

Exact steady state of a one-end driven XXZ spin chain with boundary field

Vladislav Popkov^{1,2} and Tomaž Prosen^{1,3}

¹*Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia*

²*Department of Physics, University of Wuppertal, Gausstraße 20, 42119 Wuppertal, Germany*

³*Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia*



(Received 13 May 2026; accepted 17 August 2026; published 23 September 2026)

We find an exact nonequilibrium steady state of an open dissipatively driven XXZ spin- $\frac{1}{2}$ chain with source or sink spin bath at one end and an arbitrarily oriented boundary field at the other end.

DOI: [10.1103/2cq3-66ny](https://doi.org/10.1103/2cq3-66ny)

One of the fascinating discoveries in the field of integrable quantum systems exposed to locally acting dissipation at the boundaries is that many properties of their nonequilibrium steady state (NESS), including the complete NESS density operator itself, can be expressed in matrix product ansatz (MPA) form with an infinite-dimensional auxiliary space [1]. This MPA construction led to further discoveries, even in the field of fully coherent quantum systems (i.e., without dissipation), e.g., of quasilocal charges [2,3]. These complemented the set of strictly local symmetries, previously known from the quantum inverse scattering method [4] and were instrumental in explaining high-temperature spin ballistic transport [2]. The MPA-NESS formalism was originally formulated for a one-dimensional setup which describes a coherent evolution in the bulk and dissipative coupling at both boundaries [3]. However the MPA appears to have a rather straightforward generalization to a mixed, or hybrid, case, where the dissipation acts at one boundary only, and at the other boundary it is substituted with a coherent external field term.

The goal of our short communication is a simple algebraic derivation of the MPA of NESS for a hybrid scenario, where the underlying coherent model is a paradigmatic XXZ spin- $\frac{1}{2}$ Heisenberg Hamiltonian (see below). Our method is straightforwardly generalizable to other Yang-Baxter integrable spin chains with hybrid driving.

We aim at finding a unique NESS of a quantum spin chain with a dissipator at the left end and arbitrary coherent driving (a boundary field) at the other end, described by the Lindblad master equation as follows:

$$\frac{\partial \rho}{\partial t} = -i[H, \rho] + \gamma \mathcal{D}_{\sigma_1^+}[\rho] = \mathcal{L}[\rho],$$

$$H = \frac{1}{2} \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z) + g_N,$$

$$\mathcal{D}_F[\rho] = 2F\rho F^\dagger - \{\rho, F^\dagger F\}. \quad (1)$$

We use $\sigma_n^\pm = (\sigma_n^x \pm i\sigma_n^y)/2$. Here $g_N = \vec{g} \cdot \vec{\sigma}_N$ is an arbitrary field, locally acting at the rightmost spin N of a chain. A similar scenario for the special choice $\vec{g} = (g, 0, 0)$ was considered in a recent paper [5] and treated via a quite technical “isolated defect operator” method of Ref. [1], resting upon the validity of cubic algebraic relations for the Lax operator components. As a result, the origin of some main results of Ref. [5], e.g., of a fundamental recurrence (S.14) in the Supplemental Material of Ref. [5], is difficult to trace, without resorting to symbolic computer-assisted calculus. Within our algebraic approach, the same recurrence, and, in addition, for completely general field $\vec{g}$, appears as a consequence of the fundamental relations for $U_q[SU(2)]$ quantum group generators S_z and $S_\pm$.

The Lax operator of the XXZ spin-chain with dissipation [3,6], in terms of $SU_q(2)$ generators S_z and $S_\pm$, reads

$$L_n = \sum_{\alpha=x,y,z} L^\alpha \sigma_n^\alpha = \begin{pmatrix} [S_z]_q & S_+ \\ S_- & -[S_z]_q \end{pmatrix},$$

$$[S_z]_q = \sum_{k=0}^{\infty} [s-k]_q |k\rangle \langle k|,$$

$$S_+ = \sum_{k=0}^{\infty} [k+1]_q |k\rangle \langle k+1|,$$

$$S_- = \sum_{k=0}^{\infty} [2s-k]_q |k+1\rangle \langle k|,$$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (2)$$

where $s \in \mathbb{C}$ is a representation parameter (will be specified later), and q is related to the XXZ anisotropy parameter Δ by

$$\Delta = \frac{q + q^{-1}}{2} \quad (3)$$

and is real or unimodular in the easy-axis $|\Delta| > 1$ regime or the easy-plane $|\Delta| < 1$ regime, respectively. S_z and $S_\pm$ satisfy canonical commutation relations of $U_q[SU(2)]$, while $|0\rangle$ is the lowest weight vector of the irrep $\langle 0|S_- = 0$.

It follows that L_n satisfies a divergence relation [6]:

$$\left[\frac{1}{2} h_{n,n+1}, L_n L_{n+1} \right] = A_n L_{n+1} - L_n A_{n+1}, \quad (4)$$

$$A_n = \frac{q^{S_z} + q^{-S_z}}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5)$$

where $h_{n,n+1} = \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \frac{q+q^{-1}}{2} \sigma_n^z \sigma_{n+1}^z$ is the density of the XXZ Hamiltonian with the uniaxial anisotropy Δ [see Eq. (3)]. The unique time-independent solution of Eq. (1), $\mathcal{L}[\rho_{\text{NESS}}] = 0$, the so-called dark state, or NESS, is postulated as [1]

$$\rho_{\text{NESS}} \sim \Omega_N \Omega_N^\dagger, \quad \Omega_N = \langle \mathbf{u}_l | L_1 L_2 \dots L_N | \mathbf{u}_r \rangle. \quad (6)$$

Writing $\Omega_N^\dagger = \langle \mathbf{u}_l^* | M_1 M_2 \dots M_N | \mathbf{u}_r^* \rangle$, where $M_n = \sum_\alpha M^\alpha \sigma_n^\alpha$ and $M^\alpha = (L^\alpha)^*$, we rewrite the above as

$$\rho_{\text{NESS}} \sim \langle \mathbf{u}_l | \otimes \langle \mathbf{u}_l^* | \mathbb{L}_1 \dots \mathbb{L}_N | \mathbf{u}_r \rangle \otimes | \mathbf{u}_r^* \rangle, \quad (7)$$

$$\mathbb{L}_n = L_n \otimes_a M_n,$$

where $\otimes_a$ denotes the tensor product in the auxiliary space while the usual matrix product is taken in the physical space.

From Eq. (4), we obtain a divergence condition for M_n ,

$$\left[\frac{1}{2} h_{n,n+1}, M_n M_{n+1} \right] = B_n M_{n+1} - M_n B_{n+1}, \quad (8)$$

$$B_n = -A_n^*,$$

which leads to a divergence condition for $\mathbb{L}_n$,

$$\left[\frac{1}{2} h_{n,n+1}, \mathbb{L}_n \mathbb{L}_{n+1} \right] = \mathbb{A}_n \mathbb{L}_{n+1} - \mathbb{L}_n \mathbb{A}_{n+1}, \quad (9)$$

$$\mathbb{A}_n = A_n M_n + L_n B_n,$$

and consequently to

$$\left[\frac{1}{2} \sum_{n=1}^{N-1} h_{n,n+1} + g_N, \mathbb{L}_1 \dots \mathbb{L}_N \right] = \mathbb{A}_1 \otimes \mathbb{L}^{\otimes N-1} - \mathbb{L}^{\otimes N-1} \otimes (\mathbb{A}_N - [g_N, \mathbb{L}_N]).$$

The dark state condition $\mathcal{L}[\rho_{\text{NESS}}] = 0$ for open XXZ Hamiltonian with the arbitrary right boundary field g_N is satisfied via imposing two local conditions, for the left boundary and for the right boundary:

$$\langle \mathbf{u}_l | \otimes \langle \mathbf{u}_l^* | (-i\mathbb{A}_1 + \gamma \mathcal{D}_{\sigma_1^+} [\mathbb{L}_1]) = 0, \quad (10)$$

$$(\mathbb{A}_N - [g_N, \mathbb{L}_N]) | \mathbf{u}_r \rangle \otimes | \mathbf{u}_r^* \rangle = 0. \quad (11)$$

Equation (10) is satisfied by setting $\langle \mathbf{u}_l | = \langle 0 |$ (the lowest weight state in the auxiliary space) and an appropriate choice of the representation parameter s in Eq. (2) tuned to dissipation strength γ via

$$\gamma = i \frac{q^s + q^{-s}}{2[s]_q} \quad (12)$$

(see Refs. [3,6]). For the isotropic limit $\Delta = 1$, or $q = 1$, Eq. (12) reduces to $\gamma = i/s$. For treating Eq. (11), we first

calculate, summing over repeated indices a, b, c, d , and e ,

$$\begin{aligned} [g, \mathbb{L}] &= [g_a \sigma^a, L^b \sigma^b M^c \sigma^c] = g_a L^b M^c [\sigma^a, i\epsilon_{bcd} \sigma^d] \\ &= -2g_a L^b M^c \epsilon_{bcd} \epsilon_{ade} \sigma^e = 2g_a L^b M^c (\delta_{ab} \delta_{ce} - \delta_{be} \delta_{ca}) \sigma^e \\ &= 2(\vec{g} \cdot \vec{L}) M - 2L(\vec{g} \cdot \vec{M}) \end{aligned}$$

(a crucial point in the calculation is L_n being traceless in the physical space). Accounting for Eqs. (8) and (9), and using the Hermiticity of the field term g_N , Eq. (11) reduces to

$$\left(A - 2 \sum_{\alpha=x,y,z} g_\alpha L^\alpha \right) | \mathbf{u}_r \rangle = 0, \quad (13)$$

which is a three-point recurrence relation for the coefficients of $| \mathbf{u}_r \rangle = \sum_{j=0}^\infty u_j | j \rangle$. However, since $\langle \mathbf{u}_l | = \langle 0 |$, only the coefficients u_j with $j \leq N$ give nonzero contribution to NESS (6). Explicitly, denoting $g_\pm = g_x \pm i g_y$ and $a_j = \frac{1}{2}(q^{s-j} + q^{j-s})$, the recurrence relation (13) for the components reads

$$\begin{aligned} \left(-\frac{a_j}{2} + [s-j]_q g_z \right) u_j + \frac{g_-}{2} [2s-j+1]_q u_{j-1} \\ + \frac{g_+}{2} [j+1]_q u_{j+1} = 0, \quad j = 0, 1, 2, \dots, \\ u_{-1} = 0, \quad u_0 = 1. \end{aligned} \quad (14)$$

The condition $u_{-1} = 0$ is needed for a formal validity of Eq. (14) for $j = 0$ and $u_0 = 1$ can be chosen without losing generality.

The recurrence (14) is always well defined, except for the zero field $\vec{g} = 0$ or the field along the z axis, $\vec{g} = (0, 0, g_z)$. In the latter case the assumption $u_0 = 1$ becomes invalid, and the $\mathbf{u}_r$ components should be chosen as $u_j = \delta_{j,N}$: the corresponding $\Omega_N \sim (\sigma^+)^{\otimes N}$ and the unique solution of Eq. (1) is the vacuum pure state $\rho_{\text{NESS}} = |\uparrow\uparrow\dots\uparrow\rangle\langle\uparrow\uparrow\dots\uparrow|$. The bracketed expression in Eq. (13) for this special case becomes a diagonal operator, while the pure NESS follows from the Yamamoto criterion [7]: the state $|\uparrow\uparrow\dots\uparrow\rangle$ is an eigenstate of both the Hamiltonian and of the quantum jump operator σ_1^+ in Eq. (1).

We conclude by summarizing our main result.

The unique NESS of Eq. (1), $\mathcal{L}[\rho_{\text{NESS}}] = 0$ with $g_N = g_+ \sigma_N^- + g_- \sigma_N^+ + g_z \sigma_N^z$, is given by

$$\begin{aligned} \rho_{\text{NESS}} &= \frac{\Omega_N \Omega_N^\dagger}{\text{tr}(\Omega_N \Omega_N^\dagger)}, \\ \Omega_N &= \langle 0 | L_1 L_2 \dots L_N | \mathbf{u}_r \rangle, \\ | \mathbf{u}_r \rangle &= \sum_{j=0}^\infty u_j | j \rangle, \end{aligned} \quad (15)$$

with L_n from Eq. (2), s from Eq. (12), q from Eq. (3), and coefficients u_j given by the linear recurrence (14).

Remark 1. The NESS of a modified Liouvillian (1) with boundary field g_N removed and another boundary dissipator added, $\mathcal{L}[\rho] \rightarrow \mathcal{L}[\rho] + \gamma \mathcal{D}_{\sigma_N^-}[\rho]$, is given by the same expression (15) with $| \mathbf{u}_r \rangle = | 0 \rangle$.

Remark 2. Replacing the left dissipator $\mathcal{D}_{\sigma_1^+}$ in Eq. (1) by $\mathcal{D}_{\sigma_1^-}$, one should transpose the Lax operator in Eq. (2). All the calculations are then performed in full analogy. In that case, and for a special choice of the boundary field, $\{g_x, g_y, g_z\} = \{\Omega/2, 0, 0\}$, one recovers Eq. (S.14) from the Supplemental Material of Ref. [5], and Eq. (6) is equivalent to Eq. (4) of Ref. [5]. It would be interesting to investigate how a dissipative phase transition, proposed in Ref. [5], and derived there for weak dissipation, is affected by the inclusion of a longitudinal z component in the boundary field. This study is out of our present scope.

V.P. and T.P. acknowledge support by ERC Advanced Grant No. 101096208–QUEST and by Research Program P1-0402 and Grant No. N1-0368 of the Slovenian Research and Innovation Agency (ARIS). V.P. is also supported by the Deutsche Forschungsgemeinschaft through DFG Project No. KL645/20-2.

Data availability. There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

-
- [1] T. Prosen, Exact nonequilibrium steady state of a strongly driven open XXZ chain, *Phys. Rev. Lett.* **107**, 137201 (2011).
 - [2] T. Prosen and E. Ilievski, Families of quasilocal conservation laws and quantum spin transport, *Phys. Rev. Lett.* **111**, 057203 (2013).
 - [3] T. Prosen, Matrix product solutions of boundary driven quantum chains, *J. Phys. A: Math. Theor.* **48**, 373001 (2015).
 - [4] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University, Cambridge, England, 1997).
 - [5] M. Yao, A. Lingenfelter, R. Belyansky, D. Roberts, and A. A. Clerk, Hidden time-reversal in driven XXZ spin chains: Exact solutions and new dissipative phase transitions, *Phys. Rev. Lett.* **134**, 130404 (2025).
 - [6] D. Karevski, V. Popkov, and G. M. Schütz, Exact matrix product solution for the boundary-driven Lindblad XXZ chain, *Phys. Rev. Lett.* **110**, 047201 (2013).
 - [7] N. Yamamoto, Parametrization of the feedback Hamiltonian realizing a pure steady state, *Phys. Rev. A* **72**, 024104 (2005).