes the following discussion. A gap opening in the electronic bands has been measured in different kagome compounds [22–30]. The correspondence between these experimental findings and the derived mass terms, when present, is discussed in the dedicated sections.

##### A. Haldane's model on kagome lattice

| $\mathcal{M}_{002}$ | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ |
|---------------------|---------------|---------------|---------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|                     | ✗             | ✓             | ✗             | ✗     | ✗     | ✗     | ✗     | ✗     | ✗     | ✓     | ✓     | ✓     | ✓     | ✓     |

In this section, we explore the effect of a spatially dependent magnetic field  $\mathcal{B}(\mathbf{r}) = \mathcal{B}(\mathbf{r})\mathbf{z}$  perpendicular to the 2D plane. It is chosen such that there is zero flux through the unit cell. Following the path of Haldane's work in graphene [15], we consider a magnetic field with the periodicity of the lattice. For instance, this configuration can be obtained by placing magnetic dipoles at the center of each hexagon [31]. Using Peierls substitutions, we absorb the effect of the magnetic field into the hopping amplitude  $t$ , which acquires a complex phase:  $t \rightarrow te^{i\pm\theta}$ . The kagome structure causes finite fluxes of  $\mathcal{B}$  through closed paths of nearest-neighbor hoppings. The phase  $\theta$  is therefore finite for NN paths, as opposed to graphene where next-nearest neighbor (NNN) hoppings give the first, nonzero contribution. Given  $\Phi$  as the flux of the magnetic field through the triangles, the phase is computed using the Stokes theorem:  $\theta = \frac{e}{3\hbar}\Phi$ . The sign of the acquired phase depends on the path direction, as shown in Fig. 4: hoppings as  $A \rightarrow B$ ,  $B \rightarrow C$ ,  $C \rightarrow A$  will gain a phase  $+\theta$ , while a phase  $-\theta$  is obtained for opposite hoppings. The TB Hamiltonian between nearest neighbors becomes

$$H_\theta = t \sum_i e^{-i\theta} [\hat{A}_i^\dagger \hat{B}_{i\pm\delta_{AB}} + \hat{B}_{i+\delta_{AB}}^\dagger \hat{C}_{i\pm\delta_{AC}} + \hat{C}_{i\pm\delta_{AC}}^\dagger \hat{A}_i] + \text{H.c.}, \quad (15)$$

where H.c. indicates the Hermitian conjugate. After the Fourier transform, we obtain

$$h_\theta(\mathbf{k}) = \begin{pmatrix} 0 & e^{-i\theta} \cos(\mathbf{k} \cdot \delta_{AB}) & e^{i\theta} \cos(\mathbf{k} \cdot \delta_{AC}) \\ e^{i\theta} \cos(\mathbf{k} \cdot \delta_{AB}) & 0 & e^{-i\theta} \cos(\mathbf{k} \cdot \delta_{BC}) \\ e^{-i\theta} \cos(\mathbf{k} \cdot \delta_{AC}) & e^{i\theta} \cos(\mathbf{k} \cdot \delta_{BC}) & 0 \end{pmatrix}. \quad (16)$$

We expand the Hamiltonian to linear order around the two valleys and include the spin degree of freedom. The resulting Dirac Hamiltonian reads

$$h_\theta = t \cos \theta \left[ s_0 \otimes \sigma_0 \otimes \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right] - t \sin \theta \left[ s_0 \otimes \sigma_0 \otimes \begin{pmatrix} 0 & -i & -i \\ i & 0 & i \\ i & -i & 0 \end{pmatrix} \right]. \quad (17)$$

The first term is equal to the TB Hamiltonian in the absence of a magnetic field, multiplied by a factor  $\cos \theta$ . The second term instead corresponds to  $\mathcal{M}_{002}$ , which opens a gap at the Dirac points. The strength of this perturbation is  $-t \sin \theta$ . Since  $\theta$  is proportional to the flux of  $\mathcal{B}(\mathbf{r})$ , the Hamiltonian (17) reduces to the original TB form when  $\mathcal{B}(\mathbf{r}) = 0$ . Otherwise, a gap of size  $\Delta = |t\sqrt{3} \sin \theta|$  opens at the Dirac points when the magnetic field is switched on. If NNN hoppings are taken into account, the gap is further increased. Haldane's mass term breaks time-reversal and chiral symmetry, keeping charge symmetry intact. From a geometrical perspective, all the reflection symmetries are broken while all the rotations are preserved, as one can check from Fig. 4. Boundary states are one characteristic signature of this phase. The resulting anomalous Hall effect has been experimentally revealed in different compounds with kagome structures ( $\text{Fe}_3\text{Sn}_2$ ,  $\text{TbMn}_6\text{Sn}_6$ ) [22,23].

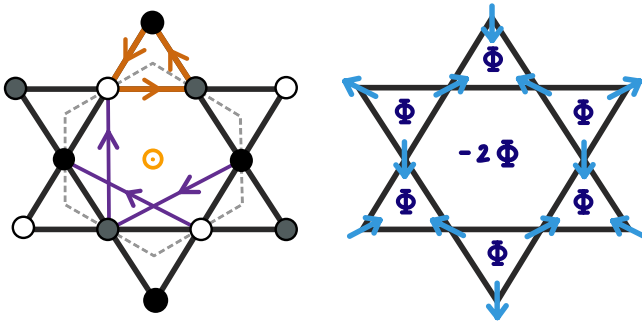


FIG. 4. Left: Kagome unit cell, with nearest neighbor (NN, orange) and second-nearest neighbor (SNN, purple) bonds. Arrows mark the direction of positive phase hoppings when Haldane's periodic magnetic field is present. A dotted, grey line encloses a surface of vanishing magnetic flux. A magnetic field of this kind results in the mass term  $\mathcal{M}_{002}$  that opens a gap at the Dirac points. Right: Magnetic fluxes  $\Phi$  through the triangles and hexagons when Haldane's magnetic field is present. In addition, we show the spin-chiral arrangement on the kagome lattice background. When electron spins are perfectly aligned to the nuclei spins (strong Hund's coupling), this chiral background order effectively acts as the Haldane's magnetic field.

### 1. Alternative realization through Hund's coupling and spin chirality

We considered the emergence of Haldane's mass by artificially applying an external, fine-tuned magnetic field. The same effect, however, can originate from the interplay between conduction electrons and a nontrivial magnetic background [32–34]. In particular, conduction electrons strongly coupled to the nuclear spins (Hund's coupling) acquire a Berry phase if their spin is forced to align with the background spins: a configuration of tilted spins in the underlying structure acts as an effective magnetic field [35]. The kagome lattice promotes one peculiar spin configuration, with a pattern of alternating spin chirality [36], represented in Fig. 4. The effective magnetic field resulting from this background configuration is identical to the magnetic field of Haldane's model and is responsible for the same gap opening in the spectrum.

### B. Spin-orbit coupling and the topological insulator

|                     | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
|---------------------|---------------|---------------|---------------|----|----|----|----|----|----|----|----|----|----|----|
| $\mathcal{M}_{102}$ | ✓             | ×             | ×             | ×  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\mathcal{M}_{202}$ | ✓             | ×             | ×             | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\mathcal{M}_{302}$ | ✓             | ×             | ×             | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |

In kagome structures, as opposed to graphene, the intrinsic SOC is relevant for nearest-neighbor hopping: NN paths do not correspond to any reflection symmetry axis of the lattice, resulting in finite angular momentum. The sign of the SOC amplitude  $\pm i\lambda$  depends on the spin and hopping direction, as shown in Fig. 4. Guo and Franz [5] explored a similar interaction, considering instead hoppings between next-nearest neighbors. However, we expect the SOC in NN hops to be dominant. The SOC Hamiltonian is given by

$$H_{\text{soc}} = \pm i \frac{\lambda}{2} \sum_{\alpha, \beta} \sum_{\langle i, j \rangle} \psi_{i\alpha}^\dagger \psi_{j\beta} (s_3)_{\alpha\beta} = \pm i \frac{\lambda}{2} \sum_{\langle i, j \rangle} (\psi_{i\uparrow}^\dagger \psi_{j\uparrow} - \psi_{i\downarrow}^\dagger \psi_{j\downarrow}),$$

where  $\pm$  is related to the hopping direction pictured in Fig. 4, and  $s_3$  is the third Pauli matrix in spin space. The explicit form in terms of the sublattice fields is

$$H_{\text{soc}} = i \frac{\lambda}{2} \sum_{i \in A} (\hat{A}_{i,\uparrow}^\dagger \hat{B}_{i \pm \delta_{AB}, \uparrow} + \hat{B}_{i \pm \delta_{AB}, \uparrow}^\dagger \hat{C}_{i \pm \delta_{AC}, \uparrow} + \hat{C}_{i \pm \delta_{AC}, \uparrow}^\dagger \hat{A}_{i, \uparrow} - \uparrow \leftrightarrow \downarrow) + \text{H.c.} \quad (18)$$

Here,  $\boxed{\uparrow \leftrightarrow \downarrow}$  indicates the corresponding term with inverted spin indices. After Fourier transform, this reads

$$H_{\text{soc}} = \sum_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}}^\dagger h_{\text{soc}}(\mathbf{k}) \hat{\Phi}_{\mathbf{k}},$$

$$h_{\text{soc}}(\mathbf{k}) = i\lambda s_3 \otimes \begin{pmatrix} 0 & \cos(\mathbf{k} \cdot \delta_{AB}) & -\cos(\mathbf{k} \cdot \delta_{AC}) \\ -\cos(\mathbf{k} \cdot \delta_{AB}) & 0 & \cos(\mathbf{k} \cdot \delta_{BC}) \\ \cos(\mathbf{k} \cdot \delta_{AC}) & -\cos(\mathbf{k} \cdot \delta_{BC}) & 0 \end{pmatrix}.$$

Here the field operators are defined as  $\hat{\Phi}_{\mathbf{k}} = (\hat{A}_{\mathbf{k}\uparrow}, \hat{B}_{\mathbf{k}\uparrow}, \hat{C}_{\mathbf{k}\uparrow}, \hat{A}_{\mathbf{k}\downarrow}, \hat{B}_{\mathbf{k}\downarrow}, \hat{C}_{\mathbf{k}\downarrow})^T$ . Evaluating the Bloch Hamiltonian  $h_{\text{soc}}(\mathbf{k})$  at the two Dirac points and including the valley degree of freedom, one obtains

$$\mathcal{H}_{\text{soc}} = i\frac{\lambda}{2} s_3 \otimes \sigma_0 \otimes \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad (19)$$

which corresponds to the gap-opening term  $\mathcal{M}_{302}$ . SOC for second-nearest-neighbor hopping corresponds to the same mass term [5], with a different SOC amplitude  $\lambda'$ . Since  $\lambda$  and  $\lambda'$  have opposite signs, the gap amplitude is reduced as  $|\lambda| - |\lambda'|$ . Time reversibility is preserved, as well as all reflections and rotations. No  $x$  or  $y$  spin couplings are present in the case of planar kagome lattice because of the  $\hat{z} \rightarrow -\hat{z}$  reflection symmetry. If a curvature is present, however, the angular momentum vector acquires nonzero components in the  $x$  and  $y$  directions, resulting in additional terms proportional to  $\mathcal{M}_{102}$  and  $\mathcal{M}_{202}$ . Such couplings respect time-reversal symmetry but break certain lattice symmetries which were preserved in the planar case. Guo and Franz demonstrated the presence of gapless edge states in the SO phase, which characterize it as a topological insulator [5]. Gap opening at the Dirac cones due to spin-orbit coupling has been observed in CoSn and Co<sub>3</sub>Sn<sub>2</sub>S<sub>2</sub> kagome compounds using angle-resolved photoemission spectroscopy [24,25].

### C. (Resonating) plaquette ordered phases

|                     | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
|---------------------|---------------|---------------|---------------|----|----|----|----|----|----|----|----|----|----|----|
| $\mathcal{M}_{010}$ | ✓             | ✓             | ✓             | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\mathcal{M}_{020}$ | ✓             | ✓             | ✓             | ×  | ✓  | ×  | ×  | ✓  | ✓  | ×  | ✓  | ×  | ✓  | ×  |

We address two families of mass terms which are off-diagonal in valley space. The first family consists of  $\mathcal{M}_{010}$  and  $\mathcal{M}_{020}$ , which are local, spin-independent terms. In graphene, O-type Kekulé distortion of the lattice is known to open a gap [14]. Its periodicity is given by the reciprocal of  $\mathbf{G} = \mathbf{K}_{D1} - \mathbf{K}_{D4}$ , resulting in valley-mixing bilinear terms. In kagome, nevertheless, hopping amplitude anisotropy due to such distortion is not responsible for gap opening. However, when the coupling within the rings is strong, a periodic charge ordering emerges. This is known as the plaquette ordered phase, pictured in Fig. 5 and previously observed in KV<sub>3</sub>Sb<sub>5</sub> [37]. To first order, the effect of such background ordering is described by a mean-field potential of the form

$$V(\mathbf{r}) = \text{Re}(\xi) \cos(\mathbf{r} \cdot \mathbf{G}) + \text{Im}(\xi) \sin(\mathbf{r} \cdot \mathbf{G}), \quad (20)$$

whose Fourier transform reads

$$\delta H = \frac{\text{Re}(\xi)}{2} \sum_{\mathbf{k}} [\hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k}-\mathbf{G}} + \hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k}+\mathbf{G}}] + \frac{\text{Im}(\xi)}{2i} \sum_{\mathbf{k}} [\hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k}-\mathbf{G}} - \hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k}+\mathbf{G}}] + \boxed{\hat{A} \leftrightarrow \hat{B}} + \boxed{\hat{A} \leftrightarrow \hat{C}}. \quad (21)$$

At the Dirac points, these terms are proportional to  $\mathcal{M}_{010}$  and  $\mathcal{M}_{020}$ . All the quantum symmetries (time-reversal, charge, chiral symmetry) are preserved. This phase is marked by translational symmetry breaking. Its enlarged unit cell that can be probed in Bragg scattering experiments. The second

|                     | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
|---------------------|---------------|---------------|---------------|----|----|----|----|----|----|----|----|----|----|----|
| $\mathcal{M}_{110}$ | ×             | ×             | ✓             | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\mathcal{M}_{210}$ | ×             | ×             | ✓             | ×  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\mathcal{M}_{310}$ | ×             | ×             | ✓             | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\mathcal{M}_{120}$ | ×             | ×             | ✓             | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\mathcal{M}_{220}$ | ×             | ×             | ✓             | ✓  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\mathcal{M}_{320}$ | ×             | ×             | ✓             | ✓  | ×  | ✓  | ✓  | ×  | ×  | ×  | ✓  | ×  | ✓  | ×  |

family of gap terms resembles the first while showing an additional spin dependence:  $\mathcal{M}_{j10}$  and  $\mathcal{M}_{j20}$ , where  $j = 1, 2, 3$ . These terms break time reversal and charge symmetry while preserving chirality.

Pollmann *et al.* [6] proposed a mechanism to spontaneously create these terms. They studied an extended Hubbard model on the kagome lattice which includes on-site and NN repulsion. In the strong-coupling limit, the effective interaction is decoupled in two terms, accounting for charge and spin exchanges, respectively. Two distinct spin-ordered phases emerge, depending on the relative strength of the two con-

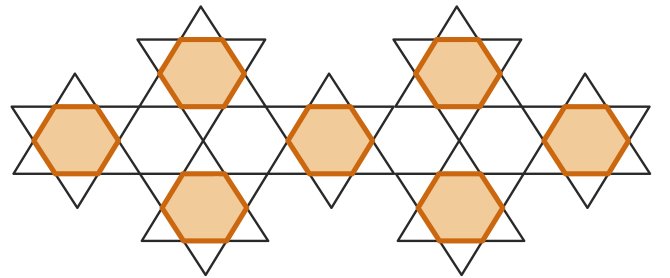


FIG. 5. O-type Kekulé dimerization pattern on kagome lattice, also referred to as a plaquette ordered phase. A mean-field approximation of the interaction with this background charge ordering results in the mass terms  $\mathcal{M}_{010}$ ,  $\mathcal{M}_{020}$ .

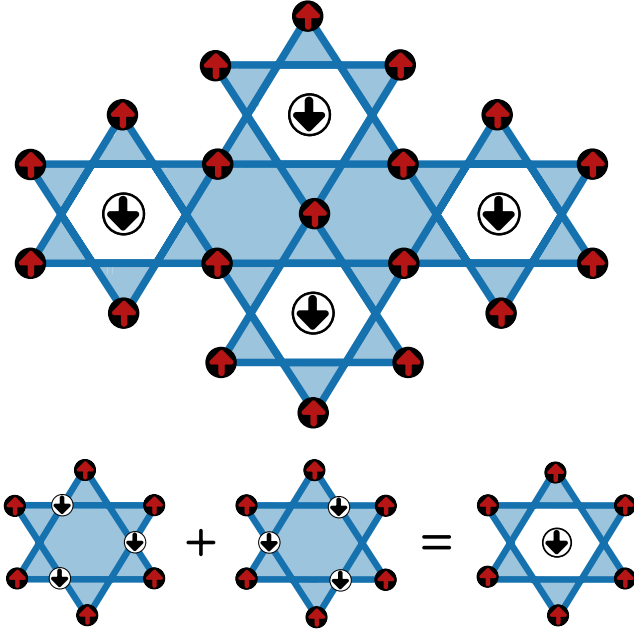


FIG. 6. Kagome lattice at 1/3 filling of strongly interacting electrons is in a resonating plaquette phase [6]. An antiferromagnetic perturbation lifts the spin degeneracy of the ground state, favoring a magnetic ordering: three resonating, parallel spins form a 3/2 spin on the hexagon, surrounded by localized spins pointing in the opposite direction. A mean-field treatment of the antiferromagnetic interaction in the presence of such ordering results in the mass terms  $\mathcal{M}_{j10}$  and  $\mathcal{M}_{j20}$ .

tributions. In the so-called short-loop phase, electrons form loops with shorter possible length, as depicted in Fig. 5. A spin-ordering within the rings is also present, which can be either ferromagnetic or antiferromagnetic (AFM), depending on the interaction strengths. Such phases resembles an O-type Kekulé dimerization pattern with spin dependence, and periodicity given by a wave vector  $\mathbf{G} = \mathbf{K}_{D1} - \mathbf{K}_{D4}$ , which results in a valley mixing. This combination of translational symmetry breaking and magnetic structure is a specific signature of these phases, revealed in Bragg scattering and neutron scattering experiments, respectively. In the resonating plaquette phase, electrons are distributed as sketched in Fig. 6: three electrons resonating on the hexagons are surrounded by three localized ones. Again, we observe antiferromagnetic spin ordering, with periodicity given by  $\mathbf{G}$ . One way to account for these phases in our framework is the mean-field approximation. A perturbative approach to the Hubbard model shows the emergence of AFM interactions. A mean-field treatment of the AFM Heisenberg Hamiltonian reduces the quadratic terms to bilinear ones [21]. If either the plaquette phase or antiferromagnetic short-loop are realized, one obtains spin-dependent, valley-mixing, on-site potential terms, which correspond to  $\mathcal{M}_{j10}$  and  $\mathcal{M}_{j20}$ , with freedom on the spin index. A detailed derivation of these mean-field calculations is reported in Appendix B. Candidate compounds for the discussed magnetic ordering are kagome antiferromagnets upon doping to obtain the desired filling, as, for instance Herbersmithite,  $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$  [38,39].

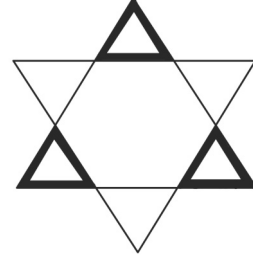


FIG. 7. Trimerization pattern of the kagome lattice: thick (thin) lines depict stronger (weaker) bonds. The resulting distortion in hopping amplitude opens a gap at Dirac points, taking the form of the mass term  $\mathcal{M}_{032}$ .

#### D. Dimerization and trimerization

|                     | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
|---------------------|---------------|---------------|---------------|----|----|----|----|----|----|----|----|----|----|----|
| $\mathcal{M}_{032}$ | ✓             | ×             | ×             | ✓  | ×  | ✓  | ✓  | ×  | ×  | ×  | ✓  | ×  | ✓  | ×  |

We consider a trimerized phase [19], represented in Fig. 7. Such a structure is given by bonds with alternating values of hopping amplitude  $t \rightarrow t \pm \eta$ , and it is sometimes called the breathing kagome phase. These formations, with modified unit cells, are present in different compounds, such as  $\text{Fe}_3\text{Sn}_2$ ,  $\text{Nb}_3\text{X}_8$  ( $\text{X} = \text{Cl}, \text{Br}, \text{I}$ ) [27–29], and  $\text{Nb}_3\text{TeCl}_7$  [30].

The correction to the Bloch Hamiltonian Eq. (3) due to trimerization is

$$h_\eta(\mathbf{k}) = 2\eta i \begin{pmatrix} 0 & \sin(\mathbf{k} \cdot \delta_{AB}) & \sin(\mathbf{k} \cdot \delta_{AC}) \\ -\sin(\mathbf{k} \cdot \delta_{AB}) & 0 & \sin(\mathbf{k} \cdot \delta_{BC}) \\ -\sin(\mathbf{k} \cdot \delta_{AC}) & -\sin(\mathbf{k} \cdot \delta_{BC}) & 0 \end{pmatrix}, \quad (22)$$

with respective field operator  $\hat{\Psi}_\mathbf{k} = (\hat{A}_\mathbf{k}, \hat{B}_\mathbf{k}, \hat{C}_\mathbf{k})^T$ . A second term is added to the original tight-binding Hamiltonian, which can be expanded around Dirac points:

$$\mathcal{H}_\eta = 2\eta i \left[ s_0 \otimes \sigma_3 \otimes \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \right], \quad (23)$$

where the spin and valley degrees of freedom have been included. This Hamiltonian corresponds to the gap-opening term  $\mathcal{M}_{032}$ , which breaks sublattice symmetry while keeping time reversal intact. The same gap term is obtained with a direction-dependent dimerization pattern, given by a  $\boldsymbol{\eta} = (\eta_{AB}, \eta_{AC}, \eta_{BC})$  as long as  $\boldsymbol{\eta} \neq (0, 0, 0)$ .

There is a correspondence between gap-opening deformations in honeycomb and kagome lattices. In graphene, a gap is opened by a staggered, on-site potential, while in kagome by lattice dimerization. It similarly occurs for O-type Kekulé distortions: graphene is gapped because of the hopping amplitude distortion, while kagome materials because of the on-site periodic potential. This relation is not accidental: kagome lattice at 1/3 filling can be mapped into a dimer model on the honeycomb lattice [6]. Quantum states (along with bilinear terms) which are localized on the sites are mapped into states

TABLE II. Kagome gap terms and their symmetries. The terms are identified by the indices  $i, j, k$ , according to the representation  $m\mathcal{M}_{ijk} = m s_i \otimes \sigma_j \otimes \tau_k$ , where  $s, \sigma$ , and  $\tau$  are Pauli matrices, respectively, acting on the spin, valley, and sublattice degrees of freedom (for the definition of the latter, see main text).  $E_i$  and  $R_j$  are, respectively, the reflection and rotation operators as defined in Fig. 3.

| Symmetries of the kagome mass terms $\mathcal{M}_{ijk}$ |        |       |        |        |    |    |    |    |    |    |    |    |    |    |    |
|---------------------------------------------------------|--------|-------|--------|--------|----|----|----|----|----|----|----|----|----|----|----|
| Idx $\{ijk\}$                                           | Parity | T-rev | Charge | Chiral | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
| $\{0,0,2\}$                                             | ✓      | ×     | ✓      | ×      | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\{0,1,0\}$                                             | ✓      | ✓     | ✓      | ✓      | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\{0,2,0\}$                                             | ×      | ✓     | ✓      | ✓      | ×  | ✓  | ×  | ×  | ✓  | ✓  | ×  | ✓  | ×  | ✓  | ×  |
| $\{0,3,2\}$                                             | ×      | ✓     | ×      | ×      | ✓  | ×  | ✓  | ✓  | ×  | ×  | ×  | ✓  | ×  | ✓  | ×  |
| $\{1,1,0\}$                                             | ✓      | ×     | ×      | ✓      | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\{2,1,0\}$                                             | ✓      | ×     | ×      | ✓      | ×  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\{3,1,0\}$                                             | ✓      | ×     | ×      | ✓      | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\{1,2,0\}$                                             | ×      | ×     | ×      | ✓      | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\{2,2,0\}$                                             | ×      | ×     | ×      | ✓      | ✓  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\{3,2,0\}$                                             | ×      | ×     | ×      | ✓      | ✓  | ×  | ✓  | ✓  | ×  | ×  | ×  | ✓  | ×  | ✓  | ×  |
| $\{1,0,2\}$                                             | ✓      | ✓     | ×      | ×      | ×  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\{2,0,2\}$                                             | ✓      | ✓     | ×      | ×      | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  |
| $\{3,0,2\}$                                             | ✓      | ✓     | ×      | ×      | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  | ✓  |
| $\{1,3,2\}$                                             | ×      | ×     | ✓      | ×      | ✓  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\{2,3,2\}$                                             | ×      | ×     | ✓      | ×      | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\{3,3,2\}$                                             | ×      | ×     | ✓      | ×      | ×  | ✓  | ×  | ×  | ✓  | ✓  | ×  | ✓  | ×  | ✓  | ×  |

which live on the bonds and vice versa. This relation holds equivalently for gapping terms.

### E. Spin-orbit coupling in trimerized kagome lattice

|                     | $\mathcal{T}$ | $\mathcal{C}$ | $\mathcal{S}$ | E1 | E2 | E3 | E4 | E5 | E6 | R1 | R2 | R3 | R4 | R5 |
|---------------------|---------------|---------------|---------------|----|----|----|----|----|----|----|----|----|----|----|
| $\mathcal{M}_{132}$ | ×             | ✓             | ×             | ✓  | ✓  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\mathcal{M}_{232}$ | ×             | ✓             | ×             | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ×  | ✓  | ×  | ×  |
| $\mathcal{M}_{332}$ | ×             | ✓             | ×             | ×  | ✓  | ×  | ×  | ✓  | ✓  | ×  | ✓  | ×  | ✓  | ×  |

We investigate how the breathing kagome trimerization pattern affects SOC. Following the procedure of Sec. IV B, we look at the SOC in this configuration: the lattice anisotropy is translated in two different SOC amplitudes [40]  $\pm\lambda_1$  and  $\pm\lambda_2$  when they are associated to a stronger or weaker bond, respectively [32]. Again, the sign of such amplitudes depends on the hopping direction and the electron spin pointing up

(or down). One can decouple the spin-orbit interaction into a regular SOC pattern with the amplitude given by the average of  $\lambda_1, \lambda_2$  and a hopping pattern which takes into account the trimerization effects:

$$\lambda = \frac{\lambda_1 + \lambda_2}{2} \quad \delta\lambda = \frac{\lambda_1 - \lambda_2}{2}. \quad (24)$$

The first contribution is equal to Eq. (18), and therefore corresponds to the gap term  $\mathcal{M}_{302}$ . The second corresponds to

$$\delta\mathcal{H}_{\text{soc}} = i\frac{\delta\lambda}{2} \sum_{i \in A} (\hat{A}_{i,\uparrow}^\dagger \hat{B}_{i+\delta_{AB},\uparrow} - \hat{A}_{i,\uparrow}^\dagger \hat{B}_{i-\delta_{AB},\uparrow} - [\hat{B} \leftrightarrow \hat{C}] - [\hat{A} \leftrightarrow \hat{C}] - [\uparrow \leftrightarrow \downarrow]) + \text{H.c.}, \quad (25)$$

where we stress that for a fixed spin (up or down) the trimerization results in an alternating pattern of SOC strength. After Fourier transformation, the Hamiltonian reads

$$\delta\mathcal{H}_{\text{soc}}(\mathbf{k}) = \sum_{\mathbf{k}} \hat{\Phi}_{\mathbf{k}}^\dagger [s_3 \otimes \delta h_{\text{soc}}(\mathbf{k})] \hat{\Phi}_{\mathbf{k}},$$

$$\delta h(\mathbf{k})_{\text{soc}} = i\delta\lambda \begin{pmatrix} 0 & \sin(\mathbf{k} \cdot \delta_{AB}) & -\sin(\mathbf{k} \cdot \delta_{AC}) \\ -\sin(\mathbf{k} \cdot \delta_{AB}) & 0 & \sin(\mathbf{k} \cdot \delta_{BC}) \\ \sin(\mathbf{k} \cdot \delta_{AC}) & -\sin(\mathbf{k} \cdot \delta_{BC}) & 0 \end{pmatrix}. \quad (26)$$

This Hamiltonian is odd in  $\mathbf{k}$ : the expansion around the two Dirac points results in opposite signs,

$$\delta\mathcal{H}_{\text{soc}} = i\frac{\delta\lambda}{2} s_3 \otimes \sigma_3 \otimes \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (27)$$

which corresponds to the gap-opening term  $\mathcal{M}_{332}$ . As one can see from Table II and Fig. 8 (second contribution, pattern of arrows), such term is invariant under rotations of  $\pm\frac{2\pi}{3}$  and under certain reflections. This gapped phase has been theoretically investigated by Bolens and Nagaosa in Ref. [40], who demonstrated its topological nature. When the effect of trimerization is taken into account both for hopping

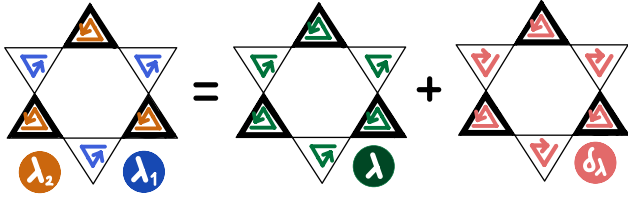


FIG. 8. Spin-orbit coupling in the trimerized phase, where thick (thin) lines represent strong (weak) bonds. Arrows mark the direction of positive amplitudes hopping for a spin up. Down spins have reversed arrows. The alternating amplitude pattern  $\lambda_{1,2}$  (depicted in blue and orange on the left-hand side) can be decoupled into a regular SOC pattern (green arrows) and a contribution from the anisotropy (pink arrows). The first term results in the mass terms  $\mathcal{M}_{j02}$  of Sec. IV B, while the second in the terms  $\mathcal{M}_{j32}$ .

amplitude and spin-orbit coupling ( $\mathcal{M}_{032}$  and  $\mathcal{M}_{332}$ , respectively), all reflection symmetries are broken. Again, a nonplanar setup gives rise to  $x$  and  $y$  components in the spin coupling, which corresponds to the gap-opening terms  $\mathcal{M}_{132}$  and  $\mathcal{M}_{232}$ .

## V. CONCLUSIONS

Motivated by recent attention on kagome materials, we investigated the properties of free electrons on such structures when their filling is  $1/3$ . In this case, a Dirac theory of massless quasiparticles emerges, like the one in graphene. We derived a family of 16 bilinear Hamiltonians which open a gap at the Dirac points. When such terms are present, the quasiparticles gain mass and the system turns into an insulator. We proceed to identify microscopic realizations of said gap terms, either through explicit perturbations or spontaneously originating from symmetry breaking. A similar discussion was present in the literature on graphene, but not similarly structured for kagome materials. In the latter, however, especially interaction-related effects could be enhanced due to the boosted fine-structure constant. Our main findings can be summarized as follows: The anomalous Hall effect (Haldane's model) and the quantum spin-Hall effect (spin-orbit coupling), originally proposed in graphene, exist in kagome materials with similar consequences, possibly with more favorable conditions: contrarily to graphene, these effects emerge on the kagome lattice already for nearest-neighbor hoppings, which indicates the latter as a prominent host for these phenomena. A different category of instabilities concerns lattice deformation. The kagome lattice at  $1/3$  filling can be mapped into a dimer model on the honeycomb lattice of graphene; remarkably, this relation emerges explicitly when comparing graphene and kagome gaps: terms living on the lattice sites of graphene have a correspondent living on the bond of kagome materials and vice versa. The simplest, gap-opening lattice deformation is represented by a dimerization of the kagome lattice, which corresponds to the staggered potential in graphene. Moreover, we considered distortion patterns with enlarged unit cells that couple the two valleys in momentum space, such as O-type Kekulé patterns. Again, such patterns open a gap in the two materials in two conjugate ways: Graphene quasiparticles become massive because of

the hopping amplitude distortion, while in kagome as on-site potential with identical periodicity. Lastly, we studied masses originating from antiferromagnetic interaction. Honeycomb lattice at  $1/2$  filling is a perfect host for Néel ordering, and a mean-field treatment reveals the trivial gapped nature of the antiferromagnetic phase. In kagome, the corner-sharing triangles frustrate antiferromagnetic orderings. In this work, however, we focus on  $1/3$  filling of the kagome lattice, with, on average, two electrons for each triangle. The presence of empty sites leaves space for different antiferromagnetic orderings. We explored the resonating plaquette phase, characterized by enlarged magnetic cells and known to emerge in the strong-coupling limit. Similarly to graphene, a mean-field treatment reveals the gapped nature of this phase

## ACKNOWLEDGMENT

The authors thank R. Ciola and R. Thomale for discussions and collaborations on related subjects.

## APPENDIX: LATTICE SYMMETRIES

Both honeycomb and kagome lattice symmetries are encoded in the wallpaper group  $p6m$ : it consists of six reflection axis, one rotation center of order six, two of order three, and three of order two [20]. In this Appendix, we will derive the operator for these geometrical transformations and study their relation with the Hamiltonian and the mass terms. In the low-energy approximation, the symmetry relations require additional care: one has to consider the starting point of the expansion and how it transforms under the symmetry. For example,  $E_1$  transforms  $\mathbf{K}_{D1}$  into  $\mathbf{K}_{D4}$ ;  $E_2$  instead keeps  $\mathbf{K}_{D1}$  unchanged (see Fig. 3). In general, these operations transform the Hamiltonian expanded around two Dirac points—let us say  $\mathbf{K}_{Di}$ ,  $\mathbf{K}_{Dj}$ —into one Hamiltonian expanded around two different Dirac points  $\mathbf{K}_{Dk}$ ,  $\mathbf{K}_{Dl}$ . While the full Hamiltonian enjoys all the symmetry of the model (any of these transformations sends  $H \rightarrow H$ ), its low-energy approximations are less symmetric, meaning that any of these might be transformed one into another under symmetry operations:  $H_{ij} \rightarrow H_{kl}$ .

### a. Gauge transformation

Before deriving the representation of the lattice symmetry transformations, we emphasize that extra care must be taken in the classification of the mass terms according to these symmetries. Rotations and reflections operators have, in fact, a natural representation in the original full Hilbert space  $\mathbb{H}_{\text{full}}$ . Therefore, it is convenient to embed the gap terms in  $\mathbb{H}_{\text{full}}$  to classify them. As mentioned in Sec. III, this is achieved by following the prescription of Eq. (14) and subsequently applying the inverse of the transformation (dependent on the Dirac point choice) originally used in the low-energy expansion to decouple the flat band from the emerging Dirac theory. However, given a certain choice for the Dirac points, the corresponding mass terms embedded in the full Hilbert space do not automatically open a gap in the spectrum of the Bloch Hamiltonian expanded around a different pair of valleys. In fact, to obtain the corresponding mass terms in  $\mathbb{H}_{\text{full}}$  for a different set of valleys, an additional gauge transformation has to be employed. This is more effectively explained with an

example: In Sec. III, we have expanded  $h(\mathbf{k})$  around  $\mathbf{K}_{D1}$ ,  $\mathbf{K}_{D4}$ , and derived the emerging Dirac theory with its corresponding mass terms. Let us focus, for instance, on the mass term  $\mathcal{M}_{002}$ , and its representation in  $\mathbb{H}_{\text{full}}$ , denoted by  $M_{002}^{14}$  (here, the upper indices indicate the valleys choice). Given the equivalence between Dirac points in the first Brillouin zone, we expect  $M_{002}^{14}$  to open a gap at all valleys. However, this turns out to not be the case and, for instance,  $H_{32} + M_{002}^{14}$  remains gapless. The reason for this seeming contradiction is that the equivalence between Dirac points is not apparent in the form of  $h(\mathbf{k})$ . In fact, upon evaluating it at equivalent Dirac points, say  $\mathbf{K}_{D1}$ ,  $\mathbf{K}_{D3}$ , we obtain two different representations:

$$h(\mathbf{K}_{D1}) = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad h(\mathbf{K}_{D3}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}. \quad (\text{A1})$$

The equivalence between certain valleys, however, is still encoded in the fact that  $h(\mathbf{K}_{D1})$  and  $h(\mathbf{K}_{D3})$  exhibit the same spectrum. This suggests that there exists a gauge transformation connecting them. In this example, it is given by

$$g_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A2})$$

such that

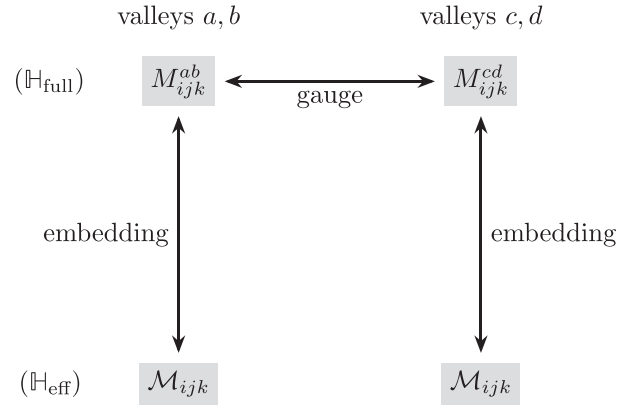
$$g_{13} \cdot h(\mathbf{K}_{D1}) \cdot g_{13}^{-1} = h(\mathbf{K}_{D3}). \quad (\text{A3})$$

Similarly, all the other transformations  $g_{ac}$  that connect any pair of Bloch Hamiltonians,  $h(\mathbf{K}_{Da})$  and  $h(\mathbf{K}_{Dc})$ , evaluated at equivalent valleys  $a, c$  can be determined. The issue presented above ( $H_{32} + M_{002}^{14}$  being gapless) is therefore solved by appropriately gauge-transforming the term  $M_{002}^{14}$  to the valleys 3, 2 [21,41]:

$$M_{002}^{32} = \begin{pmatrix} g_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & g_{42} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & g_{13} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & g_{42} \end{pmatrix} \cdot M_{002}^{14} \cdot \begin{pmatrix} g_{13}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & g_{42}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & g_{13}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & g_{42}^{-1} \end{pmatrix}, \quad (\text{A4})$$

which opens a gap in the spectrum of  $H_{32}$  [in Eq. (A4),  $\mathbf{0}$  indicates the  $3 \times 3$  null matrix]. In fact, this is true in general for all the mass terms, provided that we had appropriately chosen the  $U$  transformations to decouple the flat band depending on the Dirac point choice.<sup>2</sup> Schematically:

<sup>2</sup>The specific form of the low-energy theory Eq. (10) can be derived starting from any choice two of inequivalent Dirac points  $\mathbf{K}_{Da}$ ,  $\mathbf{K}_{Db}$ . This is done by appropriately absorbing these gauge transformations into the flat-band decoupling matrices  $U_{ab}$ , which therefore depend on the valley choice  $a, b$ . This guarantees the same family of mass terms  $\mathcal{M}_{ijk}$  independently of the chosen valleys.



### b. Reflections

The general procedure to perform reflections consists of two operations: a parity transformation (inversion) and a rotation of  $\pi$  around one axis  $\mathbf{n}$  perpendicular to the reflection axis. The spin  $\mathbf{s}$  is invariant under parity and transforms as a spin under the rotation around  $\mathbf{n}$ :

$$R_{\mathbf{n}}(\theta) = \exp(-i\theta \mathbf{n} \cdot \mathbf{s}) = \mathbb{1} \cos\left(\frac{\theta}{2}\right) - i\mathbf{n} \cdot \mathbf{s} \sin\left(\frac{\theta}{2}\right). \quad (\text{A5})$$

We will explicitly derive one reflection and one rotation operator. The procedure for the others is equivalent, and the result is reported in Table III.

$E_1$  reflects with respect to the  $y$  axis; it transforms the spin with a rotation of  $\pi$  around the  $x$  axis:  $R_x(\pi) = -is_1$ . Its action on the momenta sends  $\mathbf{k} = (k_x, k_y) \rightarrow (-k_x, k_y)$ . The valleys in the Brillouin zone are then exchanged. Sublattices transform as  $A \rightarrow B$ ,  $B \rightarrow A$ ,  $C \rightarrow C$ . The quantum operator in the Hilbert space  $\mathbb{H}_{\text{full}}$  is then

$$E_1 = -is_1 \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The low-energy Hamiltonian transforms as

$$E_1 H_{14}(k_x, k_y) E_1^{-1} = H_{36}(-k_x, k_y).$$

### c. Rotations

In this section, we consider rotation around the center of the hexagons (center of order six).

$R_1$  is an anticlockwise rotation of  $\pi/3$ . It exchanges the kagome sublattices as  $A \rightarrow C$ ,  $B \rightarrow A$ ,  $C \rightarrow B$ . Its action on the momenta transforms  $\mathbf{k} = (k_x, k_y) \rightarrow \mathbf{k}' = (\frac{k_x}{2} + \frac{\sqrt{3}}{2}k_y, -\frac{\sqrt{3}}{2}k_x + \frac{k_y}{2})$ . The full operator is given by

$$R_1 = \frac{1}{2}(\sqrt{3}s_0 - is_3) \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Hamiltonian transforms according to

$$R_1 H_{14}(\mathbf{k}) R_1^{-1} = H_{25}(\mathbf{k}').$$

## APPENDIX B: MEAN-FIELD CALCULATIONS IN THE PLAQUETTE PHASES

This Appendix is dedicated to the derivation of the mass terms of Sec. IV C.

### 1. Plaquette phase mass terms

O-type Kekulé distortions of the Kagome lattice generate dimerization patterns which alter the hopping amplitude, forming the so-called plaquettes on the hexagons. As opposed to similar phases in graphene, the hopping distortion does not open a gap in the kagome electronic bands. However, strong coupling within the rings gives rise to charge ordering that has the periodicity of the distortion  $\mathbf{G} = \mathbf{K}_{D1} - \mathbf{K}_{D4}$ . In a mean-field approximation, the Dirac fermions interact with the charge density given by

$$V(\mathbf{r}) = \langle \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \rangle \\ = \text{Re}(\xi) \cos(\mathbf{r} \cdot \mathbf{G}) + \text{Im}(\xi) \sin(\mathbf{r} \cdot \mathbf{G}).$$

The complex number  $\xi$  weights the two contributions and is related to the signs of the periodically distributed charges. For simplicity, we suppose  $\text{Im}(\xi) = 0$  and  $\text{Re}(\xi) = 1$ . The mean-field interaction is diagonal in the lattice sites and reads

$$\delta H = \sum_i \psi_i^\dagger \psi_i V(\mathbf{r}_i), \quad (\text{B1})$$

where  $\psi_i$  is the fermion operator at the lattice site  $\mathbf{r}_i$ . After Fourier transform,

$$\delta H = \frac{1}{2} \sum_i \sum_{\mathbf{k}, \mathbf{k}'} \psi_{\mathbf{k}}^\dagger \psi_{\mathbf{k}'} [e^{i\mathbf{r}_i \cdot (\mathbf{k} - \mathbf{k}' + \mathbf{G})} + e^{i\mathbf{r}_i \cdot (\mathbf{k} - \mathbf{k}' - \mathbf{G})}] \\ = \frac{1}{2} \sum_{\mathbf{k}} [\hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k} - \mathbf{G}} + \hat{A}_{\mathbf{k}}^\dagger \hat{A}_{\mathbf{k} + \mathbf{G}}] \\ + \boxed{\hat{A} \leftrightarrow \hat{B}} + \boxed{\hat{A} \leftrightarrow \hat{C}}. \quad (\text{B2})$$

These are spin-independent, valley-mixing, and sublattice-independent terms. At the Dirac points, they give rise to  $\mathcal{M}_{010}$ . With analogous derivation, one obtains  $\mathcal{M}_{020}$  by setting  $V(\mathbf{r}) = \sin(\mathbf{r} \cdot \mathbf{G})$ .

### 2. Resonating plaquette and short-loop phase mass terms

We derive a mean-field approximation of the Heisenberg Hamiltonian (antiferromagnetic interaction), assuming the spin-ordering observed in the resonating plaquette or in the antiferromagnetic short-loop phases [6]. The approximated Hamiltonian reads

$$\delta H = J \sum_{\langle i, j \rangle} \mathbf{S}_i \cdot \langle \mathbf{S}_j \rangle \\ = J \sum_{\langle i, j \rangle} (\psi_{i, \uparrow}^\dagger \psi_{i, \downarrow}^\dagger) \begin{pmatrix} m(\mathbf{r}_j) & 0 \\ 0 & -m(\mathbf{r}_j) \end{pmatrix} \begin{pmatrix} \psi_{i, \uparrow} \\ \psi_{i, \downarrow} \end{pmatrix}. \quad (\text{B3})$$

The magnetization  $m$  in a specific lattice site  $\mathbf{r}_j$  has periodicity given by  $\mathbf{G} = \mathbf{K}_{D1} - \mathbf{K}_{D4}$  (note the analogy with the plaquette phase):

$$m(\mathbf{r}_j) = \frac{1}{2} \langle \psi_{j, \uparrow}^\dagger \psi_{j, \uparrow} - \psi_{j, \downarrow}^\dagger \psi_{j, \downarrow} \rangle \\ = \text{Re}(\xi) \cos(\mathbf{r} \cdot \mathbf{G}) + \text{Im}(\xi) \sin(\mathbf{r} \cdot \mathbf{G}).$$

We momentarily assume  $m(\mathbf{r}_j) = \cos(\mathbf{r} \cdot \mathbf{G})$ . The mean-field Hamiltonian  $\delta H$  is the sum of a spin-up and spin-down contributions, which matrix element has opposite signs. For simplicity of notation, we derive the spin-up component:

$$\delta H_\uparrow = J \sum_{\langle i, j \rangle} \psi_{i, \uparrow}^\dagger \psi_{i, \uparrow} m(\mathbf{r}_i + \delta_{ij}), \\ = \frac{J}{2} \sum_{\langle i, j \rangle} \sum_{\mathbf{k}, \mathbf{k}'} \psi_{\mathbf{k}, \uparrow}^\dagger \psi_{\mathbf{k}', \uparrow} e^{i\mathbf{r}_i \cdot (\mathbf{k} - \mathbf{k}')} \\ \times [e^{i\mathbf{G} \cdot (\mathbf{r}_i + \delta_{ij})} + e^{-i\mathbf{G} \cdot (\mathbf{r}_i + \delta_{ij})}]$$

TABLE III. Reflection and rotation operators.  $s, \sigma$  are Pauli matrices acting on the spin and valley degrees of freedom, while the third component of the tensor product acts in the actual sublattice space.

|       |                                                                                                                               |       |                                                                                                                              |
|-------|-------------------------------------------------------------------------------------------------------------------------------|-------|------------------------------------------------------------------------------------------------------------------------------|
| $E_1$ | $-is_1 \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$                            | $R_1$ | $\frac{1}{2}(\sqrt{3}s_0 - is_3) \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ |
| $E_2$ | $-\frac{i}{2}(-s_1 + \sqrt{3}s_y) \otimes \sigma_1 \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ | $R_2$ | $\frac{1}{2}(s_0 - i\sqrt{3}s_3) \otimes \sigma_0 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| $E_3$ | $-\frac{i}{2}(-s_1 + \sqrt{3}s_y) \otimes \sigma_1 \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ | $R_3$ | $-is_3 \otimes \sigma_1 \otimes \mathbb{1}_3$                                                                                |
| $E_4$ | $-\frac{i}{2}(s_1 + \sqrt{3}s_y) \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$  | $R_4$ | $\frac{1}{2}(s_0 + i\sqrt{3}s_3) \otimes \sigma_0 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ |
| $E_5$ | $-\frac{i}{2}(-\sqrt{3}s_1 + s_y) \otimes \sigma_0 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ | $R_5$ | $\frac{1}{2}(\sqrt{3}s_0 + is_3) \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ |
| $E_6$ | $-\frac{i}{2}(\sqrt{3}s_1 + s_y) \otimes \sigma_0 \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$  |       |                                                                                                                              |

$$\begin{aligned}
&= \frac{J}{2} \sum_k \hat{A}_{k,\uparrow}^\dagger \hat{A}_{k+\mathbf{G},\uparrow} e^{i\mathbf{G} \cdot (\delta_{AB} + \delta_{AC})} + \text{H.c.} \\
&\quad + [\hat{A} \leftrightarrow \hat{B}] + [\hat{A} \leftrightarrow \hat{C}], \\
&= \frac{J}{2} \sum_k \hat{A}_{k,\uparrow}^\dagger \hat{A}_{k+\mathbf{G},\uparrow} + [\hat{A} \leftrightarrow \hat{B}] + [\hat{A} \leftrightarrow \hat{C}] + \text{H.c.}
\end{aligned}$$

In the last step, we computed contributions from the sublattice vectors:  $e^{i\mathbf{G} \cdot (\delta_{AB} + \delta_{AC})} = e^{i\mathbf{G} \cdot (\delta_{BA} + \delta_{BC})} = e^{i\mathbf{G} \cdot (\delta_{CA} + \delta_{CB})} = 1$ . The derivation of the spin-down contribution  $\delta H_\downarrow$  to the mean-field Hamiltonian is analogous, with magnetization of opposite signs. The complete Hamiltonian  $\delta H = \delta H_\uparrow + \delta H_\downarrow$  is sublattice independent and mixes the two valleys. The spin index can be freely chosen, and the two spin-components have opposite signs. At the Dirac points, it gives rise to  $\mathcal{M}_{j10}$ . Similarly, one obtains  $\mathcal{M}_{j20}$  by assuming a magnetization  $m(\mathbf{r}_j) = \sin(\mathbf{r} \cdot \mathbf{G})$ .

### APPENDIX C: GRAPHENE MASS TERMS

The honeycomb lattice of graphene has two sublattices (A, B). The minimal hopping Hamiltonian for spinless fermions on this structure takes the form

$$H = -t \sum_{\mathbf{r}_i \in A} (\hat{A}_i^\dagger \hat{B}_{i+\delta_1} + \hat{A}_i^\dagger \hat{B}_{i+\delta_2} + \hat{A}_i^\dagger \hat{B}_{i+\delta_3}) + \text{H.c.}, \quad (\text{C1})$$

where  $\hat{A}_i$  is the sublattice fermion operator at site  $\mathbf{r}_i$ . The sublattice vectors are given in terms of the lattice length  $a$ :

$$\delta_1 = \frac{a}{2}(1, \sqrt{3}), \quad \delta_2 = \frac{a}{2}(1, -\sqrt{3}), \quad \delta_3 = a(-1, 0).$$

TABLE IV. Graphene mass terms and their microscopic realization. Indices  $i, j, k$  in the first column label each term according to  $m\mathcal{M}_{ijk} = m s_i \otimes \sigma_j \otimes \tau_k$ . The quantum symmetries of the full massless Hamiltonian (parity, time reversal, charge, and chiral symmetry) are either preserved (✓) or broken (✗) by each term.

| Mass terms in the honeycomb lattice (graphene) |        |       |        |        |                        |           |
|------------------------------------------------|--------|-------|--------|--------|------------------------|-----------|
| Idx $ijk$                                      | Parity | T-rev | Charge | Chiral | Instability            | Reference |
| {0,0,3}                                        | ✗      | ✓     | ✗      | ✗      | Staggered potential    | [13]      |
| {0,1,1}                                        | ✓      | ✓     | ✓      | ✓      | Kekulé dimerization    | [14]      |
| {0,2,1}                                        | ✗      | ✓     | ✓      | ✓      | Kekulé dimerization    |           |
| {0,3,3}                                        | ✓      | ✗     | ✓      | ✗      | Haldane's model        | [15]      |
| {1,0,3}                                        | ✗      | ✗     | ✓      | ✗      | Antiferromagnet: x     | [16]      |
| {2,0,3}                                        | ✗      | ✗     | ✓      | ✗      | Antiferromagnet: y     |           |
| {3,0,3}                                        | ✗      | ✗     | ✓      | ✗      | Antiferromagnet: z     |           |
| {1,1,1}                                        | ✓      | ✗     | ✗      | ✓      | SOC in Kekulé          | [21]      |
| {2,1,1}                                        | ✓      | ✗     | ✗      | ✓      | SOC in Kekulé          |           |
| {3,1,1}                                        | ✓      | ✗     | ✗      | ✓      | SOC in Kekulé          |           |
| {1,2,1}                                        | ✗      | ✗     | ✗      | ✓      | SOC in Kekulé          |           |
| {2,2,1}                                        | ✗      | ✗     | ✗      | ✓      | SOC in Kekulé          |           |
| {3,2,1}                                        | ✗      | ✗     | ✗      | ✓      | SOC in Kekulé          |           |
| {1,3,3}                                        | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: x | [17]      |
| {2,3,3}                                        | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: y |           |
| {3,3,3}                                        | ✓      | ✓     | ✗      | ✗      | Spin-orbit coupling: z |           |

In Fourier space, the Hamiltonian reads

$$H = -t \sum_{\mathbf{k} \in \text{BZ}} \Psi_{\mathbf{k}}^\dagger \begin{pmatrix} 0 & \sum_j e^{ik \cdot \delta_j} \\ \sum_j e^{-ik \cdot \delta_j} & 0 \end{pmatrix} \Psi_{\mathbf{k}}, \quad (\text{C2})$$

with field operator  $\hat{\Psi}_{\mathbf{k}} = (\hat{A}_{\mathbf{k}}, \hat{B}_{\mathbf{k}})^T$ . The Bloch matrix defines two electronic bands that touch at Dirac cones (1/2 filling). As in kagome semimetals, the effective theory at these points describes Dirac, massless quasiparticles [42]. In analogy with the derivation of Sec. II, we can expand the Bloch Hamiltonian around two independent Dirac points (valley degree of freedom) and include the spin independence:

$$\mathcal{H}_{\square}(\mathbf{k}) = v_F(k_x s_0 \otimes \sigma_0 \otimes \tau_1 + k_y s_0 \otimes \sigma_3 \otimes \tau_2). \quad (\text{C3})$$

Here  $s_i$ ,  $\sigma_i$ , and  $\tau_i$  are the Pauli matrices acting on the spin, valley, and sublattice subspace. The Fermi velocity  $v_F = -\frac{3}{2}at$  depends on the lattice length  $a$  and on the hopping amplitude  $t$ . This Hamiltonian and the equivalent for Kagome semimetals [Eq. (7)] enjoy identical symmetries, discussed in the main text.

Because of an equivalent Dirac structure, the gap-opening mechanism described in Sec. III allows us to identify a family of mass terms for graphene, in perfect analogy with Kagome semimetals. However, since the two Dirac Hamiltonians have different explicit forms in terms of Pauli matrices, they give rise to distinct sets of mass terms: the gap-opening effects at the microscopic level are generally distinct in graphene and kagome semimetals. Microscopic realizations of graphene masses have been largely explored [13–18], together with their symmetry classification [12]. A resume of this analysis is presented in Table IV. We refer to Ref. [21] for detailed calculations and discussions.

- [1] M. Kang, S. Fang, L. Ye, H. C. Po, J. Denlinger, C. Jozwiak, A. Bostwick, E. Rotenberg, E. Kaxiras, J. G. Checkelsky *et al.*, Topological flat bands in frustrated kagome lattice CoSn, *Nat. Commun.* **11**, 4004 (2020).
- [2] M. Norman, *Colloquium: Herbertsmithite and the search for the quantum spin liquid*, *Rev. Mod. Phys.* **88**, 041002 (2016).
- [3] L. Savary and L. Balents, Quantum spin liquids: A review, *Rep. Prog. Phys.* **80**, 016502 (2017).
- [4] C. Mielke III, Y. Qin, J.-X. Yin, H. Nakamura, D. Das, K. Guo, R. Khasanov, J. Chang, Z. Wang, S. Jia *et al.*, Nodeless kagome superconductivity in LaRu<sub>3</sub>Si<sub>2</sub>, *Phys. Rev. Mater.* **5**, 034803 (2021).
- [5] H.-M. Guo and M. Franz, Topological insulator on the kagome lattice, *Phys. Rev. B* **80**, 113102 (2009).
- [6] F. Pollmann, K. Roychowdhury, C. Hotta, and K. Penc, Interplay of charge and spin fluctuations of strongly interacting electrons on the kagome lattice, *Phys. Rev. B* **90**, 035118 (2014).
- [7] X. Feng, Y. Zhang, K. Jiang, and J. Hu, Low-energy effective theory and symmetry classification of flux phases on the kagome lattice, *Phys. Rev. B* **104**, 165136 (2021).
- [8] S. M. Young and C. L. Kane, Dirac semimetals in two dimensions, *Phys. Rev. Lett.* **115**, 126803 (2015).
- [9] S. Rao, Weyl semi-metals: A short review, *arXiv:1603.02821*.
- [10] N. Armitage, E. Mele, and A. Vishwanath, Weyl and dirac semimetals in three-dimensional solids, *Rev. Mod. Phys.* **90**, 015001 (2018).
- [11] I. Mazin, H. O. Jeschke, F. Lechermann, H. Lee, M. Fink, R. Thomale, and R. Valentí, Theoretical prediction of a strongly correlated Dirac metal, *Nat. Commun.* **5**, 4261 (2014).
- [12] C. Chamon, C.-Y. Hou, C. Mudry, S. Ryu, and L. Santos, Masses and Majorana fermions in graphene, *Phys. Scr.* **T146**, 014013 (2012).
- [13] G. W. Semenoff, Condensed-matter simulation of a three-dimensional anomaly, *Phys. Rev. Lett.* **53**, 2449 (1984).
- [14] C. Chamon, Solitons in carbon nanotubes, *Phys. Rev. B* **62**, 2806 (2000).
- [15] F. D. M. Haldane, Model for a quantum Hall effect without Landau levels: Condensed-matter realization of the “parity anomaly”, *Phys. Rev. Lett.* **61**, 2015 (1988).
- [16] G. W. Semenoff, Chiral symmetry breaking in graphene, *Phys. Scr.* **T146**, 014016 (2012).
- [17] C. L. Kane and E. J. Mele, Quantum spin Hall effect in graphene, *Phys. Rev. Lett.* **95**, 226801 (2005).
- [18] J.-S. Park and H. J. Choi, Band-gap opening in graphene: A reverse-engineering approach, *Phys. Rev. B* **92**, 045402 (2015).
- [19] R. Ciola, K. Pongsangangan, R. Thomale, and L. Fritz, Chiral symmetry breaking through spontaneous dimerization in kagomé metals, *Phys. Rev. B* **104**, 245138 (2021).
- [20] R. J. Flores, Classifying spaces for wallpaper groups, in *Geometric Group Theory* (Springer, Basel, 2007), pp. 51–64.
- [21] S. Ciceri, Mass generation in graphene and kagome metals, Master’s thesis, Utrecht University, 2023.
- [22] L. Ye, M. Kang, J. Liu, F. von Cube, C. R. Wicker, T. Suzuki *et al.*, Massive Dirac fermions in a ferromagnetic kagome metal, *Nat. Commun.* **555**, 638 (2018).
- [23] Y. Okamura, K. Shoriki, Y. Nomura, Y. Fujishiro, A. Kitaori, N. Kanazawa, R. Arita, Y. Tokura, and Y. Takahashi, Observation of anomalous Hall resonance of massive Dirac fermions in topological kagome-lattice magnet, *npj Quantum Mater.* **8**, 57 (2023).
- [24] Z. Liu, M. Li, Q. Wang, G. Wang, C. Wen, K. Jiang, X. Lu, S. Yan, Y. Huang, D. Shen *et al.*, Orbital-selective Dirac fermions and extremely flat bands in frustrated kagome-lattice metal cosn, *Nat. Commun.* **11**, 4002 (2020).
- [25] D. F. Liu, E. K. Liu, Q. N. Xu, J. L. Shen, Y. W. Li, D. Pei, A. J. Liang, P. Dudin, T. K. Kim, C. Cacho, Y. F. Xu, Y. Sun, L. X. Yang, Z. K. Liu, C. Felser, S. S. P. Parkin, and Y. L. Chen, Direct observation of the spin–orbit coupling effect in magnetic Weyl semimetal Co<sub>3</sub>Sn<sub>2</sub>S<sub>2</sub>, *npj Quantum Mater.* **7**, 11 (2022).
- [26] K. Nakayama, Y. Li, T. Kato, Multiple energy scales and anisotropic energy gap in the charge-density-wave phase of the kagome superconductor CsV<sub>3</sub>Sb<sub>5</sub>, *Phys. Rev. B* **104**, L161112 (2021).
- [27] S. Regmi, T. Fernando, Y. Zhao, A. P. Sakhyia, G. Dhakal, I. Bin Elius, H. Vazquez, J. D. Denlinger, J. Yang, J.-H. Chu *et al.*, Spectroscopic evidence of flat bands in breathing kagome semiconductor Nb<sub>3</sub>I<sub>8</sub>, *Commun. Mater.* **3**, 100 (2022).
- [28] Y. Xie, K. Ji, J. He, X. Shen, D. Wang, and J. Zhang, Manipulation of topology by electric field in breathing kagome lattice, *Phys. Rev. Lett.* (2025), doi:10.1103/29ht-pwyt.
- [29] C. M. Pasco, I. El Baggari, E. Bianco, L. F. Kourkoutis, and T. M. McQueen, Tunable magnetic transition to a singlet ground state in a 2D van der Waals layered trimerized kagomé magnet, *ACS Nano* **13**, 9457 (2019).
- [30] H. Zhang, Z. Shi, Z. Jiang, M. Yang, J. Zhang, Z. Meng, T. Hu, F. Liu, L. Cheng, Y. Xie *et al.*, Topological flat bands in 2D breathing-kagome lattice Nb<sub>3</sub>TeCl<sub>7</sub>, *Adv. Mater.* **35**, 2301790 (2023).
- [31] G. Liu, S.-L. Zhu, S. Jiang, F. Sun, and W. M. Liu, Simulating and detecting the quantum spin Hall effect in the kagome optical lattice, *Phys. Rev. A* **82**, 053605 (2010).
- [32] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Anomalous Hall effect, *Rev. Mod. Phys.* **82**, 1539 (2010).
- [33] Y. Taguchi, Y. Oohara, H. Yoshizawa, N. Nagaosa, and Y. Tokura, Spin chirality, Berry phase, and anomalous Hall effect in a frustrated ferromagnet, *Science* **291**, 2573 (2001).
- [34] J. Ye, Y. B. Kim, A. J. Millis, B. I. Shraiman, P. Majumdar, and Z. Tešanović, Berry phase theory of the anomalous Hall effect: Application to colossal magnetoresistance manganites, *Phys. Rev. Lett.* **83**, 3737 (1999).
- [35] F. D. M. Haldane, Berry curvature on the Fermi surface: Anomalous Hall effect as a topological Fermi-liquid property, *Phys. Rev. Lett.* **93**, 206602 (2004).
- [36] K. K. Kolincio, M. Hirschberger, J. Masell, T. H. Arima, N. Nagaosa, and Y. Tokura, Kagomé lattice promotes chiral spin fluctuations, *Phys. Rev. Lett.* **130**, 136701 (2023).
- [37] Y.-X. Jiang, J.-X. Yin, M. M. Denner, N. Shumiya, B. R. Ortiz, G. Xu, Z. Guguchia, J. He, M. S. Hossain, X. Liu, J. Ruff, L. Kautzsch, S. S. Zhang, G. Chang, I. Belopolski, Q. Zhang, T. A. Cochran, D. Multer, M. Litskevich, Z.-J. Cheng *et al.*, Unconventional chiral charge order in kagome superconductor K<sub>v</sub>3Sb<sub>5</sub>, *Nat. Mater.* **20**, 1353 (2021).
- [38] P. Mendels and F. Bert, Gapless ground state in the archetypal quantum kagome antiferromagnet ZnCu<sub>3</sub>(OH)<sub>6</sub>Cl<sub>2</sub>, *J. Phys. Soc. Jpn.* **79**, 011001 (2010).

- [39] T.-H. Han, J. S. Helton, S. Chu, D. G. Nocera, J. A. Rodriguez-Rivera, C. Broholm, and Y. S. Lee, Fractionalized excitations in the spin-liquid state of a kagome-lattice antiferromagnet, *Nature (London)* **492**, 406 (2012).
- [40] A. Bolens and N. Nagaosa, Topological states on the breathing kagome lattice, *Phys. Rev. B* **99**, 165141 (2019).
- [41] M. Massaro, Strain engineering in graphene and kagomé materials, Master's thesis, Utrecht University, 2023.
- [42] A. H. C. Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, *Rev. Mod. Phys.* **81**, 109 (2009).