

Theory of planar ballistic SNS junctions at $T = 0$ Edouard B. Sonin^{*}*Racah Institute of Physics, Hebrew University of Jerusalem, Givat Ram, Jerusalem 9190401, Israel*

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This Letter presents the theory of planar ballistic superconductor–normal metal–superconductor junctions at $T = 0$ for any normal layer thickness L taking into account phase gradients in superconducting leads. The current-phase relation was derived in the model of the steplike pairing potential analytically and is exact in the limit of large ratio of the Fermi energy to the superconducting gap. At small L (short junction) the obtained current-phase relation is essentially different from that in the previous theory neglecting phase gradients. It was confirmed by recent numerical calculations and was observed in the experiment on short InAs nanowire Josephson junctions. The analysis resolves the problem with the charge conservation law in the steplike pairing potential model.

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Introduction. The ballistic superconductor–normal metal–superconductor (SNS) junction has already been studied more than a half-century. Starting from the pioneer papers [1–3], they used the self-consistent field method [4]. In this method an effective pairing potential is introduced, which transforms the second-quantization Hamiltonian with the electron interaction into an effective Hamiltonian quadratic in creation and annihilation electron operators. The effective Hamiltonian can be diagonalized by the Bogolyubov–Valatin transformation.

The effective Hamiltonian is not gauge invariant, and the theory using it violates the charge conservation law. The charge conservation law is restored if one solves the Bogolyubov–de Gennes equations together with the self-consistency equation for the pairing potential.

The widely accepted existing theory [1–3,5–8], which is the background of the present work, introduced some concepts and used some assumptions simplifying the analysis:

(1) Starting from the original paper of Andreev [5], instead of solving the self-consistency equation, it was assumed that there is a gap Δ of constant modulus $\Delta_0 = |\Delta|$ in the superconducting layers and zero gap inside the normal layer. This means that the proximity effect is ignored. Further we call it the steplike pairing potential model.

(2) The effective masses and Fermi velocities were assumed to be the same in all layers. At this assumption, the normal scattering is absent, and only the Andreev scattering is possible at the superconductor–normal metal interface.

(3) Another outcome of the assumption of equal masses and Fermi velocities was that the original second-order differential Bogolyubov–de Gennes equations are reduced to the first-order equations with boundary conditions imposed only on the wave function, but not on its derivative.

(4) It was assumed that not only the absolute value Δ_0 but also phases were constant in superconducting leads (no phase gradients). Constant phases in two leads, however, were different, as shown in Fig. 1(a).

Under these assumptions the problem is reduced to quadratures. There is the *ab initio* exact expression for the current through the junction via the sum over all Andreev bound states and the integral over all continuum states. But these sums and integrals are rather complicated for analytical and even numerical calculation because of an oscillating integrand and the necessity to calculate a small difference of large terms. The rather sophisticated formalism of the temperature Green’s function was used [1,2,7].

At the phase profile in Fig. 1(a) the charge conservation law is violated since the current flows only inside the normal layer. But it was believed that the charge conservation law can be restored by so small phase gradients in leads that this cannot affect calculations ignoring the gradients. This suggestion is true for a weak link, inside which the phase varies much faster than in the leads. But planar SNS junctions are not weak links at zero temperature [9], and phase gradients in the leads do affect the current in the normal layer. Thus, one should determine currents in the normal and superconducting layers self-consistently.

In Refs. [9,10] the assumption of constant phases in superconducting leads was revised for planar SNS junctions, in which transverse cross sections of all layers are the same. We shall use the adjective “planar” even for one-dimensional (1D) wires when the plane cross section becomes a point. The motivation was an intention to restore the charge conservation law WITHIN the steplike pairing potential model instead of postponing this for more sophisticated calculations taking into account the self-consistency equation.

The following gap profile with the constant gradient $\nabla\varphi$ in leads was considered [Fig. 1(b)]:

$$\Delta = \begin{cases} \Delta_0 e^{i\theta_0 + i\nabla\varphi x}, & x > L/2 \\ 0, & -L/2 < x < L/2, \\ \Delta_0 e^{i\nabla\varphi x} & x < -L/2. \end{cases} \quad (1)$$

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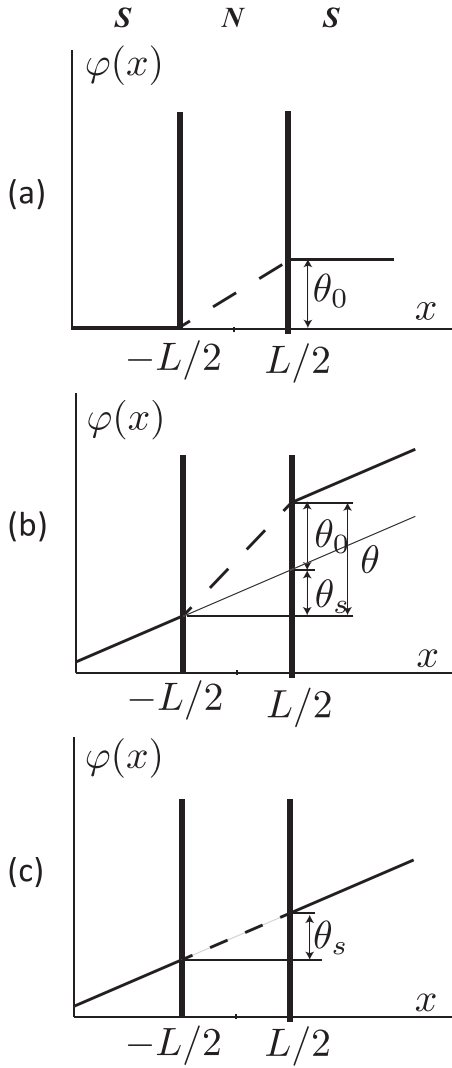


FIG. 1. The phase variation across the SNS junction. (a) The vacuum current produced by the vacuum phase θ_0 . The current is confined to the normal layer. (b) The superposition of the vacuum current and the condensate current determined by the superfluid phase $\theta_s = L\nabla\varphi$. The phase $\theta = \theta_0 + \theta_s$ is the Josephson phase. (c) The condensate current produced by the phase gradient $\nabla\varphi$ in the superconducting layers. In all layers the electric current is equal to env_s .

The phase profile is determined by two phases θ_0 and $\theta_s = L\nabla\varphi$. There is a mathematically correct exact analytical solution of the Bogolyubov–de Gennes equations for any choice of θ_0 and θ_s . But we filter these solutions by the requirement of the charge conservation law. The strict conservation law was replaced by a softer condition that, at least, total currents deep in all layers are the same. The state with the phase gradients in leads [Fig. 1(b)] can be obtained from the state without gradients [Fig. 1(a)] by the Galilean transformation. Galilean invariance of the ballistic SNS junction despite broken translational invariance has been already noticed by Bardeen and Johnson [3] and confirmed in Ref. [9] in the steplike pairing potential model. In Ref. [10] the Galilean invariance was demonstrated for an arbitrary gap profile under the conditions that the ratio of Δ_0 to the Fermi energy is small and

the Andreev reflection is the only mechanism of scattering. The Galilean transformation of the ground state ($\theta_s = \theta_0 = 0$) yields the state with the phase profile shown in Fig. 1(c). In this state the total current J in all layers is equal to the same current J_s as in a uniform superconductor:

$$J = J_s = env_s = J_0 \frac{\theta_s}{\pi} = J_0 \frac{\theta}{\pi}, \quad (2)$$

where

$$J_0 = \frac{\pi en\hbar}{2mL} = \frac{e\hbar k_f}{mL}, \quad (3)$$

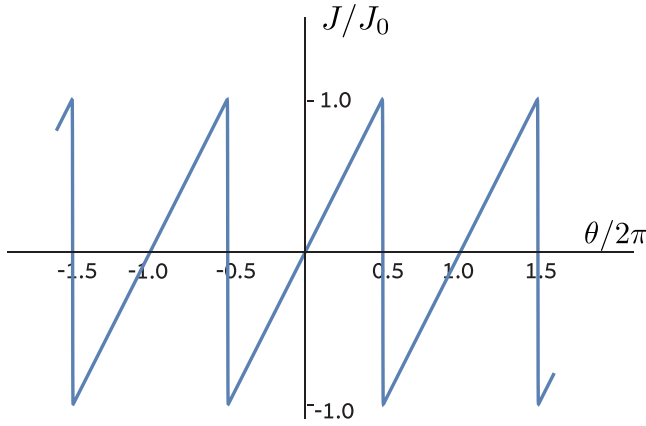
e is the electron charge, m is the electron mass, n is the electron density, $v_s = \frac{\hbar}{2m}\nabla\varphi$ is the superfluid velocity, and k_f is the 1D Fermi wave number. The phase θ_s and J_s were called the superfluid phase and the condensate current, respectively [9]. Since the Galilean invariance takes place for any gap variation in space, the linear current-phase relation (CPR), Eq. (2), is valid beyond the steplike pairing potential model, and the more accurate approach taking into account the self-consistency equation cannot modify it. This agrees with numerical calculations by Riedel *et al.* [11]. They found that although the pairing potential amplitude smoothly varied across interfaces between layers, the phase gradient remained strictly constant along the whole junction as in Fig. 1(c). Recent numerical calculations by Krekels *et al.* [12] also confirmed this (see further discussion of this work in the end of the Letter). But nobody paid attention that these numerical calculations contradict the phase profile with constant phases in leads [Fig. 1(a)] assumed in the existing theory.

The phase θ_0 and the current J_v produced by this phase were called the vacuum phase and the vacuum current, respectively [9]. The current J_v is calculated with all Andreev levels being unoccupied. Since the current J_v flows only in the normal layer and violates the charge conservation law, it should be compensated by the current produced by quasi-particles at Andreev levels. It was called excitation current J_q [9]. At zero temperature the excitation current appears at the critical current determined by the Landau criterion that the energy of the lowest Andreev level reaches 0. The CPR is derived from the condition that the total current flowing only in the normal layer vanishes: $J_v + J_q = 0$.

Summarizing, contrary to the existing theory, which connected the total current only with the vacuum current, the condensate current can flow through the planar SNS junction without violation of the charge conservation law at the same phase profile as in a uniform superconductor. However, despite the essential difference in the physical picture of the charge transport through the junction, at zero temperature both theories predicted in the limit $L \rightarrow \infty$ (long junction) the same sawtooth CPR (Fig. 2) given by Eq. (2) at $-\pi < \theta < \pi$. Coincidence of the two CPRs in the limit $L \rightarrow \infty$ led to the wrong conclusion that there is no difference between the condensate and the vacuum currents (see discussion in Refs. [13,14]).

Up to now the analysis [9,10] was focused on long junctions with L much larger than the coherence length

$$\zeta_0 = \frac{\hbar^2 k_f}{m\Delta_0}. \quad (4)$$

FIG. 2. Sawtooth CPR ($T = 0, L \rightarrow \infty$).

The present Letter addresses the whole diapason of L down to $L = 0$. Remarkably, at zero temperature it is possible to obtain a simple CPR analytically for any L . The difference between the two theories grows with decreasing L and is maximal at $L = 0$. In the limit $L = 0$ the normal layer disappears, and the SNS junction becomes a uniform superconductor. The theory neglecting phase gradients contradicts this evident assertion.

The analysis in the Letter mostly addresses the 1D case (single quantum channel). For the extension of the calculation on multidimensional [two-dimensional (2D) and three-dimensional (3D)] systems currents are integrated over transverse wave-vector components.

Bogolyubov–de Gennes equations, Galilean invariance, and the energies of Andreev levels. The Bogolyubov–de Gennes equations for the wave function

$$\psi(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \quad (5)$$

with the energy ε are

$$\begin{aligned} \varepsilon u &= -\frac{\hbar^2}{2m}(\nabla^2 + k_f^2)u + \Delta v, \\ \varepsilon v &= \frac{\hbar^2}{2m}(\nabla^2 + k_f^2)v + \Delta^* u. \end{aligned} \quad (6)$$

As in the previous investigations, it is assumed that the Fermi energy $\varepsilon_f = \frac{\hbar^2 k_f^2}{2m}$ is much larger than the gap Δ_0 . Then only Andreev reflection is possible, and there is no significant change of the quasiparticle momentum after reflection. Wave functions are superpositions of plane waves with wave numbers only close to either $+k_f$, or $-k_f$. These plane waves describe quasiparticles, which will be called rightmovers (+) and leftmovers (−). After transformation of the wave function,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} e^{\pm i k_f x}, \quad (7)$$

the second-order terms in gradients, $\nabla^2 \tilde{u}$ and $\nabla^2 \tilde{v}$, can be neglected for small Δ_0/ε_f , and the second-order Bogolyubov–de Gennes equations are reduced to the equations of the first

order in gradients [5]:

$$\begin{aligned} \varepsilon \tilde{u} &= \mp \frac{i\hbar^2 k_f}{m} \nabla \tilde{u} + \Delta \tilde{v}, \\ \varepsilon \tilde{v} &= \pm \frac{i\hbar^2 k_f}{m} \nabla \tilde{v} + \Delta^* \tilde{u}. \end{aligned} \quad (8)$$

Let us demonstrate Galilean invariance of the Bogolyubov–de Gennes equations when the wave functions are superpositions of only rightmovers, or only of leftmovers [10]. Suppose that we found the Bogolyubov–de Gennes function $\begin{pmatrix} \tilde{u}_0 \\ \tilde{v}_0 \end{pmatrix}$ with the energy ε_0 for an arbitrary profile of the superconducting gap $\Delta(x)$. One can check that the wave function

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \tilde{u}_0 e^{i\nabla\varphi x/2} \\ \tilde{v}_0 e^{-i\nabla\varphi x/2} \end{pmatrix} \quad (9)$$

with constant gradient $\nabla\varphi$ satisfies Eq. (8) where the gap $\Delta(x)$ is replaced by $\Delta(x)e^{i\nabla\varphi x}$ and the energy is

$$\varepsilon = \varepsilon_0 \pm \frac{\hbar^2 k_f}{2m} \nabla\varphi = \varepsilon_0 \pm \hbar k_f v_s = \varepsilon_0 \pm \frac{\hbar^2 k_f}{2mL} \theta_s. \quad (10)$$

Thus, the Galilean transformation produces the same Doppler shift in the energy as in a uniform superconductor. The Galilean transformation transforms the current j_i in any i th state (either bound or continuum) to the current $j_i + en_i v_s$, where n_i is the density in the i th state. Summation over all bound and continuum states yields that the Galilean transformation added the same condensate current $J_s = en v_s$ in all layers (since the density n is the same in all layers), which does not violate the charge conservation law. Our derivation is valid for any profile of the gap in the space and remains valid if Δ vanishes in some part of the space.

In the absence of phase gradients, the energies of Andreev levels are known from numerous previous publications. They are determined by the equation (see, e.g., Bardeen and Johnson [3], or Eq. (2) of Thuneberg [7])

$$\varepsilon_{0\pm}(s, \theta_0) = \frac{\hbar^2 k_f}{2mL} \left[2\pi s + 2 \arccos \frac{\varepsilon_{0\pm}(s, \theta_0)}{\Delta_0} \pm \theta_0 \right], \quad (11)$$

where s is an integer. For a moving condensate, the energy

$$\varepsilon_{\pm}(s, \theta) = \frac{\hbar^2 k_f}{2mL} \left[2\pi s + 2 \arccos \frac{\varepsilon_{0\pm}(s, \theta_0)}{\Delta_0} \pm \theta \right] \quad (12)$$

differs from the energy $\varepsilon_{0\pm}(s, \theta_0)$ by the Doppler shift.

For our derivation of the CPR, only the lowest Andreev level of leftmovers (electrons moving in the direction opposite to the current direction) with the energy $\varepsilon_{0-}(0, \theta_0)$ in the frame moving with the condensate is needed:

$$\varepsilon_0(\theta_0) = \frac{\hbar^2 k_f}{2mL} \left[2 \arccos \frac{\varepsilon_0(\theta_0)}{\Delta_0} - \theta_0 \right]. \quad (13)$$

The energy of this level in the laboratory frame is

$$\varepsilon(\theta) = \frac{\hbar^2 k_f}{2mL} \left[2 \arccos \frac{\varepsilon_0(\theta_0)}{\Delta_0} - \theta \right]. \quad (14)$$

In the limit $L \rightarrow 0$, the $s = 0$ Andreev level of leftmovers is the only Andreev level with the energy [15]

$$\varepsilon_0(\theta_0) = \Delta_0 \cos \frac{\theta_0}{2}. \quad (15)$$

The current in the Andreev state is determined by the canonical relation connecting it with the derivative of the energy with respect to the phase:

$$j_{\pm}(s, \theta_0) = \frac{2e}{\hbar} \frac{\partial \varepsilon_{0\pm}(s, \theta_0)}{\partial \theta_0}. \quad (16)$$

The factor 2 takes into account that θ_0 is the phase of a Cooper pair but not of a single electron.

The current $j_{\pm}(s)$ is a current produced by a quasiparticle created at the s th state. In unoccupied Andreev states the vacuum current is two times less and has a sign opposite to the sign of $j_{\pm}(s)$ [10]. At $L \rightarrow 0$ the vacuum current in the Andreev state with the energy Eq. (15) is

$$j_v(\theta_0) = -\frac{e}{\hbar} \frac{\partial \varepsilon_0(\theta_0)}{\partial \theta_0} = \frac{e\Delta_0}{2\hbar} \sin \frac{\theta_0}{2}. \quad (17)$$

Current-phase relation. At Josephson phase θ smaller than the critical value θ_{cr} (see below) the condensate current is the only current in all layers, and the CPR is given by Eq. (2).

The phase θ_{cr} is determined from the Landau criterion: at $\theta = \theta_{cr}$ the energy of the lowest Andreev level $\varepsilon(\theta_0)$ in the laboratory coordinate frame reaches zero. At $\theta > \theta_{cr}$ a nonzero vacuum current appears since $\theta_0 \neq 0$. The CPR must be derived from the condition $\varepsilon(\theta_0) = 0$, which allows occupation of the lowest Andreev level. At $\varepsilon(\theta_0) = 0$ Eqs. (10) and (14) give relations

$$\varepsilon_0(\theta_0) - \frac{\hbar^2 k_f}{2mL} \theta_s = 0, \quad \arccos \frac{\varepsilon_0(\theta_0)}{\Delta_0} = \frac{\theta}{2}. \quad (18)$$

Together with Eq. (2) connecting the total current J with θ_s , this yields the exact CPR

$$J = J_{cr} \cos \frac{\theta}{2}, \quad (19)$$

where

$$J_{cr} = \frac{2e\Delta_0}{\pi \hbar} \quad (20)$$

is the critical current in the superconducting leads determined from the Landau criterion (depairing current).

The critical phase θ_{cr} is determined by the condition that the two CPRs [Eqs. (2) and (19)] give the same current:

$$\theta_{cr} - \frac{2mL\Delta_0}{\hbar^2 k_f} \cos \frac{\theta_{cr}}{2} = \theta_{cr} - \frac{2L}{\xi_0} \cos \frac{\theta_{cr}}{2} = 0. \quad (21)$$

The part of the CPR at $\theta < \theta_{cr}$ can be called condensate current branch. In this branch the condensate current is the only current through the junction ($\theta_0 = 0$, $\theta = \theta_s$), and the phase distribution is the same as in a uniform superconductor [Fig. 1(c)]. At $\theta > \theta_{cr}$ the vacuum current and the excitation currents appear ($\theta_0 \neq 0$ and $\theta_s \neq \theta$), but their sum vanishes, as required by the charge conservation law. Along this branch the phase slip occurs when the phase difference across the junction loses 2π . So, the branch can be called phase slip branch. The CPRs for 1D junctions are shown for $L = 0$ in Fig. 3(a) (solid line) and for $L/\xi_0 = 1/2$ in Fig. 3(b) (curve 1).

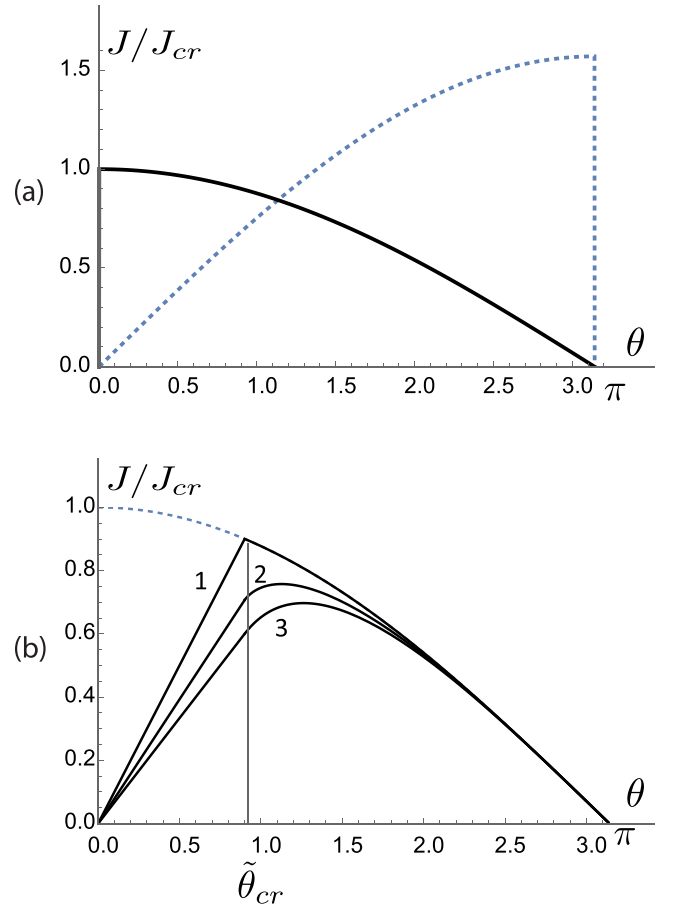


FIG. 3. Current-phase relations at $T = 0$. (a) $L = 0$. The solid line shows the current-phase relation valid for any dimensionality of the junction. The current-phase relation in the theory neglecting phase gradients in leads is shown by the dashed line. (b) $L = \xi/2$. The curves 1, 2, and 3 are the current-phase relations for 1D, 2D, and 3D junctions, respectively. In the 1D case the length $\tilde{\xi}$ coincides with ξ_0 .

The condensate current branch at zero phase in Fig. 3(a) is vertical because at $L = 0$ the junction becomes a uniform superconductor, in which there are no phase jumps at currents less than critical. The phase branch at $L = 0$ describes a phase slip in a uniform superconductor.

When the normal layer thickness L grows the slope of the condensate current branch and the Josephson critical current (the maximal current across the junction) decrease, while the critical phase θ_{cr} approaches π . Eventually, the CPR is described by the sawtooth current-phase curve (Fig. 2) for a very long junction with the small Josephson critical current $J_0 \propto 1/L$.

For derivation of the CPR it was not necessary to make complicated calculations of sums and integrals determining the vacuum current. Whatever its value is, at $T = 0$ the vacuum current in the phase slip branch is compensated by the excitation current in the partially occupied lowest Andreev level. The values of these currents separately are necessary only if one wants to know how the occupation number of this level varies along the phase slip branch.

For multidimensional systems currents calculated for a single 1D channel must be integrated over wave vectors $\mathbf{k}_\perp$ transverse to the current direction keeping in mind that $k_f = \sqrt{k_F^2 - k_\perp^2}$. Here k_F is the Fermi radius of a multidimensional system. The integration operation is $\int_{-k_F}^{k_F} \frac{dk_\perp}{2\pi} \dots$ in the 2D case and $\int_0^{k_F} \frac{k_\perp dk_\perp}{2\pi} \dots$ in the 3D case. After integration, currents become current densities.

At the condensate current branch the linear CPR remains valid after integration, but one should replace the 1D density n by the 2D or 3D density. The condensate current branch extends up the phase $\tilde{\theta}_{cr}$ determined by the equation

$$\tilde{\theta}_{cr} - \frac{2L}{\tilde{\zeta}} \cos \frac{\tilde{\theta}_{cr}}{2} = 0, \quad (22)$$

which is similar to Eq. (21) for 1D junctions with k_f replaced by k_F and the 1D coherence length ζ_0 [Eq. (4)] replaced by

the coherence length

$$\tilde{\zeta} = \frac{\hbar^2 k_F}{m \Delta_0} \quad (23)$$

for multidimensional junctions. At $\theta = \tilde{\theta}_{cr}$ the transition to the phase slip branch occurs only at the maximal $k_f = k_F$. In all other channels with $k_f < k_F$, the condensate current branch extends up to phases larger than $\tilde{\theta}_{cr}$. Thus, at $\theta > \tilde{\theta}_{cr}$ we have a mixture of channels with the condensate current branch at $k_f < k_c$ and with the phase slip branch at $k_f > k_c$. Here

$$k_c = \frac{2L \cos \frac{\theta}{2}}{\tilde{\zeta} \theta} k_F. \quad (24)$$

Finally, integration over all channels yields

$$\begin{aligned} J &= \frac{2e\Delta_0}{\pi^2 \hbar} \cos \frac{\theta}{2} \int_0^{\sqrt{k_F^2 - k_c^2}} dk + \frac{e\hbar}{\pi^2 mL} \theta \int_{\sqrt{k_F^2 - k_c^2}}^{k_F} \sqrt{k_F^2 - k^2} dk \\ &= J_{cr} \theta \left(\frac{\cos \frac{\theta}{2}}{2\theta} \sqrt{1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{\tilde{\zeta}^2 \theta^2}} + \frac{\tilde{\zeta}}{4L} \arctan \frac{\frac{2L \cos \frac{\theta}{2}}{\tilde{\zeta} \theta}}{\sqrt{1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{\tilde{\zeta}^2 \theta^2}}} \right) \end{aligned} \quad (25)$$

for the 2D junction and

$$J = \frac{e\Delta_0}{\pi^2 \hbar} \cos \frac{\theta}{2} \int_0^{\sqrt{k_F^2 - k_c^2}} k dk + \frac{e\hbar}{2\pi^2 mL} \theta \int_{\sqrt{k_F^2 - k_c^2}}^{k_F} \sqrt{k_F^2 - k^2} 2\pi k dk = J_{cr} \cos \frac{\theta}{2} \left(1 - \frac{4L^2 \cos^2 \frac{\theta}{2}}{3\tilde{\zeta}^2 \theta^2} \right) \quad (26)$$

for the 3D junction. These expressions show that in the limit $L = 0$ the expression for the ratio J/J_{cr} in multidimensional junctions does not differ from that in 1D junctions, and the plot J/J_{cr} vs θ [solid line in Fig. 3(a)] describes the CPR for junctions of any dimensionality. But the critical current given by Eq. (20) for 1D junctions must be replaced by the critical current densities for multidimensional junctions:

$$J_{cr} = \begin{cases} \frac{2e\Delta_0 k_F}{\pi^2 \hbar} & \text{2D case} \\ \frac{e\Delta_0 k_F^2}{2\pi^2 \hbar} & \text{3D case.} \end{cases} \quad (27)$$

The CPRs for 2D and 3D junctions at $L/\tilde{\zeta} = 1/2$ are shown in Fig. 3(b) (curves 2 and 3) together with the CPR for a 1D junction (curve 1). There is a cusp in the 1D CPR in the critical point $\theta = \tilde{\theta}_{cr}$, which is smeared out in the 2D and 3D cases. In 2D and 3D junctions the first derivative (slope) of the current-phase curve is continuous at $\theta = \tilde{\theta}_{cr}$, but the critical point is still nonanalytic with jumps in a higher derivative.

Conclusions and discussion. The Letter demonstrates that at $T = 0$ phase gradients in the superconducting leads commonly ignored in the past are important for planar ballistic SNS junctions for any thickness L of the normal layer. The CPR was derived analytically and is exact in the steplike pairing potential model for the Fermi energy much larger than the gap Δ_0 . A more realistic (but much more complicated) theory taking into account

the self-consistency equation can quantitatively modify the phase slip branch of the CPR at the Josephson phase more than the critical value determined from the Landau criterion. But the condensate current branch at the Josephson phase less than critical value was obtained from the Galilean invariance and is valid beyond the steplike pairing potential model.

The theory taking into account phase gradients in the superconducting leads resolves the problem of charge conservation law within the steplike pairing potential model. This refutes the opinion [7] that the charge conservation law cannot be restored without solving the self-consistency equation.

The difference between the previous theory and ours is especially large for short junctions. One can see this in Fig. 3(a) showing the CPR for $L = 0$. The solid line shows the CPR obtained in the present work. The dashed line shows the CPR obtained ignoring gradients in superconducting leads, i.e., assuming that $\theta_s = 0$ and $\theta = \theta_0$. Within this approach, the total current at $L = 0$ is

$$J = 2j_v(\theta) = \frac{e\Delta_0}{\hbar} \sin \frac{\theta}{2}, \quad (28)$$

where $j_v(\theta)$ is the vacuum current in the single Andreev level [Eq. (17)]. The factor 2 takes into account two spins.

Equation (28) was originally derived by Kulik and Omel'yanchouk [15] for a superconducting bridge between bulk leads. It was widely accepted and discussed in various

papers and reviews [see, e.g., Eq. (7) in Ref. [7], Eq. (177) in Ref. [8], or Eq. (3.3) in Ref. [6]]. The present analysis demonstrates that the CPR Eq. (28) is invalid for planar SNS junctions at $L \rightarrow 0$ [16], when the junction becomes a uniform superconductor.

According to Eq. (28), the critical Josephson current is always at the phase π , i.e., the CPR is forward skewed (the current maximum is located at the phase exceeding the phase $\pi/2$ of the current maximum of the sinusoidal CPR). But at the derivation of Eq. (28) the charge conservation law was violated. According to our analysis repairing this flaw, in short junctions the CPR is backward skewed (the current maximum is at the phase less than $\pi/2$).

In the past the possibility of backward-skewed CPR was revealed in the theory of dirty SNS junctions at temperatures close to critical (see Sec. 2.10 in [17] and Sec. V.A.4 in [18]). It was explained by gap suppression close to the normal layer determined from the Ginzburg-Landau theory. This resulted in a nonlinear phase shift close to the junction. Far from the junction the phase was constant. In contrast, our work revealed the backward-skewed CPR at $T = 0$ in ballistic junctions at constant gap. In this case the charge conservation requires constant phase gradients in superconducting leads.

Spanton *et al.* [19] observed the backward-skewed CPR in short InAs nanowire junctions. They explained this by the effect of Coulomb interaction. According to Refs. [17,18] and this Letter, the backward-skewed CPR is possible also in models without interaction. But our theory is not directly applicable to their experiment on 1D normal bridges between 3D leads, which are not planar junctions.

Krekels *et al.* [12] compared our theory with their numerical calculations using a more accurate theory taking into account the self-consistency equation. They concluded that numerical results qualitatively well agree with the backward-skewed CPR obtained in our analysis [cf. Fig. 3(b) in this Letter and Fig. 3(a) in [12]] and disagree with the theory ignoring phase gradients in leads.

The steplike pairing potential model is a strongly idealized model. Still the continuous interest in it during 60 years is justified. As was noted by Bardeen and Johnson [3] about this model, “it is important to understand physically the nature of the supercurrent flow for such an ideal problem without complications.”

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Data availability. No data were created or analyzed in this study.

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