

Continuous topological phase transition between $\mathbb{Z}_2$ topologically ordered phases

Qi Zhang¹ and Wen-Tao Xu^{2,3,*}

¹*Beijing National Laboratory for Condensed Matter Physics and Institute of Physics,
Chinese Academy of Sciences, Beijing 100190, China*

²*Technical University of Munich, TUM School of Natural Sciences, Physics Department, 85748 Garching, Germany*

³*Munich Center for Quantum Science and Technology (MCQST), Schellingstrasse 4, 80799 München, Germany*



(Received 16 August 2025; revised 30 October 2025; accepted 12 November 2025; published 8 December 2025)

Topological phase transitions beyond anyon condensation remain poorly understood. A notable example is the transition between the toric code (TC) and double semion (DS) phases, which has two distinct $\mathbb{Z}_2$ topological orders in $(2+1)$ dimensions $[(2+1)\text{D}]$. Previous studies reveal that the transition between them can be either first order or via an intermediate phase, thus the existence of a direct continuous transition between them remains a long-standing problem. Motivated by the fact that both phases can arise from condensing distinct anyons in the $\mathbb{Z}_4$ topological order, we introduce a perturbed $\mathbb{Z}_4$ quantum double (QD) model to study the TC-DS transition. We confirm the existence of a continuous $(2+1)\text{D}$ XY* transition between the TC and DS phases by mapping it to a two-coupled quantum Ising model. Importantly, using the condensation order parameters and the area law coefficients of the Wilson loops, we further reveal that $\mathbb{Z}_4$ anyons, fractionalized from the $\mathbb{Z}_2$ topological orders, become deconfined at the transition between $\mathbb{Z}_2$ topologically ordered phases. Our results open a path toward developing a theoretical framework for topological phase transitions beyond anyon condensation.

DOI: [10.1103/1bjs-mf2l](https://doi.org/10.1103/1bjs-mf2l)

Introduction. Topological phases of matter, first discovered in the fractional quantum Hall effect [1,2], have revolutionized our understanding of quantum many-body systems. Owing to their potential applications in quantum error correction and fault-tolerant quantum computing [3,4], topological phases have become a central focus of both theoretical and experimental research. Exactly solvable models with topologically ordered ground states have been constructed to study these phases [4–6], and some of them have been experimentally realized using quantum processors and quantum simulators [7–10]. As phases of matter beyond the conventional Ginzburg-Landau-Wilson (GLW) paradigm of spontaneous symmetry breaking, topologically ordered states pose significant challenges for studying their phase transitions within the framework of the GLW paradigm. Nonetheless, topological phase transitions driven by anyon condensation can be partially understood using the GLW paradigm, because they can be mapped, via duality transformations, to spontaneous symmetry-breaking phase transitions characterized by local order parameters [11–16]. Such topological phase transitions have been extensively studied using various models and methods [14,15,17–28].

However, topological phase transitions that are *not* induced by anyon condensation remain poorly understood. A notable

example is the transition between the toric code (TC) [4] and double semion (DS) phases [6], which represent two distinct $\mathbb{Z}_2$ topological orders with incompatible anyons. By interpolating the fixed-point TC and DS string-net models using a one-parameter Hamiltonian, the transition has been studied using exact diagonalization [29] and quantum Monte Carlo [30,31], the former method indicates a first-order transition, while the latter method suggests the existence of an intermediate gapless phase, as shown in Fig. 1(a). Additionally, a fine-tuned transition between the TC and DS phases can be realized using deformed projected entangled pair states (PEPS) with suitable tuning parameters [20,22,32,33]. Furthermore, a field-theoretic argument has been proposed for a continuous transition between the TC and DS phases in the presence of $\text{SU}(2)$ symmetry [34]. Despite these efforts, how to concretely realize a direct continuous transition between the TC and DS phases using a generic Hamiltonian remains a long-standing open question. If such a continuous transition exists, the nature of the critical point, particularly how the incompatible anyons of the two $\mathbb{Z}_2$ topological orders reconcile at the critical point, remains unclear.

In this letter, we confirm the existence of a direct continuous TC-DS transition and provide insights into its nature. Rather than considering a model interpolating directly between fixed points of the TC and DS phases [29–31], we propose a perturbed $\mathbb{Z}_4$ quantum double (QD) model based on the observation that both phases can emerge from the $\mathbb{Z}_4$ topological order via anyon condensation [20,22,35]. To determine its phase diagram and the quantum criticality of the continuous TC-DS transition, we introduce a duality transformation that maps the perturbed $\mathbb{Z}_4$ QD model to a two-coupled $(2+1)$ -dimensional $[(2+1)\text{D}]$ quantum Ising

*Contact author: wen-tao.xu@tum.de

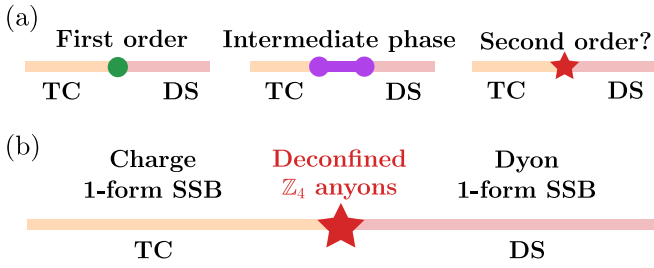


FIG. 1. Schematic illustration of the motivation and main results. (a) Previous works [29–31] reveal that the transition between the TC and DS phases can be first order or via an intermediate phase. The existence of a direct continuous transition between them is an open question. (b) We confirm a direct continuous TC-DS transition point, where the adjacent TC and DS phases spontaneously break two different 1-form symmetries. We further reveal deconfined $\mathbb{Z}_4$ anyons at their transition point.

model, and approximate its ground states using variationally optimized infinite PEPS (iPEPS) [36,37] with automatic differentiation [38].

Using the condensation order parameters and the area law coefficients of the Wilson loop expectation values, we identify that the continuous TC-DS transition belongs to the XY^* universality class. Interestingly, as shown in Fig. 1(b), the continuous TC-DS transition resembles a deconfined quantum critical point [39,40]—a direct continuous transition between two phases that spontaneously break different symmetries and exhibit fractionalized excitations—in the following aspects: First, the perturbed $\mathbb{Z}_4$ QD model has two distinct 1-form symmetries, which are looplike generalized symmetries. One 1-form symmetry is broken spontaneously in the TC phase, while the other one is broken spontaneously in the DS phase. Second, some anyons in the $\mathbb{Z}_2$ topological orders are fractionalized and become deconfined at the continuous TC-DS transition point. Our results suggest that, at the critical point between topological phases that cannot be connected via anyon condensation, fractionalized anyons that are confined in both adjacent phases can emerge. This opens a path toward developing a general theoretical framework for topological phase transitions beyond anyon condensation.

A perturbed $\mathbb{Z}_4$ quantum double model. Because the model interpolating between the fixed-point Hamiltonians of the TC and DS phases [29,30] is difficult to analyze analytically and simulate numerically, we alternatively propose a perturbed $\mathbb{Z}_4$ QD model to realize the topological phase transition between the TC and DS phases [41]. The fixed-point Hamiltonian for the $\mathbb{Z}_4$ QD phase on the square lattice is [4]

$$\mathcal{H}_{\text{QD}} = - \left(\sum_v A_v + \sum_p B_p + \text{H.c.} \right), \quad (1)$$

where the vertex and plaquette operators, A_v and B_p , are illustrated in Fig. 2(a), and are defined using the $\mathbb{Z}_4$ qudit operators: $X = \sum_{n=0}^3 |n\rangle \langle (n+1) \bmod 4|$, $Z = \sum_{n=0}^3 i^n |n\rangle \langle n|$. Electric charges $\{e, e^2, e^3\}$ correspond to the eigenvalues $\{i, -1, -i\}$ of A_v , while magnetic fluxes $\{m, m^2, m^3\}$ correspond to the same eigenvalues of B_p . Their composites, $e^n m^k$ ($n, k \in \{1, 2, 3\}$), are known as dyons. Notice that charges

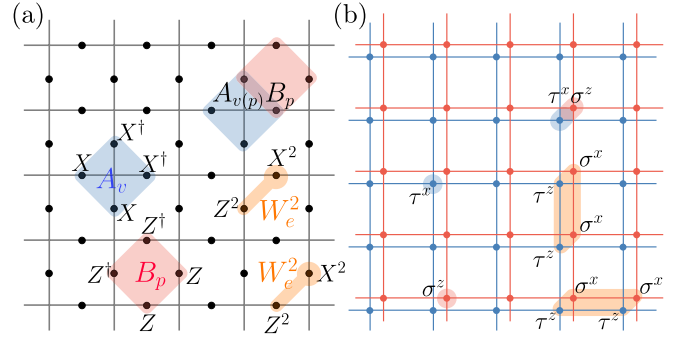


FIG. 2. Perturbed $\mathbb{Z}_4$ QD model and duality transformation. (a) The multisite local Hamiltonian terms of the perturbed $\mathbb{Z}_4$ QD model, where e of W_e^2 is the edge where the X^2 operator is located. (b) The perturbed $\mathbb{Z}_4$ QD model is mapped to a two-coupled Ising model, where the blue (red) lattice is the primal (dual) lattice, and we shift the dual lattice to overlap with the primal lattice.

(fluxes) are located on the primal (dual) lattice, and we adopt the convention in Ref. [35] that dyons live on the dual lattice. Further details on the anyon content of the $\mathbb{Z}_2$ topological order are provided in the Supplemental Material (SM) [42].

According to anyon condensation theory [43–46], the $\mathbb{Z}_4$ QD model can undergo phase transitions to the TC phase by condensing the charge e^2 [47], and to the DS phase by condensing the dyon $e^2 m^2$ [20,22]. Microscopically, the transition to the TC phase can be realized by a Hamiltonian $\mathcal{H}_{\text{QD}} - h_z \sum_e Z_e^2$ with $h_z \gtrsim 0.657$ [48]. In contrast, dyon condensation is less straightforward. Interestingly, the fixed-point DS model can be constructed using $\mathbb{Z}_4$ qudit operators [35],

$$\mathcal{H}_{\text{DS}} = - \sum_p (A_{v(p)} B_p + A_{v(p)} B_p^\dagger + \text{H.c.}) - \sum_e W_e^2, \quad (2)$$

where $v(p)$ denotes the vertex in the southwest of and adjacent to the plaquette p [see Fig. 2(a)]. The first term penalizes *semionic* dyon excitations $e^n m^k$ with odd n, k [49]. The last term $W_e^2 = X_e^2 Z_{e'}^2$ induces the condensation of the *bosonic* dyon $e^2 m^2$, where e' denotes the nearest-neighboring edge in the southwest of the edge e .

By combining $\mathcal{H}_{\text{QD}}$, $\mathcal{H}_{\text{DS}}$, and the perturbation terms $-\sum_e X_e^2$ and $-\sum_e Z_e^2$, we propose the following perturbed $\mathbb{Z}_4$ QD model to study topological phase transitions between the TC and DS phases:

$$\begin{aligned} H_{\text{QD}} = & - \left(\sum_v A_v + \sum_p B_p + \text{H.c.} \right) - h_x \sum_e X_e^2 - h_z \sum_e Z_e^2 \\ & - \frac{1}{2} \sum_p (A_{v(p)} B_p + A_{v(p)} B_p^\dagger + \text{H.c.}) - h_w \sum_e W_e^2. \end{aligned} \quad (3)$$

The Hamiltonian possesses an extensive set of conserved quantities: $[A_v^2, H_{\text{QD}}] = 0$ for all vertices, and $[B_p^2, H_{\text{QD}}] = 0$ for all plaquettes. Products of these operators generate the so-called 1-form symmetries [50–52], represented by the non-local loop operators $\prod_{e \in C} Z_e^2$, $\prod_{e \in \hat{C}} W_e^2$, $\prod_{e \in \hat{C}} X_e^2$, where C and $\hat{C}$ denote loops on the original and dual lattices, respectively. Since different topological orders can be understood

TABLE I. Condensation order parameters and area law coefficients. S ($\hat{S}$) denotes a semi-infinite long string on the primal (dual) lattice, C ($\hat{C}$) denote the a loop on the primal (dual) lattice, R denotes the set of sites surround by the loop C ($\hat{C}$), and a_R is the area of R . The column “Def. in H_{QD} ” lists the definition of the condensation order parameters and area law coefficients in the $\mathbb{Z}_4$ QD model, while “Def. in H_{dual} ” for the two-coupled Ising model. The columns “ $\mathbb{Z}_4$ QD,” “DS,” “TC,” and “Trivial” show values of these quantities in the respective phases.

	Label	Def. in H_{QD}	Def. in H_{dual}	$\mathbb{Z}_4$ QD	DS	TC	Trivial
Condensation	$\langle e^2 \rangle$	$ \langle \prod_{e \in S} Z_e^2 \rangle ^{1/2}$	$ \langle \tau_i^z \rangle $	0	0	>0	>0
	$\langle e^2 m^2 \rangle$	$ \langle \prod_{e \in S} W_e^2 \rangle ^{1/2}$	$ \langle \tau_i^z \sigma_i^x \rangle $	0	>0	0	>0
	$\langle m^2 \rangle$	$ \langle \prod_{e \in S} X_e^2 \rangle ^{1/2}$	$ \langle \sigma_i^x \rangle $	0	0	0	>0
	$\alpha(e)$	$\frac{\ln \langle \prod_{e \in C} Z_e^{\gamma(e)} \rangle}{-a_C}$	$\frac{\ln \langle \prod_{i \in R} \sigma_i^z \rangle}{-a_R}$	0	>0	0	>0
Confinement	$\alpha(em)$	$\frac{\ln \langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} Z_e^{\gamma(e')} \rangle}{-a_{\hat{C}}}$	$\frac{\ln \langle \prod_{i \in R} \tau_i^x \sigma_i^z \rangle}{-a_R}$	0	0	>0	>0
	$\alpha(m)$	$\frac{\ln \langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} \rangle}{-a_{\hat{C}}}$	$\frac{\ln \langle \prod_{i \in R} \tau_i^x \rangle}{-a_R}$	0	>0	>0	>0

as distinct patterns of higher-form symmetry breaking [53], the 1-form symmetries can be used to characterize different topological phases.

Condensation order parameters and area law coefficients. We use nonlocal order parameters to detect spontaneous 1-form symmetry breaking, which distinguishes different topological phases. One class of such nonlocal order parameters includes $\langle \prod_{e \in S} Z_e^2 \rangle$, $\langle \prod_{e \in \hat{S}} W_e^2 \rangle$, and $\langle \prod_{e \in \hat{S}} X_e^2 \rangle$ [54,55], where S ($\hat{S}$) denotes the open strings on the original (dual) lattices. They can be viewed as segments of closed loops C ($\hat{C}$), as detailed in SM [42]. Because the anyons e^2 , $e^2 m^2$, and m^2 are created at the end points of the string operators, their nonzero expectation values imply the corresponding anyons condense, as summarized in Table I. We therefore call them condensation order parameters.

Another class of nonlocal order parameters is defined using Wilson loop operators $\langle \prod_{e \in C} Z_e^{\gamma(e)} \rangle$, $\langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} Z_e^{\gamma(e')} \rangle$, and $\langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} \rangle$, where e' denotes the edge immediately southwest of e , and the exponents $\gamma(e)$, $\hat{\gamma}(e)$, taking values 1 or 3, depend on the edge orientations [35,42]. It is well known that the expectation values of the Wilson loop operators can detect the confinement of anyons [56,57]. Let us focus on $\langle \prod_{e \in C} Z_e^{\gamma(e)} \rangle$, where the loop operator creates a pair of anyons e and e^3 and then annihilates them. The expectation value satisfies $\langle \prod_{e \in C} Z_e^{\gamma(e)} \rangle \propto e^{-\alpha(e)a_C}$ for a large loop C that surrounds the area a_C , where $\alpha(e)$ is the area law coefficient. When $\alpha(e) > 0$, it is the area law, indicating that e is confined. If $\alpha(e) = 0$, it implies that $\langle \prod_{e \in C} Z_e^{\gamma(e)} \rangle$ decays at most as the perimeter of the loop C , i.e., the perimeter law, indicating that e is deconfined. Therefore, we can use the area law coefficient $\alpha(e)$ to distinguish the deconfinement and confinement of e . A similar argument applies to $\langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} Z_e^{\gamma(e')} \rangle$ and $\langle \prod_{e \in \hat{C}} X_e^{\hat{\gamma}(e)} \rangle$, which is summarized in Table I.

Mapping to a two-coupled (2 + 1)D quantum Ising model. The 1-form symmetries not only characterize different topological phases but also simplify the perturbed $\mathbb{Z}_4$ QD model in Eq. (4) via a duality transformation. It is well known that the $\mathbb{Z}_2$ lattice gauge theory with the exact 1-form Wilson or 't Hooft loop symmetry can be mapped to the transverse-field Ising model [58] via the Wegner duality transformation [11,56]. In the same spirit, we propose the following duality

transformation for the perturbed $\mathbb{Z}_4$ QD model,

$$A_v + A_v^\dagger \rightarrow 2\tau_v^x, \quad B_p + B_p^\dagger \rightarrow 2\sigma_p^z, \\ X_e^2 \rightarrow \sigma_p^x \sigma_{p'}^x, \quad Z_e^2 \rightarrow \tau_v^z \tau_{v'}^z,$$

where $\{\sigma_p^\alpha\}$ ($\{\tau_v^\alpha\}$) with $\alpha \in \{x, y, z\}$ is a set of Pauli matrices of $\mathbb{Z}_2$ qubits applying on the site of the dual (primal) lattice. As shown in Fig. 2(b), by shifting the dual lattice such that it completely overlaps the primal lattice, and relabeling v , v' and p , p' using i , j , the perturbed $\mathbb{Z}_4$ QD model in Eq. (3) is mapped to a two-coupled (2 + 1)D quantum Ising model:

$$H_{\text{dual}} = -2 \sum_i (\tau_i^x + \sigma_i^z + \tau_i^x \sigma_i^z) \\ - h_x \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - h_z \sum_{\langle ij \rangle} \tau_i^z \tau_j^z \\ - h_w \sum_{\langle ij \rangle} \tau_i^z \tau_j^z \sigma_i^x \sigma_j^x. \quad (4)$$

When taking the boundary condition into account, i.e., consider the system on a torus, the duality mapping becomes $H_{\text{QD}} = \bigoplus_{b_x^\sigma, b_y^\sigma, b_x^\tau, b_y^\tau} \frac{1 + \prod_i \sigma_i^z}{2} \frac{1 + \prod_i \tau_i^x}{2} H_{\text{dual}}(b_x^\sigma, b_y^\sigma, b_x^\tau, b_y^\tau)$, where $b_x^\sigma, b_y^\sigma, b_x^\tau, b_y^\tau$ are boundary conditions in the x and y directions of the σ or τ spins which can be either periodic (+) or antiperiodic (−), and the antiperiodic boundary condition means that the ferromagnetic couplings along some noncontractible loops are changed to the antiferromagnetic couplings. In the SM [42], we further analyze that the symmetry sectors and boundary conditions induced by the duality transformation do not affect the evaluation of order parameters. If the term $\sum_i \tau_i^x \sigma_i^z$ is absent and $h_x = h_z$, it becomes a standard (2 + 1)D quantum Ashkin-Teller model. It is obvious that the phase diagram of H_{dual} is symmetric about $h_x = h_z$, because the model is invariant under the exchange of $\sigma_i^x \leftrightarrow \tau_i^z$, $\sigma_i^z \leftrightarrow \tau_i^x$, and $h_x \leftrightarrow h_z$. Furthermore, its phase diagram is also symmetric about $h_z = h_w$, because $U H_{\text{dual}}(h_x, h_z, h_w) U^\dagger = H_{\text{dual}}(h_x, h_w, h_z)$, where the unitary transformation is $U = \prod_i (\frac{1 + \sigma_i^x}{2} + \tau_i^x \frac{1 - \sigma_i^x}{2})$. Since duality transformations preserve the structure of phase diagrams, we can directly deduce the phase diagram of the $\mathbb{Z}_4$ QD model H_{QD} from that of H_{dual} .

The nonlocal condensation order parameters and the Wilson loop expectation values of the perturbed $\mathbb{Z}_4$ QD model

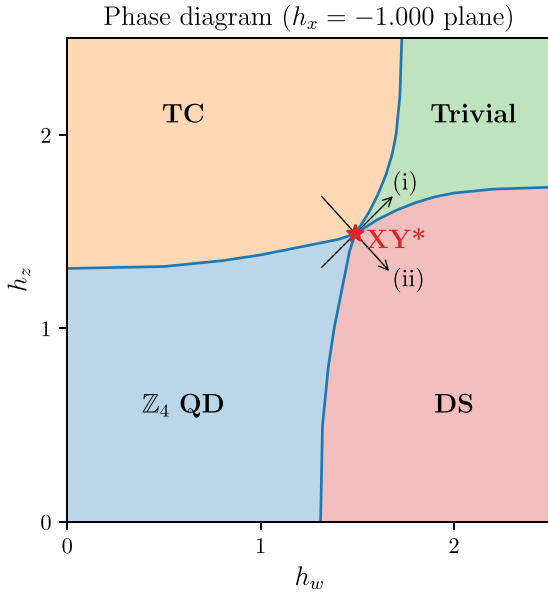


FIG. 3. Phase diagram of the perturbed $\mathbb{Z}_4$ QD model on the $h_x = -1$ plane. The model has three tuning parameters: h_x , h_z , and h_w . The solid blue lines indicate the 3D Ising* transitions. The red star marks the multicritical point, which belongs to the XY* universality class. Arrows (i) and (ii) denote the lines $h_z = h_w$ and $h_w = 2.980 - h_z$, respectively, both crossing the multicritical point.

can be mapped to the local order parameters and the mem-
branelike disorder parameters of the two-coupled quantum

Ising model [59,60] (see SM [42] and Table I). This allows us to extract condensation order parameters and the area law coefficients directly from the ground states of the two-coupled Ising model, which can be approximated using C_{4v} -symmetric iPEPS [36–38]. Compared to other methods such as quantum Monte Carlo, iPEPS is more suitable to evaluate nonlocal order parameters, and the method of extracting area law coefficients from iPEPS is proposed in Ref. [61]. By comparing the numerical results of the condensation order parameters and the area law coefficients with Table I, we obtain the phase diagrams for various h_x planes (see SM [42]), one of which is shown in Fig. 3. It includes the $\mathbb{Z}_4$ QD, TC, DS, and trivial phases, and their phase boundaries are described by the 3D Ising* universality class describing 1-form symmetry-breaking transitions [62]. Next, we focus on the multicritical point, especially the continuous TC-DS transition.

Continuous topological phase transition between the $\mathbb{Z}_4$ QD phase and the trivial phase. By tuning h_x to negative values, we find that a continuous TC-DS phase transition becomes possible. We focus on the $h_x = -1$ plane (Fig. 3), where a continuous TC-DS transition emerges at the junction of four phases: the $\mathbb{Z}_4$ QD, TC, DS, and trivial phases. Since the phase diagram is symmetric about $h_z = h_w$ [line (i) in Fig. 3] and the continuous TC-DS transition must occur at a point on this line, we first analyze the condensation order parameters and area law coefficients along this line.

Figure 4(a) shows that the area law coefficients $\alpha(e)$, $\alpha(em)$, and $\alpha(m)$ are zero when $h_w = h_z < h_c \approx 1.490$, but they are nonzero when $h_w = h_z > h_c$, implying e^n , $e^n m^k$, and

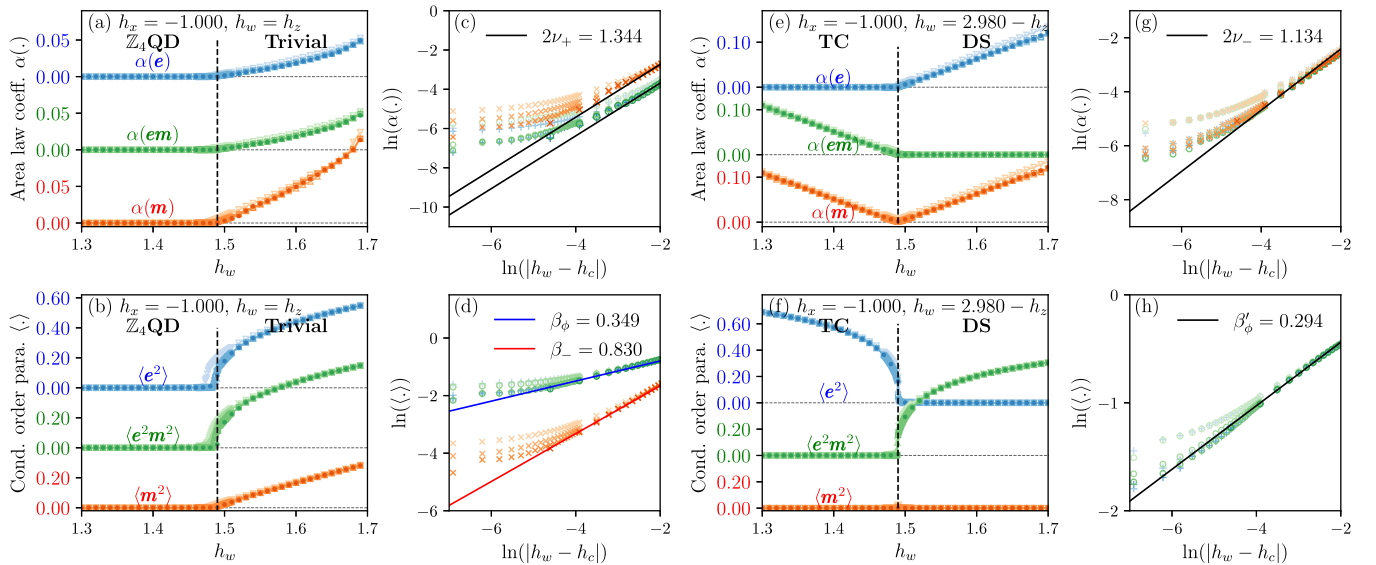


FIG. 4. Condensation order parameters and area law coefficients near and at the continuous TC-DS transition point $(h_x, h_z, h_w) \approx (-1.000, 1.490, 1.490)$. Different colors correspond to different order parameters, and the colors from light to dark correspond to the bond dimensions $(D, \chi) = (4, 100), (5, 100), (6, 100), (7, 100)$, where D is the iPEPS bond dimension and χ is the bond dimension of the corner transfer matrix renormalization group method contracting the iPEPS [63–65]. The vertical dashed line marks the location of the multicritical point ($h_c = 1.490$). From the XY* universality class, we have three order parameter critical exponents β_ϕ , β'_ϕ , β_- and two correlation length critical exponents ν_+ and ν_- . (a) Area law coefficients along the line (i) in Fig. 3. (b) Condensation order parameters along the line (i). (c) Double-log plot of the area law coefficients in (a), where the slope of the straight lines is $2\nu_+$. (d) Double-log plot of the condensation order parameters in (b), where the slopes of the straight lines are β_ϕ and β_- , respectively. (e) Area law coefficients along the line (ii) in Fig. 3. (f) Condensation order parameters along the line (ii). (g) Double-log plot of the area law coefficients in (e), where the slope of the straight lines is $2\nu_-$. (h) Double-log plot of the condensation order parameters in (f), where the slope of the straight lines is β'_ϕ .

$\mathbf{m}^k$ ($n, k \in \{1, 3\}$) are confined. Moreover, Fig. 4(b) shows that the condensation order parameters $\langle \mathbf{e}^2 \rangle$, $\langle \mathbf{e}^2 \mathbf{m}^2 \rangle$, and $\langle \mathbf{m}^2 \rangle$ are zero when $h_w = h_z < h_c$, but they are nonzero when $h_w = h_z > h_c$, implying $\mathbf{e}^2$, $\mathbf{e}^2 \mathbf{m}^2$, and $\mathbf{m}^2$ condense. By comparing the numerical results in Figs. 4(a) and 4(b) with Table I, we conclude the perturbed $\mathbb{Z}_4$ QD model is in the $\mathbb{Z}_4$ QD phase when $h_w = h_z < h_c$ and the trivial phase when $h_w = h_z > h_c$.

Since the Ising* lines meet at the multicritical point, a natural expectation of the effective theory describing the physics near and at the multicritical point is the $O(2)$ ϕ^4 field theory, given by the Lagrangian $\mathcal{L} = (\partial\phi)^2/2 + m^2\phi^2/2 + g\phi^4/(4!) + \dots$, where $\phi = (\phi_0, \phi_1)$ is a two-component order parameter field corresponding to the condensation order parameters $\langle \mathbf{e}^2 \mathbf{m}^2 \rangle$ and $\langle \mathbf{e}^2 \rangle$. From the lattice model perspective, the duality transformation gauges the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the two-coupled Ising model, such that the perturbed $\mathbb{Z}_4$ QD model is obtained. We should also gauge the $\mathbb{Z}_2$ symmetry $\phi_i \leftrightarrow -\phi_i$ in the $O(2)$ ϕ^4 field to get the effective field theory describing the multicritical point of the perturbed $\mathbb{Z}_4$ QD model, and the gauging orbifolds the ϕ_i fields, i.e., ϕ_i can only appear in pairs and ϕ_i and $-\phi_i$ become identical [62]. Therefore, the multicritical point of the two-coupled Ising (perturbed $\mathbb{Z}_4$ quantum double) model corresponds to the Wilson-Fisher fixed point of the ungauged (gauged) $O(2)$ ϕ^4 field theory, namely the 3D XY (XY*) conformal field theory [66,67] (see SM [42] for more discussions on the distinctions between the XY and XY* theories). We can predict critical exponents of the condensation order parameters and the area law coefficients from the field theory [42]. In Fig. 4(c), the area law coefficients match the correlation length critical exponent $\nu_+ = 1/(3 - \Delta_+)$, where Δ_+ is the scaling dimension of the relevant field $\phi^2 = \phi_0^2 + \phi_1^2$ deriving to the gapped phases along $h_w = h_z$. Figure 4(d) shows that the three condensation order parameters exhibit two distinct critical exponents $\beta_\phi = \Delta_\phi \nu_+$ and $\beta_- = \Delta_- \nu_+$, where Δ_ϕ and Δ_- are the scaling dimensions of the fields ϕ_0 (or ϕ_1) and $\phi_0^2 - \phi_1^2$, which correspond to $\langle \mathbf{e}^2 \mathbf{m}^2 \rangle$ (or $\langle \mathbf{e}^2 \rangle$) and $\langle \mathbf{m}^2 \rangle$, separately. The numerical results imply that the multicritical point belongs to the XY* universality class.

Continuous topological phase transition between the toric code and double semion phases. Finally, we investigate the path $h_x = -1, h_w = 2h_c - h_z$ [line (ii) in Fig. 3], which traverses the TC phase, the DS phase, and the multicritical point, and is orthogonal to $h_w = h_z$ [line (i)]. The area law coefficients in Fig. 4(e) indicate that $\mathbf{e}$ and $\mathbf{e}^3$ are confined when $h_w > h_z$ and $\mathbf{e}^n \mathbf{m}^k$ with $n, k \in \{1, 3\}$ are confined when $h_w < h_z$, while $\mathbf{m}$ and $\mathbf{m}^3$ are confined whenever $h_w \neq h_z$. According to the condensation order parameters in Fig. 4(f), $\mathbf{e}^2 \mathbf{m}^2$ condenses when $h_w > h_z$, while $\mathbf{e}^2$ condenses when $h_w < h_z$. By comparing the numerical results in Figs. 4(e) and 4(f) with Table I, we conclude that along the line (ii), the system is in the TC phase when $h_z > h_w$ and in the DS phase when $h_w > h_z$.

As shown in Fig. 4(g), we find that the area law coefficients exhibit a critical exponent consistent with the correlation length exponent $\nu_- = 1/(3 - \Delta_-)$, where Δ_- is the scaling dimension of the relevant field $\phi_0^2 - \phi_1^2$ deriving the multicritical point to the TC or DS phases. Figure 4(h) shows that the condensation order parameter exhibits the critical exponent

$\beta'_\phi = \Delta_\phi \nu_-$. The distinct critical exponents from the lines (i) and (ii) in Fig. 3 further confirm that the multicritical point belongs to the XY* universality class, and a detailed field theory analysis is provided in SM [42].

Interestingly, the continuous TC-DS transition exhibits features that resemble deconfined quantum critical points [39,40]—a direct continuous transition between two phases that spontaneously break different symmetries and exhibit fractionalized excitations—in the following aspects: First, from Fig. 4(f), $\langle \mathbf{e}^2 \rangle = 0$ in the DS phase while $\langle \mathbf{e}^2 \mathbf{m}^2 \rangle = 0$ in the TC phase, implying the gapped TC and DS phases spontaneously break two distinct 1-form symmetries, $\prod_{e \in \mathcal{C}} Z_e^2$ and $\prod_{e \in \hat{\mathcal{C}}} W_e^2$, respectively. Second, the anyons $\mathbf{m}$ and $\mathbf{m}^3$ are confined in both the TC and DS phases, but they are deconfined at the continuous TC-DS transition point, indicated by the area law coefficient $\alpha(\mathbf{m})$ shown in Fig. 4(e); since $\mathbf{m} \times \mathbf{m} = \mathbf{m}^3 \times \mathbf{m}^3 = \mathbf{m}^2$ is a well-defined anyon in both TC and DS phases, $\mathbf{m}$ and $\mathbf{m}^3$ are fractionalized anyons with respect to the $\mathbb{Z}_2$ topological order. A difference between the continuous TC-DS transition point and deconfined quantum critical points is that two broken 1-form symmetries $\prod_{e \in \mathcal{C}} Z_e^2$ and $\prod_{e \in \hat{\mathcal{C}}} W_e^2$ do not have mixed anomalies.

Discussions and outlook. In this letter, we propose a perturbed $\mathbb{Z}_4$ QD model to study the topological phase transitions between the TC and DS phases, from which we find that a direct continuous phase transition exists between these two $\mathbb{Z}_2$ topologically ordered phases. We identify that the continuous TC-DS transition belongs to the XY* universality class, where deconfined $\mathbb{Z}_4$ anyons emerge. Our findings imply that if we consider a model without any explicit $\mathbb{Z}_4$ structure but exhibiting a continuous TC-DS transition, fractionalized anyons can emerge at the transition. We emphasize that the TC-DS transition is not a fine-tuned result, since it is a continuous transition line in the plane defined by $h_x < 0.8$ and $h_z = h_w$, and we further verified this conclusion through the analysis of $O(2)$ field theory [42]. Since the TC and DS phases are dual to two different $\mathbb{Z}_2$ symmetry protected topological (SPT) phases in $(2+1)\text{D}$ [68], the multicritical point can also be interpreted as a topological critical point between two $\mathbb{Z}_2$ SPT phases. We note that a topological phase transition with an exact continuous symmetry can realize a deconfined quantum critical point [69]. It is therefore interesting to consider a model that exhibits the continuous TC-DS transition while maintaining an exact continuous symmetry at the critical point. Our results shed light on constructing a general theoretical framework for continuous topological phase transitions beyond anyon condensation, i.e., considering emergent anyons to reconcile two incompatible topological orders.

Since the TC-DS transition can be simulated on a digital quantum computer [33], it would be interesting to experimentally realize this transition and measure the corresponding anyonic order parameters. In addition, one may consider adding more perturbations such as $-h'_x \sum_e X_e + \text{H.c.}$, $-h'_z \sum_e Z_e + \text{H.c.}$ to the perturbed $\mathbb{Z}_4$ QD model in Eq. (3). These terms explicitly break the exact 1-form symmetries $\prod_{e \in \mathcal{C}} Z_e^2$, $\prod_{e \in \hat{\mathcal{C}}} X_e^2$, hence the duality transformation and nonlocal order parameters no longer work. In such

cases, the Fredenhagen-Marcu order parameter can be used to detect anyon condensation [55,70,71], and an adiabatic approach can be used to investigate confinement [72]. Alternatively, the broken exact 1-form symmetry could be restored via quantum error correction [73].

Acknowledgments. The authors appreciate valuable discussions with L. Wang on numerical computations and support

in providing computational resources. W.-T.X. thanks R.-Z. Huang for helpful comments and acknowledges support from the Munich Quantum Valley, which is supported by the Bavarian state government with funds from the Hightech Agenda Bayern Plus.

Data availability. The data that support the findings of this article are openly available [74].

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