

Umklapp correction to Landau damping and conditions for nontrivial modifications to quantum critical transport

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We compute the particle-hole bubble for an Ising-nematic metal when the critical Fermi surface approaches the Brillouin zone boundary for $d = 2$ dimensions. We find two qualitatively distinct contributions: (i) the standard antipodal piece, which gives $\Pi_{\text{ATP}}(\mathbf{q}, i\Omega) \propto \Omega/q$, and (ii) an additional umklapp piece from electrons near the zone boundary, which gives $\Pi_{\text{U}}(\mathbf{q}, i\Omega) \propto \Omega^\alpha$ at the minimum umklapp momentum $q \approx \Delta_q$, with $\alpha = 2/3$ or $1/2$ depending on the temperature T . At high T when $\alpha = 1/2$, the minimum T for the activation of linear/quasilinear in T resistivity, which is expected to be $T_U \propto \Delta_q^3$ from $z = 3$ criticality, could potentially get reduced to $T_U \propto \Delta_q^4$ due to the $\sqrt{\Omega}$ term. We discuss why due to fermion self-energy effects, we find only one hyperspecific scenario where this possibility might be realized and offer some evidence for the reduction of T_U at the level of a scattering rate estimate for this case. For $d = 3$ the umklapp contribution gives $\Pi_{\text{U}} \sim \Omega$ irrespective of T ; therefore T_U is not modified in this case.

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I. INTRODUCTION

There is a peculiar aspect baked into Landau's Fermi liquid theory. On one hand, the derivation of Landau's mass renormalization formula that connects the effective mass of the quasiparticle to Landau parameters (that characterize fermion interactions) explicitly requires Galilean invariance and in fact there is no analogous derivation for fermions on a crystal lattice [1,2]. On the other hand, the T^2 resistivity that is said to be the hallmark of Fermi liquid metallicity requires umklapp backscattering, which is only possible on a lattice [3,4]. This aspect is easy to understand as a cloud of electrons in an external electric field glides across effortlessly, as there is no impedance to the motion, leading to a perfect conductor. It does not matter how the electrons interact with one another and whether or not quasiparticles exist. In reality, the presence of impurities and a crystal lattice circumvent this conclusion and allow for currents to decay, leading to nonzero resistivity.

It has been argued that umklapp is not general enough to explain the much less well understood strange metal phenomena. The interesting transport properties do not survive down to zero temperatures, and a spatially disordered interaction was proposed as the mechanism for strange metal behavior [5,6]. In recent works [7,8], strange metal behavior was observed on a numerical simulation of the periodic Anderson model in $d = 3$ dimensions (the microscopic model for heavy fermions) in the vicinity of (but not right at) the quantum

critical point. Since the model lives on a lattice and has no disorder whatsoever, it shows completely intrinsic (disorder-free) strange metal behavior. It was shown in [9] that the spatially disordered interaction of [5,6] does not lead to strange metallicity for $d > 2$, and therefore it is not unreasonable to assume that lattice effects are at play as suggested at the end of Ref. [8].

The effective model for the heavy fermion systems at criticality involves a conduction and a spinon Fermi surface interacting via hybridization fluctuations [10–13]. The model lives on a continuum and therefore an interplay of potential disorder and hybridization is required as the mechanism of $T \ln T$ resistivity, which is at odds with the results from Refs. [7,8], which is a clean model on a lattice. Thus the effects of umklapp scattering on the clean effective model become a worthwhile question to investigate.

In two dimensions, umklapp transport due to a critical scalar field was thoroughly investigated in [14,15], where a linear/quasilinear in T resistivity was found above a transition temperature T_U . A relativistic bosonic mode coupled to a Fermi surface (FS) experiences Landau damping, which is captured by a dressed bosonic propagator of the form [16,17]

$$D^{-1}(\mathbf{q}, i\Omega) = q^2 + \Omega^2 + r - \Pi(\mathbf{q}, i\Omega). \quad (1)$$

There are obvious overlaps and differences between this simple Ising-nematic model and critical heavy fermions. The differences come from the fact that the hybridization fluctuation model has a light c electron and a heavy f electrons with generically mismatched Fermi surfaces that give rise to additional energy-momentum scales absent from the Ising-nematic system with a single FS. The overlap comes from the fact that the boson propagators in both cases are nearly identical at criticality, which is why here we focus on the simpler model.

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Within the random phase approximation, $\Pi(\mathbf{q}, i\Omega)$ is built up of particle-hole bubbles summed over the entire FS. The standard derivation of $\Pi(\mathbf{q}, i\Omega)$ starts by noting that for any $\mathbf{q}$, the dominant contribution comes from the points on the FS which have tangents parallel to $\mathbf{q}$. For convex Fermi surfaces there are only two such points, called the antipodal patch contributions, and the calculation gives ($q = |\mathbf{q}|$)

$$\Pi_{\text{ATP}}(\mathbf{q}, i\Omega) \propto |\Omega|/q \sim \gamma_{\mathbf{q}}|\Omega|, \quad (2)$$

which dominates over the Ω^2 term in $D(\mathbf{q}, i\Omega)$ and in the $r \rightarrow 0$ limit gives rise to the $z = 3$ criticality [16,17]. The derivation assumes that the FS is small and far away from the Brillouin zone (BZ) boundaries.

When the FS is large enough to approach the BZ boundary, umklapp scattering channels open: an electron near the zone face can scatter across the boundary to the opposite side of the FS. It was noted in [14] that such processes “modify the coefficient $\gamma_{\mathbf{q}}$ of the Landau damping term . . . This can lead to a breakdown of the $z = 3$ quantum critical scaling, which relies on the relation $\gamma_{\mathbf{q}} \sim 1/q$.” In the current work, we pursue this line of reasoning further.

We perform an explicit patch calculation of the umklapp contribution to $\Pi(\mathbf{q}, i\Omega) = \Pi_{\text{ATP}}(\mathbf{q}, i\Omega) + \Pi_{\text{U}}(\mathbf{q}, i\Omega)$ for $d = 2$ critical Ising-nematic metal. We find that the umklapp piece does not simply renormalize $\gamma_{\mathbf{q}}$. Instead, for momentum transfer q close to umklapp momenta Δ_q , it introduces a qualitatively new frequency dependence $\Pi_{\text{U}}(q \sim \Delta_q, i\Omega) \propto \Omega^\alpha$. For $z = 3$ criticality, the competition between fermion self-energy $\Sigma(i\omega)$ vs $i\omega$ gives rise to an emergent temperature scale T_{NFL} below which $\Sigma(i\omega)$ dominates [14]. We show that unlike Π_{ATP} , which does not change whether it is evaluated using bare or dressed fermions [17], above T_{NFL} (the regime of interest in [14]) we find $\Pi_{\text{U}}(q \sim \Delta_q, i\Omega) \propto \sqrt{\Omega}$. Below T_{NFL} , the fermion self-energy weakens this contribution to $\Pi_{\text{U}}(q \sim \Delta_q, i\Omega) \propto \Omega^{2/3}$. We show that for $q \sim 0$, Π_{U} does not kick in due to threshold requirements.

As a consequence, $z = 3$ is stable at $T \rightarrow 0$ for any finite Δ_q . At higher T , due to the effects of $\Pi_{\text{U}} \propto \sqrt{\Omega}$, there is a possibility that the minimum temperature for the activation of critical umklapp transport which requires $q \sim \Delta_q$ is lowered from $T_{\text{U}} \propto \Delta_q^3$ (as suggested purely from $z = 3$ criticality [14,15]) to $T_{\text{U}} \propto \Delta_q^4$. However, the possibility of this reduction explicitly requires that fermion self-energy effects become important only at temperatures much below the reduced temperature $T_{\text{U}} \propto \Delta_q^4$.

This gives us conditions that the model parameters need to satisfy for this possibility to be realized, and we discuss the necessary conditions such that the lowered activation temperature scale emerges self-consistently. If the conditions are not satisfied then without needing any calculation whatsoever we know for certain that Π_{U} cannot lead to any reduction of T_{U} .

We show that despite the potentially dominating $\sqrt{\Omega}$ behavior, it is only in the so-called naive large- N limit [18] (with N fermion flavors and a single scalar field) studied in [14] that these conditions are satisfied (due to strong suppression of T_{NFL} at large N). For this model what we do not provide is a rigorous explicit transport calculation of the form in [14] that would demonstrate conclusively that T_{U} is reduced, as this lies much outside the scope of this work. However, for this

model we do show that there is a strong relative enhancement of the scattering rate of fermions close to the BZ boundaries at the reduced $T_{\text{U}} \propto \Delta_q^4$ due to the effects of Π_{U} . Since these scattering rates feed directly into the memory matrix elements used in rigorous evaluation of resistivity, we believe this to be good evidence in support of the conjecture that resistivity gets enhanced at lower T_{U} .

We also show that for $d = 3$, $\Pi_{\text{U}} \sim \Omega$ irrespective of T ; therefore it never competes with Π_{ATP} . So despite the original motivation, the heavy fermion criticality remains completely unmodified by the correction.

The article is structured as follows, in Sec. II we evaluate the umklapp contribution to the boson self-energy. Then in Sec. III, we discuss the conditions under which Π_{U} can possibly lead to a modification of T_{U} , which is shown at the level of scattering rate in Sec. IV. In Sec. V we discuss higher-order corrections to our one-loop calculation. Finally, we end in Sec. VI, outlining why we expect that for physically realistic systems, the effects of Π_{U} would be benign even in the case of $d = 2$.

II. PARTICLE-HOLE BUBBLE

We evaluate the simple particle-hole (p-h) bubble using the free fermion propagator [17]

$$\begin{aligned} \Pi(\mathbf{q}, i\Omega) &= -g^2 \int \frac{d^2k d\omega}{(2\pi)^3} \frac{1}{i(\omega + \Omega) - \varepsilon_{\mathbf{k}+\mathbf{q}}} \cdot \frac{1}{i\omega - \varepsilon_{\mathbf{k}}} \\ &= g^2 \int \frac{d^2k}{(2\pi)^2} \frac{n_F(\varepsilon_{\mathbf{k}+\mathbf{q}}) - n_F(\varepsilon_{\mathbf{k}})}{i\Omega + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}}}, \end{aligned} \quad (3)$$

where g is the fermion boson coupling strength $g \int \phi c^\dagger c$ and $n_F(x)$ is the Fermi distribution function.

The imaginary part of the self-energy for real frequencies is obtained by $i\Omega \rightarrow \Omega + i\eta$, giving

$$\begin{aligned} \text{Im } \Pi(\mathbf{q}, \Omega) &= -\frac{g^2}{4\pi} \int d^2k [n_F(\varepsilon_{\mathbf{k}+\mathbf{q}}) - n_F(\varepsilon_{\mathbf{k}})] \delta(\Omega + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}}). \end{aligned} \quad (4)$$

Performing a Taylor expansion of $n_F(\varepsilon + \Omega) - n_F(\varepsilon) \approx \Omega \partial_\varepsilon n_F$ (valid for $\Omega \ll \varepsilon_F$ the Fermi energy) and taking $T \rightarrow 0$ so that $\partial_\varepsilon n_F \rightarrow -\delta(\varepsilon)$ gives

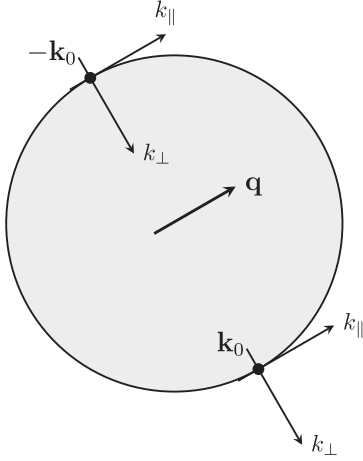
$$\frac{\text{Im } \Pi(\mathbf{q}, \Omega)}{\Omega} = \frac{g^2}{4\pi} \int d^2k \delta(\varepsilon_{\mathbf{k}}) \delta(\Omega + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}}). \quad (5)$$

The first δ function restricts $\mathbf{k}$ to the FS and the second enforces energy conservation. Generically, these two constraints fix $\mathbf{k}$ to isolated points, and the result is a sum over such solutions weighted by inverse Jacobians.

A. Antipodal patch contribution

We briefly review the standard calculation to establish the mechanism that produces $1/q$. The dominant contribution comes from the two antipodal points $\pm \mathbf{k}_0$ in Fig. 1, where $\mathbf{v}_F \perp \mathbf{q}$. Without loss of generality we take $\mathbf{q} = (0, q_\parallel)$. Expanding near $+\mathbf{k}_0$ with $\mathbf{k} = \mathbf{k}_0 + (\delta k_\perp, \delta k_\parallel)$ gives

$$\varepsilon_{\mathbf{k}} = v_F \delta k_\perp + \frac{\kappa}{2} \delta k_\parallel^2, \quad \varepsilon_{\mathbf{k}+\mathbf{q}} = v_F \delta k_\perp + \frac{\kappa}{2} (\delta k_\parallel + q_\parallel)^2. \quad (6)$$

FIG. 1. The two antipodal patches contributing to Π_{ATP} .

The Fermi velocities at both $\mathbf{k}$ and $\mathbf{k} + \mathbf{q}$ are v_F . Consequently, the linear terms $v_F \delta k_\perp$ cancel in the energy difference, giving

$$\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}} = -\kappa q_\parallel \delta k_\parallel - \frac{\kappa}{2} q_\parallel^2. \quad (7)$$

The first δ function $\delta(\varepsilon_{\mathbf{k}})$ fixes $\delta k_\perp = -(\kappa/2v_F)\delta k_\parallel^2$, and the second δ function then fixes $\kappa q_\parallel \delta k_\parallel = \Omega - \kappa q_\parallel^2/2$, which always has a solution. Since the energy difference depends only on $\delta k_\parallel$, the Jacobian of the second δ function is

$$\mathcal{J}_{\text{ATP}} = \left| \frac{\partial(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}})}{\partial(\delta k_\parallel)} \right| = \kappa |q_\parallel|. \quad (8)$$

This produces a singular factor $1/(\kappa |q_\parallel|)$ after integration over $\delta k_\parallel$. Including both antipodal points and accounting for the Jacobian of the first δ function $1/v_F$ that comes from the $\delta k_\perp$ integral gives us

$$\frac{\text{Im } \Pi_{\text{ATP}}(\mathbf{q}, \Omega)}{\Omega} = \frac{g^2}{2\pi v_F \kappa |q_\parallel|}. \quad (9)$$

This $1/|q_\parallel|$ gives $\gamma_{\mathbf{q}} \propto 1/|\mathbf{q}|$ and $z = 3$. The cancellation of linear energy terms is a direct consequence of the parallel velocities at the connected points. With the linear terms gone, the constraint falls on the quadratic variable $\delta k_\parallel$, whose Jacobian vanishes at $q_\parallel = 0$.

B. Umklapp contribution

Without loss of generality, we take $\mathbf{q} = (q_x, q_y)$ with $q_x, q_y \geq 0$. Consider the FS point $\mathbf{K}_F^x = (k_F, 0)$ closest to the BZ boundary along $\hat{x}$. An electron in the vicinity of $\mathbf{K}_F^x$ can scatter by $\mathbf{q}$ with $q_x \approx \Delta_q$, wrapping across the BZ to land in the vicinity of $\mathbf{K}_F^x + (\Delta_q, 0) = (k_F + \Delta_q, 0) \equiv \tilde{\mathbf{K}}_F^x$. Due to periodicity we have $\tilde{\mathbf{K}}_F^x = (k_F + \Delta_q, 0) = (-k_F, 0)$ in the eyes of the dispersion relation, which then gives $\varepsilon_{\mathbf{K}_F^x} = \varepsilon_{\tilde{\mathbf{K}}_F^x} = 0$.

The Fermi velocity at $\mathbf{K}_F^x$ is $+v_F \hat{x}$ and at $\tilde{\mathbf{K}}_F^x$ it is $-v_F \hat{x}$ (by inversion symmetry). We set $\mathbf{k} = \mathbf{K}_F^x + (\delta k_x, \delta k_y)$ expressed in a way that allows for a simple Taylor expansion of the

dispersion around the zero energy point $\mathbf{K}_F^x$ as

$$\varepsilon_{\mathbf{k}} = v_F \delta k_x + \frac{\kappa}{2} \delta k_y^2. \quad (10)$$

The other choice is more subtle. We use

$$\begin{aligned} \mathbf{k} + \mathbf{q} &= \mathbf{K}_F^x + (\delta k_x + q_x, \delta k_y + q_y) \\ &= \tilde{\mathbf{K}}_F^x + (\delta k_x + q_x - \Delta_q, \delta k_y + q_y) \\ &\Rightarrow \mathbf{k} + \mathbf{q} = \tilde{\mathbf{K}}_F^x + (\delta k_x + \delta q_x, \delta k_y + q_y), \end{aligned} \quad (11)$$

where we have implicitly assumed that $\mathbf{k} + \mathbf{q}$ is close to $\tilde{\mathbf{K}}_F^x$, which allows for a Taylor expansion of the dispersion about $\tilde{\mathbf{K}}_F^x$ as

$$\tilde{\varepsilon}_{\mathbf{k}+\mathbf{q}} = -v_F(\delta k_x + \delta q_x) + \frac{\kappa}{2}(\delta k_y + q_y)^2. \quad (12)$$

This expansion is justified when $\delta q_x \equiv q_x - \Delta_q \ll \Delta_q$, and the tilde is there to avoid confusion and keep it explicit that the expansion is about the umklapp point. The correctness of the sign of $-v_F$ can be verified by noting that as either k_x or q_x increase, we go deeper into the umklapp FS, thereby lowering the energy and vice versa. The linearization requires $v_F \neq 0$, which holds for $\Delta_q > 0$ when the FS does not hit the zone boundary.

With these ingredients in place the energy difference now gives

$$\varepsilon_{\mathbf{k}} - \tilde{\varepsilon}_{\mathbf{k}+\mathbf{q}} = 2v_F \delta k_x + v_F \delta q_x - \kappa \delta k_y q_y - \frac{\kappa}{2} q_y^2. \quad (13)$$

The crucial feature is $2v_F \delta k_x$, since the antiparallel velocities cause the linear terms to add rather than cancel. Substituting $2v_F \delta k_x = -\kappa \delta k_y^2$ from $\delta(\varepsilon_{\mathbf{k}})$ and plugging in $\delta(\Omega + \varepsilon_{\mathbf{k}} - \tilde{\varepsilon}_{\mathbf{k}+\mathbf{q}})$ gives

$$\Omega + v_F \delta q_x - \kappa \delta k_y q_y - \frac{\kappa}{2} q_y^2 - \kappa \delta k_y^2 = 0, \quad (14)$$

and we immediately see the δk_y^2 floating around in the energy-conserving constraint compared to Eq. (7), which is the root cause of the difference from the antipodal contributions.

For $q_y = 0$, this reduces to

$$\kappa \delta k_y^2 = \Omega + v_F \delta q_x \equiv \Omega - \Omega_0(q_x), \quad (15)$$

where we define the threshold frequency

$$\Omega_0(q_x) \equiv -v_F \delta q_x = v_F(\Delta_q - q_x). \quad (16)$$

The constraint (15) has real solutions $\delta k_y = \pm \sqrt{(\Omega - \Omega_0)/\kappa}$ only when $\Omega > \Omega_0$. The Jacobian is

$$\mathcal{J}_U = \left| \frac{\partial}{\partial \delta k_y} (\kappa \delta k_y^2 - \Omega + \Omega_0) \right| = 2\kappa |\delta k_y| = 2\sqrt{\kappa(\Omega - \Omega_0)}. \quad (17)$$

Summing over the two roots gives us

$$\frac{\text{Im } \Pi_U}{\Omega} \Big|_{q_y=0} = \frac{g^2}{4\pi v_F} \cdot \frac{\Theta(\Omega - \Omega_0)}{\sqrt{\kappa(\Omega - \Omega_0)}}, \quad (18)$$

where the v_F comes from the δk_x integral over $\delta(\varepsilon_{\mathbf{k}})$. This is a square-root threshold singularity as $\text{Im } \Pi/\Omega$ diverges as $1/\sqrt{\Omega - \Omega_0}$ as $\Omega \rightarrow \Omega_0^+$.

At the umklapp resonance $q_x = \Delta_q \Rightarrow \Omega_0 = 0$, this becomes

$$\text{Im } \Pi_U(\Delta_q, \Omega) \propto \sqrt{\Omega}. \quad (19)$$

For $q_y \neq 0$, Eq. (14) gives

$$\kappa \delta k_y^2 + \kappa \delta k_y q_y + \frac{\kappa}{2} q_y^2 = \Omega + v_F \delta q_x. \quad (20)$$

Completing the square gives $\kappa \tilde{\delta k}_y^2 = \Omega - \Omega_0(\mathbf{q})$, where $\tilde{\delta k}_y \equiv \delta k_y + q_y/2$ and $\Omega_0(\mathbf{q}) \equiv v_F(\Delta_q - q_x) + \frac{\kappa}{4} q_y^2 = \tilde{\epsilon}_q - \frac{\kappa}{4} q_y^2$. Equation (18) maintains the same structure:

$$\frac{\text{Im } \Pi_U(\mathbf{q}, \Omega)}{\Omega} = \frac{g^2}{4\pi v_F} \cdot \frac{\Theta(\Omega - \Omega_0(\mathbf{q}))}{\sqrt{\kappa(\Omega - \Omega_0(\mathbf{q}))}}. \quad (21)$$

In the above discussion, we considered an umklapp bubble in the vicinity of $\mathbf{K}_F^x$ with $q_x \approx \Delta_q$. We also need to account for the contribution from $\mathbf{K}_F^y = (0, k_F)$, which requires $q_y \approx \Delta_q$. Needless to say, the analysis remains exactly the same with the substitution $x \leftrightarrow y$ throughout. This means for a given $\mathbf{q} = (q_x, q_y)$, adding the contributions from $\mathbf{K}_F^x$ and $\mathbf{K}_F^y$,

$$\frac{\text{Im } \Pi_U(\mathbf{q}, \Omega)}{\Omega} = \frac{g^2}{4\pi v_F} \left[\frac{\Theta(\Omega - \Omega_0(\mathbf{q}))}{\sqrt{\kappa(\Omega - \Omega_0(\mathbf{q}))}} + \frac{\Theta(\Omega - \Omega_0(\bar{\mathbf{q}}))}{\sqrt{\kappa(\Omega - \Omega_0(\bar{\mathbf{q}}))}} \right], \quad (22)$$

with $\bar{\mathbf{q}} = (q_y, q_x)$.

C. Extension to three dimensions

The key features of the calculation generalize to three dimensions (3D) with modifications to the dimensionality of the transverse integration. In 3D, the FS is a two-dimensional (2D) surface, and the patch expansion near an umklapp point involves two transverse momenta ($\delta k_y, \delta k_z$) rather than one (δk_y) in 2D. After the FS constraint fixes the longitudinal momentum δk_x , the energy conservation becomes a constraint on ($\delta k_y, \delta k_z$) as

$$\kappa_1 \delta k_y^2 + \kappa_2 \delta k_z^2 = \Omega - \Omega_0(\mathbf{q}), \quad (23)$$

where κ_1, κ_2 are the two principal curvatures at the umklapp point. This is an ellipse in ($\delta k_y, \delta k_z$) space and gives

$$\int d\delta k_y d\delta k_z \delta(\kappa_1 \delta k_y^2 + \kappa_2 \delta k_z^2 - E) = \frac{\pi}{\sqrt{\kappa_1 \kappa_2}} \Theta(E). \quad (24)$$

Therefore, in 3D the umklapp contribution to $\text{Im } \Pi/\Omega$ has a step-function threshold and no square-root singularity:

$$\text{Im } \Pi_U^{3D}/\Omega \sim \Theta(\Omega - \Omega_0). \quad (25)$$

This implies that for critical heavy fermion systems, the umklapp self-energy Π_U plays no serious role. This conclusion does not change even when $\text{Im } \Pi_U^{3D}$ is evaluated using dressed fermion propagators, as can be seen at the end of Appendix A.

III. IMPLICATIONS FOR TRANSPORT NEAR CRITICALITY

As was noted in [14,15], near Ising-nematic criticality $r \rightarrow 0$, the minimum umklapp momenta Δ_q gives a dynamically generated temperature scale T_U , which decides the crossover from $\rho(T) \sim T^2$ to $\rho(T) \sim T$ as T increases from 0. After incorporating Π_U and the boson thermal mass, the fully dressed boson propagator written in terms of real frequency Ω for the values of $\mathbf{q}$ when $\Omega_0(\mathbf{q}) = 0$ is

$$D^{-1}(\mathbf{q}, \Omega) = q^2 + aT - \eta \frac{i\Omega}{q} - i\lambda\sqrt{\Omega}. \quad (26)$$

For $q \sim \Delta_q$, this furnishes additional energy scales. Balancing $\eta\Omega/q$ with bare q^2 gives the $z = 3$ crossover scale $T_3 \sim \eta^{-1} \Delta_q^3$ [14,15]. Comparing $\lambda\sqrt{\Omega}$ with the bare kinetic term q^2 gives $T_4 \sim \lambda^{-2} \Delta_q^4$. Balancing the boson thermal mass with the bare kinetic term gives $T_m \sim a^{-1} \Delta_q^2$. And finally, comparing the two pieces of boson self-energy $\eta\Omega/q$ vs $\lambda\sqrt{\Omega}$ gives $T_2 \sim \lambda^2 \Delta_q^2 \eta^{-2}$. References [14,15] only have T_3 as their relevant temperature scale, giving $T_U \sim T_3$, and there are no analogs of T_4, T_2 , and T_m .

Making use of Δ_q as the source of a small parameter furnishes us a hierarchy of scales:

$$T_4 \ll T_3 \ll T_2 \sim T_m. \quad (27)$$

Now the question is, Under what circumstances is the above relative T scale meaningful and consistent, and when can we expect the reduced temperature scale T_4 to act as the crossover scale T_U ?

A. Memory matrix transport

In Ref. [14], the authors evaluated umklapp transport resistivity at criticality and found a crossover temperature $T_U \sim T_3$ (T_0 in their notation) which *a priori* demonstrates that T_4 plays no role. Here we discuss why this disagreement comes about.

They parametrized the boson propagator as $D^{-1} = r_{\mathbf{q}} + \gamma_{\mathbf{q}}|\Omega|$ and suspected that umklapp could modify $\gamma_{\mathbf{q}}$ and thereby alter $z = 3$. Although $\gamma_{\mathbf{q}} \sim 1/q$ can be calculated for spherical dispersions, they set $\gamma_{\mathbf{q}} \sim \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) \delta(\epsilon_{\mathbf{k}+\mathbf{q}})$, as it allowed them to numerically capture the nonuniversal effects by performing the summation for general lattice dispersion $\epsilon_{\mathbf{k}}$. A direct comparison with Eq. (5) gives $\gamma_{\mathbf{q}} \sim \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) \delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \Omega)$. Ignoring Ω inside the second δ function is unproblematic for capturing the antipodal $\gamma_{\mathbf{q}} \sim 1/q$ contribution, as can be seen from the discussion following Eq. (7).

Our result shows that the umklapp correction does *not* take the form of an Ω independent $\gamma_{\mathbf{q}}$. Instead, it introduces a qualitatively new term $\sqrt{\Omega}$ in the boson self-energy at $q \sim \Delta_q$ which explicitly requires keeping Ω in $\delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \Omega)$, as can be seen from the discussion around Eq. (15). This term is not captured by the simple $\gamma_{\mathbf{q}}|\Omega|$ ansatz of [14] and does not reflect in the transport properties they evaluate. Redoing the memory matrix analysis with $\gamma_{\mathbf{q}} \sim \sum_{\mathbf{k}} \delta(\epsilon_{\mathbf{k}}) \delta(\epsilon_{\mathbf{k}+\mathbf{q}} - \Omega)$ would hopefully capture the effects of Π_U at the level of transport, but this lies much outside the scope of this work.

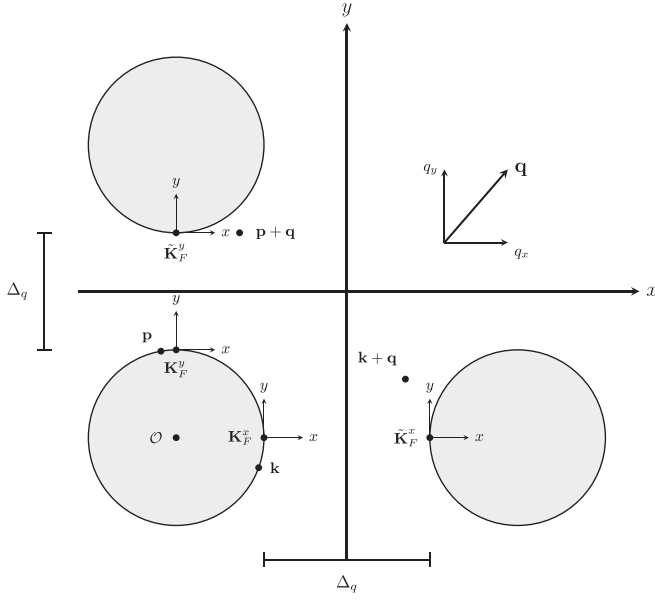


FIG. 2. The two umklapp contributions to Π_U . $\mathcal{O}$ is the center of the original FS. Also shown are the two umklapp partners accessible via momentum transfer $\mathbf{q}$. The proportions are greatly exaggerated for visual clarity. The analysis in the main text focuses on umklapp along the x direction as the steps for the y direction are exactly the same.

As a sanity check for the validity of the calculations performed until now, we plot the full $\Pi/\Omega \propto \sum_{\mathbf{k}} \delta(\varepsilon_{\mathbf{k}}) \delta(\varepsilon_{\mathbf{k}+\mathbf{q}} - \Omega)$ from Eq. (5) as a function of Ω for the dispersion from [14] numerically and compare it with $\Pi = \Pi_{\text{ATP}} + \Pi_U$ from the analytical calculations shown above. We set $\mathbf{q} = (\Delta_q, 0)$, which ensures that $\Omega_0(\mathbf{q}) = 0$. Now for a fixed $\mathbf{q}$, purely from the antipodal contribution we expect Π_{ATP}/Ω to be a constant in Ω , whereas $\Pi = \Pi_{\text{ATP}} + \Pi_U$ would give rise to an additional $1/\sqrt{\Omega}$ from Eq. (18), whose effects can be seen clearly in Fig. 3.

B. Bottlenecks

In evaluating $\Pi(\mathbf{q}, \Omega)$ we have only used free fermion propagators. The dressed boson propagator gives rise to a fermion self-energy $\Sigma(i\omega)$, which for non-Fermi liquids (NFL) gives rise to a temperature scale T_{NFL} below which $\Sigma(i\omega)$ dominates over the bare $i\omega$ of the fermion propagator, and the quasiparticle picture is invalidated [16,17].

An important point to note is that due to the momentum activation threshold, Π_U does not modify fermion self-energy in any meaningful way. When evaluating $\Sigma(i\omega)$ using the fully dressed $D(\mathbf{q}, i\Omega)$, including both Π_{ATP} and Π_U , the effects of Π_U kick in only when $q \sim \Delta_q$. In this regime the boson propagator is effectively gapped with a mass $\sim \Delta_q^2$, which at worst only gives Fermi-liquid-like self-energy corrections with prefactors that depend on Δ_q , but it never really competes with $\Sigma(i\omega)$ and the dressed NFL propagator remains completely intact. Now reevaluating Π_{ATP} using the dressed fermion propagator gives back the same Ω/q result [17].

Reevaluating Π_U using the dressed NFL fermion propagator weakens the contribution from

$$\Pi_U \sim \sqrt{\Omega} \longrightarrow \Omega^{2/3}. \quad (28)$$

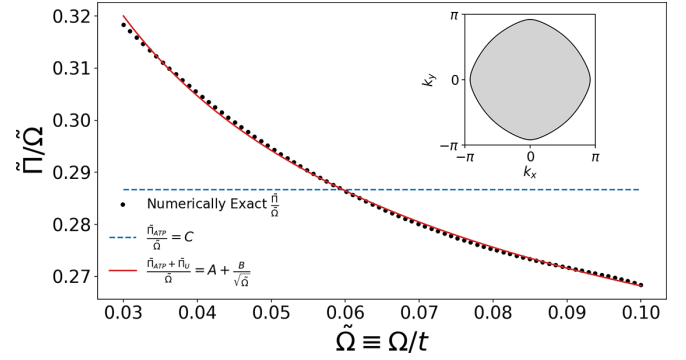


FIG. 3. The frequency divided boson self-energy as a function of boson frequency for fixed $\mathbf{q} = (\Delta_q, 0)$. The tildes ensure that everything is dimensionless. The dispersion is from [14], $\varepsilon_{\mathbf{k}} = -2t(\cos(k_x) + \cos(k_y)) - 4t'\cos(k_x)\cos(k_y) - \mu$. We choose $(t, t', \mu) = (1.0, 0.3, 1.1)$, where the accompanying FS is in the inset. For values of $\tilde{\Omega}$ smaller than the ones shown in the plot, deviations kick in due to the numerical implementation of the Dirac δ function, which can be remedied by a smaller regulator at the expense of longer runtime. For values of $\tilde{\Omega}$ larger than the ones shown in the plot we again see deviations, but those are due to the breakdown of the patch approximation and the $\tilde{\Omega}^{-1/2}$ behavior is not expected for such high frequencies. Within the practical and theoretical bound we see a nice agreement of the exact result with the calculation of Π_U performed in this work.

A hand-wavy way of seeing this is as follows. Ignoring Ω_0 for the moment, we saw that $\Pi_U/\Omega \propto 1/\sqrt{\Omega}$. In Eq. (21) the square root comes from the integral over δk_y^2 , and the Ω inside the square root comes from the $i\omega$ kinetic term in the free fermion propagator. In the NFL regime we can ignore the $i\omega$ term and worry only about $\Sigma(i\omega)$ in the dressed fermion propagator to obtain

$$\Pi_U \propto \frac{\Omega}{\sqrt{\Sigma(i\Omega)}}. \quad (29)$$

This argument is fairly schematic, does not hold under technical scrutiny, and has only been presented to show transparently why the correct answer (derived in Appendix A) looks the way it does.

Now for $(d, z) = (2, 3)$ criticality, $\Sigma(i\Omega) \sim \Omega^{2/3}$, which immediately gives $\Pi_U \sim \Omega^{2/3}$. Even at $q \sim \Delta_q$ and very low Ω , when $\Omega^{2/3}$ dominates over Ω/q , $\Omega^{2/3} \sim \Delta_q^2$ still gives $\Omega \sim \Delta_q^3$, so $z = 3$ criticality remains intact even at higher values of q for $T \rightarrow 0$.

C. Consistency requirements

For the lowered temperature T_4 to exist robustly we need the following condition:

$$T_{\text{NFL}} \ll T_4, \quad (30)$$

where T_{NFL} is the temperature scale above which the bare $i\omega$ dominates over $\Sigma(i\omega)$ in the dressed fermion propagator. Working above T_{NFL} justifies the use of free fermion propagators, leading to $\Pi_U \sim \sqrt{\Omega}$, which then gives rise to T_4 . If $T_4 \ll T_{\text{NFL}}$, then T_4 and T_2 never show up, since below T_{NFL} we have $\Pi_U \sim \Omega^{2/3}$, which gives rise to the exact same T_3 as was obtained from Ω/q up to numerical prefactors.

We use the fact that for spherical FS, $v_F = k_F/m$, $\kappa = m^{-1}$, and $\varepsilon_F \sim k_F^2/m$, where k_F is the Fermi momenta and m is the fermion mass. This gives $T_4 \sim (\Delta_q/k_F)^4 \varepsilon_F^3 g^{-4}$ and $T_3 \sim (\Delta_q/k_F)^3 \varepsilon_F^2 g^{-2}$, since $\lambda \sim g^2 m^{3/2} k_F^{-1}$ and $\eta \sim g^2 m^2 k_F^{-1}$, as can be seen from Eq. (22), and $T_2 \sim (\Delta_q/k_F)^2 \varepsilon_F$, having ignored factors of 2 and π , etc.

Now we have to check when exactly is $T_{\text{NFL}} \ll T_4$.

1. Yukawa-SYK model

For the Yukawa-Sachdev-Ye-Kitaev model (Y-SYK) model $T_{\text{NFL}} \sim g^4 \varepsilon_F^{-1}$ [17], and the requirement $T_{\text{NFL}} \ll T_4 \Rightarrow g^2 \varepsilon_F^{-1} \ll \Delta_q/k_F$ is completely inconsistent with (and in fact the exact opposite of) $T_4 \ll T_3 \Rightarrow \Delta_q/k_F \ll g^2 \varepsilon_F^{-1}$. Thus in this large- N limit, the following condition does not hold consistently:

$$T_{\text{NFL}} \ll T_4 \ll T_3. \quad (31)$$

When $\Delta_q/k_F \ll g^2 \varepsilon_F^{-1}$, the condition that actually holds is

$$T_4 \ll T_3 \ll T_{\text{NFL}}. \quad (32)$$

But just because a condition is consistent does not mean it is physically meaningful. As discussed up above, below T_{NFL} , we have $\Pi_U \sim \Omega^{2/3}$ and $\Pi_{\text{ATP}} \sim \Omega/q$, both of which give T_3 as the physically relevant scale and T_4 does not show up in this case, so the possibility of $T_U \sim T_4$ can be rejected immediately.

The other consistent possibility comes from $\Delta_q/k_F \gg g^2 \varepsilon_F^{-1}$, giving

$$T_{\text{NFL}} \ll T_2 \ll T_3 \ll T_4. \quad (33)$$

But in this case, $g^2 \varepsilon_F^{-1}$ acts as the small parameter, making T_2 the smallest scale and in fact, above T_2 , Π_{ATP} strongly dominates over Π_U , so we would still expect $T_U \sim T_3$ for this case.

2. Naive large- N limit

In the naive large- N limit with N species of fermions and a single bosonic field we have $T_{\text{NFL}} \sim g^4 \varepsilon_F^{-1} N^{-3}$ [14], which is adequately suppressed for large enough N . Now requiring $T_{\text{NFL}} \ll T_4$ immediately gives $N^{-3/4} g^2 \varepsilon_F^{-1} \ll \Delta_q/k_F$, which can be satisfied consistently with $T_4 \ll T_3 \Rightarrow \Delta_q/k_F \ll g^2 \varepsilon_F^{-1}$. This is precisely the large- N limit and temperature regime studied in [14]. In this case the following condition holds consistently,

$$T_{\text{NFL}} \ll T_4 \ll T_3 \ll T_2, \quad (34)$$

which suggests the possibility that $T_U \sim T_4$, but this needs to be confirmed using a more careful transport analysis. It is called the naive large- N limit due to the technical problems that occur for $T \ll T_{\text{NFL}}$ as the N^{-1} expansion becomes ill defined [18,19] and other methods are needed as a remedy [20,21].

3. Matrix large- N limit

In this case the scalar field comes with N^2 matrix valued components and gives rise to $T_{\text{NFL}} \sim g^4 \varepsilon_F^{-1} N$ [22]. Since both Π_{ATP} and Π_U are $O(N^{-1})$ for this model we immediately get $T_3 \sim O(N)$, $T_4 \sim O(N^2)$, and $T_2 \sim O(1)$.

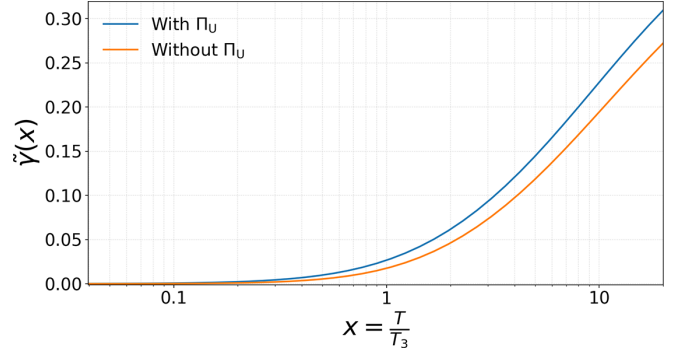


FIG. 4. The scattering rate of the electron at $\mathbf{K}_F^x$ in terms of dimensionless temperature for the Y-SYK model above T_{NFL} . The boson thermal mass has been included here with $T_m = T_4$ set by hand. Here $\Delta_q/k_F = \alpha g^2 \varepsilon_F^{-1}$ with $\alpha = 10$, which sets $T_2 = \alpha^{-1} T_3 \ll T_3 \ll T_4 = \alpha T_3$. The three tick labels shown on the x axis correspond to $T = T_2$, T_3 , and T_4 , respectively. The effect of Π_U is only minute and quantitative.

Although this case is the most messy since we have three different small parameters N^{-1} , Δ_q/k_F , and g^2/ε_F , for large enough N the lowest T scale is always T_2 , above which Π_{ATP} completely dominates Π_U and again, we expect nothing very interesting.

IV. SCATTERING RATE

For SYK and naive large- N models for temperatures above T_{NFL} , we reanalyze the scattering rate in [15] with two modifications. We include the boson thermal mass aT and include Π_U in the boson propagator. For simplicity, we focus on the scattering rate of the electron that originally sits at $\mathbf{K}_F^x$ to scatter and land in the vicinity of $\tilde{\mathbf{K}}_F^x$ right on the umklapp FS from Fig. 2, as was the case at the start of [15]. Since the scattering rate obtained in [15] did not involve the boson mass, for $T > T_3$ an indefinite linear in T scattering rate was obtained. It was at the level of the quantum Boltzmann equation along with the relaxation time approximation that it became clear that there is another scale $T_1 \propto \gamma_0 \Delta_q$ set by the potential disorder γ_0 , above which the linear in T resistivity saturates to a $\sqrt{T}$ dependence.

The exact same simple scattering rate when evaluated including the boson thermal mass gives

$$\tilde{\gamma}(T) \approx T \int_{-\infty}^{\infty} \frac{d\phi}{q_\phi^2 + aT} \propto \frac{T}{\sqrt{\Delta_q^2 + aT}}, \quad (35)$$

which immediately gives a scale $T_m \sim a^{-1} \Delta_q^2$, above which the scattering rate becomes $\sqrt{T}$. In the accompanying plots we arbitrarily set T_m as exactly the same as the largest relevant T scale.

In Fig. 4, for $T_2 < T < T_3$, the effects of Π_U seem extremely benign and only quantitative as discussed earlier for the Y-SYK model. The physically appropriate choice would be $\Delta_q/k_F = 0.1$, $g^2 \varepsilon_F = 0.01$ but in the evaluation it is the relative ratios that matter, so we set $\Delta_q/k_F = 10$, $g^2 \varepsilon_F^{-1} = 1.0$

In Fig. 5 for the naive large- N model, we can see that due to the presence of Π_U , the scattering rate gets enhanced at

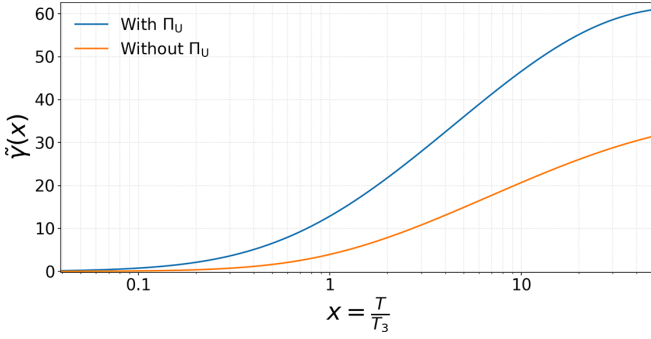


FIG. 5. The same plot for the naive large- N limit. The boson thermal mass has been included in both cases with $T_2 = T_m$ for simplicity, which leads to $\sqrt{x}$ saturation at large x . Here $\Delta_q/k_F = \alpha g^2 \varepsilon_F^{-1}$, with $\alpha = 0.1$, which sets $T_4 = \alpha T_3 \ll T_3 \ll T_2 = T_m = \alpha^{-1} T_3$. The three tick labels shown on the x axis correspond to $T = T_4$, T_3 , and T_2 , respectively. Π_U leads to a substantial enhancement of the scattering rate beyond T_4 .

a much lower $T \sim T_4$ compared to what is expected without Π_U , where we see an upturn only around T_3 . At higher temperatures, both scattering rates saturate due to the boson thermal mass. The details regarding the evaluation of these plots can be found in Appendix B.

V. CORRECTIONS TO ONE-LOOP RESULTS

There are deep structural reasons why $\Pi_{\text{ATP}} \sim \Omega/q$ retains its form irrespective of evaluation using free or dressed fermion propagator and higher-order corrections to one-loop calculation are regular (though they can be nonanalytic) [19,23,24]. This comes from the fact that for small momentum transfers $q \sim 0$, there is charge conservation in each FS patch, and that the low-energy scattered fermions $\mathbf{k}$, $\mathbf{k} + \mathbf{q}$ are nearly collinear. Therefore even though individual correction diagrams can be singular, the constraints from the above conditions lead to precise cancellations that leave only regular corrections intact.

The situation for umklapp contribution is drastically different since fermions hop across the BZ boundary, there is no FS patch charge conservation for the individual process (although, of course, global charge conservation remains intact), and the scattered fermions $\mathbf{k}$, $\mathbf{k} + \mathbf{q}$ for $q \sim \Delta_q$ are nearly *antiparallel*, which necessitates a closer look at higher-order corrections to one-loop results.

Here we focus on the following concrete problem of frequency dependence of higher-order corrections to $\Pi_U(i\Omega)$ for the specific case of $\mathbf{q} = (\Delta_q, 0)$ (as it does away with the complications of $\Omega_0(\mathbf{q})$). This is very closely related to the calculations performed in [23,25,26], which looked at frequency-dependent corrections to boson self-energy at $\mathbf{q} = (0, 0)$ boson momenta, as it is directly connected to optical conductivity.

Firstly we distinguish between $\Pi_U^0 \sim \int G_0 \cdot G_0 \propto \sqrt{\Omega}$ and $\Pi_U \sim \int G \cdot G \propto \Omega^{2/3}$ and closely follow [26], where we can see the corrections to boson self-energy having been classified into three categories: self-energy (SE), Maki-Thompson (MT), and Aslamazov-Larkin (AL) contributions.

Now interestingly, the most complicated AL corrections are the ones that can be seen to be least problematic. This is because one can see that for an external boson momenta q , there is always at least one internal boson momenta with q dependence as $\int \cdots D(q+l) D(l) dl$, so for our case of $q \sim \Delta_q \Rightarrow D^{-1}(q+l) \sim \Delta_q^2 + \cdots$ is effectively gapped so it cannot give a singular contribution and even its regular contribution cannot be as strong as contributions due to SE and MT corrections, both of which feature internal boson lines with purely low momentum dependence therefore are always critical.

Now an important takeaway from [23,25,26] is that both SE and MT corrections have the same frequency scaling, and it is the numerical prefactors that make them cancel each other off.

For both SE and MT contributions, we find the correction to the bare bubble:

$$\delta \Pi_U^{0, \text{MT}}(i\Omega), \delta \Pi_U^{0, \text{SE}}(i\Omega) \propto \Omega^{1/6} \quad (36)$$

and

$$\delta \Pi_U^{\text{MT}}(i\Omega), \delta \Pi_U^{\text{SE}}(i\Omega) \propto \Omega^{2/3}, \quad (37)$$

where the calculation is presented in Appendix C. We know for Π_{ATP} that the surviving corrections coming from MT, SE, and AL to the bare fermion bubble $\int G_0 G_0$ goes as $\Omega^{1/3}$ [23,25], so it is interesting to see that the same correction for the umklapp bubble is the square root of the antipodal correction $\sqrt{\Omega^{1/3}} \sim \Omega^{1/6}$.

We add that the SE correction to the dressed bubble is somewhat meaningless, as the dressed bubble already fully accounts for the fermion self-energy effects, but this correction just shows self-consistency of the dressed $\Pi_U \propto \Omega^{2/3}$.

We immediately see that the corrections to the bare bubble $\Omega^{1/6} \gg \Omega^{1/2}$ for small Ω , which is in sync with our calculation that shows that at low frequencies $\sqrt{\Omega} \rightarrow \Omega^{2/3}$, so the corrections have to take over at low enough Ω .

VI. CONCLUSION

We computed the umklapp contribution to the particle-hole polarization bubble for $d = 2$ Ising-nematic metal whose Fermi surface approaches the Brillouin zone boundary. The key result is that umklapp processes do not simply renormalize the Landau damping coefficient $\gamma_q \sim 1/q$ as was previously posited [14]. Instead, they introduce a qualitatively distinct contribution $\Pi_U \propto \Omega^\alpha$, where $\alpha = 2/3$ for $T \ll T_{\text{NFL}}$ and $\alpha = 1/2$ for $T \gg T_{\text{NFL}}$ at the minimum umklapp momentum $q \sim \Delta_q$, separated from $q = 0$ by a frequency threshold $\Omega_0(\mathbf{q})$. For $T \ll T_{\text{NFL}}$, both $\Omega^{2/3}$ and Ω/q give $z = 3$. This conclusion does not hold for model deformations with generalized boson dispersion [27] $q^2 \rightarrow q^{1+\epsilon}$, where additional competing scales are generated. The schematic formula (in the non-Fermi-liquid regime)

$$\Pi_U(\Omega) \propto \frac{\Omega}{\sqrt{\Sigma(i\Omega)}} \quad (38)$$

immediately gives a first-order insight into the potential importance of Π_U .

The impact of Π_U is strongly controlled by the nature of the large- N limit. For Yukawa-SYK and matrix large- N

models, within the scope of this work the effects of Π_U seem inconsequential. For the naive large- N model there is a possibility for a lower temperature T_4 acting as the crossover scale for the activation of critical umklapp transport compared to T_3 as is previously expected. We've shown this to be the case at the level of simple scattering rate, but whether this actually modifies T_U depends on a transport calculation that goes beyond the present work.

For physically realistic systems, $N \sim O(1)$, so the energy scale estimates made for the Y-SYK model seem most realistic, as there we do not have prefactors of N artificially suppressing or inflating any T scale, which leads us to believe that for realistic systems the effects of Π_U would be benign.

In $d = 3$ the additional transverse momentum integration removes the square-root dependence to $\Pi_U^{3D} \sim \Omega$, which does not compete with the antipodal Ω/q damping. Consequently, despite the original physical motivation for the current work [7,8,10–12], we find that the umklapp correction plays no significant role for three-dimensional critical systems such as heavy-fermion metals near a Kondo-breakdown transition.

ACKNOWLEDGMENT

We thank Stefan Kehrein for important feedback on the manuscript.

DATA AVAILABILITY

The data that support the findings of this article are openly available [28].

APPENDIX A: UMKLAPP SELF-ENERGY AT LOW TEMPERATURES

We start with the self-consistent patch calculation of the antipodal piece as it makes the subsequent calculations more illuminating. Starting with $\mathbf{q} = (0, q_{\parallel})$,

$$\varepsilon_{\mathbf{k}} = v_F \delta k_{\perp} + \frac{\kappa}{2} \delta k_{\parallel}^2$$

$$\Pi_U(\mathbf{q}, i\Omega_m) = -g^2 T \sum_{\omega_n} \int \frac{d^2 k}{(2\pi)^2} \frac{1}{(W(i\omega_n) - v_F \delta k_x - \frac{\kappa}{2} \delta k_y^2)(W(i(\omega_n + \Omega_m)) + v_F(\delta k_x + \delta q_x) - \frac{\kappa}{2}(\delta k_y + q_y)^2)}. \quad (\text{A6})$$

The first factor comes from the propagator at the patch near $\mathbf{K}_F^x$, the second from the umklapp partner $\tilde{\mathbf{K}}_F^x$ from Fig. 2.

The integrand takes the form

$$\frac{1}{(\alpha - v_F \delta k_x)(\beta + v_F \delta k_x)} = \frac{-1}{v_F^2} \cdot \frac{1}{(\delta k_x - \alpha/v_F)(\delta k_x + \beta/v_F)}, \quad (\text{A7})$$

where the two poles in the complex δk_x plane

$$P_1 = \frac{\alpha}{v_F}, \quad \text{Im}(P_1) = \frac{\text{Im } W(i\omega_n)}{v_F} \propto \text{sgn}(\omega_n), \quad (\text{A8})$$

$$\varepsilon_{\mathbf{k}+\mathbf{q}} = v_F \delta k_{\perp} + \frac{\kappa}{2}(\delta k_{\parallel} + q_{\parallel})^2, \quad (\text{A1})$$

we can write down the antipodal part of the boson self-energy bubble as [17]

$$\Pi_{\text{ATP}}(\mathbf{q}, i\Omega_m) = -2g^2 T \sum_{\omega_n} \int \frac{d^2 k}{4\pi^2} \frac{1}{(W(i\omega_n) - v_F \delta k_{\perp} - \frac{\kappa}{2} \delta k_{\parallel}^2)} \cdot \frac{1}{(W(i\omega_n + i\Omega_m) - v_F k_{\perp} - \frac{\kappa}{2}(\delta k_{\parallel} + q_{\parallel})^2)}, \quad (\text{A2})$$

where we use a placeholder $W(i\omega) \equiv i\omega - \Sigma(i\omega)$ to avoid clutter. Performing the integral over $\delta k_{\perp}$ gives

$$\Pi_{\text{ATP}}(\mathbf{q}, i\Omega_m) = \frac{-ig^2 T}{v_F} \sum_{\omega_n} \int \frac{dk_y}{(2\pi)} \frac{\text{sgn}(\omega_n + \Omega_m) - \text{sgn}(\omega_n)}{W(i\omega_n + i\Omega_m) - W(i\omega_n) - \frac{\kappa}{2}q_{\parallel}^2 - \kappa q_{\parallel} k_{\parallel}}, \quad (\text{A3})$$

and performing the contour integral over $\delta k_{\parallel}$ gives [17]

$$\begin{aligned} \Pi_{\text{ATP}}(\mathbf{q}, i\Omega_m) &= \frac{g^2 T}{2\kappa v_F |q_{\parallel}|} \sum_{\omega_n} \text{sgn}(\Omega_m) \\ &\quad \times [\text{sgn}(\omega_n + \Omega_m) - \text{sgn}(\omega_n)] \\ &= -\frac{g^2 |\Omega_m|}{2\pi \kappa v_F |q_{\parallel}|}. \end{aligned} \quad (\text{A4})$$

For the umklapp contribution the fermion dispersion that acts as the starting point is

$$\begin{aligned} \varepsilon_{\mathbf{k}} &= v_F \delta k_x + \frac{\kappa}{2} \delta k_y^2 \\ \tilde{\varepsilon}_{\mathbf{k}+\mathbf{q}} &= -v_F(\delta k_x + \delta q_x) + \frac{\kappa}{2}(\delta k_y + q_y)^2. \end{aligned} \quad (\text{A5})$$

As explained in Sec. III B, despite this umklapp contribution, the fermion self-energy $\Sigma(i\omega_n)$ remains unchanged. For the $d = 2$ Yukawa-SYK model $\Sigma(i\omega) = -iB \text{sgn}(\omega)|\omega|^{2/3}$ with $\sqrt{3}B = g^{4/3} \kappa^{1/3} (v_F/2\pi)^{-2/3}$ [17]. The umklapp particle-hole bubble in Matsubara frequencies is

$$\begin{aligned} P_2 &= -\frac{\beta}{v_F}, \\ \text{Im}(P_2) &= -\frac{\text{Im } W(i(\omega_n + \Omega_m))}{v_F} \propto -\text{sgn}(\omega_n + \Omega_m), \end{aligned} \quad (\text{A9})$$

are defined in terms of

$$\begin{aligned} \alpha &\equiv W(i\omega_n) - \frac{\kappa}{2} \delta k_y^2, \\ \beta &\equiv W(i(\omega_n + \Omega_m)) + v_F \delta q_x - \frac{\kappa}{2}(\delta k_y + q_y)^2. \end{aligned} \quad (\text{A10})$$

For $\omega_n > 0$, we pick up the residue at P_1 :

$$\oint \frac{d\delta k_x}{2\pi} \frac{-1}{v_F^2} \cdot \frac{1}{(\delta k_x - P_1)(\delta k_x - P_2)} = \frac{-1}{v_F^2} \cdot \frac{i}{P_1 + \beta/v_F} = \frac{-i}{v_F(\alpha + \beta)}. \quad (\text{A11})$$

For $\omega_n < -\Omega_m$, we pick up the residue at $P_2 = -\beta/v_F$:

$$\frac{-1}{v_F^2} \cdot \frac{i}{-\beta/v_F - P_1} = \frac{-1}{v_F^2} \cdot \frac{i}{-(\alpha + \beta)/v_F} = \frac{+i}{v_F(\alpha + \beta)}. \quad (\text{A12})$$

Combining both sectors, the δk_x integral gives

$$\int \frac{d\delta k_x}{2\pi} G_1 \cdot G_2 = \frac{-i [\text{sgn}(\omega_n) + \text{sgn}(\omega_n + \Omega_m)]}{2v_F(\alpha + \beta)}. \quad (\text{A13})$$

The factor $\frac{1}{2}[\text{sgn}(\omega_n + \Omega_m) + \text{sgn}(\omega_n)]$ equals ± 1 when both signs agree, and zero otherwise. The denominator can be

written as ($\delta \tilde{k}_y \equiv \delta k_y + q_y/2$):

$$\begin{aligned} \alpha + \beta &= W(i\omega_n) + W(i(\omega_n + \Omega_m)) + v_F \delta q_x - \kappa \delta k_y^2 \\ &\quad - \kappa \delta k_y q_y - \frac{\kappa}{2} q_y^2 \\ &= W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2, \end{aligned} \quad (\text{A14})$$

which gives us

$$\begin{aligned} \Pi_U(\mathbf{q}, i\Omega_m) &= \frac{ig^2 T}{2v_F} \sum_{\omega_n} [\text{sgn}(\omega_n) + \text{sgn}(\omega_n + \Omega_m)] \\ &\quad \times \int \frac{d\delta \tilde{k}_y}{2\pi} \frac{1}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2}. \end{aligned} \quad (\text{A15})$$

Taking $\Omega_m > 0$ allows us to write ($\tilde{\omega}_n \equiv -\omega_n - \Omega_m$):

$$\begin{aligned} \sum_{\omega_n} [\text{sgn}(\omega_n) + \text{sgn}(\omega_n + \Omega_m)] f(\omega_n) &= \sum_{\omega_n > 0} [2]f(\omega_n) + \sum_{\omega_n < -\Omega_m} [-2]f(\omega_n) = \sum_{\omega_n > 0} [2]f(\omega_n) + \sum_{\tilde{\omega}_n > 0} [-2]f(-\tilde{\omega}_n - \Omega_m) \\ &= 2 \sum_{\omega_n > 0} f(\omega_n) - f(-\omega_n - \Omega_m). \end{aligned} \quad (\text{A16})$$

This allows us to write

$$\begin{aligned} \Pi_U(\mathbf{q}, i\Omega_m) &= \frac{ig^2 T}{v_F} \sum_{\omega_n > 0} \int \frac{d\delta \tilde{k}_y}{2\pi} \frac{1}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2} \\ &\quad - \frac{1}{W(-i(\omega_n + \Omega_m)) + W(-i\omega_n) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2} \\ &= \frac{ig^2 T}{v_F} \sum_{\omega_n > 0} \int \frac{d\delta \tilde{k}_y}{2\pi} (2i) \text{Im} \left[\frac{1}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2} \right] \\ &= -\frac{2g^2 T}{v_F} \text{Im} \left[\sum_{\omega_n > 0} \int \frac{d\delta \tilde{k}_y}{2\pi} \frac{1}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}) - \kappa \delta \tilde{k}_y^2} \right] \\ &= -\frac{2g^2 T}{v_F} \text{Im} \left[\sum_{\omega_n > 0} \frac{-i}{2\sqrt{\kappa(W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q}))}} \right] \\ &= \frac{g^2 T}{v_F \sqrt{\kappa}} \text{Re} \left[\sum_{\omega_n > 0} \frac{1}{\sqrt{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q})}} \right], \end{aligned} \quad (\text{A17})$$

where we have used the fact that $W(-i\omega_n) = W^*(i\omega_n)$ since W is purely imaginary, and in performing the contour integral over $\delta \tilde{k}_y$, we have used the fact that $\text{Im}[W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \Omega_0(\mathbf{q})] > 0$ for $\omega_n, \Omega_m > 0$. The above expression gives insight into the origin of the threshold discussed in Sec. II B. In the extreme case of large $\Omega_0(\mathbf{q})$, the piece in the summation is almost purely imaginary, dominated by $-\Omega_0(\mathbf{q})$ inside the square root for small ω_n , so the real part becomes negligible and at large ω_n , the Ω_m dependence is washed away anyway. The summation over ω_n is divergent

and the physically meaningful Ω_m -dependent piece is captured by $\Delta \Pi_U(\mathbf{q}, i\Omega_m) \equiv \Pi_U(\mathbf{q}, i\Omega_m) - \Pi_U(\mathbf{q}, 0)$.

To gain insight into the scaling behavior of Π_U , we take $\delta q_x = q_y = 0 \Rightarrow \Omega_0(\mathbf{q}) = 0$ and approximate the Matsubara summation with continuous integrals to obtain

$$\begin{aligned} \Delta \Pi_U(\mathbf{q} = (\Delta_q, 0), i\Omega) &\propto \text{Re} \left[\int_0^\infty d\omega [W(i\omega) \right. \\ &\quad \left. + W(i(\omega + \Omega))]^{-1/2} - [2W(i\omega)]^{-1/2} \right]. \end{aligned} \quad (\text{A18})$$

We first approximate $W(i\omega) \approx -\Sigma(i\omega) \sim iB\omega^{2/3}$ deep in the non-Fermi-liquid regime. Then we rescale $\omega = \Omega\tilde{\omega}$ to find immediately that

$$\Pi_U(\mathbf{q} = (\Delta_q, 0), i\Omega) \sim \frac{g^2 \Omega^{2/3}}{v_F \sqrt{\kappa B}}, \quad (\text{A19})$$

which is what we discussed in Sec. III B. At higher temperatures it is safe to approximate $W(i\omega) \approx i\omega$, in which case the frequency rescaling remains the same, $\omega = \Omega\tilde{\omega}$, giving

$$\Pi_U(\mathbf{q} = (\Delta_q, 0), i\Omega) \propto \sqrt{\Omega}. \quad (\text{A20})$$

We also discuss what happens in three dimensions. When all is said and done, we end up with an expression that looks like

$$\begin{aligned} \Pi_U^{3D}(\mathbf{q} = (\Delta_q, 0), i\Omega_m) &\propto \int d\omega \cdots \int \frac{d\delta k_y d\delta k_z}{2\pi} \\ &\times \frac{1}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - \kappa \delta k_y^2 - \kappa \delta k_z^2}, \end{aligned} \quad (\text{A21})$$

which on moving to polar coordinates becomes

$$\begin{aligned} \Pi_U^{3D}(\mathbf{q} = (\Delta_q, 0), i\Omega_m) &\propto \int d\omega \\ &\cdots \int \frac{r dr}{W(i\omega_n) + W(i(\omega_n + \Omega_m)) - r^2}, \end{aligned} \quad (\text{A22})$$

which is now a simple logarithmic integral. How $W(i\omega)$ scales with ω does not influence how $\Pi_U^{3D}(i\Omega)$ scales with Ω . We still need to rescale $\omega = \Omega\tilde{\omega}$, and that gives us back the same

result as found in Sec. II C:

$$\Pi_U^{3D}(\mathbf{q} = (\Delta_q, 0), i\Omega) \propto \Omega. \quad (\text{A23})$$

APPENDIX B: SCATTERING RATE ESTIMATE

We start with the dressed boson propagator:

$$D^{-1}(\mathbf{q}, \Omega) = q^2 + aT - \eta \frac{i\Omega}{q} - i\lambda \frac{\Omega}{\sqrt{\Omega - \Omega_0(\mathbf{q})}}, \quad (\text{B1})$$

from which we can immediately write down the spectral function as

$$B(\mathbf{q}, \Omega) = \frac{\eta\Omega/q + \lambda \frac{\Omega}{\sqrt{\Omega - \Omega_0(\mathbf{q})}}}{(q^2 + aT)^2 + \left(\eta\Omega/q + \lambda \frac{\Omega}{\sqrt{\Omega - \Omega_0(\mathbf{q})}}\right)^2}. \quad (\text{B2})$$

We reuse the scattering rate estimate formula from [15]

$$\begin{aligned} \tilde{\gamma}(T) &\propto T \int_0^T d\Omega \int_{-\phi_{\max}}^{\phi_{\max}} d\phi \frac{B(q\phi, \Omega)}{\Omega} \\ &= T \int_0^T d\Omega \int_{-\phi_{\max}}^{\phi_{\max}} d\phi \\ &\times \frac{\eta q\phi^{-1} + \lambda(\Omega - \Omega_0(\mathbf{q}))^{-1/2}}{(q^2 + aT)^2 + \Omega^2(\eta q\phi^{-1} + \lambda(\Omega - \Omega_0(\mathbf{q}))^{-1/2})^2}. \end{aligned} \quad (\text{B3})$$

We choose T_3 as the temperature scale relative to which we evaluate the dimensionless scattering in terms of the $x \equiv T/T_3$ rate, which gives us

$$\tilde{\gamma}(x) = x \int_0^x d\tilde{\Omega} \int_{-\phi_{\max}}^{\phi_{\max}} d\phi \frac{\alpha^3 \tilde{q}_\phi^{-1} + \alpha^{3/2}(\tilde{\Omega} + \alpha^{-3} \sin^2(\phi)/4)^{-1/2}}{(\tilde{q}_\phi^2 + \alpha_M x)^2 + \tilde{\Omega}^2(\alpha^3 \tilde{q}_\phi^{-1} + \alpha^{3/2}(\tilde{\Omega} + \alpha^{-3} \sin^2(\phi)/4)^{-1/2})^2}. \quad (\text{B4})$$

In the above formula $\tilde{\Omega} \equiv T_3\Omega$, $\tilde{q}_\phi \equiv q_\phi/k_F$, $q_\phi^2 \equiv q(\theta = 0, \theta' = \pi - \phi) = k_F^2 + (\Delta_q + k_F)^2 - 2k_F(\Delta_q + \Delta_q)\cos(\phi)$, $\alpha \equiv [\Delta_q/k_F]/[g^2\varepsilon_F^{-1}]$, and $\cos(\phi_{\max}) \equiv k_F/(k_F + \Delta_q)$ where the corresponding angles and momentum definitions can be seen in Fig. 6.

We also need to use the fact that $\Omega_0(\mathbf{q}) \equiv v_F(\Delta_q - q_x) + \frac{\kappa}{4}q_y^2 = \tilde{\varepsilon}_\mathbf{q} - \frac{\kappa}{4}q_y^2$. When the scattering out electron lies exactly on the umklapp FS, we have $\tilde{\varepsilon}_\mathbf{q} = 0 \Rightarrow -\Omega_0(\mathbf{q}) = \frac{\kappa}{4}k_F^2 \sin^2(\phi)$, since $q_y = k_F \sin(\phi)$.

The analogous formula with the boson thermal mass but without Π_U from [15] is

$$\tilde{\gamma}(x) = x \int_0^x d\tilde{\Omega} \int_{-\phi_{\max}}^{\phi_{\max}} d\phi \frac{\alpha^3 \tilde{q}_\phi^{-1}}{(\tilde{q}_\phi^2 + \alpha_M x)^2 + \tilde{\Omega}^2(\alpha^3 \tilde{q}_\phi^{-1})^2}. \quad (\text{B5})$$

The value of α_M is set by the specific model of our choosing. For the SYK model, since we set $T_m = T_4$ this gives us $\alpha_M = \alpha$. For the naive large- N model, since we set $T_m = T_2$ this gives us $\alpha_M = \alpha^3$.

APPENDIX C: BOSON SELF-ENERGY CORRECTION ESTIMATES

We evaluate these corrections with the full fermion propagators, keeping $W(i\omega) \equiv i\omega - \Sigma(i\omega)$, which will allow us to interpolate between corrections to free fermion Π_U^0 or Π_U by setting $W(i\omega) \approx i\omega$, $-\Sigma(i\omega)$ above and below T_{NFL} , respectively.

1. MT correction

We start with [25]

$$\begin{aligned} \delta\Pi_U^{\text{MT}}(\mathbf{q} = (\Delta_q, 0), i\Omega) &\propto \int d\omega d\nu dk_x dk_y dl_x dl_y G(\mathbf{k}, i\omega) G(\mathbf{k} + \mathbf{q}, i\omega + i\Omega) \\ &\times G(\mathbf{k} + \mathbf{l}, i\omega + i\nu) G(\mathbf{k} + \mathbf{q} + \mathbf{l}, i\omega + i\Omega + i\nu) D(\mathbf{l}, i\nu). \end{aligned} \quad (\text{C1})$$

To proceed further, we make the following assumptions: (i) D is nearly independent of momenta in the x direction [25], which allows us to do the k_x, l_x integrals and use Eq. (A13),

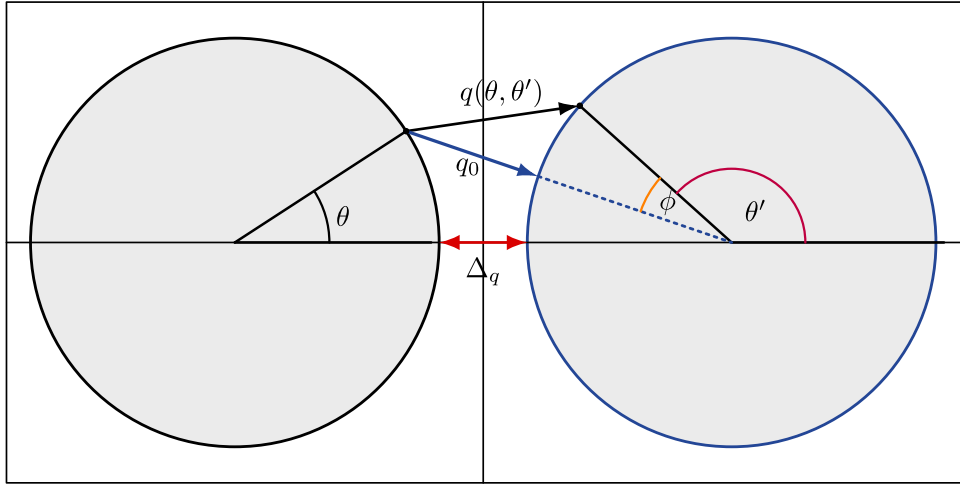


FIG. 6. The parametrization of the FS used to evaluate the scattering rate. The point $\mathbf{K}_F^x$ corresponds to the electron at $\theta = 0$, which sets $\theta' + \phi = \pi$. For this electron the scattering momenta $q_\phi \equiv q(\theta = 0, \theta' = \pi - \phi)$.

(ii) that l_y dependence in fermion propagator is ignored, which allows us to immediately perform $\int dl_y D(\mathbf{l}, i\nu) \propto \nu^{-1/3}$, which is essentially giving up on momentum conservation for the moment. Stressing over momentum conservation is important for correct prefactors, but for frequency scaling it is not essential (at least for this problem). When all is said and done, we end up with

$$\begin{aligned} \delta\Pi_U^{\text{MT}}(\mathbf{q} = (\Delta_q, 0), i\Omega) & \\ & \propto \int d\omega d\nu dk_y \frac{1}{W(i\omega) + W(i\omega + i\Omega) - \kappa k_y^2} \\ & \times \frac{1}{W(i\omega) + W(i\omega + i\Omega + i\nu) - \kappa k_y^2} \nu^{-1/3}. \end{aligned} \quad (\text{C2})$$

Irrespective of T , we have to scale $\omega \propto \Omega \tilde{\omega}$ and $\nu \propto \Omega \tilde{\nu}$, which gives $\int d\omega d\nu \nu^{-1/3} \propto \Omega^2 \cdot \Omega^{-1/3}$.

Now for $T \gg T_{\text{NFL}}$ we have $W(i\omega) \approx i\omega$, so the two denominators give Ω^{-2} and we need to scale $k_y \rightarrow \sqrt{\Omega} \tilde{k}_y$, finally giving us

$$\delta\Pi_U^{0,\text{MT}} \propto \sqrt{\Omega} \cdot \Omega^{-1/3} = \Omega^{1/6}. \quad (\text{C3})$$

For $T \ll T_{\text{NFL}}$ we have $W(i\omega) \approx i\omega^{2/3}$, so the two denominators give $\Omega^{-4/3}$, and we need to scale $k_y \rightarrow \Omega^{1/3} \tilde{k}_y$ to obtain

$$\delta\Pi_U^{\text{MT}} \propto \Omega^2 \cdot \Omega^{-1/3} \cdot \Omega^{1/3} \cdot \Omega^{-4/3} = \Omega^{2/3}. \quad (\text{C4})$$

2. SE correction

We only look at the frequency dependence of a single correction piece, keeping in mind that all other pieces have exactly the same frequency dependence.

The piece we look at is [26]

$$\begin{aligned} \delta\Pi_U^{\text{SE}}(\mathbf{q} = (\Delta_q, 0), i\Omega) & \propto \int d\omega dk_x dk_y \Sigma(i\omega + i\Omega) \\ & \times G^2(\mathbf{k} + \mathbf{q}, i\omega + i\Omega) G(\mathbf{k}, \omega). \end{aligned} \quad (\text{C5})$$

Here the frequency scaling can be peeled off very simply without the need for any further massaging of the expression.

We set $\omega \propto \Omega \tilde{\omega}$ throughout.

Now for $T \gg T_{\text{NFL}}$ we have $W(i\omega) \approx i\omega$, so the three denominators give Ω^{-3} and we need to scale $k_x \propto \Omega \tilde{k}_x$ and $k_y \propto \sqrt{\Omega} \tilde{k}_y$, finally giving us

$$\delta\Pi_U^{0,\text{SE}} \propto \Omega^2 \sqrt{\Omega} \cdot \Omega^{2/3} \Omega^{-3} = \Omega^{1/6}. \quad (\text{C6})$$

For $T \ll T_{\text{NFL}}$ we have $W(i\omega) \approx i\omega^{2/3}$, so the three denominators give $[\Omega^{2/3}]^{-3}$ and we need to scale $k_x \rightarrow \Omega^{2/3} \tilde{k}_x$ and $k_y \rightarrow \Omega^{1/3} \tilde{k}_y$ to obtain

$$\delta\Pi_U^{\text{SE}} \propto \Omega \cdot \Omega^{2/3} \cdot \Omega^{1/3} \cdot \Omega^{-2} \cdot \Omega^{2/3} = \Omega^{2/3}. \quad (\text{C7})$$

- [1] P. Coleman, *Introduction to Many-Body Physics* (Cambridge University Press, 2015).
- [2] L. D. Landau, The theory of a Fermi liquid, *Sov. Phys. JETP* **3**, 920 (1957).
- [3] S. A. Hartnoll and D. M. Hofman, Locally critical resistivities from Umklapp scattering, *Phys. Rev. Lett.* **108**, 241601 (2012).
- [4] S. A. Hartnoll, A. Lucas, and S. Sachdev, *Holographic Quantum Matter* (MIT Press, 2018).
- [5] H. Guo, A. A. Patel, I. Esterlis, and S. Sachdev, Large- n theory of critical Fermi surfaces. II. Conductivity, *Phys. Rev. B* **106**, 115151 (2022).

- [6] A. A. Patel, H. Guo, I. Esterlis, and S. Sachdev, Universal theory of strange metals from spatially random interactions, *Science* **381**, 790 (2023).
- [7] A. Gleis, S.-S. B. Lee, G. Kotliar, and J. von Delft, Emergent properties of the periodic Anderson model: A high-resolution, real-frequency study of heavy-fermion quantum criticality, *Phys. Rev. X* **14**, 041036 (2024).
- [8] A. Gleis, S.-S. B. Lee, G. Kotliar, and J. von Delft, Dynamical scaling and Planckian dissipation due to heavy-fermion quantum criticality, *Phys. Rev. Lett.* **134**, 106501 (2025).

- [9] Y.-L. Wang, X.-H. Ge, and S.-J. Sin, Linear- t resistivity from spatially random vector coupling, *Phys. Rev. B* **111**, 115135 (2025).
- [10] I. Paul, C. Pépin, and M. R. Norman, Multiscale fluctuations near a Kondo breakdown quantum critical point, *Phys. Rev. B* **78**, 035109 (2008).
- [11] I. Paul, C. Pépin, and M. R. Norman, Kondo breakdown and hybridization fluctuations in the Kondo-Heisenberg lattice, *Phys. Rev. Lett.* **98**, 026402 (2007).
- [12] I. Paul, C. Pépin, and M. R. Norman, Equivalence of single-particle and transport lifetimes from hybridization fluctuations, *Phys. Rev. Lett.* **110**, 066402 (2013).
- [13] E. E. Aldape, T. Cookmeyer, A. A. Patel, and E. Altman, Solvable theory of a strange metal at the breakdown of a heavy Fermi liquid, *Phys. Rev. B* **105**, 235111 (2022).
- [14] X. Wang and E. Berg, Scattering mechanisms and electrical transport near an Ising nematic quantum critical point, *Phys. Rev. B* **99**, 235136 (2019).
- [15] P. A. Lee, Low-temperature t -linear resistivity due to Umklapp scattering from a critical mode, *Phys. Rev. B* **104**, 035140 (2021).
- [16] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, 2011).
- [17] S. Sachdev, *Quantum Phases of Matter* (Cambridge University Press, 2023).
- [18] D. Chowdhury, A. Georges, O. Parcollet, and S. Sachdev, Sachdev-Ye-Kitaev models and beyond: Window into non-Fermi liquids, *Rev. Mod. Phys.* **94**, 035004 (2022).
- [19] S.-S. Lee, Low-energy effective theory of Fermi surface coupled with U(1) gauge field in $2 + 1$ dimensions, *Phys. Rev. B* **80**, 165102 (2009).
- [20] I. Mandal and S.-S. Lee, Ultraviolet/infrared mixing in non-Fermi liquids, *Phys. Rev. B* **92**, 035141 (2015).
- [21] D. Dalidovich and S.-S. Lee, Perturbative non-Fermi liquids from dimensional regularization, *Phys. Rev. B* **88**, 245106 (2013).
- [22] J. A. Damia, S. Kachru, S. Raghu, and G. Torroba, Two-dimensional non-Fermi-liquid metals: A solvable large- n limit, *Phys. Rev. Lett.* **123**, 096402 (2019).
- [23] Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, Gauge-invariant response functions of fermions coupled to a gauge field, *Phys. Rev. B* **50**, 17917 (1994).
- [24] M. A. Metlitski and S. Sachdev, Quantum phase transitions of metals in two spatial dimensions. I. Ising-nematic order, *Phys. Rev. B* **82**, 075127 (2010).
- [25] Z. D. Shi, D. V. Else, H. Goldman, and T. Senthil, Loop current fluctuations and quantum critical transport, *SciPost Phys.* **14**, 113 (2023).
- [26] Y. Gindikin and A. V. Chubukov, Fermi surface geometry and optical conductivity of a two-dimensional electron gas near an Ising-nematic quantum critical point, *Phys. Rev. B* **109**, 115156 (2024).
- [27] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil, Controlled expansion for certain non-Fermi-liquid metals, *Phys. Rev. B* **82**, 045121 (2010).
- [28] V. Mishra, Scattering rates, self-energy codes, and the data set are available upon reasonable request, Zenodo, 2026, <https://doi.org/10.5281/zenodo.19814477>.