

Enhancing decoding performance using efficient error learning

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(Received 23 July 2025; revised 19 February 2026; accepted 26 February 2026; published 27 May 2026)

Lowering the resource overhead needed to achieve fault-tolerant quantum computation is crucial to building scalable quantum computers. We show that adapting conventional maximum likelihood (ML) decoders to a small subset of efficiently learnable physical error characteristics can significantly improve the logical performance of a quantum error-correcting code. Specifically, we leverage error information obtained from efficient characterization methods based on cycle error reconstruction (CER), which yields Pauli error rates on the n qubits of an error-correcting code. Although the total number of Pauli error rates needed to describe a general noise process is exponentially large in n , we show that only a few of the largest few Pauli error rates are needed and that a heuristic technique can complete the Pauli error distribution for ML decoding from this restricted dataset. Using these techniques, we demonstrate significant performance improvements for decoding quantum codes under a variety of physically relevant error models. For instance, with CER data that constitute merely 1% of the Pauli error rates in the system, we achieve a 10 times gain in performance compared with the case where decoding is based solely on the fidelity of the underlying noise process. Our conclusions underscore the promise of recent error characterization methods for improving quantum error correction and lowering overheads.

DOI: [10.1103/vtj7-nh6l](https://doi.org/10.1103/vtj7-nh6l)

I. INTRODUCTION

Large-scale quantum computers must be designed to ensure reliable computation in the presence of noise. The theory of fault-tolerant quantum computation makes it possible to encode quantum information and perform an arbitrarily large quantum circuit [1,2], at the expense of a substantial overhead in terms of the number of physical qubits needed to store a logical qubit and the number of physical devices required to execute a logical operation.

The overhead poses a significant challenge to realizing the guarantees of fault-tolerant quantum computation using current hardware. Parallel advances have been made in various aspects of a fault-tolerant scheme to reduce the overhead required [3,4] to achieve a suppression of noise that affects the logical information, i.e., the logical error rate. These include the development of new quantum error-correcting codes [5,6], and enhancing the error correction capabilities for existing ones [7–10].

Most decoding algorithms make simplified assumptions about the underlying error model, potentially reducing the performance of quantum error correction (QEC). Ideally, we aim to optimize decoding to the experimentally measurable characteristics of the specific noise process affecting our physical system. Some recent work has taken this direction. In Refs. [7,11–15], surface code decoders were adapted to an inherent bias [5,8,16] between the probabilities of X -type and Z -type errors and to the correlated

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errors in Ref. [17] observed in some hardware devices. A bias is only one of the exponentially many features that describe the noisy dynamics of a quantum system. In some cases, the decoders have been adapted to specific instances of non-Pauli noise, including rotation errors [18–20], correlated errors [21], amplitude damping noise [22], erasure [5,23,24], leakage [25–27], and circuit-level noise for fault tolerance [28,29]. In general, fully adapting a decoding algorithm to the microscopic details of an underlying physical noise process is achieved by maximum likelihood (ML) decoders [30–35]. Logical error suppression with an ML decoder indicates the best achievable performance for a code and error-model pair. However, estimating this quantity presents challenges, depending on the family of codes. Firstly, ML decoding is computationally intractable for general stabilizer codes; concatenated codes are among the few families of quantum error-correcting codes known to have an efficient ML decoder [36]. Secondly, an ML decoder requires precise knowledge of the underlying physical error model, which normally requires an exponential number $\mathcal{O}(4^n)$ of parameters to represent and an exponential number of experiments to learn. Unfortunately, process tomography methods to fully characterize an error process are neither scalable nor generally accurate [37–39]. The problem of inefficient error representation can be overcome with recursive techniques for concatenated codes, and the problem of exponential experimental overhead can be overcome thanks to the discovery of efficient error-learning techniques [40–42], both of which are described below.

In this paper, we introduce and demonstrate a method that achieves a significant enhancement in the performance of quantum error correction through a decoding strategy informed by error model information obtained through efficient state-of-the-art error-learning experiments. We show that the suppression of logical error rates consistently improves with the amount of noise characterization data available, achieving an order-of-magnitude improvement even when the calibration data include only a small fraction of the total parameters required to fully characterize all the Pauli error rates in the underlying noise process. Our findings rely on a class of scalable Pauli-error-learning techniques based on the Pauli eigenvalue estimation ideas of Emerson *et al.* [40] and the cycle benchmarking protocol of Erhard *et al.* [41], which culminate in the analysis and robustness guarantees of the cycle error reconstruction (CER) protocol implemented in Ref. [43] and described in Ref. [42]. The CER protocol enables efficient estimation of a subset of the Pauli error rates of an error channel to within multiplicative precision. The feasibility of these error-learning methods has already been experimentally demonstrated across several platforms [40–44], and most recently in Ref. [45] in which this approach was applied to learn a subset of the Pauli error rates on the physical qubits that was small enough to be scalable but large enough to

reliably infer the resulting logical error rate in a transversal controlled-NOT (CNOT) gate implemented on a 16-qubit trapped-ion quantum computer. Similarly, our proposed method does not rely on learning all of the (exponentially many) parameters of the error channel, but requires only a very limited subset of Pauli error rate data characterizing the K -largest Pauli error rates in a noise process, where $K \ll 4^n$. The practical relevance of this input dataset can be justified on physical grounds by postprocessing ideas that extend Pauli eigenvalue estimation [40] and cycle benchmarking [41], to show that the K -largest Pauli error rates on n physical qubits can be extended to all error rates in the system using a number of experiments proportional to K , given certain strong but plausible assumptions on the (otherwise unknown) error model [46,47].

Our methodology is twofold. First, we design a heuristic technique to reconstruct a complete set of Pauli error probabilities from limited error characterization data. Second, we use an ML decoder using the reconstructed Pauli error distribution. Our tools are designed for the family of concatenated quantum codes [48,49], which are among the few families of quantum error-correcting codes known to have an efficient ML decoder [36].

This paper is organized as follows. Section II provides the necessary background and essential mathematical foundations to lay the foundation for the main contribution of this paper, which is detailed in Sec. III. In Sec. IV C, we demonstrate the efficacy of our approach by numerically estimating the reduction in logical error rates achieved for concatenated Steane codes under various realistic error models. The findings highlight the potential of cleverly using a small portion of the CER data to significantly enhance performance, surpassing the results obtained without adapting the decoding techniques to the underlying error model. Finally, in Sec. V, we provide concluding remarks and outline interesting future directions.

II. BACKGROUND

This section provides the technical background necessary to obtain a thorough understanding of our results. It is organized as follows. We begin with a brief introduction to the mathematical modeling of physical noise processes. Then, in Sec. II B, we discuss an important method of estimating the probabilities of Pauli errors in an error model, which sets the context for the research problem we want to address. Finally, in Sec. II C, we recall some basic knowledge of quantum error correction for concatenated codes using the stabilizer formalism.

A. Physical error processes

Markovian processes encompass a wide range of physically motivated noise mechanisms. Typically, Markovian noise arises from a system's interactions with its environment, and is mathematically described by completely

positive trace-preserving (CPTP) maps [50–53]. These maps, denoted as $\mathcal{E}$, transform any quantum state ρ into its *noisy* counterpart, $\mathcal{E}(\rho)$.

An interesting case of CPTP maps is unitary errors:

$$\mathcal{E}_U(\rho) = U\rho U^\dagger. \quad (1)$$

Another popular class of CPTP maps is *Pauli channels*, where the noisy state $\mathcal{E}(\rho)$ is described as a mixture of any of the Pauli matrices applied to ρ :

$$\mathcal{E}_P(\rho) = \sum_{P \in \mathcal{P}_n} \text{Prob}(P) P \rho P, \quad (2)$$

Coherent errors and Pauli errors constitute two extremes of the noise spectrum described by CPTP maps.

We can describe the effect of a general CPTP map as

$$\mathcal{E}(\rho) = \sum_{P, Q \in \mathcal{P}_n} \chi_{P, Q}(\mathcal{E}) P \rho Q, \quad (3)$$

where $\chi_{P, Q}(\mathcal{E})$ are complex numbers that correspond to a unique pair of n -qubit Pauli matrices $P, Q \in \mathcal{P}_n$. The Hermitian matrix $\chi(\mathcal{E})$ with unit trace is known as the χ -matrix representation of $\mathcal{E}$ [54–56]. We will often refer to the diagonal entries of the χ -matrix, $\{\chi_{P, P}(\mathcal{E})\}_{P \in \mathcal{P}_n}$, as Pauli error rates. For a Pauli channel in Eq. (2), $\chi_{P, P}$ denotes the probability of a Pauli error P .

Various metrics exist to quantify the impact of noise described by a CPTP map [57–60]. Among these, average fidelity

$$F_g = \int d\psi \text{Tr}(|\psi\rangle\langle\psi| \mathcal{E}(|\psi\rangle\langle\psi|)) \quad (4)$$

is one of the most common choices, particularly because there are efficient experimental protocols to estimate it. We will use a closely related quantity, denoted by $\epsilon(\mathcal{E})$, referred to as the “process infidelity” in Refs. [41, 61–63], to quantify the *physical error rate*:

$$\epsilon(\mathcal{E}) = \frac{3}{2}(1 - F_g). \quad (5)$$

It can be shown that $\epsilon(\mathcal{E}) = 1 - \chi_{0,0}(\mathcal{E})$ [64, 65].

B. Error learning and cycle error reconstruction

An efficient noise characterization approach should focus on acquiring (only) the key parameters necessary to understand the critical effects of the underlying noise process on fault-tolerance schemes. One such important parameter is the average fidelity of a noise process, denoted by $\epsilon(\mathcal{E})$, which can be efficiently estimated by means of randomized benchmarking (RB) experiments [66–69]. While RB experiments provide information about

the fidelity averaged over a set of Clifford operations, learning more fine-grained error information is critical to optimizing control and predicting error correction performance. In particular, it was critical to devise methods to learn (1) more fine-grained error properties than just the fidelity $\epsilon(\mathcal{E})$, which is only one of the $\mathcal{O}(4^{2n})$ parameters that completely specify an n -qubit physical noise process $\mathcal{E}$, and (2) the error properties associated with specific cycles of interest (e.g., parallel sets of gate operations) rather than an error map averaged over a set of Clifford operations.

The solution to (1) was developed in 2007 by Emerson *et al.* [40], who demonstrated that symmetrizing (or twirling) a quantum channel, e.g., an error channel or an “error oracle,” via $\mathcal{P}_n$ simplifies the channel to a *diagonal* Pauli-transfer matrix (PTM), i.e., a Pauli channel. Moreover, the individual eigenvalues of that channel can then be efficiently learned (i.e., experimentally measured) by propagation of a Pauli operator through the twirled channel. This enables a protocol for learning the eigenvalues of the error channel. Moreover, the full set of Pauli operators can be generated by applying local Clifford operations drawn from $\mathcal{C}_1^{\otimes n}$ to the $|0\rangle^{\otimes n}$ state and applying their inverse to the output state. When the local Clifford operators are sampled randomly, this is equivalent to a local Clifford twirl of the error channel, which creates degeneracies in the PTM eigenvalues. Emerson *et al.* [40] further demonstrated that the error probabilities of the PTM can be related to the learned eigenvalues via an invertible linear transformation, and showed that *efficient* learning of error probabilities of the PTM is possible via postprocessing by making reasonable assumptions on the correlations present in the error model. Emerson *et al.* [40] showed, as an example, that efficient learning of all low-weight error probabilities is possible assuming permutation symmetry over the qubits, which is equivalent to measuring only the Hamming weight of the output strings under the Pauli-channel learning experiment. These ideas were then demonstrated experimentally on a three-qubit nuclear magnetic resonance system [40].

The solution to (2) was developed in the cycle benchmarking protocol of Erhard *et al.* [41] by their generalizing the error amplification ideas of RB to the setting of Pauli-channel learning. In particular, Erhard *et al.* [41] showed that by repeating a fixed cycle of interest under randomized compiling [70], one can apply the methods of Ref. [40] to learn the eigenvalues of the error channel *associated with a specific cycle of interest*. In this way, the cycle benchmarking method creates a Pauli channel in place of the memory channel, or the “oracle” assumed in the Pauli-channel learning protocol of Emerson *et al.* [40]. As a result, the eigenvalues and error probabilities of the error channel of a specific cycle can be learned to multiplicative precision in a state preparation and measurement-free way.

This protocol was then demonstrated experimentally on a ten qubit ion trap [41].

These ideas have been combined into the cycle benchmarking technique known as cycle error reconstruction [42]: an experimental protocol to measure the individual 4^n Pauli eigenvalues and Pauli error rates (of an error map $\mathcal{E}$ associated with any cycle or any circuit block of interest) to within multiplicative precision and with rigorous robustness guarantees. Appendix A provides an overview of CER. These protocols are practical and efficient, and have been demonstrated experimentally on multiple small-to-medium-scale quantum computing platforms [40–45]. Subsequent work [46,47] leverages the solutions to (1) and (2) and adds additional postprocessing steps to justify, under assumptions on the allowed error model correlations, direct estimation of the K -largest Pauli error rates of an error channel $\mathcal{E}$ using a number of experiments that scales linearly with K [47].

In this work we apply and extend the ideas described above with a focus on enabling the efficient learning of the errors that are relevant in the context of enhancing decoding for quantum error correction.

C. Quantum error correction

In this section, we outline the essential concepts of quantum error correction using concatenated codes within the stabilizer formalism. Our focus is on correcting errors described by a general CPTP map, especially including non-Pauli error. For a more elaborate explanation, see Appendix B and the references cited therein.

An $[[n, k, d]]$ stabilizer code [71] is given by the common eigenspace with an eigenvalue $+1$ of $n - k$ commuting operators $S_1, \dots, S_{n-k}$, called “stabilizer generators.” The group $\mathcal{S} = \langle S_1, \dots, S_{n-k} \rangle$ is called the “stabilizer group.” The encoded version of a k -qubit state is the n -qubit state denoted by $\bar{\rho}$. For the results in this paper, we limit ourselves to codes that encode only one logical qubit, i.e., $k = 1$.

An $[[n, k]]$ stabilizer group can be used to derive a canonical choice of a basis for the n -qubit Pauli group. To start with, we have $n - k$ stabilizer generators for $\mathcal{S}$. We define the group of logical operators for a code, denoted by $\mathcal{L} = \mathcal{N}(\mathcal{S})/\mathcal{S}$, where $\mathcal{N}(\mathcal{S})$ is the normalizer of the stabilizer group in the Pauli group. Note that $\mathcal{L}$ consists of $2k$ generators. The remaining $n - k$ generators, labeled $T_1, \dots, T_{n-k}$, form the group of *pure errors* [30,72,73], denoted by $\mathcal{T}$, where $\mathcal{T} = \mathcal{N}(\mathcal{L})/\mathcal{S}$. Consequently, an n -qubit Pauli error E can be written as a product of three terms:

$$E = T \cdot L \cdot S \quad (6)$$

where $S \in \mathcal{S}$, $L \in \mathcal{L}$, and $T \in \mathcal{T}$.

QEC in the stabilizer formalism is a twofold procedure. The first step includes measuring each of the stabilizer generators on the noisy state $\mathcal{E}(\bar{\rho})$. The outcomes of the measurement form the error syndrome, denoted by $\vec{s}$. While T is entirely fixed by the syndrome $\vec{s}$: $T \equiv T_s$, the problem of determining the most likely logical operator, denoted by L_s , is called “maximum likelihood decoding”:

$$L_{\vec{s}} = \operatorname{argmax}_{L \in \mathcal{L}} \operatorname{Prob}(L | s),$$

where the probability of a logical operator $\operatorname{Prob}(L | s)$ is given by

$$\operatorname{Prob}(L | s) = \sum_{S \in \mathcal{S}} \operatorname{Prob}(T_s \cdot L \cdot S).$$

When selecting a decoder for a quantum error-correcting code, one must balance efficiency and accuracy. Although ML decoding is theoretically optimal, it remains computationally intractable [30]. In contrast, *minimum-weight* decoding is an alternative suboptimal strategy with lower thresholds but offers high efficiency, as it can be implemented with the use of a lookup table [74,75].

With a slight loss of generality, we assume that syndrome measurement and the application of the recovery operation prescribed by the decoder L_s are executed perfectly. Under this perfect quantum error correction assumption, the combined effect of noise and quantum error correction preserves the codespace. We refer to this composite process as an effective channel and characterize it by its direct action on encoded states, described by a k -qubit CPTP map, denoted as $\mathcal{E}_1^s$. The *average effective channel*, $\bar{\mathcal{E}}_1$, is then given by

$$\bar{\mathcal{E}}_1 = \sum_{\vec{s} \in \mathbb{Z}_2^{n-k}} \operatorname{Prob}(\vec{s}) \mathcal{E}_1^s,$$

which quantifies the average strength of noise that remains in the encoded information after quantum error correction. We define the *logical error rate* as the infidelity of $\bar{\mathcal{E}}_1$.

Concatenation is a well-established technique for constructing a family of codes with increasing sizes that exhibit a threshold [71,76–78]. This approach involves recursively encoding the physical qubits of a quantum error-correcting code into another quantum error-correcting code. Performing this process ℓ times results in the *level- ℓ* concatenated code. For simplicity, we consider the same code at each level. Appendix B 2 outlines the computation of the average effective channel $\bar{\mathcal{E}}_\ell$, which characterizes the combined impact of noise and QEC on the level- ℓ concatenated code.

III. WORKING WITH PARTIAL NOISE CHARACTERIZATION INFORMATION

Recall that a maximum likelihood decoder defined in Eq. (B7) in Appendix B 1 requires complete knowledge of the error model—namely, the 4^n Pauli error rates for an $[[n, k, d]]$ quantum error-correcting code. However, in a practical scenario, we assume that only a handful of these correspond to the K -largest Pauli error rates estimated with the use of techniques developed in Ref. [47], where $K \ll 4^n$. In what follows, we use $\mathcal{N}_K$ to denote the set of Pauli errors that correspond to the K -largest Pauli error rates, χ_K to refer to the respective error rates, and use $\mathcal{D}_K$ to denote a decoder that computes a recovery operation conditioned on an error syndrome s , given χ_K .

The limited data, constituting only K Pauli error rates, are insufficient to directly use an ML decoder. We need to first complete the Pauli error distribution by assigning probabilities to those errors in $\mathcal{P}_n$ that are not in $\mathcal{N}_K$. Hence, given an error syndrome, we design $\mathcal{D}_K$ to apply a twofold strategy:

- (1) Apply a heuristic algorithm to obtain the error rates of all 4^n Pauli error rates from $\mathcal{N}_K$ affecting a code block.
- (2) Apply an ML decoding strategy given the error syndrome and the heuristically reconstructed Pauli error distribution from step (1).

A special case of the limited error characterization data is where CER is used to estimate only the probabilities of the single-qubit errors are known. A common assumption used to infer the probabilities of higher-weight errors is that errors are independently and identically distributed across qubits. Under this assumption, the probability of a high-weight error is determined by the product of the probabilities of the single-qubit errors that form the tensor product representation of the multiqubit error. Inspired by this approach, we design a greedy heuristic algorithm that leverages the probabilities of Pauli errors in $\mathcal{N}_K$ to estimate the distribution of all 4^n Pauli errors affecting the code. Our heuristic algorithm, which we call “uncorrelated split search” (USS), operates on the following principle. Consider an n -qubit Pauli error P that is not included in $\mathcal{N}_K$. Note that P can be expressed as $P = P_1 \cdot P_2$ for various choices of Pauli errors P_1 and P_2 acting on disjoint subspaces. In essence, P_1 acts on a subspace of dimension 2^m , while P_2 acts on a subspace of dimension $2^{(n-m)}$. For each of these decompositions, our estimate of the probability of P is merely the product of the probabilities of P_1 and P_2 : $\text{Prob}(P) = \text{Prob}(P_1)\text{Prob}(P_2)$. Each bipartition of the n physical qubits into sets B_1 and B_2 corresponds to an error $P_1 \equiv P_1(B_1)$ with nontrivial support in B_1 and another error $P_2 \equiv P_2(B_2)$ with nontrivial support in B_2 . Denoting the set of bipartitions of n qubits as $\mathcal{B}$, we find that the overall probability of the error P adopts a straightforward

sum-product structure:

$$\text{Prob}(P) = \sum_{B_1, B_2 \in \mathcal{B}(P)} \text{Prob}(P_1(B_1))\text{Prob}(P_2(B_2)). \quad (7)$$

Physically speaking, we are assuming that the error P can occur whenever both P_1 and P_2 occur. It should be noted that there are several ways to combine two n -qubit errors to produce P . However, our heuristic ignores the possibilities where the errors in the decomposition have overlapping supports. Since P_1 and P_2 act on fewer than n qubits, in a physically plausible error model we are more likely to find the error probabilities of P_1 and P_2 in the CER data provided $\mathcal{N}_K$. If they are not found, we recursively apply the aforementioned strategy to both P_1 and P_2 until their probabilities are found in $\mathcal{N}_K$.

While it is reasonable to assume that $\mathcal{N}_K$ contains all single-qubit errors, this is not always the case when $K < 3n + 1$ or if the physical noise model exhibits strong correlations. In such scenarios, we might encounter a single-qubit error P_1 whose probability cannot be inferred from $\mathcal{N}_K$. To address this, we establish the base case where $K = 0$, corresponding to $\mathcal{N}_0 = I$. This implies that we always know the average physical error rate $\epsilon(\mathcal{E})$. On the basis of this assumption, we use a simple ansatz to assign the probability of any single-qubit error P :

$$\text{Prob}(P) = \frac{1 - (1 - \epsilon)^{1/n}}{3} (1 - \epsilon^{1/n})^{n-1}, \quad (8)$$

where $(1 - \epsilon)^{1/n}$ is the fidelity of a single qubit depolarizing channel, assuming that $\mathcal{E}$ is an independent and identically distributed depolarizing channel with average physical error rate $\epsilon(\mathcal{E})$. We are now ready to describe the formal algorithm of the USS method stated in Algorithm 1.

It is important to note that every Pauli error $P \in \mathcal{P}_n$ is visited only once when it is assigned a probability. Once its probability is assigned, it is added to the set $\hat{\mathcal{N}}$. After this, its probability is simply looked up from $\hat{\mathcal{N}}$ whenever needed. At the end of the algorithm, the set $\hat{\chi}_K$ should contain all the 4^n Pauli error probabilities. Hence, the time complexity of our USS method in deriving the entire Pauli error distribution is linear in the number of Pauli errors.

Note that the USS technique can be applied to any dataset containing the $K \leq 4^n$ Pauli error rates. Flammia and Wallman [46] presented general methods for estimating error rates for Pauli errors from an arbitrary set, such as low-weight Pauli errors. Subsequently, high-weight errors are estimated by assuming a correlation length within the system.

The USS technique produces a complete set of 4^n Pauli error rates that can serve as inputs to a decoder. In this work, we focus on a scenario where these 4^n rates feed into an ML decoder for a concatenated code built from code blocks of size n . However, the applicability of

ALGORITHM 1. The pseudocode describes the uncorrelated split search strategy, a greedy algorithm used to estimate the probabilities of all 4^n Pauli errors, leveraging the probabilities of K Pauli errors from CER. We call the function below with inputs $\mathcal{N}_K, \chi_K$. Upon completion, $\hat{\mathcal{N}}$ will include all 4^n Pauli errors, and $\hat{\chi}_K$ will include their respective probabilities, which will serve as input to the ML decoder.

```

1: procedure UNCORRELATED SPLIT SEARCH( $P, \hat{\mathcal{N}}, \hat{\chi}_K$ )
2:   if  $P$  is found in  $\hat{\mathcal{N}}$  then
3:     return  $\chi_{P,P}(\mathcal{E})$ 
4:   else if  $P$  is a single qubit error then
5:      $\epsilon_0 \leftarrow 1 - (1 - \chi_{I,I})^{1/n}$ 
6:      $p_\star \leftarrow \frac{\epsilon_0}{3}(1 - \epsilon_0)^{n-1}$ 
7:   else
8:      $p_\star \leftarrow 0$ 
9:     for  $B_1, B_2 \in \mathcal{B}(P)$  do
10:       $p_1 \leftarrow \text{UNCORR SPLIT SEARCH}(P_1(B_1), \hat{\mathcal{N}}, \hat{\chi}_K)$ 
11:       $p_2 \leftarrow \text{UNCORR SPLIT SEARCH}(P_2(B_2), \hat{\mathcal{N}}, \hat{\chi}_K)$ 
12:       $p_\star \leftarrow p_\star + p_1 p_2$ 
13:   Add the error  $P_i$  and its probability  $p_\star$  to  $\hat{\mathcal{N}}$ .
14:   return  $p_\star$ 
    
```

the USS technique is not limited to ML decoding. The same microscopic Pauli error rates can be supplied to any decoder that accepts such fine-grained error information, such as weighted minimum-weight perfect-matching decoders [79] and belief-propagation decoders [80,81].

IV. RESULTS

Our goal is to demonstrate the importance of CER data and the *uncorrelated split search* technique in improving the performance of the ML decoder for concatenated codes. We achieve this by presenting numerical evidence across a broad range of realistic error models. In particular, we investigate a case where only the largest K Pauli error rates are extracted with the use of CER. Our aim is to prove that our methods are widely applicable to various coherent and incoherent errors characterized by CPTP maps, going beyond error models that can be closely approximated by Pauli errors.

Our numerical analysis focuses on concatenated Steane codes; however, our conclusions remain qualitatively similar for other families of concatenated codes as well. We use n to denote the size of a code block in the concatenated code; $n = 7$ for the Steane code. The total number of physical qubits in the level- ℓ concatenated Steane code is n^ℓ . So, a realistic error model for the concatenated Steane code is described by a CPTP map on n^ℓ qubits. Recall that the n^ℓ physical qubits of a level- ℓ concatenated code belong to $n^{\ell-1}$ disjoint code blocks. Since the encoding circuit of the concatenated code shown in Fig. 4 resembles a tree where every layer is an identical encoding circuit, we assume that noise acts in an independent and identical fashion on all the $n^{\ell-1}$ code blocks of the concatenated code [19,60,82].

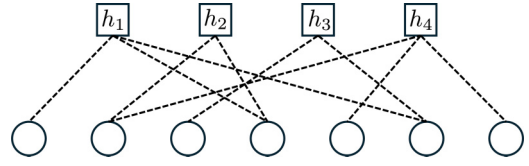


FIG. 1. A factor graph used for constructing an error in the CG1D error model. Each square node specifies a local interaction supported on the qubits of a Steane code block, represented by circular nodes that are incident to the respective square node. In the example shown, the local interactions denoted by h_1, h_2, h_3 , and h_4 are supported on qubits $\{1, 4, 6\}$, $\{2, 4\}$, $\{3, 6\}$, and $\{2, 6, 7\}$, respectively. The sum of these local interactions is an n -qubit Hamiltonian denoted by H which is used to construct a unitary error $U = \exp\left(-it \sum_{i=1}^4 h_i\right)$.

A. Error models

We want to numerically investigate the performance gain in QEC achieved by leveraging a small fraction of the CER data and the USS algorithm across a broad class of physically motivated error models. While the independent and identically distributed Pauli error model facilitates code analysis and design, extension beyond the Pauli framework is essential to accurately capturing realistic noise [73,83–85]. In most experimental implementations, noise is characterized by a quantum channel that exhibits spatial correlations and varies between qubits. Additionally, noise correlations arise from the engineering challenge of addressing individual qubits without inadvertently affecting their neighbors [21,86–89].

A broad class of realistic noise processes can be physically modeled by considering qubit interactions governed by a Hamiltonian H [90–93]. The resulting unitary noise is given by $U = \exp(iHt)$, where $t \geq 0$ serves as a tunable parameter controlling the noise strength. Of particular interest are Hamiltonians expressed as a sum of local interactions: $H = \sum_i h_i$, where h_i are random Hermitian matrices of dimension 2^m for $m \leq n$. Such local interactions are characteristic of Hamiltonians commonly studied in many-body physics. These error models, referred to as coarse-grained one-dimensional (CG1D) error models in Refs. [46,94], are effectively represented by the factor graph, as shown in Fig. 1, which illustrates the support of these local interactions. In particular, CG1D models have been shown to provide an accurate description of the physical noise processes that affect certain Google devices [94].

We perform numerical studies on random instances of the CG1D error model by generating random Hamiltonians describing interactions between the qubits of a Steane code block in a concatenated code. To construct a random Hamiltonian, we sample local Hamiltonians supported on a subset of the seven qubits in the Steane code block. We

choose the sizes of these subsets from a Poisson distribution whose mean is fixed to reflect a mean correlated length λ in a system. By setting λ to be greater than 1, we can sample a large number of local interactions that involve multiple qubits.

B. Quantitative estimation of the performance gain

We want to demonstrate a performance improvement with the ML decoder under a variety of noise models as described earlier with the amount of error characterization data from CER supplied to the decoder. However, first, we want to establish a baseline for comparison by defining a reference scenario representing the performance of a decoder $\mathcal{D}_1$, which has access only to the average fidelity of the underlying physical noise process. Consequently, for any given physical noise process, we intend to compare two metrics: (1) the performance of a decoder $\mathcal{D}_K$ for $K > 0$ against (2) the performance of $\mathcal{D}_1$. We gauge this comparison qualitatively using Δ_ℓ for a level- ℓ concatenated code, which is defined as

$$\Delta_\ell(K) = \frac{\bar{r}_\ell(\mathcal{E}; \mathcal{D}_1)}{\bar{r}_\ell(\mathcal{E}; \mathcal{D}_K)}, \quad (9)$$

where we have used $\bar{r}_\ell(\mathcal{E}; \mathcal{D}_K)$ to denote the average logical error rate for the level- ℓ concatenated code, where $\mathcal{E}$ is the physical noise process and $\mathcal{D}_K$ is the decoder used. In what follows we refer to $\Delta_\ell(K)$ as “gain” [95].

Note that a gain larger than 1 signifies a relative performance gain that is achieved by the ML decoder that has access to K Pauli error rates in $\mathcal{N}_K$, in conjunction with the USS method in Algorithm 1. Across a range of physically relevant error models, we consistently observe a gain exceeding 1, which consistently improves with the fraction of CER information accessible by the decoder. Notably, even when only 1% of the total CER information is available to the decoder, the observed gain is approximately 10.

Note that the gain defined in Eq. (9) strictly depends on the microscopic details of the underlying physical noise process $\mathcal{E}$. However, for a broader analysis of typical performance improvements across a range of random instances of $\mathcal{E}$, we examine the gain within an ensemble of random physical noise processes having similar levels of physical noise strength, quantified by their error rates. We categorize these physical noise channels on the basis of their error rates falling within specific intervals, forming what we call a “bin”: $\mathfrak{B} = \mathcal{E} : b_1 \leq r(\mathcal{E}) < b_2$. For each bin, we report the median gain observed over all physical noise processes in the bin, since it reflects the most frequently observed gain in the corresponding ensemble of physical noise processes. In our graphical presentations, we distinguish bins with different colors and provide the average physical as well as logical error rate of the physical noise processes in each bin, along with the size of the bin.

C. Performance analysis

Figure 2 shows gains for the level-2 concatenated Steane code achieved with an ML decoder that leverages Pauli error rates in $\mathcal{N}(K)$ along with the USS method for several values of K . The trends in Fig. 2 are drawn from an ensemble of 336 unitary errors drawn from a CG1D error model. Each unitary error is constructed by our exponentiating a Hamiltonian multiplied by a time parameter t , chosen between 0.001 and 0.1. The Hamiltonian is composed of random local Hamiltonians that are supported on random subsets of qubits whose sizes are sampled from the Poisson distribution with mean 2. The mean is chosen to be 2 in order to introduce a higher frequency of many-body interaction terms in the n -qubit Hamiltonian.

All of the random unitary errors are organized into bins according to their physical infidelities. We use a distinct color to identify a bin and provide the average physical as well as logical error rate corresponding to the unitary errors in each bin and the size of the bin in the legend. The performance gain reported on the y axis in Fig. 2 is the median of the gains associated with the unitary errors in the respective bins, while the error bars denote the first quartiles, indicating that there are instances of physical noise processes where the performance gain is substantially higher than the median. We observe that even with as low as 1% of all Pauli error rates within a code block, i.e., $K = 163$, we are able to achieve a typical performance gain of 10 times over $\mathcal{D}_1$, which has access only to the average physical error rate.

While Fig. 2 highlights the performance gains achieved under CG1D error models, we have observed similar improvements across a broad range of noise models. These include coherent errors from random unitary circuits with one- and two-qubit gates, as well as Markovian noise represented by n -qubit CPTP maps, which are constructed by composing multiple one- and two-qubit quantum channels. A detailed discussion of these results is given in Appendixes C and D. The outstanding observation across all the error models considered is a 10 times performance gain achieved with just 1% of the available Pauli error rates in the CER data—specifically, the 164 largest Pauli error rates. These results underscore the versatility of CER data and the effectiveness of the USS in QEC for general Markovian noise.

While our results demonstrate that K can be as small as 1% of the full CER data, we do not claim, nor do we presently know whether, there is a general scaling law relating K to the system size n . In contrast, in the setting where the CER data consist of all Pauli error rates up to a fixed weight w , as discussed in Appendix E, the number of included Pauli terms admits a clear scaling: the size of the corresponding CER data grows as $\mathcal{O}(n^w)$.

Each curve in Fig. 2 shows the median logical error rate over an ensemble of coherent-error channels whose

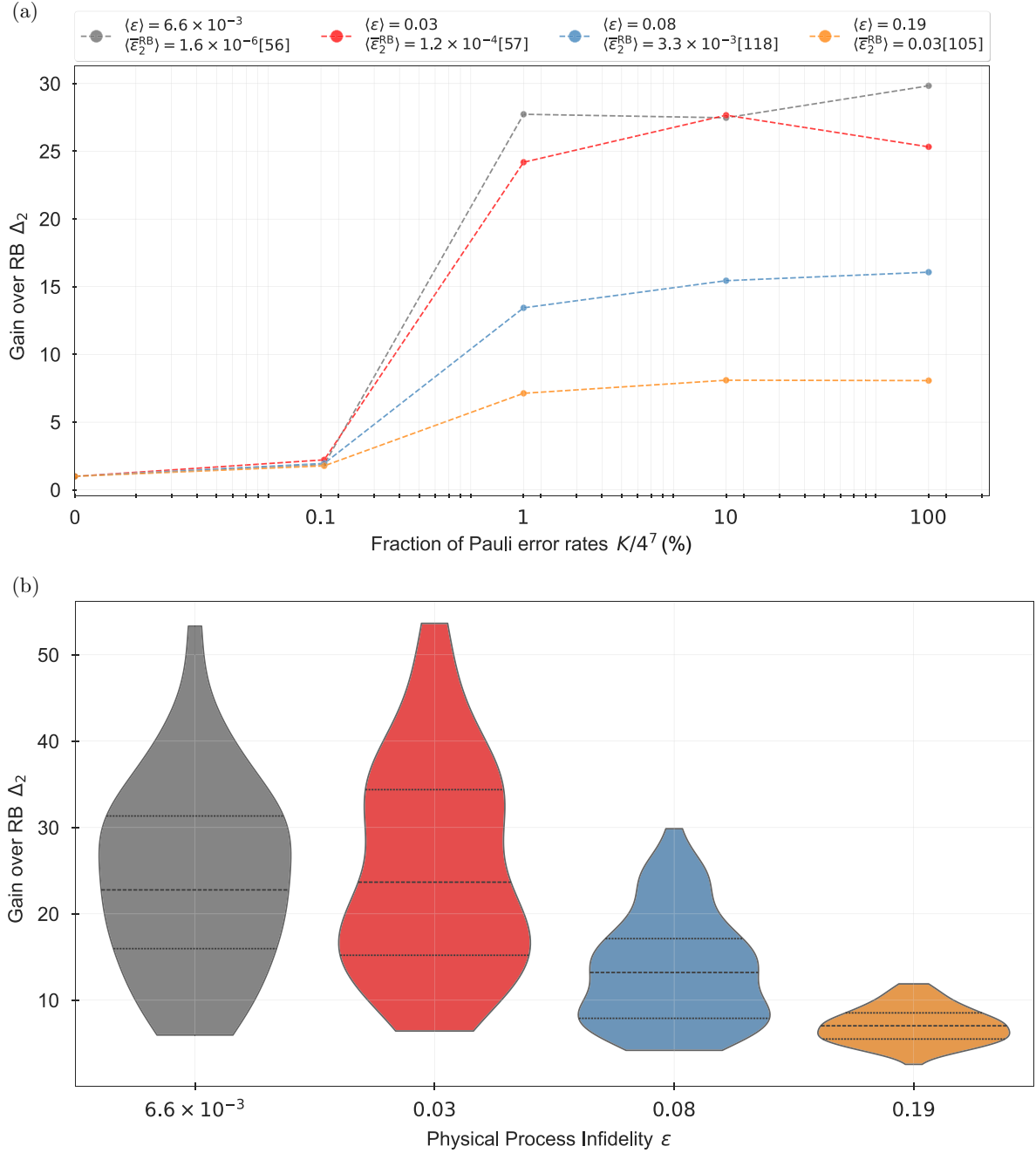


FIG. 2. Performance enhancement in the QEC capabilities of the Steane code obtained by use of partial error characterization data using CER and using the USS method outlined in Algorithm 1 as a function of the number of Pauli error rates estimated from CER. This enhancement is quantified by $\Delta_2 = \bar{r}_2(\mathcal{E}; \mathcal{D}_1) / \bar{r}_2(\mathcal{E}; \mathcal{D}_K)$. (a) Δ_2 based on numerical simulations of QEC under random unitary noise processes arising from the CG1D error model. These random unitary errors are grouped into bins with similar physical infidelities, and the median performance gain for each bin is shown. Each bin is distinguished by a unique color, as indicated in the legend, which also details the number of errors in each bin and their average physical and logical infidelities. In all but the extremely-high-noise regime, the ML decoder achieves a performance gain of approximately 10 times when it assumes knowledge of merely approximately 1% of the available error characterization data from CER, specifically 163 Pauli error rates. (b) Variation in performance gains across individual physical noise processes within the four bins considered in (a) when the ML decoder has access to those 163 Pauli error rates that constitute the 1% of the error data from CER. The width of a violin-shaped region at a specific height (Δ_2) is proportional to the number of noise processes that exhibited the corresponding performance gain. The horizontal dashed lines represent the first quartile, median, and third quartile of the performance gains, respectively. While most of performance gains fall between 15 times and 30 times, gains in the low-noise regime can reach as high as 50 times.

physical process infidelities lie within the range indicated in the legend. Although the channels in a given ensemble share similar process infidelities, they can still differ significantly in other features that fully determine the underlying unitary noise. It is well established that process infidelity alone does not fully characterize QEC performance [60]. As a result, a channel with slightly lower process infidelity may nevertheless produce a higher logical error rate than a channel with higher infidelity, depending on the specific structure of its unitary error. This effect can lead to non-monotonic behavior in the logical error rates, as observed in Fig. 2(a), as well as crossings between curves associated with different infidelity bins.

The performance improvements depicted in Fig. 2 are attributed to the combined impact of the partial CER data regarding the underlying noise process and the USS algorithm (Algorithm 1). In the following analysis, we aim to isolate the individual contributions of the partial CER data and the USS algorithm to these gains. For an n -qubit noise process $\mathcal{E}$, we introduce three key notations: (1) the full vector of all 4^n Pauli error rates, denoted as $\chi(\mathcal{E})$; (2) the vector of length 4^n containing K nonzero values, specifically the Pauli error rates from CER, labeled $\chi_K(\mathcal{E})$; and (3) the vector incorporating K Pauli error rates from CER, with the remainder assigned by the USS algorithm, represented as $\hat{\chi}_K(\mathcal{E})$.

If $\chi(\mathcal{E})$ closely resembles $\chi_K(\mathcal{E})$, this suggests that the CER data are primarily responsible for the performance enhancement of the ML decoder. To quantify the contribution of the USS algorithm, we estimate the total variation distance (TVD) between the distributions in (1) and (2), denoted by $\Delta_T(\chi(\mathcal{E}), \chi_K(\mathcal{E}))$. This value will be compared with the TVD between $\chi(\mathcal{E})$ and $\hat{\chi}_K(\mathcal{E})$. If the latter is smaller, this indicates that the USS algorithm plays a crucial role in enhancing the decoder's understanding of the original noise process beyond the available CER data. As shown in Table I, for the ensemble of CG1D error models considered in Fig. 2, $\text{TVD}(\chi(\mathcal{E}), \chi_K(\mathcal{E}))$ is approximately 5–10 times larger than $\text{TVD}(\chi(\mathcal{E}), \hat{\chi}_K(\mathcal{E}))$ for $K = 163$, which represents roughly 1% of the available CER data. This shows that the USS algorithm plays a key role in achieving the observed performance gains alongside CER.

V. CONCLUSION

In this work, we have demonstrated how the rich set of noise data that can be acquired efficiently with the use of CER can be used directly to improve logical error rates. We demonstrated a method to harness only the K -largest Pauli error rates derived from CER to develop an efficient heuristic to reconstruct all of the remaining Pauli error rates. This classical postprocessing method significantly reduces the experimental cost associated with estimating the error rates for the ML decoder. By integrating our heuristic into the maximum likelihood decoder, we numerically estimated

the suppression in the optimal logical error rate achievable for the given error model. Testing our methods for a wide variety of realistic error models, from coherent errors to incoherent errors, we found a performance gain of at least an order of magnitude when compared with a scenario where only the average error rate of the underlying physical error model is given. The consistent improvement observed across diverse error models underscores the promise of our heuristic approach in completing the Pauli error distribution from limited CER data, regardless of the specific error model chosen.

Although concatenated codes are extensively used in proofs of the fault-tolerance accuracy threshold theorem [96], these are less appealing from a practical standpoint. This is largely attributed to the large weights of the stabilizer generators, resulting in significant errors when an error syndrome is measured. Our focus has been on concatenated codes with increasing distance as levels grow, albeit encoding only a single logical qubit. However, this is not a strict limitation of our methods, and recent studies [77,78] have shown constructions of concatenated codes that have an increasing rate with the number of levels, and concatenated schemes with topological codes [97] to achieve fault tolerance. Nonetheless, the most promising candidates [4,98–101] for quantum error-correcting codes to achieve fault tolerance with a low resource overhead lie within the realm of quantum low-density parity check (LDPC) codes, whose stabilizer generators have a weight that is at most a constant independent of the size of the code. Only a handful of quantum LDPC code families are known to have decoders that can be adapted to the properties of the underlying error model. For instance,

TABLE I. Role of the USS algorithm in enhancing the performance of the ML decoder for the ensemble of CG1D error models, as reported in Fig. 2. The second column represents the TVD between the distribution of Pauli error rates in the noise process, $\chi(\mathcal{E})$, and the distribution of the leading $K = 163$ error rates extracted by CER, denoted as $\chi_K(\mathcal{E})$. The third column shows the TVD between $\chi(\mathcal{E})$ and the distribution generated by the USS algorithm, $\hat{\chi}_K(\mathcal{E})$, which is based on the CER data comprising the leading K Pauli error rates. While the TVDs differ significantly depending on individual instances of the random CG1D error model, we have averaged the results across subsets (bins) of the ensemble, grouped by their process infidelities. The first column represents the average physical error rate for a bin in Fig. 2. The results show that, on average, the USS algorithm improves the decoder's knowledge of the underlying noise process, beyond what is provided by the CER data alone.

$\langle \epsilon \rangle$	$\text{TVD}(\chi(\mathcal{E}), \chi_K(\mathcal{E}))$	$\text{TVD}(\chi(\mathcal{E}), \hat{\chi}_K(\mathcal{E}))$
6.6×10^{-3}	0.001	0.007
0.03	0.006	0.027
0.08	0.017	0.081
0.19	0.037	0.188

Ref. [102] presents a variant of the standard minimum-weight perfect-matching algorithm in surface codes [102] to account for the asymmetry in the probabilities of the single qubit Pauli errors. In some cases, ML decoding algorithms for surface codes based on tensor networks have been adapted to experimental features of the underlying CPTP map [103,104]. Analyzing performance gains for adapting a decoder for quantum LDPC codes to limited CER data remains an interesting problem for future research.

The key conceptual implication of our work is that logical error rates can be significantly suppressed when the decoder is provided with even a small fraction of the *a priori* error rates. If we consider interpreting these results in the context of fault-tolerant QEC, then this would entail accounting for a number of error sources, where memory errors in our analysis serve as a proxy for the *cumulative* errors arising from imperfections in each of the components in fault-tolerant implementations, including the transversal gates on the data qubits, the syndrome extraction circuits across the data and ancillary qubits, the direct projection-valued measure (PVM) measurement of the ancillary qubits, and the recovery operations on the data qubits.

However, for an in-depth analysis of various types of errors in the QEC process, we need to consider the broader setting of circuit-level noise, where physical errors are more appropriately described by fault paths, which map to effective Pauli errors and, ultimately, to logical cosets of the code. Recent work has demonstrated the feasibility and practicality of error learning for particular subsets of the above-mentioned components of fault tolerance [45,105]. ML decoding remains a comparison of logical coset probabilities, although computing these probabilities becomes substantially more challenging due to the enlarged error space. Practical approaches may therefore combine partial characterization of effective Pauli errors induced by noisy circuits with heuristic extrapolation schemes, providing a principled approximation to ML decoding under circuit-level noise. We leave a detailed exploration of these ideas to future work.

Our results build a compelling case that the problem of estimating the *cumulative* errors in the fault-tolerant setting presents a key opportunity for optimizing decoder performance. Our work demonstrates that even a modest amount of the experimentally accessible error characterization data can yield significant improvements in decoder performance.

ACKNOWLEDGMENTS

This research was undertaken thanks in part to funding from the Canada First Research Excellence Fund. Research was partially sponsored by the ARO under Grant No. W911NF-21-1-0007. S.D.B. acknowledges support

from the Australian Research Council via the Centre of Excellence for Engineered Quantum Systems (Project No. CE170100009). A.J. acknowledges support from ERC Starting Grant No. 101163189 and the UK Research and Innovation Future Leaders Fellowship scheme (Grant No. MR/X023583/1).

DATA AVAILABILITY

All data used to generate the figures in this work are publicly available [106]. The repository also contains links to external storage for datasets that exceed GitHub size limits, along with documentation describing the correspondence between datasets and figures.

APPENDIX A: CYCLE ERROR RECONSTRUCTION

In this appendix, we review the CER method [41,42] to estimate (i.e., learn) the Pauli error rates of a CPTP map to multiplicative precision.

We have already encountered the χ -matrix representation of a CPTP map in Eq. (3). An alternative representation is obtained by expressing the input state ρ as a linear combination of Pauli matrices. The CPTP map $\mathcal{E}$ can be described as an affine transformation, denoted by $\Lambda(\mathcal{E})$, on the vector space of Pauli matrices. The $4^n \times 4^n$ real matrix $\Lambda(\mathcal{E})$ is known as the “Pauli-Liouville representation” [107]. Each row and each column corresponds to a unique pair of n -qubit Pauli matrices $P, Q \in \mathcal{P}_n$:

$$\Lambda_{P,Q}(\mathcal{E}) = \text{Tr}(\mathcal{E}(P) \cdot Q). \quad (\text{A1})$$

Recall the simple yet versatile technique of estimating the average fidelity of a CPTP map using experimental protocols, known as “randomized benchmarking” [68,69]. The core concept of the standard RB protocol lies in using the two-design property of the Clifford group [67], leading us to rewrite the process fidelity of a CPTP map $\mathcal{E}$ in Eq. (4) as

$$\epsilon(\mathcal{E}) = 1 - \frac{1}{|\mathcal{C}_n|} \sum_{C \in \mathcal{C}_n} \text{Tr}(|0\rangle\langle 0| \cdot C^\dagger \mathcal{E}(C\rho C^\dagger)C), \quad (\text{A2})$$

where $\mathcal{C}_n$ is the n -qubit Clifford group. This formulation suggests an experimental design where we initialize a fixed state $|0\rangle$, apply m random Clifford gates C sampled from $\mathcal{C}_n$, followed by an inverse to return to the initial state in the absence of noise, and then measure in the Z basis. The probability of obtaining $+1$, denoted by P_s , is related to the average fidelity of the underlying CPTP $\mathcal{E}$ as

$$P_s = A_0 \epsilon^m(\mathcal{E}), \quad (\text{A3})$$

where A_0 is a constant related to state preparation and measurement errors.

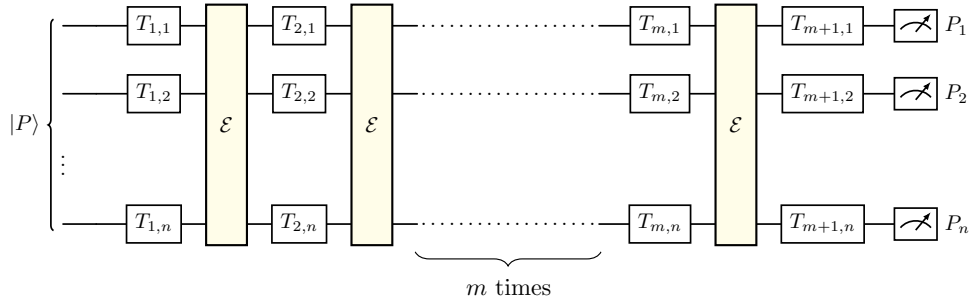


FIG. 3. CER protocol to estimate a Pauli error rate. The circuit consists of m cycles of the noise $\mathcal{E}$ followed by random Pauli gates $T_i \in \mathcal{P}_n$ expressed as $T_i = T_{i,1} \otimes T_{i,2} \otimes \dots \otimes T_{i,n}$ to perform twirling. Initialization of the input in a Pauli eigenstate $|P\rangle$ and measuring at the end in the P basis yields a decay law to estimate $\chi_{P,P}(\mathcal{E})$.

Note that the quantity measured by RB, $\epsilon(\mathcal{E})$, is only one of the $\mathcal{O}(4^{2n})$ parameters that specify $\mathcal{E}$. There are several noise processes of similar fidelity that lead to vastly different logical error rates for quantum error-correcting schemes [60]. CER provides a more detailed insight into reconstructing $\mathcal{E}$ efficiently. In contrast to RB, which provides a single number $r(\mathcal{E})$, CER estimates all the 4^n Pauli error rates in $\mathcal{E}$.

Let us define a CPTP map $\tau[\mathcal{E}]$ by averaging the map obtained by conjugating both the input and output of $\mathcal{E}$ by a Pauli operator P , over all n -qubit Pauli operators. More precisely, for any state ρ ,

$$\tau[\mathcal{E}](\rho) = \frac{1}{|\mathcal{P}_n|} \sum_{P \in \mathcal{P}_n} P \mathcal{E}(P \rho P) P, \quad (\text{A4})$$

where $\mathcal{P}_n$ denotes the n -qubit Pauli group (modulo phases). The map $\tau[\mathcal{E}]$, called the “Pauli twirl of $\mathcal{E}$ ” [40,68], is itself a Pauli channel due to a straightforward observation from representation theory. Furthermore, $\mathcal{E}$ and $\tau[\mathcal{E}]$ have identical Pauli error rates. We can now use a technique akin to RB to extract the Pauli error rates of $\mathcal{E}$, which is based on the following observation. The effect of $\tau[\mathcal{E}]$ on an eigenstate of a Pauli operator $P \in \mathcal{P}_n$ with eigenvalue λ_P , denoted by $|P\rangle$, is entirely determined by $\Lambda_{P,P}(\mathcal{E})$ defined in Eq. (A1). Hence, measuring P on $\tau[\mathcal{E}] (|P\rangle \langle P|)$ yields λ_P with probability that obeys a decay law similar to Eq. (A3) but with $r(\mathcal{E})$ replaced by $\chi_{P,P}(\mathcal{E})$. Hence, by choosing the input to be each of the 4^n Pauli eigenstates, we can estimate all the corresponding Pauli error rates of $\mathcal{E}$. Figure 3 presents a schematic overview of the CER protocol. It is important to note that Fig. 3 represents a specific instance of the CER protocol tailored for estimating the Pauli error rates of a memory error process such as $\mathcal{E}$. We skip detailing the general technique [42] since it is not directly pertinent to this paper.

Flammia and Wallman [46] showed that all of the Pauli error rates can be estimated with the use of $\mathcal{O}(n2^n)$ measurements in total and with use of the Walsh-Hadamard transformation to convert the eigenvalues to the Pauli error

rates. While CER offers a scalable and efficient technique to extract each of the 4^n Pauli error rates for a n -qubit CPTP map, the exponential growth in their number makes it impractical to estimate all of them within polynomial time. This leads us to another development, presented in Ref. [47] where the study authors propose a method to estimate the K -largest Pauli error rates of a CPTP map $\mathcal{E}$ using a number of experiments that scale linearly with K . Our work leverages the result in Ref. [47].

APPENDIX B: QUANTUM ERROR CORRECTION THEORY FOR NON-PAULI NOISE

Recall that an $[[n, k, d]]$ stabilizer code [71] $\mathcal{Q}$ is defined as a 2^k -dimensional subspace of $\mathbb{C}_2^n$, such that every element of $\mathcal{Q}$ is an eigenstate with eigenvalue $+1$ of $n - k$ commuting Pauli operators called the “stabilizer generators.”

Before describing the QEC strategy, we first introduce the necessary notation. Let ρ denote a single qubit state that is encoded into an $[[n, k, d]]$ stabilizer code whose encoding circuit [108] is U , yielding $\bar{\rho}$ given by

$$\bar{\rho} = U \cdot \rho \otimes |0\rangle \langle 0|^{n-1} \cdot U^\dagger. \quad (\text{B1})$$

A noisy encoded state is modeled by the action of an n -qubit CPTP map $\mathcal{E}$ on $\bar{\rho}$, resulting in $\mathcal{E}(\bar{\rho})$.

The strategy for dealing with general noise is inspired by the Pauli case. In the simple case where $\mathcal{E}$ is a Pauli error model described in Eq. (2), we note that any Pauli error E on the encoded state may be detected by observing the outcomes of measuring each of the stabilizer generators, or equivalently, by examining the commutation relations between E and each of $\{g_1, \dots, g_{n-k}\}$. The binary sequence $s \in \mathbb{Z}_2^{n-k}$, where $s_i = -1$ when $[E, g_i] = 0$ and $s_i = 1$ when $[E, g_i] \neq 0$, is called the “error syndrome.” Clearly, there are only 2^{n-k} distinct error syndromes, whereas there are 4^n distinct n -qubit Pauli errors. So, many Pauli errors will have the same error syndrome. The Pauli channel now induces a probability distribution on the error syndromes, where the probability of an error syndrome s , denoted by

$\text{Prob}(s)$, is given by the sum of probabilities of all Pauli errors that result in the error syndrome s .

The case of non-Pauli errors is somewhat more complicated. Measuring each stabilizer generator g_i on $\mathcal{E}(\rho)$ yields an error syndrome s with probability $\text{Prob}(s)$ given by

$$\text{Prob}(s) = \text{Tr}(\Pi_s \cdot \mathcal{E}(\bar{\rho})), \quad (\text{B2})$$

where ρ is the encoded state and Π_s is a projector corresponding to the syndrome measurement:

$$\Pi_s = \prod_{i=1}^{n-k} \frac{\mathbb{I} + (-1)^{s_i}}{2}. \quad (\text{B3})$$

Expressing the noisy state $\mathcal{E}(\bar{\rho})$ using the χ -matrix in Eq. (3), we find that

$$\text{Prob}(s) = \sum_{P, Q \in \mathcal{P}_s} \chi_{P, Q}(\mathcal{E}) \text{Tr}(\Pi_s \cdot P \cdot \mathcal{E}(\bar{\rho}) \cdot Q), \quad (\text{B4})$$

where $\mathcal{P}_s$ is the set of all Pauli errors whose error syndrome, obtained from commutation relations with stabilizer generators, is s . In contrast to a Pauli error model, in non-Pauli error the probability of an error syndrome is not independent of the input state. Additionally, computing $\text{Prob}(s)$ from Eq. (B4) requires time proportional to 2^n , unlike the simple case of Pauli channels, where it takes time only quadratic in n . After measurement of an error syndrome, the state of the system transforms to

$$\frac{\Pi_s \cdot \mathcal{E}(\bar{\rho}) \cdot \Pi_s}{\text{Prob}(s)}. \quad (\text{B5})$$

1. Maximum likelihood decoding

Before we define the decoding problem, it is worth analyzing some structure of the Pauli group induced by the stabilizer group. Note that there are nontrivial Pauli errors $E \neq \mathbb{I}$ that have a trivial syndrome: $s = 00 \dots 0$. These errors belong to $\mathcal{N}(\mathcal{S})$: the normalizer of the stabilizer group $\mathcal{S}$ within the Pauli group $\mathcal{P}_n$,

$$\mathcal{N}(\mathcal{S}) = \{E \in \mathcal{P}_n : E \cdot S \cdot E \in \mathcal{S}\}.$$

Clearly, $\mathcal{S} \subset \mathcal{N}(\mathcal{S})$, which encompasses errors that leave all encoded states unchanged. However, errors in $\mathcal{L} = \mathcal{N}(\mathcal{S})/\mathcal{S}$, referred to as “logical errors,” are those causing nontrivial transformations to the encoded states but cannot be detected by the error syndrome. Finally, $\mathcal{T} = \mathcal{P}_n/\mathcal{N}(\mathcal{S})$, referred to as “pure errors” [30,72,73], denotes classes of errors where each class corresponds to a unique error syndrome. Consequently, we can now express any Pauli error $E \in \mathcal{P}_n$ as a product: $E = T \cdot L \cdot S$, where $T \in \mathcal{T}$, $L \in \mathcal{L}$, and $S \in \mathcal{S}$.

In the decomposition of a Pauli error shown in Eq. (6), note that each pure error generator anticommutes with a unique stabilizer generator. In other words, the error syndrome tells us precisely the pure error generators that make up the T element of the error:

$$T = \prod_{i=1}^{n-k} T_i^{s_i}. \quad (\text{B6})$$

While T is entirely fixed by learning the error syndrome s through measurements, $T \equiv T_s$, it remains to determine L and S to completely specify a Pauli error. The choice of S is not important, as two errors that differ in this decomposition by a stabilizer are effectively indistinguishable. However, the choice of the logical operator $L \in \mathcal{L}$ is crucial—applying a wrong choice will not return us to the right encoded state after QEC, marking its failure.

The task of determining L is a statistical inference problem known as ML decoding [30,31], stated as follows. Given an error syndrome s and an error model $\mathcal{E}$, compute $L_s \in \mathcal{L}$ such that

$$L_s = \underset{L \in \mathcal{L}}{\text{argmax}} \text{Prob}(L|s), \quad (\text{B7})$$

where $\text{Prob}(L|s)$ is the probability of a logical error, defined as the sum of all errors in the coset $T_s \cdot L \cdot \mathcal{S}$:

$$\text{Prob}(L|s) = \sum_{S \in \mathcal{S}} \text{Prob}(T_s \cdot L \cdot S). \quad (\text{B8})$$

Note that the probabilities of Pauli errors $P = T_s \cdot L \cdot S$ are derived from a given error model. If $\mathcal{E}$ is a Pauli error model, this is straightforward; $\text{Prob}(P) = \chi_{P, P}(\mathcal{E})$. In the general case, including non-Pauli error, we derive $\text{Prob}(P)$ from the Pauli twirl of $\mathcal{E}$: $\tau[\mathcal{E}]$. So $\text{Prob}(P) = \chi_{P, P}(\tau[\mathcal{E}])$.

An alternative decoding strategy involves selecting a logical operator that maximizes the probability of a specific error for a given choice of S , i.e.,

$$L_s = \underset{L \in \mathcal{L}, S \in \mathcal{S}}{\text{argmax}} \text{Prob}(T_s L S).$$

For an independent and identically distributed depolarizing noise model with error rate p per qubit, the operation $T_s L_s S$ represents the lowest-weight Pauli error that is consistent with the observed syndrome $\vec{s}$. This approach defines the *minimum-weight decoder* [30].

The output of the decoder, L_s , can be combined with T_s and any stabilizer S to yield a recovery operation: $T_s \cdot L_s \cdot S$. After applying the recovery operation on the n -qubit state in Eq. (B5), we find

$$\bar{\sigma} = \frac{(T_s \cdot L_s \cdot S \cdot \Pi_s \cdot \mathcal{E}(\bar{\rho}) \cdot \Pi_s \cdot S \cdot L_s \cdot T_s)}{\text{Prob}(s)}. \quad (\text{B9})$$

Since syndrome measurement and recovery involve active quantum controls, errors are inevitable during these steps.

for the logical corrections for each of the code blocks at level 2. For this, we use Eq. (B8), where the Pauli error model to be used to derive the probabilities of physical I, X, Y, Z errors is replaced by the distribution of logical errors from the level-1 code blocks. We proceed in this iterative fashion until the final level, where the distribution of logical operators for the code block at level ℓ is computed. Ultimately, the output of the ML decoder is the logical operator with the highest probability, described in Eq. (B7). Applying this logical operator allows us to compute an effective channel for the level- ℓ concatenated code using Eqs. (B9)–(B11). Figure 5 provides a schematic for the functioning of the ML decoder for concatenated codes.

For a concatenated code with ℓ levels, we can follow the recipe described above to compute the effective logical channel conditioned on the observed syndrome, denoted $\mathcal{E}_\ell^s$. The average effective channel $\bar{\mathcal{E}}_\ell$, which we refer to in short as $\bar{\mathcal{E}}_\ell$, is obtained by averaging over all possible syndrome outcomes, similar to Eq. (B11). The average logical error rate is then defined as the infidelity of $\bar{\mathcal{E}}_\ell$. However, the number of syndrome outcomes grows exponentially with the number of levels ℓ , making an exact computation of the average effective channel impractical. For instance, even at $\ell = 2$, there are approximately 10^{12} possible syndrome outcomes. Appendix F details efficient sampling techniques for estimating the average logical error rate for concatenated codes, bypassing the need to explicitly enumerate all syndrome outcomes.

APPENDIX C: RANDOM UNITARY CIRCUITS

Coherent errors constitute an important class of physically relevant error models, arising from various scenarios, such as imperfect control during logical operations or unintended qubit interactions within a code. These are particularly malicious for any quantum code because of two key characteristics. Firstly, they accumulate more rapidly than incoherent errors. Coherent errors, being rotations, add linearly, unlike incoherent errors, which accumulate only quadratically [115,116]. Secondly, since most quantum codes are tailored to correct incoherent errors such as depolarizing noise, their effectiveness decreases when faced with coherent errors [18]. For most of the realistic hardware, it has been observed that errors are often correlated in spatial regions as well as temporal regions. With a loss of generality, we want to focus on time-independent yet spatially correlated unitary errors. For instance, when we apply a two-qubit entangling gate such as a CNOT gate between two qubits in a code, we observe that a faulty implementation leaves behind correlated errors [117,118]. In superconducting architectures, crosstalk is a major source of noise, which is often of similar magnitude as the single qubit errors [119]. Since unitary errors take a specific form depending on the choice of hardware device under consideration, we analyze random correlated unitary

error models so that our outstanding conclusions apply to any realistic hardware.

In this appendix, we showcase gains for unitary errors resulting from a random unitary circuit of a fixed depth [120,121]. Recall that the gain refers to a relative performance gain, defined in Eq. (9), of the decoder with access to K error rates in the CER data, over a decoder that assumes access to the average physical error rate alone.

We consider a model for coherent errors wherein an n -qubit unitary error U is composed of at most d random unitary errors $U_1, \dots, U_d$, each acting nontrivially on at most $k \leq n$ qubits. For convenience, let us label the support of U_i by the set $\mathfrak{S}_i \subseteq \{1, \dots, n\}$. We model each of the unitary errors U_i by first choosing $\mathfrak{S}_i$ and then sampling a random unitary matrix of dimension $2^{|\mathfrak{S}_i|}$. Typically, a physical system has a mean correlation length λ , indicating that most errors in $U_1, \dots, U_d$ are supported on λ qubits. Additionally, we expect that the frequency of k -qubit errors decreases exponentially with k . These assumptions are reflected in our model by sampling $\{|\mathfrak{S}_1|, |\mathfrak{S}_2|, \dots, |\mathfrak{S}_d|\}$ from a Poisson distribution with mean λ . Once $|\mathfrak{S}_i|$ is chosen, we assign $\mathfrak{S}_i$ to a random subset of $\{1, \dots, n\}$ of the given size. Finally, the random unitary matrix U_i is obtained by exponentiating a random Hermitian matrix H_i of dimension $2^{|\mathfrak{S}_i|}$ in the form $U_i = \exp(i H_i t)$, where t is a virtual time parameter that is fixed to acquire a handle on the strength of noise in the system. To introduce significant correlations in our system, we set $\lambda = 2$ to generate random unitary errors in this model. Note that for these correlated error models, a minimum-weight decoder is likely to fail since multiqubit errors are not exponentially suppressed compared with single-qubit errors. Furthermore, we expect that learning the actual probabilities of the multiqubit errors from the limited CER data would be crucial since they would deviate significantly from our ansatz for the multiqubit error probabilities assigned with use of the USS method in Algorithm 1.

Figure 6 shows gains for the level-2 concatenated Steane code achieved with an ML decoder that leverages Pauli error rates in $\mathcal{N}(K)$ along with the USS method for several values of K . The trends in Fig. 6 are drawn from an ensemble of 336 random unitary errors. To generate each unitary error, we build a random unitary circuit of depth 12 by choosing random unitary gates supported on random subsets of the qubits in the Steane code block whose sizes are distributed according to the Poisson distribution with mean 2. Recall that each unitary gate is obtained by exponentiating a random Hamiltonian multiplied by a constant t , providing a handle on the noise strength. To generate the random channels shown in Fig. 6 we have considered six distinct values of t the range $[0.25, 0.65]$. These numbers are chosen to obtain the desired spectrum in the fidelities of the resulting n -qubit unitary errors.

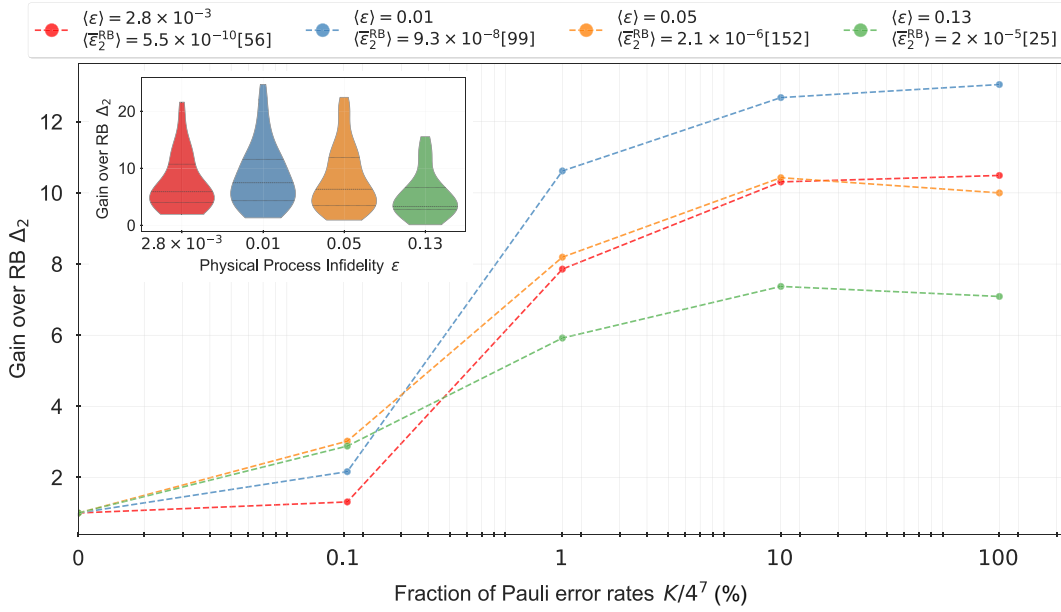


FIG. 6. Performance improvements achieved by the ML decoder for concatenated Steane codes, quantified by $\Delta_2 = \bar{r}_2(\mathcal{E}; \mathcal{D}_1)/\bar{r}_2(\mathcal{E}; \mathcal{D}_K)$, under unitary errors, with differing fractions of CER data provided to the decoder. Each unitary error is generated from a random unitary circuit of depth at most 9, consisting of random single-qubit, two-qubit, and three-qubit gates. The error maps are grouped into bins as described in Fig. 2. A typical performance gain of 5 times can be achieved in most noise regimes, leveraging merely 1% of the error characterization data from CER, i.e., 164 Pauli error rates. The inset shows the variation of performance gains for the individual channels in each of the bins when only 1% of the error data is available to the ML decoder. While the median performance gains in the bins are 5 times, we see that for some unitary errors, the performance gains can be as high as 20 times.

APPENDIX D: GENERIC MARKOVIAN ERRORS

A slightly inclusive notion of noise processes that includes decoherence effects is given by CPTP maps. An n -qubit CPTP map can be described with the use of at most $\mathcal{O}(2^{n^2})$ operators:

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger, \quad (\text{D1})$$

where K_i are positive matrices, called “Kraus operators” [53], and satisfy

$$\sum_i K_i^\dagger K_i = \mathbb{I}.$$

The χ -matrix representation of $\mathcal{E}$ shown in Eq. (3) can be derived by expressing the Kraus operators in Eq. (D1) as linear combinations of Pauli matrices.

An important subclass of CPTP maps is *noncatastrophic channels*, which are characterized by $\epsilon(\mathcal{E}) \geq 1/2$ and $u(\mathcal{E}) \geq 1/2$, where $u(\mathcal{E})$ is the unitarity [122] of the CPTP map $\mathcal{E}$. In this study, we focus specifically on noncatastrophic CPTP maps, which we refer to as “realistic errors.” It can be shown that a noncatastrophic channel $\mathcal{E}$ can be accurately described with the use of only one Kraus operator [61]: $\mathcal{E}(\rho) \simeq K \rho K^\dagger$, where K is the *leading Kraus operator*. Expressing K in terms of its polar decomposition yields $K = PU$, where U is a unitary error,

while P is a projector, denoting an incoherent error. This decomposition highlights a crucial point: realistic CPTP maps can be effectively described as a combination of a unitary error and an incoherent error. We exploit this idea to sample realistic error models by composing a unitary error $\mathcal{E}_U$ in Eq. (1) with a Pauli channel $\mathcal{E}_P$ in Eq. (2).

We construct random n -qubit CPTP maps by composing at most m CPTP maps on $k \leq n$ qubits. To derive one such CPTP map, we first sample a number u from a Poisson distribution with mean λ , indicating a mean correlation length in the system. Then we compute a random subset of n qubits of size u , denoted by $\mathfrak{S}_i$, and generate two maps supported on $\mathfrak{S}_i$. One is a random unitary error expressed as $\mathcal{E}_U(\rho) = \exp(iHt)\rho \exp(-iHt)$, where H is a random Hermitian matrix of dimension 2^u . The other is a random Pauli channel $\mathcal{E}_P$ of fixed average physical error rate, obtained by our choosing a random vector of $4^u - 1$ real numbers denoting the probabilities of all the nontrivial Pauli errors supported on $\mathfrak{S}$. Finally, we compose $\mathcal{E}_U$ and $\mathcal{E}_P$ to obtain one instance of a random noncatastrophic CPTP map. Composing several of these random CPTP maps yields an n -qubit CPTP map.

Similarly to the analysis of unitary errors, we examine typical performance gains for random CPTP maps categorized into bins on the basis of their physical infidelities. Figure 7 shows approximately 10 times performance gains achieved for the level-2 concatenated Steane code across

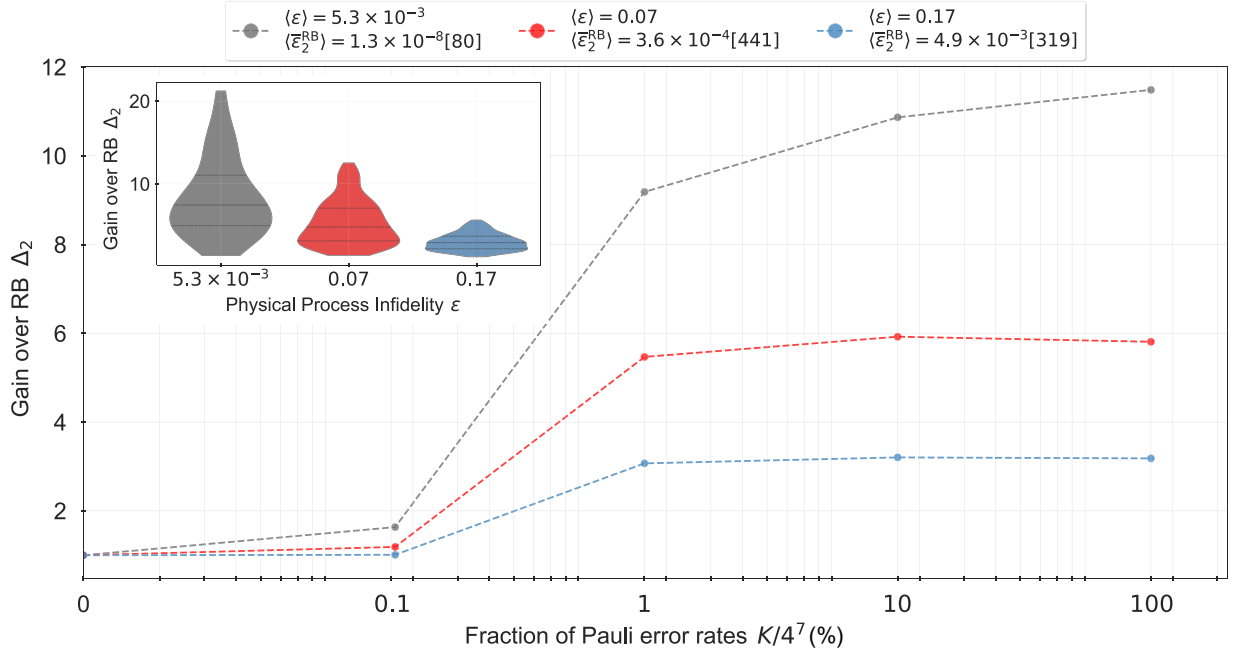


FIG. 7. Performance improvements achieved by the ML decoder for concatenated Steane codes, quantified by $\Delta_2 = \bar{r}_2(\mathcal{E}; \mathcal{D}_1)/\bar{r}_2(\mathcal{E}; \mathcal{D}_K)$, under random CPTP noise processes, with differing fractions of CER data provided to the decoder. These maps are grouped into bins using a similar strategy as described in Figs. 2 and 6. Across most noise regimes, a 5 times to 10 times performance gain is observed, even when only 1% of the CER data is made available to the ML decoder. Each violin-shaped region in the inset corresponds to a bin identified by its respective color. The width of the region at a specific height (Δ_2) is proportional to the number of random CPTP maps that exhibited the corresponding performance gain when only 1% of the error characterization data from CER is made available to the ML decoder. While the median performance gain across channels in a bin typically ranges between 5 times and 10 times, individual maps exhibit performance gains as high as 20 times.

several ensembles of noncatastrophic CPTP maps, with the use of only 1% of the Pauli error probabilities in the CER data.

Together with the unitary cases shown in Figs. 2 and 6, the results in Fig. 7 underscore the significance of CER data and demonstrate the versatility of the uncorrelated split search method in achieving substantial performance gains.

APPENDIX E: k -BODY NOISE RECONSTRUCTION

Recall that the goal of this paper is to highlight the critical importance of noise calibration, even when it constitutes as little as 1% of the total available spectrum, for substantially enhancing the performance of the ML decoder. Our numerical findings in Sec. IV C are derived on the basis of a particular selection of limited error characterization information from CER—representing the K -largest Pauli error rates in the underlying physical noise process, which can be obtained with the use of experimental protocols detailed in Refs. [46,94]. However, the scope of our techniques for assigning the rates for the remaining errors, using the uncorrelated split search algorithm, is agnostic to the choice for the limited CER information. In

this appendix, we stress this point using a different rationale for selecting partial CER data based on the Hamming weight of Pauli errors, known as “ k -body noise reconstruction,” or k -NR as referenced in Refs. [41,42]. Here we describe our partial calibration data for an n -qubit CPTP map as comprising Pauli error rates of all Pauli errors with a weight of at most k , denoted by $\chi_k(\mathcal{E})$. Note that the size of $\chi_w(\mathcal{E})$ equals N_w , as shown in Fig. 8, which scales exponentially with w . For example, $k = 3$ denotes calibration data consisting of N_3 Pauli error rates, corresponding to single-qubit, two-qubit, and three-qubit Pauli errors. Figure 8 illustrates the performance gains achieved with increasing amounts of k -NR data, where the remaining Pauli error probabilities are assigned with use of the uncorrelated split search algorithm. Notably, we observe approximately 5 times performance improvement for unitary errors when the decoder is provided with single-qubit and two-qubit error rates, and approximately 10 times improvement for CG1D errors. To have a fair comparison of the two different rationales for selecting the limited CER data given to the ML decoder, we used the same ensemble of physical noise processes in Fig. 8 as in the numerical studies for Figs. 2 and 6.

In most physical models, it is typically expected that interactions involving a significant number of qubits tend

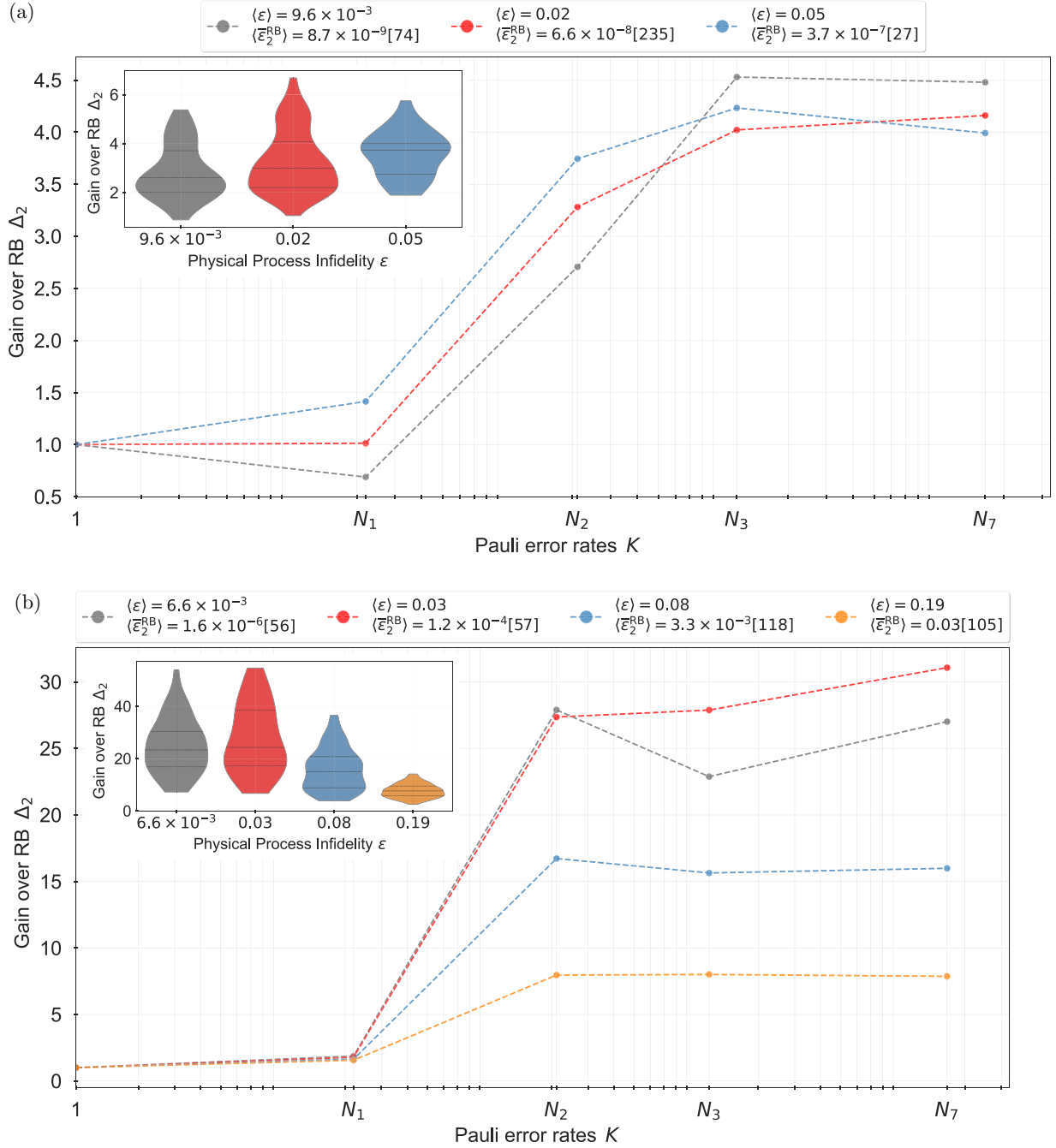


FIG. 8. Significance of calibration data from k -NR in enhancing the performance of an ML decoder for concatenated Steane codes under two distinct models for coherent errors. In (a), unitary errors are generated from random unitary circuits with depths up to 9, comprising random single-qubit and two-qubit gates. In contrast, in (b), the CG1D error model is used to generate the errors. The x axis represents the number of Pauli error rates K accessible to the decoder, specifically all the Pauli error rates for Pauli errors of weight up to w for $0 \leq w \leq 3$, denoted by N_w . Performance improvements are quantified with the use of $\Delta_2 = \bar{r}_2(\mathcal{E}; \mathcal{D}_1) / \bar{r}_2(\mathcal{E}; \mathcal{D}_K)$. In (a), for each bin containing random noise processes with similar average physical error rates, the median performance gain is observed to be approximately 5 times. For the CG1D error model in (b), the median gains are significantly larger, typically around 30 times. As shown in Figs. 2 and 6, the inset plots detail the individual performance gains for all noise processes across the bins when the ML decoder is provided with all single-qubit and two-qubit error rates. Notably, these individual gains can far exceed the median values. Overall, where calibration data are sourced from k -NR, the performance gains are not as pronounced compared with when the calibration data comprise the K -largest Pauli error rates. This disparity arises from the prevalence of correlated errors in the physical noise model, which causes some low-weight errors to occur less frequently than certain higher-weight errors.

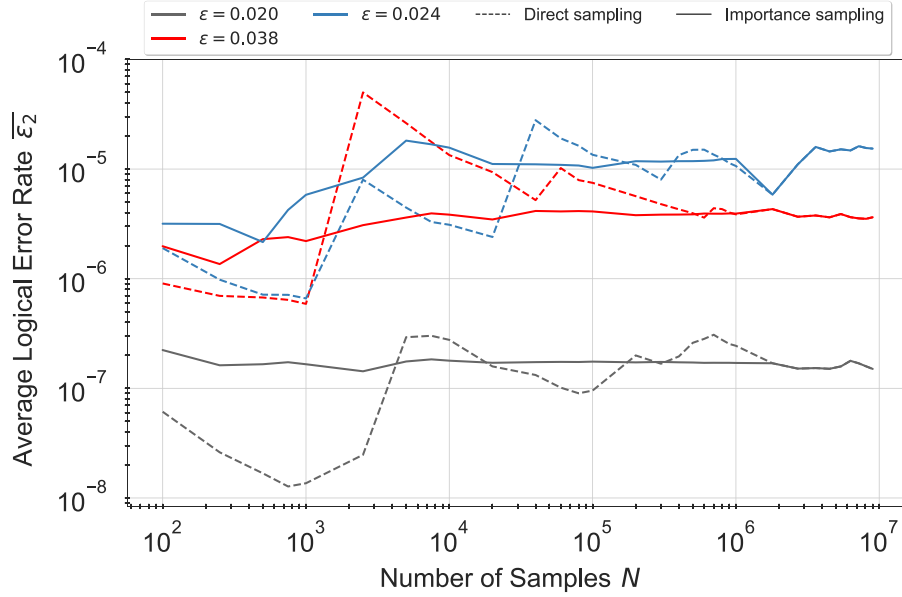


FIG. 9. Rapid convergence of Monte Carlo estimates of the average logical error rate for the level-2 concatenated Steane code obtained with the importance sampling technique. The underlying physical noise process is the unitary error model introduced in Sec. C. The estimate from the importance sampler converges to the true average logical error rate, which is determined by direct sampling after a very large number of samples. We average the logical error rate estimates over all random unitary error ensembles, characterized by their average physical error rates. For most physical noise strengths indicated by the physical error rates, we observe that the importance sampler converges with only 1/100 of the samples required by the direct sampler. Similar results can be found in Refs. [60,124].

to be weaker than those involving fewer qubits. Essentially, this implies that the likelihood of Pauli errors decreases exponentially with error weight once it exceeds a certain correlation length λ within the system. This rule of thumb supports the use of k -NR data for decoding purposes. However, when $k \leq \lambda$, it is possible that some Pauli errors of higher weight are more probable than certain errors of lower weight. Another case where the effective correlation length might increase is when noise accumulates over time, i.e., under composition. In such scenarios, it is better to choose the k -largest error rates rather than systematically choosing errors by increasing weights. This is evident from the larger performance improvements observed in the composition of the unitaries in Fig. 6 versus Fig. 8(a). In contrast, the absence of inherent composition in the error model leads to the performance gains from the k -NR data depicted in Fig. 8(b) being similar to those obtained by choosing the leading Pauli error rates for the same physical noise processes within the CG1D error model.

APPENDIX F: ESTIMATING THE AVERAGE LOGICAL ERROR

We want to outline the techniques used to estimate the average logical error rate for the concatenated Steane code. Recall from our discussion in Sec. II C that the logical error rate conditioned on a syndrome measurement

outcome is simply given by the infidelity of the effective channel: $r(\mathcal{E}_\ell^s)$, where ℓ denotes the concatenation level. Equation (B11) defines the average effective channel where the sum is over all error syndromes of the level- ℓ concatenated code. Recall that a level- ℓ concatenated code is an $[[n^\ell, 1, d^\ell]]$ code, where n is the total number of physical qubits in a code block and d is its distance. For the level-2 concatenated Steane code, we find $n = 7$ and $d = 3$, implying that there are $2^{n^\ell - 1}$ error syndromes. Even for $\ell = 2$, the number is more than 10^{14} . Hence, it is infeasible to compute the average logical error rate exactly. We instead estimate the average logical fidelity by a Monte Carlo sampling [60,65,114] of the syndrome distribution $\text{Prob}(s)$ given in Eq. (B2). We denote the sampling estimate of $r(\mathcal{E}_\ell)$ by $r(\hat{\mathcal{E}}_\ell)$, given by

$$r(\hat{\mathcal{E}}_\ell) = \frac{1}{N} \sum_{\hat{s}} r(\mathcal{E}_\ell^{\hat{s}}), \quad (\text{F1})$$

where N is the number of syndrome samples drawn from the syndrome distribution $\text{Prob}(s)$. Although it may seem straightforward to obtain $r(\hat{\mathcal{E}}_\ell)$ in this way, there are subtleties that cause estimates to converge very slowly to the true average logical error rate due to the presence of outlier syndromes that strongly impact the average [60,65,123]. Therefore, we use an importance sampling technique in which we sample error syndromes from a different distribution $Q(s)$ that assigns a higher weight to outlier syndromes

and corrects the infidelity found for the sampled syndrome s by a ratio $\text{Prob}(s)/Q(s)$. In our simulations, we choose $Q(s) = \text{Prob}(s)^\alpha$, where α is selected such that the uncorrectable errors, for which the average infidelity is close to 1, have a weight greater than a fixed threshold [124]. Figure 9 demonstrates how the average logical error rate estimated from the importance sampling method converges to the true average estimated from the direct sampling method using a very large number of samples. We showed explicit convergence of the logical error rate estimates in the two sampling methods under a few unitary channels modeled by random quantum circuits in Sec. C. Clearly, the importance sampling method estimates the true average with far fewer samples. In this work, our numerical methods to estimate performance gains from different decoding strategies under various error models use the importance sampling method.

- [1] P. W. Shor, in *Proceedings of 37th Conference on Foundations of Computer Science (FOCS)* (IEEE, Burlington, Vermont, USA, 1996).
- [2] J. Preskill, in *Introduction to Quantum Computation and Information*, edited by H.-K. Lo, T. Spiller, and S. Popescu (World Scientific, 1998), p. 213, chapter 8.
- [3] A. Paetznick, Resource optimization for fault-tolerant quantum computing, Ph.D. thesis, University of Waterloo, <https://uwspace.uwaterloo.ca/items/3d9f1202-e0c2-474d-b9ec-c7b41a0d8ace>, 2014.
- [4] D. Gottesman, Fault-tolerant quantum computation with constant overhead, *Quantum Inf. Comput.* **14**, 1339 (2014).
- [5] K. Sahay, J. Jin, J. Claes, J. D. Thompson, and S. Puri, High-threshold codes for neutral-atom qubits with biased erasure errors, *Phys. Rev. X* **13**, 041013 (2023).
- [6] S. Bravyi, A. W. Cross, J. M. Gambetta, D. Maslov, P. Rall, and T. J. Yoder, High-threshold and low-overhead fault-tolerant quantum memory, *Nature* **627**, 778 (2024).
- [7] D. K. Tuckett, S. D. Bartlett, and S. T. Flammia, Ultrahigh error threshold for surface codes with biased noise, *Phys. Rev. Lett.* **120**, 050505 (2018).
- [8] J. P. Bonilla Ataides, D. K. Tuckett, S. D. Bartlett, S. T. Flammia, and B. J. Brown, The XZZX surface code, *Nat. Commun.* **12**, 2172 (2021).
- [9] C. A. Pattison, M. E. Beverland, M. P. da Silva, and N. Delfosse, Improved quantum error correction using soft information, *ArXiv:2107.13589*.
- [10] O. Higgott, T. C. Bohdanowicz, A. Kubica, S. T. Flammia, and E. T. Campbell, Improved decoding of circuit noise and fragile boundaries of tailored surface codes, *Phys. Rev. X* **13**, 031007 (2023).
- [11] S. Huang, M. Newman, and K. R. Brown, Fault-tolerant weighted union-find decoding on the toric code, *Phys. Rev. A* **102**, 012419 (2020).
- [12] J. Lee, J. Park, and J. Heo, Rectangular surface code under biased noise, *Quantum Inf. Process.* **20**, 231 (2021).
- [13] U. Azad, A. Lipińska, S. Mahato, R. Sachdeva, D. Bhoumik, and R. Majumdar, Surface code design for asymmetric error channels, *IET Quantum Commun.* **3**, 174 (2022).
- [14] Q. Xu, N. Mannucci, A. Seif, A. Kubica, S. T. Flammia, and L. Jiang, Tailored XZZX codes for biased noise, *Phys. Rev. Res.* **5**, 013035 (2023).
- [15] Y. Kim, J. Kang, and Y. Kwon, Design of quantum error correcting code for biased error on heavy-hexagon structure, *Quantum Inf. Process.* **22**, 230 (2023).
- [16] S. Puri, L. St-Jean, J. A. Gross, A. Grimm, N. E. Frattini, P. S. Iyer, A. Krishna, S. Touzard, L. Jiang, A. Blais, S. T. Flammia, and S. M. Girvin, Bias-preserving gates with stabilized cat qubits, *Sci. Adv.* **6**, eaay5901 (2020).
- [17] D. Bluvstein, H. Levine, G. Semeghini, T. T. Wang, S. Ebadi, M. Kalinowski, A. Keesling, N. Maskara, H. Pichler, M. Greiner, V. Vuleta, and M. D. Lukin, A quantum processor based on coherent transport of entangled atom arrays, *Nature* **604**, 451 (2022).
- [18] S. Bravyi, M. Englbrecht, R. Koenig, and N. Peard, Correcting coherent errors with surface codes, *npj Quantum Inf.* **4**, 55 (2018).
- [19] C. Chamberland, J. Wallman, S. Beale, and R. Laflamme, Hard decoding algorithm for optimizing thresholds under general Markovian noise, *Phys. Rev. A* **95**, 042332 (2017).
- [20] M. Gutiérrez, L. Svec, A. Vargo, and K. R. Brown, Approximation of realistic errors by Clifford channels and Pauli measurements, *Phys. Rev. A* **87**, 030302(R) (2013).
- [21] N. H. Nickerson and B. J. Brown, Analysing correlated noise on the surface code using adaptive decoding algorithms, *Quantum* **3**, 131 (2019).
- [22] A. Jayashankar, M. D. H. Long, H. K. Ng, and P. Mandayam, Achieving fault tolerance against amplitude-damping noise, *Phys. Rev. Res.* **4**, 023034 (2022).
- [23] A. Kubica, A. Haim, Y. Vaknin, H. Levine, F. Brandão, and A. Retzker, Erasure qubits: Overcoming the T_1 limit in superconducting circuits, *Phys. Rev. X* **13**, 041022 (2023).
- [24] Y. Wu, S. Kolkowitz, S. Puri, and J. D. Thompson, Erasure conversion for fault-tolerant quantum computing in alkaline earth Rydberg atom arrays, *Nat. Commun.* **13**, 4657 (2022).
- [25] P. Aliferis and B. M. Terhal, Fault-tolerant quantum computation for local leakage faults, *Quantum Inf. Comput.* **7**, 139 (2007).
- [26] A. G. Fowler, Coping with qubit leakage in topological codes, *Phys. Rev. A* **88**, 042308 (2013).
- [27] K. C. Miao, M. McEwen, J. Atalaya, D. Kafri, L. P. Pryadko, A. Bengtsson, A. Opremcak, K. J. Satzinger, Z. Chen, P. V. Klimov *et al.*, Overcoming leakage in quantum error correction, *Nat. Phys.* **19**, 1780 (2023).
- [28] C. Chamberland and P. Ronagh, Deep neural decoders for near term fault-tolerant experiments, *Quantum Sci. Technol.* **3**, 044002 (2018).
- [29] D. Biswas, G. M. Vaidya, and P. Mandayam, Noise-adapted recovery circuits for quantum error correction, *Phys. Rev. Res.* **6**, 043034 (2024).
- [30] P. S. Iyer and D. Poulin, Hardness of decoding quantum stabilizer codes, *IEEE Trans. Inf. Theory* **61**, 5209 (2015).
- [31] M.-H. Hsieh and F. Le Gall, NP-hardness of decoding quantum error-correction codes, *Phys. Rev. A* **83**, 052331 (2011).
- [32] H. Jung, J. Kang, and J. Ha, in *Proceedings of the 11th International Conference on Information and*

- Communication Technology Convergence (ICTC)* (Jeju Island, South Korea, 2020), p. 227.
- [33] S. Bravyi, M. Suchara, and A. Vargo, Efficient algorithms for maximum likelihood decoding in the surface code, *Phys. Rev. A* **90**, 032326 (2014).
 - [34] T. J. Yoder, Maximum-likelihood decoding of quantum codes, U.S. Patent filed by International Business Machines Corporation, <https://patents.google.com/patent/US11736122B1/en> (2023).
 - [35] L. P. Pryadko, On maximum-likelihood decoding with circuit-level errors, *Quantum* **4**, 304 (2020).
 - [36] D. Poulin, Optimal and efficient decoding of concatenated quantum block codes, *Phys. Rev. A* **74**, 052333 (2006).
 - [37] Y. S. Weinstein, T. F. Havel, J. Emerson, N. Boulant, M. Saraceno, S. Lloyd, and D. G. Cory, Quantum process tomography of the quantum Fourier transform, *J. Chem. Phys.* **121**, 6117 (2004).
 - [38] D. Suess, Ł. Rudnicki, T. O. Maciel, and D. Gross, Error regions in quantum state tomography: Computational complexity caused by geometry of quantum states, *New J. Phys.* **19**, 093013 (2017).
 - [39] R. Levy, D. Luo, and B. K. Clark, Classical shadows for quantum process tomography on near-term quantum computers, *Phys. Rev. Res.* **6**, 013029 (2024).
 - [40] J. Emerson, M. Silva, O. Moussa, C. Ryan, M. Laforest, J. Baugh, D. G. Cory, and R. Laflamme, Symmetrized characterization of noisy quantum processes, *Science* **317**, 1893 (2007).
 - [41] A. Erhard, J. J. Wallman, L. Postler, M. Meth, R. Stricker, E. A. Martinez, P. Schindler, T. Monz, J. Emerson, and R. Blatt, Characterizing large-scale quantum computers via cycle benchmarking, *Nat. Commun.* **10**, 5347 (2019).
 - [42] A. Carignan-Dugas, D. Dahlen, I. Hincks, E. Ospadov, S. J. Beale, S. Ferracin, J. Skanes-Norman, J. Emerson, and J. J. Wallman, The error reconstruction and compiled calibration of quantum computing cycles, *ArXiv:2303.17714*.
 - [43] S. J. Beale *et al.*, True-Q, 2020, <https://doi.org/10.5281/zenodo.3945250>
 - [44] A. Hashim, R. K. Naik, A. Morvan, J.-L. Ville, B. Mitchell, J. M. Kreikebaum, M. Davis, E. Smith, C. Iancu, K. P. O'Brien, I. Hincks, J. J. Wallman, J. Emerson, and I. Siddiqi, Randomized compiling for scalable quantum computing on a noisy superconducting quantum processor, *Phys. Rev. X* **11**, 041039 (2021).
 - [45] N. Fazio, R. Freund, D. Sannamoth, A. Steiner, C. D. Marciniak, M. Rispler, R. Harper, T. Monz, J. Emerson, and S. D. Bartlett, Characterizing physical and logical errors in a transversal CNOT via cycle error reconstruction, *Phys. Rev. X* **16**, 011030 (2026).
 - [46] S. T. Flammia and J. J. Wallman, Efficient estimation of Pauli channels, *ACM Trans. Quantum Comput.* **1**, 1 (2020).
 - [47] R. Harper, W. Yu, and S. T. Flammia, Fast estimation of sparse quantum noise, *PRX Quantum* **2**, 010322 (2021).
 - [48] E. Knill and R. Laflamme, Concatenated quantum codes, *ArXiv:quant-ph/9608012*.
 - [49] T. Jochym-O'Connor and R. Laflamme, Using concatenated quantum codes for universal fault-tolerant quantum gates, *Phys. Rev. Lett.* **112**, 010505 (2014).
 - [50] M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10**, 285 (1975).
 - [51] K. Kraus, A. Böhm, J. Dollard, and W. Wootters, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, Lecture Notes in Physics (Springer, Berlin, 1983).
 - [52] I. L. Chuang and M. A. Nielsen, Prescription for experimental determination of the dynamics of a quantum black box, *J. Mod. Opt.* **44**, 2455 (1997).
 - [53] D. A. Lidar, Lecture notes on the theory of open quantum systems, *ArXiv:1902.00967*.
 - [54] Y. Nambu and K. Nakamura, On the matrix representation of quantum operations, *ArXiv:quant-ph/0504091*.
 - [55] A. Gilchrist, D. R. Terno, and C. J. Wood, Vectorization of quantum operations and its use, *ArXiv:0911.2539*.
 - [56] C. J. Wood, J. D. Biamonte, and D. G. Cory, Tensor networks and graphical calculus for open quantum systems, *Quantum Inf. Comput.* **15**, 759 (2015).
 - [57] M. Gutiérrez and K. R. Brown, Comparison of a quantum error-correction threshold for exact and approximate errors, *Phys. Rev. A* **91**, 022335 (2015).
 - [58] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, 2018).
 - [59] R. Kueng, D. M. Long, A. C. Doherty, and S. T. Flammia, Comparing experiments to the fault-tolerance threshold, *Phys. Rev. Lett.* **117**, 170502 (2016).
 - [60] P. Iyer and D. Poulin, A small quantum computer is needed to optimize fault-tolerant protocols, *Quantum Sci. Technol.* **3**, 030504 (2018).
 - [61] A. Carignan-Dugas, M. Alexander, and J. Emerson, A polar decomposition for quantum channels (with applications to bounding error propagation in quantum circuits), *Quantum* **3**, 173 (2019).
 - [62] K. Boone, Concepts and methods for benchmarking quantum computers, Ph.D. thesis, University of Waterloo, <https://uwspace.uwaterloo.ca/items/72401eee-0981-4aa6-ac7b-ae3e873df07a>, 2021.
 - [63] T. Proctor, K. Young, A. D. Baczewski, and R. Blume-Kohout, Benchmarking quantum computers, *Nat. Rev. Phys.* **7**, 105 (2025).
 - [64] N. Johnston and D. W. Kribs, Quantum gate fidelity in terms of Choi matrices, *J. Phys. A: Math. Theor.* **44**, 495303 (2011).
 - [65] P. Iyer Sridharan, Une analyse critique de la correction d'erreurs quantiques pour du bruit réaliste, Ph.D. thesis, Université de Sherbrooke, Sherbrooke, Québec, Canada, also published as "A critical analysis of quantum error correction methods for realistic noise processes", <http://hdl.handle.net/11143/14194>, 2018.
 - [66] J. Emerson, R. Alicki, and K. Życzkowski, Scalable noise estimation with random unitary operators, *J. Opt. B: Quantum Semiclassical Opt.* **7**, S347 (2005).
 - [67] C. Dankert, R. Cleve, J. Emerson, and E. Livine, Exact and approximate unitary 2-designs and their application to fidelity estimation, *Phys. Rev. A* **80**, 012304 (2009).
 - [68] E. Magesan, J. M. Gambetta, and J. Emerson, Scalable and robust randomized benchmarking of quantum processes, *Phys. Rev. Lett.* **106**, 180504 (2011).
 - [69] E. Magesan, J. M. Gambetta, and J. Emerson, Characterizing quantum gates via randomized benchmarking, *Phys. Rev. A* **85**, 042311 (2012).

- [70] J. J. Wallman and J. Emerson, Noise tailoring for scalable quantum computation via randomized compiling, *Phys. Rev. A* **94**, 052325 (2016).
- [71] D. E. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, <https://doi.org/doi:10.7907/rzr7-dt72>, 1997.
- [72] E. Pelchat and D. Poulin, Degenerate Viterbi decoding, *IEEE Trans. Inf. Theory* **59**, 3915 (2013).
- [73] S. Beale, Modeling and managing noise in quantum error correction, Ph.D. thesis, University of Waterloo, <https://hdl.handle.net/10012/20047>, 2023.
- [74] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland, Surface codes: Towards practical large-scale quantum computation, *Phys. Rev. A* **86**, 032324 (2012).
- [75] R. Acharya *et al.*, Quantum error correction below the surface code threshold, *Nature* **638**, 920 (2025).
- [76] E. Knill and R. Laflamme, Concatenated quantum codes, *ArXiv:quant-ph/9608012*.
- [77] S. Yoshida, S. Tamiya, and H. Yamasaki, Concatenate codes, save qubits, *npj Quantum Inf.* **11**, 88 (2025).
- [78] H. Yamasaki and M. Koashi, Time-efficient constant-space-overhead fault-tolerant quantum computation, *Nat. Phys.* **20**, 247 (2024).
- [79] O. Higgott, PyMatching: A Python package for decoding quantum codes with minimum-weight perfect matching, *ACM Trans. Quantum Comput.* **3**, 1 (2022).
- [80] J. M. Renes, Belief propagation decoding of quantum channels by passing quantum messages, *New J. Phys.* **19**, 072001 (2017).
- [81] Y.-H. Liu and D. Poulin, Neural belief-propagation decoders for quantum error-correcting codes, *Phys. Rev. Lett.* **122**, 200501 (2019).
- [82] B. Rahn, A. C. Doherty, and H. Mabuchi, Exact performance of concatenated quantum codes, *Phys. Rev. A* **66**, 032304 (2002).
- [83] Y. Tomita and K. M. Svore, Low-distance surface codes under realistic quantum noise, *Phys. Rev. A* **90**, 062320 (2014).
- [84] A. S. Darmawan and D. Poulin, Tensor-network simulations of the surface code under realistic noise, *Phys. Rev. Lett.* **119**, 040502 (2017).
- [85] S. Resch and U. R. Karpuzcu, Benchmarking quantum computers and the impact of quantum noise, *ACM Comput. Surv.* **54**, 1 (2021).
- [86] B. D. Clader, C. J. Trout, J. P. Barnes, K. Schultz, G. Quiroz, and P. Titum, Impact of correlations and heavy tails on quantum error correction, *Phys. Rev. A* **103**, 052428 (2021).
- [87] M. Sarovar, T. Proctor, K. Rudinger, K. Young, E. Nielsen, and R. Blume-Kohout, Detecting crosstalk errors in quantum information processors, *Quantum* **4**, 321 (2020).
- [88] K. Rudinger, C. W. Hogle, R. K. Naik, A. Hashim, D. Lobser, D. I. Santiago, M. D. Grace, E. Nielsen, T. Proctor, S. Seritan, S. M. Clark, R. Blume-Kohout, I. Siddiqi, and K. C. Young, Experimental characterization of crosstalk errors with simultaneous gate set tomography, *PRX Quantum* **2**, 040338 (2021).
- [89] D. Hangleiter, Crosstalk diagnosis for the next generation of quantum processors, *Quantum Views* **4**, 46 (2020).
- [90] A. Valenti, E. van Nieuwenburg, S. Huber, and E. Greplova, Hamiltonian learning for quantum error correction, *Phys. Rev. Res.* **1**, 033092 (2019).
- [91] E. van den Berg and P. Wocjan, Techniques for learning sparse Pauli-Lindblad noise models, *Quantum* **8**, 1556 (2024).
- [92] R. Alicki, Quantum error correction fails for Hamiltonian models, *Fluct. Noise Lett.* **6**, C23 (2006).
- [93] G. Consani and P. A. Warburton, Effective Hamiltonians for interacting superconducting qubits: Local basis reduction and the Schrieffer–Wolff transformation, *New J. Phys.* **22**, 053040 (2020).
- [94] R. Harper and S. T. Flammia, Learning correlated noise in a 39-qubit quantum processor, *PRX Quantum* **4**, 040311 (2023).
- [95] We recognize that an alternative reference for comparison, in place of the performance of $\mathcal{D}_1$, is the performance of the minimum-weight decoder, denoted $\mathcal{D}_{\text{MWD}}$. Despite its widespread use [19,74], the minimum-weight decoder is generally considered suboptimal compared with $\mathcal{D}_1$ because it accounts for the most likely error instead of the most likely logical coset [125,126]. Consequently, our estimation of gain is a conservative one. Specifically, when we set the numerator in Eq. (9) to $\mathcal{D}_{\text{MWD}}$, we observe that the gain Δ_ℓ is several orders of magnitude higher than the values reported in our estimates.
- [96] P. Aliferis, D. Gottesman, and J. Preskill, Quantum accuracy threshold for concatenated distance-3 codes, *Quantum Inf. Comput.* **6**, 97 (2006).
- [97] Z. Li, I. Kim, and P. Hayden, Concatenation schemes for topological fault-tolerant quantum error correction, *Quantum* **7**, 1089 (2023).
- [98] A. Krishna, A new way to use old codes, *Nat. Phys.* **20**, 182 (2024).
- [99] O. Fawzi, A. Grospellier, and A. Leverrier, Constant overhead quantum fault tolerance with quantum expander codes, *Commun. ACM* **64**, 106 (2021).
- [100] A. A. Kovalev and L. P. Pryadko, Fault tolerance of quantum low-density parity check codes with sublinear distance scaling, *Phys. Rev. A* **87**, 020304(R) (2013).
- [101] J. Roffe, Towards practical quantum LDPC codes, *Quantum Views* **5**, 63 (2021).
- [102] S. T. Spitz, B. Tarasinski, C. W. J. Beenakker, and T. E. O’Brien, Adaptive weight estimator for quantum error correction in a time-dependent environment, *Adv. Quantum Technol.* **1**, 1800012 (2018).
- [103] A. S. Darmawan, Optimal adaptation of surface-code decoders to local noise, *ArXiv:2403.08706*.
- [104] C. T. Chubb, General tensor network decoding of 2D Pauli codes, *ArXiv:2101.04125*.
- [105] E. T. Hockings, A. C. Doherty, and R. Harper, Improving error suppression with noise-aware decoding, *ArXiv:2502.21044*.
- [106] Pavithran Iyer Sridharan, Data for *arXiv:2507.08536v1*, GitHub repository, <https://github.com/paviudes/data-for-2507.08536v1.git>
- [107] E. Nielsen, J. K. Gamble, K. Rudinger, T. Scholten, K. Young, and R. Blume-Kohout, Gate set tomography, *Quantum* **5**, 557 (2021).

- [108] A. Mondal and K. K. Parhi, Quantum circuits for stabilizer error correcting codes: A tutorial, *IEEE Circuits Syst. Mag.* **24**, 33 (2024).
- [109] M. Katsuda, K. Mitarai, and K. Fujii, Simulation and performance analysis of quantum error correction with a rotated surface code under a realistic noise model, *Phys. Rev. Res.* **6**, 013024 (2024).
- [110] T. Camara, H. Ollivier, and J.-P. Tillich, in *2007 IEEE International Symposium on Information Theory* (Acropolis Congress and Exhibition Center, Nice, France, 2007), p. 811.
- [111] J. Marks, T. Jochym-O'Connor, and V. Gheorghiu, Comparison of memory thresholds for planar qudit geometries, *New J. Phys.* **19**, 113022 (2017).
- [112] A. W. Cross, Fault-tolerant quantum computer architectures using hierarchies of quantum error-correcting codes, Ph.D. thesis, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, <http://hdl.handle.net/1721.1/44407>, 2008.
- [113] A. S. Darmawan and D. Poulin, Linear-time general decoding algorithm for the surface code, *Phys. Rev. E* **97**, 051302(R) (2018).
- [114] P. Iyer, A. Jain, S. D. Bartlett, and J. Emerson, Efficient diagnostics for quantum error correction, *Phys. Rev. Res.* **4**, 043218 (2022).
- [115] E. Huang, A. C. Doherty, and S. Flammia, Performance of quantum error correction with coherent errors, *Phys. Rev. A* **99**, 022313 (2019).
- [116] D. Greenbaum and Z. Dutton, Modeling coherent errors in quantum error correction, *Quantum Sci. Technol.* **3**, 015007 (2018).
- [117] P. Zhao, K. Linghu, Z. Li, P. Xu, R. Wang, G. Xue, Y. Jin, and H. Yu, Quantum crosstalk analysis for simultaneous gate operations on superconducting qubits, *PRX Quantum* **3**, 020301 (2022).
- [118] V. Tripathi, H. Chen, M. Khezri, K.-W. Yip, E. Levenson-Falk, and D. A. Lidar, Suppression of crosstalk in superconducting qubits using dynamical decoupling, *Phys. Rev. Appl.* **18**, 024068 (2022).
- [119] A. Ketterer and T. Wellens, Characterizing crosstalk of superconducting transmon processors, *Phys. Rev. Appl.* **20**, 034065 (2023).
- [120] S. Choi, Y. Bao, X.-L. Qi, and E. Altman, Quantum error correction in scrambling dynamics and measurement-induced phase transition, *Phys. Rev. Lett.* **125**, 030505 (2020).
- [121] M. Liu, J. Liu, Y. Alexeev, and L. Jiang, Estimating the randomness of quantum circuit ensembles up to 50 qubits, *npj Quantum Inf.* **8**, 137 (2022).
- [122] J. Wallman, C. Granade, R. Harper, and S. T. Flammia, Estimating the coherence of noise, *New J. Phys.* **17**, 113020 (2015).
- [123] S. Bravyi and A. Vargo, Simulation of rare events in quantum error correction, *Phys. Rev. A* **88**, 062308 (2013).
- [124] A. Jain, P. Iyer, S. D. Bartlett, and J. Emerson, Improved quantum error correction with randomized compiling, *Phys. Rev. Res.* **5**, 033049 (2023).
- [125] Y. Yuan and C. Lu, in *2022 International Symposium on Information Theory and Its Applications (ISITA)* (IEEE, Ranzan, Japan, 2022), p. 204.
- [126] A. deMarti iOlius, P. Fuentes, R. Orús, P. M. Crespo, and J. Etxezarreta Martinez, Decoding algorithms for surface codes, *Quantum* **8**, 1498 (2024).