

Fabry-Pérot quasinormal modes for topological edge statesMarc Martí Sabaté¹ and Benjamin Vial²*Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom*

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Topological waveguides supporting quantum valley Hall edge states confine waves to interfaces and, due to topological protection, are resistant to backscattering even in the presence of defects. These topological insulators are typically studied by means of an infinite spectral problem. However, practical implementations are necessarily finite. In this work, we propose an alternative framework for analyzing topologically nontrivial states in open, finite systems. Our approach is based on a quasinormal modal expansion method, which directly characterizes the existence and excitation of these modes within the open system. The resulting spectrum is complex and discrete and fully describes the topologically nontrivial states, revealing an analogy of topological mode steering as a dispersive Fabry-Pérot cavity, with a dispersion relation closely related to that of the corresponding infinite (Floquet-Bloch) ribbon problem. Our results illustrate how topologically protected waveguiding can be understood in terms of leaky cavity modes and offers a powerful framework for analyzing finite topological devices.

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A fundamental understanding of wave energy manipulation and channeling underpins advances in electronic properties, acoustic switches, optical devices, vibration control, and filters. The ability to guide, split, and redirect waves between channels—as well as to steer them around sharp bends in a robust and lossless manner—is of significant interest across various fields of wave physics and engineering. To address these challenges, concepts from topological insulators, adapted to Newtonian wave systems, have recently generated considerable interest within the community [1–3]. In particular, topological insulators [4] have inspired devices [5–14] that exhibit robust wave propagation, immune to certain forms of disorder and imperfections [2,15].

One prominent example is the quantum valley Hall effect (QVHE), realized in photonic and phononic crystals by breaking spatial inversion symmetry to create a nontrivial,

valley-dependent band topology [16,17]. At the interface between two such crystals with inverted symmetry, valley-polarized interfacial states emerge within the bulk band gap, characterized by a well-defined valley pseudospin. These states localize energy at the interface, forming modes that exhibit negligible intervalley scattering—even at sharp corners, defects, or the ends of the *finite* valley Hall waveguide placed in an open domain—which leads to strongly suppressed backscattering as a consequence of topological robustness.

In an *infinite* valley Hall waveguide, the edge state spans a continuous range of frequencies within the bulk band gap. However, a *finite* topological waveguide of length L behaves like a Fabry-Pérot resonator for the edge modes [18]. The waveguide boundaries act to quantize the allowed edge-state frequencies. Consequently, the continuous spectrum of the infinite system is replaced by a discrete set of modes in the open finite system [19,20]. These Fabry-Pérot-like topological modes depend strongly on the interface length: they enable compact topological devices, but in practice they confine the existence of protected states to the vicinity of these discrete resonances (eigenfrequencies) [21].

While conventional theoretical approaches rely on infinite periodic structures and Floquet-Bloch theory, practical implementations are inherently finite and subject to open

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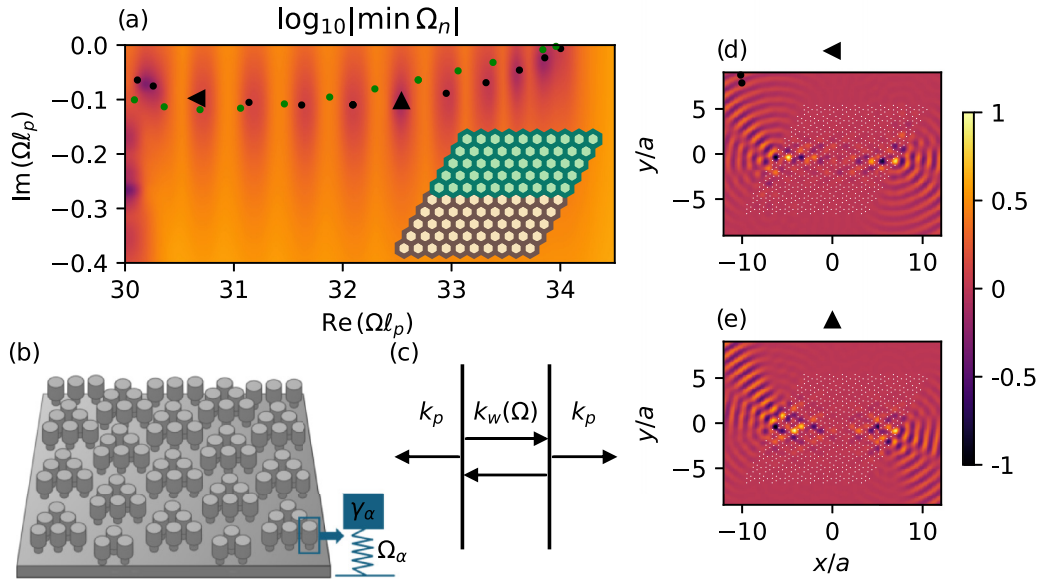


FIG. 1. Spectral properties of the finite cluster. Panel (a) shows the complex spectrum for the finite cluster shown below: green and beige correspond to chiral materials. Panel (b) is a schematic of a thin elastic plate structured with a hexagonal lattice of pillars modeled as pointlike mass-spring resonators. Panel (c) represents the reduced-order model for the QVHE interface state, which consists of a dispersive Fabry-Pérot cavity in an open space. Two of the eigenstates are depicted in panels (d) and (e) (real part of the displacement field); the eigenfield is trapped at the interface between both materials, proving that they are responsible for the interface state.

boundary conditions. Quasinormal modes (QNMs) provide a natural framework for this purpose; their discrete complex spectrum [22,23] describes both oscillation and decay. The quasinormal mode expansion method (QMEM) has been successfully applied to leaky optical cavities and resonators [24] and more recently to elastic plates and metastructures [25]. In the context of topological waveguides, QNMs provide a useful framework for quantifying how edge states form resonances and how topologically protected states exist in the vicinity of these discrete resonances, as evidenced by the Fourier spectrum in Fig. 4.

In this Letter, we use the QMEM to investigate QVHE edge states in a finite phononic crystal waveguide. Our findings reveal that the real part of the discrete complex spectrum associated with the QNMs can be viewed as a discretization of the band structure corresponding to the infinite Floquet-Bloch ribbon problem, and it offers a compelling analogy between the behavior of QNMs and that of a dispersive Fabry-Pérot resonator. Our approach facilitates the investigation of topological physics phenomena directly within finite and open systems. Specifically, we elucidate the discrete spectrum associated with this phenomenon by mapping its characteristics onto a one-dimensional Fabry-Pérot cavity in open space, where the dispersion relation inside the cavity is determined by the eigensolution of the ribbon problem. Furthermore, we show that a superposition of QNMs accurately reconstructs the QVHE in finite lattices, offering an efficient computational approach to solving the multiple-scattering problem.

We consider a two-dimensional phononic crystal composed of a hexagonal lattice of scatterers (pointlike mass-spring resonators) attached to the upper surface of a thin elastic plate as shown in Fig. 1(b). We consider only flexural waves, modeled utilizing Kirchhoff-Love thin plate theory.

Assuming harmonic time dependence, the multiple scattering problem is addressed using a Green's function approach [26,27], with complex resonances identified following the procedure outlined by Vial *et al.* [25] described in detail in the Supplemental Material [28] (see also Refs. [29–33] therein).

The design of the underlying hexagonal cells is illustrated in Fig. 2. Figure 2(a) demonstrates how C_{3v} symmetries can produce symmetry-protected degeneracies at the K point. By breaking the vertical symmetries of a cell—achieved by rotating the triangular arrangement of scatterers—this degeneracy is broken, resulting in two distinct valleys surrounding a band gap.

It is well known that such symmetry breaking [34] produces a topological phase transition [7] occurring between the lower and upper valleys, allowing the band gap (in gray) to be characterized as topologically nontrivial. Similarly, a topological phase transition occurs between chiral pairs of such cells. By rotating the scatterer arrangement positively or negatively to break vertical symmetries, one can create two distinct bulk materials. When these materials are joined, a valley-polarized edge mode arises at their interface within the topologically nontrivial band gap, giving rise to the QVHE.

These QVHE modes are zero-line modes (ZLMs), whose formation is described in Ref. [9]. As expected, when stacking such chiral pairs, two ZLMs exist in the ribbon problem, as shown in Fig. 2(c). These modes are concave up and concave down, corresponding to even and odd ZLMs, respectively, and exist along topologically distinct interfaces [35].

For a finite interface of length L , these ZLMs become quantized into a set of allowed Fabry-Pérot resonances, determined by the band structure of the ZLM permitted to exist at the interface and the value of L . Moreover, a reduced one-dimensional model can be constructed that captures only the propagation of the interface state. This is illustrated in

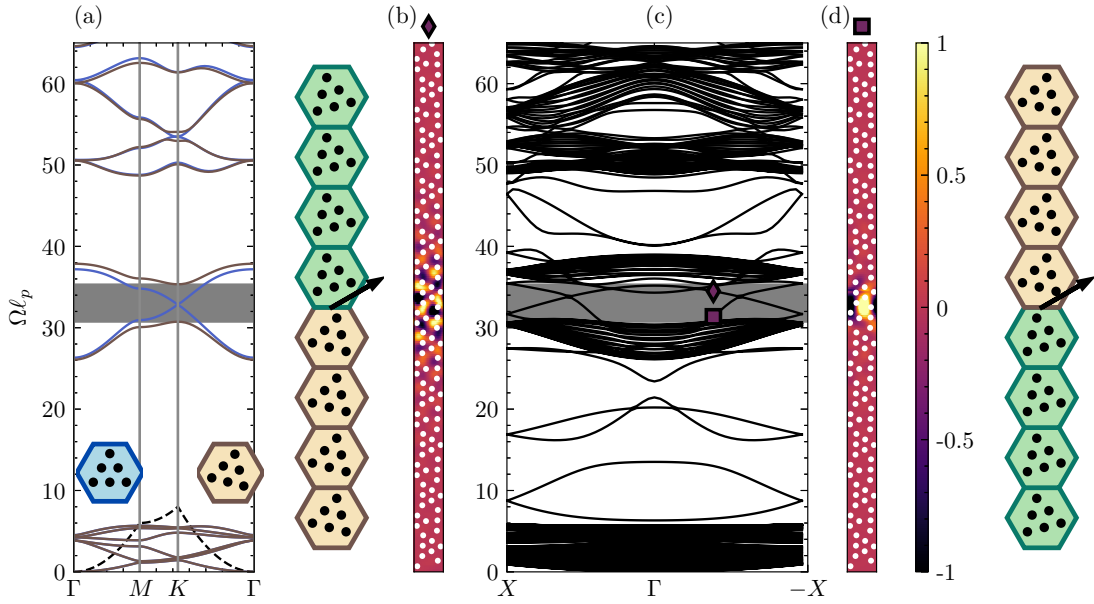


FIG. 2. Spectral features of the infinite system. Panel (a) shows the band structures for the degenerate (blue) unit cell preserving the C_{3v} symmetry and the broken symmetry (brown) unit cell. The gray shaded region corresponds to the topologically nontrivial band gap. Panel (c) shows the dispersion of the ribbon problem schematic at the left; two chiral materials (green and beige) are arranged creating an interface, and the ribbon problem is solved, showing the existence of two edge states in the band gap of the doubly periodic structure, whose eigenstates are shown in panels (b) and (d).

Fig. 1(c): the inset schematic in Fig. 1(b) shows the actual resonator cluster, with distinct colors indicating different chiral media and the interface aligned along the x direction. Figure 1(c) presents the reduced-order model, wherein the topologically nontrivial interface—within the band gap—is effectively described as a slab of dispersive material, forming an analog of a Fabry-Pérot cavity. The eigenfrequencies Ω_m associated with QNMs of a Fabry-Pérot cavity satisfy the following resonance condition [36,37]:

$$k_w(\Omega_m)L = \pi m - i \log \frac{k_p + k_w(\Omega_m)}{k_p - k_w(\Omega_m)}, \quad (1)$$

with k_w and k_p being the wave numbers associated with the waveguide and the bare plate, respectively; L being the length of the cavity; and $m \in \mathbb{Z}$. The dispersive properties of the plate are given by $\Omega = \ell_p c_p \omega_p = \ell_p c_p^2 k_p^2$, with $c_p = \sqrt{\rho h/D}$, where ℓ_p , ω_p , ρ , h , and D are the lattice parameter for the hexagonal unit cell, the angular frequency, the density, the thickness, and the bending stiffness of the plate [$D = Eh^3/12(1 - \nu^2)$, where E is the Young's modulus and ν is the Poisson ratio], respectively. Ω is a normalized angular frequency variable [26]. To estimate the dispersion relation of the waveguide [$k_w(\Omega)$], it is necessary to compute the spectral properties of the corresponding infinite system. The existence of the QVHE edge state is proven numerically in Fig. 3. For a line source positioned at the green spot in Figs. 3(e) and 3(f), the scattering field Φ is recovered using direct and modal computations. For the latter, only the QNMs found in Fig. 1(a) have been considered. Given the good agreement between both scattering fields, we claim that the QNMs found in the previous analysis are sufficient for describing the behavior of the topologically nontrivial edge state. Figure 3(a) represents

the moduli of the excitation coefficient for each QNM, given by the following expression:

$$\Phi = \sum_n \frac{u(\Omega_n)}{u(\Omega)} \frac{\Phi_n \cdot \Psi^i}{\Omega - \Omega_n} \Phi_n = \sum_n b_n(\Omega) \Phi_n, \quad (2)$$

where Ψ^i is the incident field and $u(\Omega)$ is an arbitrary function, in which $u = 1/k^3$ has been tested numerically to work optimally [25]. The excitations are Lorentzian-shaped curves centered at $\text{Re}(\Omega_n)$ and their width is inversely proportional to the quality factor of the leaky mode $Q_n = \text{Re}(\Omega_n)/2|\text{Im}(\Omega_n)|$.

Within the topologically nontrivial band gap, wave propagation is confined to the interface between the two chiral materials, as bulk propagation is prohibited. Consequently, wave transport in this frequency range can be effectively treated as a one-dimensional problem. The propagation along the waveguide is dispersive and governed by the concave-down edge state, as shown in Fig. 2(c). At both ends of the finite cluster, we assume an interface with the unstructured (bare) plate, effectively embedding the waveguide within an open homogeneous medium.

Under this assumption, and following Eq. (1), the real part of the logarithmic term can be neglected, leading to an approximately equispaced distribution of wave numbers along the waveguide. By projecting these wave numbers k_w onto the ribbon dispersion diagram, we observe that the axis is sampled at L/a discrete values, where a denotes the ribbon period and L the total length of the waveguide. This subdivision is illustrated in Fig. 4(a), where red vertical segments indicate the sampled wave numbers for a waveguide of length $L = 10a$. Green points in the same panel represent the real part of the Fourier transform of the QNM eigenfields along the interface.

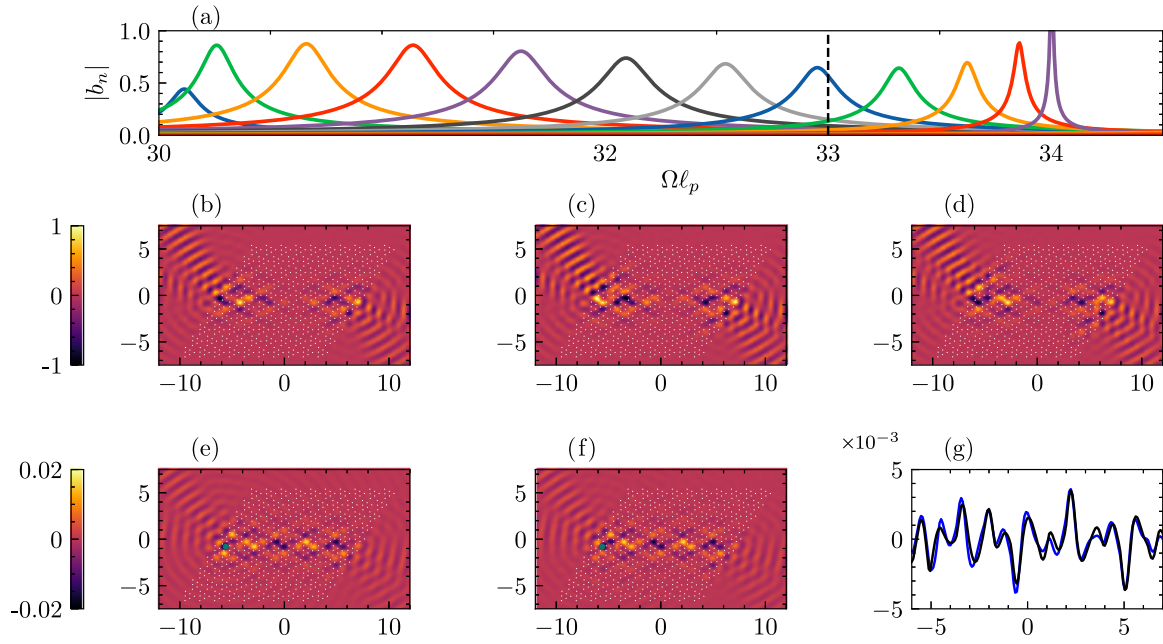


FIG. 3. Scattering simulation. Panel (a) shows the evolution in frequency of the excitation coefficients b_n , defined by Eq. (2). Each QNM is excited in a bell-shape manner, centered at the real frequency of the QNM and with a width inversely proportional to its Q factor. Panels (b), (c), and (d) depict three of the QNMs, the ones with bigger excitation coefficients for a pointlike source located at the beginning of the waveguide and frequency $\Omega = 33$. These three modes are mainly the responsible for the edge state that is recovered in the scattering simulations [panels (e) and (f)]. While panel (e) is a direct scattering computation solving the inverse of the multiple scattering matrix, panel (f) shows the reconstruction of the real part of the scattering field using a linear superposition of the three QNMs depicted above [panels (b), (c), and (d)]. Finally, panel (g) shows the scattering field at the interface between both chiral media, where blue represents the modal reconstruction, while black is the direct computation for the scattering field.

The dispersion of the QNMs aligns closely with that of the concave-down edge state derived from the ribbon configuration. Furthermore, the colormap in Fig. 4(a) displays the absolute value of the Fourier transform of the scattered field along the interface, reaffirming that the observed dispersion closely follows the curve dictated by the ribbon dispersion. The wave numbers obtained from the Fourier transform of the QNMs (green dots) do not align precisely with the red vertical segments; this discrepancy arises because the segments correspond to the ideal case of an edge state exhibiting a parabolic relation—analogueous to that of a homogeneous medium.

To improve the estimation of the QNMs using the dispersive Fabry-Pérot model, the dispersion relation of the concave-down edge state has been fitted and incorporated into Eq. (1). Specifically, the relation $k_w(\Omega) = \arccos[(\Omega - \Omega_1)/\Omega_0]/p$ is employed, where the parameters Ω_0 , Ω_1 , and p have been obtained through numerical fitting. The resulting QNMs are presented in Fig. 1(a) as green dots in the complex frequency plane, and their real part is plotted in Fig. 4(b). The mean relative error between the QNMs and their estimation using the dispersive Fabry-Pérot model [black and green points in Fig. 1(a), respectively] is 0.44%. Discrepancies between QNMs and Fabry-Pérot predictions in Fig. 1(a) arise from the geometry of our analyzed cluster, which is not a square, leading to an interface between the cluster and the continuum (bare plate) that is not perpendicular to the direction of the waveguide. Furthermore, finite-size effects coming from the boundaries in the direction perpendicular to the interface might also be expected.

By changing the length of the waveguide, we can change the distance and position of the QNMs of the cluster, according to Eq. (1). Figure 4(b) shows the evolution of the real frequency of the QNMs for different waveguide lengths L . The length L is a multiple of the length of the ribbon, so that the number of QNMs in a given finite structure will be given by the ratio between the length of the waveguide and the length of the ribbon. It is clearly seen in the picture that the number of QNMs increases with the length of the waveguide, while the frequency separation between QNMs diminishes. In the limit where L tends to ∞ , we will get QNMs accumulating in the regions where the ribbon dispersion is almost flat ($\Omega = 34$); that is to say, the density of states of the truncated system will converge to the infinite one.

Numerical studies indicate that the first few QNMs accurately approximate the wave field inside the crystalline structure, but the rate of convergence depends on where the series is evaluated in space; in particular, convergence can be slow near material interfaces [38] or in the intermediate region between the far-field and the collection of scatterers. A key caveat is the spatial region of validity: because QNMs satisfy outgoing (radiation) boundary conditions, their eigenfields typically diverge in the far field for complex eigenfrequencies with $\text{Im}(\Omega_n) < 0$. Consequently, the QNM expansion is expected to be most accurate within the structured material or collection of scatterers considered. The exterior and far-field evaluations generally require many more modes to obtain converged results. In our formulation, the QNMs are obtained from the complex eigenfrequencies of the homo-

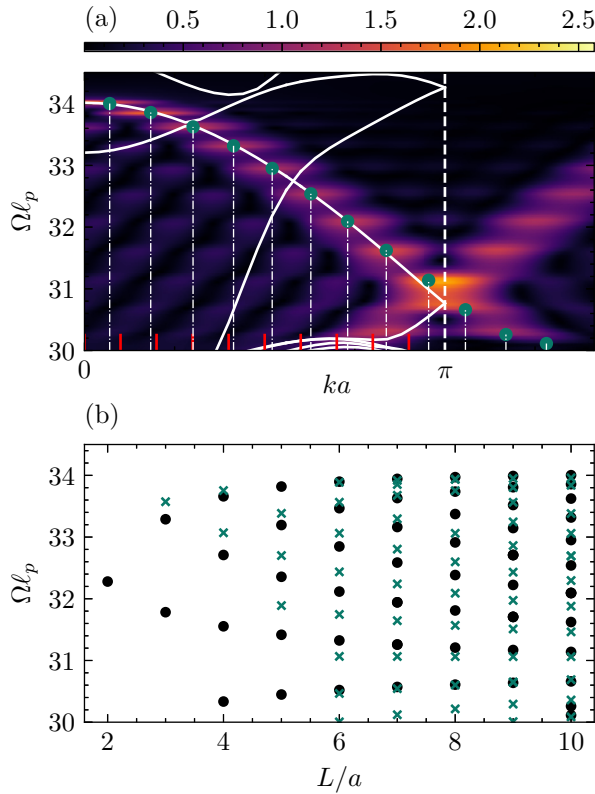


FIG. 4. Fabry-Pérot cavity analog. Panel (a) shows the superposition between the ribbon dispersion, the Fourier transform of the scattering field at the interface between both materials (colormap), and the real part of the Fourier transform of the QNMs (black crosses). Panel (b) shows the evolution in the distribution of the QNMs of the QVHE waveguide as a function of the length of the waveguide. The number of solutions and their real frequency depends on the length, following the equations of a Fabry-Pérot cavity.

geneous multiple-scattering problem, and the reconstructed field is built from the associated Green's function representation (involving Hankel functions of the first kind and first

order), whose far-field asymptotics contain factors of the form $\exp(i\Omega_n r)$ with complex Ω_n , leading to exponential spatial growth for leaky modes. Therefore, the QNM expansion converges most rapidly in and near the crystal, its convergence degrades with distance, and it may converge very slowly—or even fail to converge—in the far field. In practice, we assess convergence by increasing the number of modes retained in Eq. (2) and verifying that the reconstructed field within the chosen finite computational domain changes negligibly.

In summary, applying the QMEM to a finite phononic topological waveguide supporting valley Hall edge states establishes a simple reduced-order model for describing the complex spectra associated with topological metamaterials. The resulting QNMs are discrete, and the real part of these can be viewed as a sampling of the band structure associated with the dispersion of the infinite system (the Floquet-Bloch ribbon problem), analogous to a Fabry-Pérot cavity containing a dispersive material. These results bridge the gap between topological band theory and cavity-resonator physics and can guide the design of compact topological devices—such as filters, sensors, or lasers—that leverage both topological robustness and resonant field enhancement.

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Data availability. The data are not publicly available. The data are available from the authors upon reasonable request.

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