

Emulating time refraction at photonic time interfaces via smooth temporal transitions

Mariya Antyufeyeva^{1,2} and Victor Pacheco-Peña^{1,*}

¹*School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne, NE1 7RU, United Kingdom*

²*School of Radiophysics, Biomedical Electronics and Computer Systems, V. N. Karazin Kharkiv National University, 4, Svobody Square, Kharkiv, 61022, Ukraine*

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The introduction of time as a new degree of freedom to control wave-matter interactions has opened alternative avenues to fully control wave propagation in four dimensions (x, y, z, t). Time interfaces (rapid changes of the constitutive relationships of the medium where a wave propagates) have recently become popular, as they are the temporal analogue of spatial interfaces. While recent groundbreaking experimental work has demonstrated time interfaces from water waves, microwaves, and the optical regime, rapidly changing, for instance, the permittivity of the medium requires carefully engineered structures. Here, we study the possibility of implementing smooth temporal transitions of the permittivity and permeability of the medium (while considering time-independent impedance) to mimic the response of the time-refracted waves produced by photonic time interfaces. It is shown that indeed, as long as the signal is inside the structure during the modulation time, there are some values of rising or falling time of the smooth modulation that enable a full approximate emulation of time refraction. These results may open further avenues to explore by relaxing the speed at which the time modulation should be introduced when designing four-dimensional media.

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I. INTRODUCTION

Controlling wave-matter interactions in three-dimensional space has been at the center of the microwave and photonics community, making it possible to manipulate wave propagation using spatially engineered structures placed along the path where a wave propagates [1–12]. Recently, wave propagation in four dimensions, with time t as the extra degree of freedom, has become a prominent research topic, enabling the manipulation of waves in both space and time [13–23]. Wave propagation in temporally modulated media (in particular rapid changes using a single step function) was theoretically explored in a seminal work by Morgenthaler [3] where the electromagnetic properties of the medium were changed rapidly from one value to a different value at a time t_0 using a step function, i.e., using a time-dependent relative permittivity, $\varepsilon(t)$, and/or relative permeability, $\mu(t)$. Such a scenario is now known as a

temporal boundary or interface where, similar to a spatial interface between two materials of different constitutive parameters (such as ε , μ , impedance), a time-reflected (backward, BW) wave and a time-refracted (forward, FW) wave are excited. These two waves will then travel with a frequency different from that of the signal existing inside the medium for times $t < t_0$ [3,24].

The exploration of time interfaces has led to exciting new physical phenomena [24–28], including temporal aiming [29], temporal multilayer structures modeling higher-order transfer functions [30] and impedance transformers that can be used as temporal filters [31]; classical and quantum antireflection temporal coating [32,33]; a temporal analogue of metasurface unit cells [34]; broadband light sculpting [35]; an analogue of Faraday polarization rotation at the time interface in magnetoplasma [36]; waveguides and transmission lines [37,38]; broadband frequency translation [39]; phase-conjugation [40]; a temporal prism [41] and an inverse prism [42,43]; temporal Brewster angle [44]; temporal twistronics [45]; amplification and holding of waves [46–48]; ultraslow surface waves [47]; and propagating to evanescent wave coupling [49], among others [18,50–57]. Interestingly, experimental demonstrations of time interfaces have been reported in the past few years. Examples include time interfaces for water waves [58], microwaves [39,40] and the optical regime [59–61],

*victor.pacheco-pena@newcastle.ac.uk

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demonstrating how the field is rapidly evolving towards experimental implementations.

One of the challenges of time interfaces is that the electromagnetic properties of the medium where the wave propagates should rapidly change with a fall (rise) time smaller than the period of the signals already present in the medium. However, recent works have shown that tapered time interfaces (using smooth profiles [62] or multisteps [63]) can also be implemented to achieve, for instance, frequency conversion where the final frequency can be similar to that obtained when using a sharp time interface (as long as the final permittivity value is the same, as expected). Such an approach can be viewed as the spatial tapering used in horn antennas where reflections due to an abrupt spatial change between the guided structure and free space are minimized by ensuring a smooth change of impedance [64]. Similarly, temporal tapering [62] produces a FW wave with the BW wave being reduced and ultimately eliminated when the permittivity (and the impedance using time-independent μ) of the medium smoothly changes over time [62]. As the BW wave can be eliminated and only the FW wave remains, an interesting question is what will happen to the phase of the FW wave depending on the rise (fall) time of the smooth transition?, i.e., is there a value of rise (fall) time of a smoothly changing time-dependent material where the phase profile of the FW wave is exactly (or at least approximately) the same as the phase of the FW wave produced by step functions (time interfaces)? If such conditions exist, it would be possible to mimic time refraction fully using smoothly varying material properties, potentially leading to further practical implementations.

Inspired by the interesting opportunities and possibilities that time interfaces can open, in this work, we address this question by theoretically investigating the conditions under which the smooth profile of $\varepsilon(t)$ and $\mu(t)$ (changing simultaneously) can mimic the amplitude and phase values of the time-refracted electromagnetic wave produced by a time interface [single step functions of $\varepsilon(t)$ and $\mu(t)$]. For this, we consider values of $\varepsilon(t)$ and $\mu(t)$ such that the impedance is preserved for all times. To do this, we make use of the modal basis method, also known as the evolutionary approach to electromagnetics (EAE) in the time domain [65–69], considering a cavity surrounded by perfect electric conducting walls and filled with a time-varying medium. The $\varepsilon(t)$ and $\mu(t)$ values of the medium filling the cavity are changed over time using both a step function and a sigmoid function to modify the rise (fall) time of the modulated material. (Note that we focus on a sigmoid as this is a natural function to describe a smooth transition when compared to a rapid transition due to a step function.) As shown below, this approximation enables us to map the instantaneous frequency, amplitude, and phase of the FW wave, or the total field in our case. The analytical solutions obtained for the time part of the

field (modal amplitudes) allow us to analyze in detail its reliance on the time duration of the smooth modulation and contrast of the medium [variation of $\varepsilon(t)$ and $\mu(t)$ before and after the temporal modulation]. Interestingly, it will be shown that there are values of rise (fall) times of $\varepsilon(t)$ and $\mu(t)$ where the steady-state field oscillations of the FW wave (after the time modulation has been completely applied) have approximately the same amplitude and phase as that of the FW wave produced by a single step function of $\varepsilon(t)$ and $\mu(t)$. Full analytical expressions of these conditions are provided, demonstrating how, in applications such as frequency conversion, it is possible to closely mimic the time-refracted signal produced by a time interface using sigmoid (or smooth) functions of $\varepsilon(t)$ and $\mu(t)$. The theoretical calculations are confirmed using numerical simulations via COMSOL Multiphysics®. Our work provides further physical insights to understand wave propagation in time-varying media. In particular, the evolution of the phase when implementing smooth temporal changes of $\varepsilon(t)$ and $\mu(t)$ will be relevant to applications including space-time crystals [70] and temporal coherence [35,61].

II. BACKGROUND AND THEORY

A. Description of the problem under study

A schematic representation of the problem under study is shown in Fig. 1. We consider a cavity (surrounded by a perfect electric conductor, PEC, walls) filled with an isotropic lossless homogeneous medium defined by a time-dependent $\varepsilon(t)$ and $\mu(t)$ changing simultaneously [Fig. 1(a)]. [It is important to note that we make use of a closed system to better observe the field distribution within our structure. Moreover, and most importantly, as shown below, the smooth transitions of $\varepsilon(t)$ and $\mu(t)$ may become larger compared to the period of the incident signal. This means that the longer the transition time, the bigger the medium in space should be in a real experiment, for instance. This is because the incident signal should remain inside the time-varying medium for all the times when $\varepsilon(t)$ and $\mu(t)$ are changing. Thus, by considering a closed structure, we make sure that the signal remains confined for all times. However, the same approach can be used to analyze the case when $\varepsilon(t)$ and $\mu(t)$ change within an unbounded medium, as numerically demonstrated below.] A time interface is implemented by rapidly changing the constitutive parameters of the medium where a wave travels. This is schematically shown in Fig. 1(a) and Fig. 1(c), where time-dependent $\varepsilon(t)$ and $\mu(t)$ of the medium filling the cavity have values of ε_1 and μ_1 for times $t < t_0$, and they are rapidly changed to ε_2 and μ_2 at time $t = t_0$ (i.e., single step functions of $\varepsilon(t)$ and $\mu(t)$, where $\varepsilon_{1,2}$, $\mu_{1,2}$ are values of relative permittivity and permeability, respectively, of the medium before and after the time interface

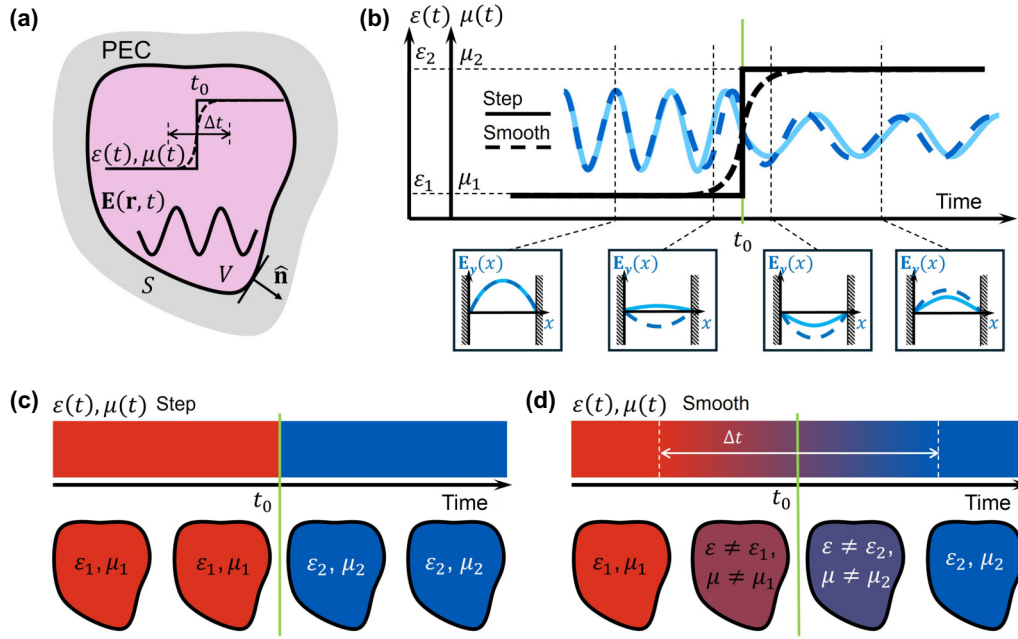


FIG. 1. Schematic representation of a time interface (single step) and smooth change of $\varepsilon(t)$ and $\mu(t)$ of the medium inside a cavity. Schematic of the geometry of a general cavity and problem statement (a), representation of the problem in time for the step and smooth change of $\varepsilon(t)$ and $\mu(t)$ and their influence on the field inside the cavity (b). Solid light blue and dashed blue lines represent the field distribution for the case of a time interface and a smooth function of $\varepsilon(t)$ and $\mu(t)$, respectively. (c),(d) Sketches of the time interface and corresponding step and smooth changes, respectively, of $\varepsilon(t)$ and $\mu(t)$.

is introduced). Moreover, as the aim is to study the possibility of emulating the time-refracted signal (FW wave) produced by time interfaces, in this manuscript it is considered that the impedance of the medium is kept constant for all times ($Z = \sqrt{\mu_1/\varepsilon_1} = \sqrt{\mu_2/\varepsilon_2}$). The case when the impedance of the medium changes over time is also an interesting scenario and may produce similar responses to those presented below (these are currently under study but fall outside the scope of this manuscript). Here, in addition to time interfaces, we study smooth transitions of $\varepsilon(t)$ and $\mu(t)$ from ε_1, μ_1 to ε_2, μ_2 , respectively [Fig. 1(d)], to theoretically establish similarities between rapid (time interface) and slow temporal modulations. Importantly, the cavity from Fig. 1 has, in general, an arbitrary shape and is geometrically defined by a volume, V , which is bounded by the closed PEC surface, S [Fig. 1(a)].

For completeness, schematic representations of $\varepsilon(t)$ and $\mu(t)$ of the medium filling the cavity at different time instants are shown in Figs. 1(c) and 1(d) for the case of a single step function of $\varepsilon(t)$ and $\mu(t)$ and a smooth transition (with transition time Δt), respectively. The impact of introducing such temporal functions of $\varepsilon(t)$ and $\mu(t)$ is schematically shown in Fig. 1(b), where it can be observed how the amplitude of the field already inside the cavity may change, but the spatial distribution (for example, the fundamental mode of the cavity) is preserved. As discussed below, the single step or smooth temporal modulation of $\varepsilon(t)$ and $\mu(t)$ will influence the amplitude and phase of

the electromagnetic field in the cavity. Mathematically, throughout this manuscript, the single step function and smooth temporal transition of the constitutive ε and μ values are defined as follows, respectively:

$$\varepsilon(t)^{\text{step}} = \varepsilon_1 + (\varepsilon_2 - \varepsilon_1)H(t - t_0) = \begin{cases} \varepsilon_1, & t < t_0, \\ \varepsilon_2, & t > t_0, \end{cases} \quad (1a)$$

$$\varepsilon(t)^{\text{smooth}} = \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{1 + e^{-\gamma(t-t_0)}}, \quad (1b)$$

where $\varepsilon(t)^{\text{step}}$ and $\varepsilon(t)^{\text{smooth}}$ are the time-dependent relative permittivity for the time interface (single step function) and smooth transition, respectively; $\mu(t) = Z^2\varepsilon(t)$; $n(t) = \sqrt{\mu(t)\varepsilon(t)} = Z\varepsilon(t)$; $H(t - t_0)$ is the Heaviside function centered at t_0 (corresponding to the time when the transition is halfway from ε_1 to ε_2 for the smooth function); Δt is the rise or fall time of the smooth transition [Eq. (1b)] defined as $\Delta t = \zeta/\gamma$, with γ as a control parameter [when γ goes to infinity, Eq. (1b) converges into Eq. (1a)] and ζ as the parameter used here to estimate the transition time. As detailed below, this parameter ensures that the smooth transition has approximately the same values of $\varepsilon(t)$ as the time interface [at times $t < t_0$ and $t > t_0$ from Eq. (1a)].

B. Theoretical approach

In this section, we present a theoretical analysis of the scenario shown in Fig. 1 using EAE in the time domain.

The theory behind expanding electromagnetic fields within cavities considering certain normal modes with orthogonality properties was introduced and has been the subject of study for the past century [65,67–69,71–76]. This approach has been used to, for instance, analyze the electromagnetic field and its interaction within a cavity filled with slowly time-varying media (with time modulations on the order of 1000 times the period of the signal inside the cavity) [72,77,78]. The EAE applied to the electromagnetic field analysis within a cavity filled with a linear homogeneous medium completely separates the spatial field distribution and its temporal evolution into two independent problems [65,69,71,72,79,80]. Based on this and following Refs. [65,69,81], in this section, we provide the key steps to solve the problem under study (Fig. 1) using the EAE in the time domain.

As it is known, when considering time interfaces, the constitutive relationships describing the interaction between the electromagnetic field and the medium inside the cavity (considering nondispersive materials where the working frequency falls away from any material resonances [3,24,28,82]) can be written as $\mathbf{D}(\mathbf{r}, t) = \epsilon_0 \epsilon(t) \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t) = \mu_0 \mu(t) \mathbf{H}(\mathbf{r}, t)$, where $\mathbf{D}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$, $\mathbf{E}(\mathbf{r}, t)$, and $\mathbf{H}(\mathbf{r}, t)$ are the electric field displacement, magnetic flux density, electric field, and magnetic field, respectively; $\mathbf{r}$ is the position vector; t is time; and ϵ_0 and μ_0 are the absolute permittivity and permeability values in free space, respectively. Here, we consider that for times $t < 0$ there is an electromagnetic signal that exists within the cavity. This source is then stopped at $t = 0$. This means that there is an initial distribution of the electromagnetic field, $\mathbf{E}(\mathbf{r}, 0) = \mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}(\mathbf{r}, 0) = \mathbf{H}_0(\mathbf{r})$ at the time when the fields start to be observed ($t = 0$). For times $t > 0$, the electromagnetic signal keeps resonating within the cavity and at time $t = t_0 > 0$ a time interface is introduced. Note that this configuration is in line with Refs. [13,18]. For simplicity, we specify the shape of the cavity as rectangular [see Fig. 1(b)], but we note that this technique can be applied to any geometry of the cavity.

Following the EAE in the time domain, in general, we are looking for the electric and magnetic fields in the cavity filled with the time-dependent medium as the expansion in the modal basis vectors [69]:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) = & \sum_{m=1}^{\infty} e'_m(t) \mathbf{E}'_m(\mathbf{r}) + \sum_{m=1}^{\infty} e''_m(t) \mathbf{E}''_m(\mathbf{r}) \\ & + \sum_{\alpha=1}^{\infty} a_{\alpha}(t) \nabla \phi_{\alpha}(\mathbf{r}), \end{aligned} \quad (2a)$$

$$\begin{aligned} \mathbf{H}(\mathbf{r}, t) = & \sum_{m=1}^{\infty} h'_m(t) \mathbf{H}'_m(\mathbf{r}) + \sum_{m=1}^{\infty} h''_m(t) \mathbf{H}''_m(\mathbf{r}) \\ & + \sum_{\beta=1}^{\infty} b_{\beta}(t) \nabla \psi_{\beta}(\mathbf{r}), \end{aligned} \quad (2b)$$

where $\mathbf{E}'_m(\mathbf{r})$ and $\mathbf{E}''_m(\mathbf{r})$ are the electric and $\mathbf{H}'_m(\mathbf{r})$ and $\mathbf{H}''_m(\mathbf{r})$ are the magnetic solenoidal modal vectors depending only on spatial coordinates and representing the divergence-free electromagnetic part of the field in the cavity. The terms $e'_m(t)$, $e''_m(t)$, $h'_m(t)$, and $h''_m(t)$ are the electric and magnetic modal amplitudes representing their evolution with time. The prime (') and double-prime (') superscripts denote the different sets of modal vectors. The terms $\nabla \phi_{\alpha}(\mathbf{r})$ and $\nabla \psi_{\beta}(\mathbf{r})$ are the electric and magnetic irrotational modal vectors, also depending only on spatial coordinates. These modal vectors account for the presence of static electromagnetic fields in the cavity [81], and $a_{\alpha}(t)$ and $b_{\beta}(t)$ are their electric and magnetic modal amplitudes, respectively. To find the modal vectors, one can consider the well-known eigenvalue problems and scalar Helmholtz equations with Dirichlet and Neumann boundary conditions, taking into account the orthonormality condition [69] as follows:

$$\begin{cases} \nabla \times \mathbf{E}'_m(\mathbf{r}) = \omega'_m \mu_0 \mathbf{H}'_m(\mathbf{r}), \\ \nabla \times \mathbf{H}'_m(\mathbf{r}) = \omega'_m \epsilon_0 \mathbf{E}'_m(\mathbf{r}), \\ (\hat{\mathbf{n}} \times \mathbf{E}'_m(\mathbf{r}))|_S = 0, \end{cases} \quad (3a)$$

$$\begin{cases} \nabla \times \mathbf{H}''_m(\mathbf{r}) = \omega''_m \epsilon_0 \mathbf{E}''_m(\mathbf{r}), \\ \nabla \times \mathbf{E}''_m(\mathbf{r}) = \omega''_m \mu_0 \mathbf{H}''_m(\mathbf{r}), \\ (\hat{\mathbf{n}} \cdot \mathbf{H}''_m(\mathbf{r}))|_S = 0, \end{cases} \quad (3b)$$

$$\begin{cases} (\nabla^2 + \kappa_{\alpha}^2) \phi_{\alpha}(\mathbf{r}) = 0, \phi_{\alpha}(\mathbf{r})|_S = 0, \\ \frac{\kappa_{\alpha}^2 \epsilon_0}{V} \int_V |\phi_{\alpha}(\mathbf{r})|^2 dV = 1, \end{cases} \quad \begin{cases} (\nabla^2 + \kappa_{\beta}^2) \psi_{\beta}(\mathbf{r}) = 0, \left(\frac{\partial}{\partial \mathbf{N}} \psi_{\beta}(\mathbf{r}) \right) \Big|_S = 0, \\ \frac{\kappa_{\beta}^2 \mu_0}{V} \int_V |\psi_{\beta}(\mathbf{r})|^2 dV = 1, \end{cases} \quad (3c)$$

$$\begin{aligned} \frac{\epsilon_0}{V} \int_V |\mathbf{E}'_m|^2 dV = \frac{\mu_0}{V} \int_V |\mathbf{H}'_m|^2 dV = \frac{\epsilon_0}{V} \int_V |\mathbf{E}''_m|^2 dV = \frac{\mu_0}{V} \int_V |\mathbf{H}''_m|^2 dV = 1, \\ \frac{\epsilon_0}{V} \int_V |\nabla \phi_{\alpha}|^2 dV = \frac{\mu_0}{V} \int_V |\nabla \psi_{\beta}|^2 dV = 1, \end{aligned} \quad (3d)$$

where $\omega'_m \neq 0$, $\omega''_m \neq 0$, $\kappa_\alpha > 0$, and $\kappa_\beta > 0$, are real eigenvalues. The eigenvalues ω'_m and ω''_m represent the eigenfrequencies of an empty cavity (i.e., where the filling material is vacuum). When the shape of the cavity is rectangular, we can write $\omega'_m = \omega''_m \equiv \omega_{pqs} = \frac{\pi}{\sqrt{\epsilon_0 \mu_0}} \sqrt{\left(\frac{p}{l_x}\right)^2 + \left(\frac{q}{l_y}\right)^2 + \left(\frac{s}{l_z}\right)^2}$, where p, q , and s are integers indicating the number of variations of the standing-wave pattern in the x, y , and z directions, respectively, and l_x, l_y , and l_z are the dimensions of the cavity along each direction [64]. The solutions to Eqs. 3(a) and 3(b) are associated with complete sets of solenoidal TE and TM modes [65,69,81].

Coming back to the description of the initial field in the cavity under consideration [$\mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$], we can reduce our calculation by considering that only one TE mode exists within the cavity and no static field is present. This means that the irrotational electric and magnetic parts of the initial fields associated with the presence of a static field are zero [83] [i.e., in the last terms from Eq. (2), $a_\alpha(0) = 0$ and $b_\beta(0) = 0$]. Due to this, the expansions from Eq. (2) for the initial fields are only limited by the first sum and only by the first term in this sum (i.e., $m = 1$). As it is known, as the magnetic field is $\pi/2$ out of phase with the electric field in a lossless resonator, the magnetic field becomes zero twice per period of field oscillation, while, at the same time, the electric field reaches its maximum within the entire volume of the cavity. It is this moment of time that we call $t = 0$ in this work and, as explained

above, it is the time when the source stops. Based on this, we can define $\mathbf{H}_0(\mathbf{r}) = 0$. With these considerations, the expressions of the electric and magnetic fields using the expansion in the modal basis vectors [Eq. (2)] are reduced to

$$\begin{aligned}\mathbf{E}(\mathbf{r}, 0) &= \mathbf{E}_0(\mathbf{r}) = e'_1(0)\mathbf{E}'_1(\mathbf{r}) = e_0\mathbf{E}'_1(\mathbf{r}), \\ \mathbf{H}(\mathbf{r}, 0) &= \mathbf{H}_0(\mathbf{r}) = 0,\end{aligned}\quad (4)$$

with $t = 0$, $e_0 = (\epsilon_0/V) \int_V \mathbf{E}_0(\mathbf{r}) \cdot \mathbf{E}'_1(\mathbf{r}) dV$. The term $e_0 \neq 0$ plays the role of the initial condition of the solenoidal electric modal amplitude, $e'_m(t)$ [65], when $m = 1$. Equation (4) then represents the modal expansion of the initial field at any point $\mathbf{r}$ inside the cavity. Naturally, the initial conditions for all modal amplitudes, except $e'_1(t)$, are zero, as we are only dealing with the fundamental TE mode, as mentioned above.

Regarding the modal amplitudes $e'_m(t)$, $e''_m(t)$, $a_\alpha(t)$, $h'_m(t)$, $h''_m(t)$, and $b_\beta(t)$ for times $t > 0$ from Eq. (2), as they describe the time evolution of the electric and magnetic fields at each point in space within the volume of the cavity, they can be obtained by projecting Maxwell's equations onto the modal basis vectors. A detailed description of a similar procedure was presented in Refs. [65,79]. The application of this procedure for the current problem yields the following equations for finding the modal amplitudes (also called evolutionary equations in the literature [65,67–69,72,80,81]):

$$\begin{cases} \frac{d}{dt} \epsilon(t) e'_m(t) - \omega'_m h'_m(t) = 0, & e'_1(0) = e_0, \quad e'_{m \neq 1}(0) = 0, \quad m = 1, 2, \dots, \\ \frac{d}{dt} \mu(t) h'_m(t) + \omega'_m e'_m(t) = 0, & h'_m(0) = 0, \quad m = 1, 2, \dots, \end{cases} \quad (5a)$$

$$\begin{cases} \frac{d}{dt} \epsilon(t) e''_m(t) - \omega''_m h''_m(t) = 0, & e''_m(0) = 0, \quad m = 1, 2, \dots, \\ \frac{d}{dt} \mu(t) h''_m(t) + \omega''_m e''_m(t) = 0, & h''_m(0) = 0, \quad m = 1, 2, \dots, \end{cases} \quad (5b)$$

$$\frac{d}{dt} \epsilon(t) a_\alpha(t) = 0, \quad a_\alpha(0) = 0, \quad \alpha = 1, 2, \dots, \quad (5c)$$

$$\frac{d}{dt} \mu(t) b_\beta(t) = 0, \quad b_\beta(0) = 0, \quad \beta = 1, 2, \dots \quad (5d)$$

The solution of the homogeneous linear differential equation with constant coefficients accompanied with zero initial conditions yields zero sought functions [84]. Thus, over the sets of evolutionary equations from Eq. (5), only Eq. (5a) for $m = 1$ has a nonzero solution. This fact simplifies the electric and magnetic field expansion from

Eq. (2) in the considered cavity in a similar fashion to the initial field decomposition from Eq. (4). Based on this, the electric and magnetic fields for the scenario presented in Fig. 1 are determined as

$$\mathbf{E}(\mathbf{r}, t) = e'_1(t)\mathbf{E}'_1(\mathbf{r}), \quad \mathbf{H}(\mathbf{r}, t) = h'_1(t)\mathbf{H}'_1(\mathbf{r}), \quad (6)$$

where $\mathbf{E}'_1(\mathbf{r})$ and $\mathbf{H}'_1(\mathbf{r})$ are the electric and magnetic field distributions of the fundamental TE mode in the cavity, determined by Eq. (3a), and $e'_1(t)$ and $h'_1(t)$ are the temporal amplitudes of the electric and magnetic fields described by

the system of evolutionary equations:

$$\begin{cases} \frac{d}{dt}\varepsilon(t)e'_1(t) - \omega'_1 h'_1(t) = 0, & e'_1(0) = e_0, \\ \frac{d}{dt}\mu(t)h'_1(t) + \omega'_1 e'_1(t) = 0, & h'_1(0) = 0. \end{cases} \quad (7)$$

The description of an efficient approach to solve systems like Eq. (7) can be found in Refs. [65,67,81], and using those steps, the solution of the system from Eq. (7) is obtained as follows:

$$e'_1(t) = \frac{\varepsilon(0)}{\varepsilon(t)} e_0 \cos[\omega'_1 \tau(t)], \quad (8a)$$

$$h'_1(t) = -\frac{\varepsilon(0)}{Z\varepsilon(t)} e_0 \sin[\omega'_1 \tau(t)], \quad (8b)$$

where $\mu(t) = Z^2 \varepsilon(t)$ and $\tau(t) = \int_0^t \frac{du}{Z\varepsilon(u)}$ is an auxiliary function (u is the variable of integration over time). Thus, Eq. (8) represent a full analytical expression of the temporal behavior of the electromagnetic fields at any point $\mathbf{r}$ within the cavity. Now, let us denote the time-varying amplitude in Eq. (8) as $\varepsilon(0)/\varepsilon(t) = A(t)$ and the phase as $\varphi(t) = \omega'_1 \tau(t)$. Importantly, this follows the definition of the instantaneous frequency $[\Omega(t)]$, as defined in Ref. [3]; i.e., $\Omega(t) = \frac{d\varphi(t)}{dt} = \omega'_1 \frac{d\tau(t)}{dt} = \omega'_1 \frac{1}{Z\varepsilon(t)} = \frac{\omega'_1}{n(t)}$. Since the goal of solving the problem is to estimate the contribution of the smooth transition of $\varepsilon(t)$ and $\mu(t)$ [Eq. (1b)] compared to a time interface [Eq. (1a)], the parameters $A(t)$, $\tau(t)$, and instantaneous frequency $\Omega(t)$ for the “step” and “smooth” changes of $\varepsilon(t)$ and $\mu(t)$ for times $t < t_0$ (before the time interface) and $t > t_0$ (after the time interface) can be calculated as

$$A(t)^{\text{step}} = \begin{cases} 1, & t < t_0, \\ \varepsilon_1/\varepsilon_2, & t > t_0, \end{cases} \quad \Omega(t)^{\text{step}} = \begin{cases} \frac{\omega'_1}{Z\varepsilon_1}, & t < t_0, \\ \frac{\omega'_1}{Z\varepsilon_2}, & t > t_0, \end{cases} \quad (9a)$$

$$\tau(t)^{\text{step}} = \begin{cases} \frac{t}{\varepsilon_1 Z}, & t < t_0, \\ \frac{1}{\varepsilon_2 Z} \left\{ t + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} t_0 \right\}, & t > t_0, \end{cases}$$

$$A(t)^{\text{smooth}} = \begin{cases} \frac{(\varepsilon_2 + \varepsilon_1 e^{\gamma t_0})(1 + e^{\gamma(t_0-t)})}{(1 + e^{\gamma t_0})(\varepsilon_2 + \varepsilon_1 e^{\gamma(t_0-t)})}, & t < t_0, \\ \frac{(\varepsilon_2 + \varepsilon_1 e^{\gamma t_0})(1 + e^{-\gamma(t-t_0)})}{(1 + e^{\gamma t_0})(\varepsilon_2 + \varepsilon_1 e^{-\gamma(t-t_0)})}, & t > t_0, \end{cases} \quad \Omega(t)^{\text{smooth}} = \begin{cases} \frac{\omega'_1}{Z} \frac{1 + e^{-\gamma(t_0-t)}}{\varepsilon_1 + \varepsilon_2 e^{-\gamma(t_0-t)}}, & t < t_0, \\ \frac{\omega'_1}{Z} \frac{1 + e^{-\gamma(t-t_0)}}{\varepsilon_2 + \varepsilon_1 e^{-\gamma(t-t_0)}}, & t > t_0, \end{cases} \quad (9b)$$

$$\tau(t)^{\text{smooth}} = \begin{cases} \frac{1}{\varepsilon_1 Z} \left\{ t - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} \left[\frac{1}{\gamma} \ln \left(\frac{\varepsilon_2 e^{-\gamma(t_0-t)} + \varepsilon_1}{\varepsilon_2 e^{-\gamma t_0} + \varepsilon_1} \right) \right] \right\}, & t < t_0, \\ \frac{1}{\varepsilon_2 Z} \left\{ t + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} t_0 - \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} \left[\frac{1}{\gamma} \ln \left(\frac{\varepsilon_2 + \varepsilon_1 e^{-\gamma(t-t_0)}}{\varepsilon_2 e^{-\gamma t_0} + \varepsilon_1} \right) \right] \right\}, & t > t_0, \end{cases}$$

As observed, the parameters from Eq. (9a) correspond to those obtained by Morgenthaler in Ref. [3] and widely known in the literature, considering the case when the impedance of the medium is constant. Regarding the terms describing the smooth transition [Eq. (9b)], it can be seen how for longer times ($t \gg t_0$) the final frequency of the

electromagnetic signal inside the cavity is approximately the same as that of a step function, as expected. Next, we present an in-depth analysis of the conditions under which the full expressions from Eqs. (9a) and (9b) for the time interface and smooth transition, respectively, may approximately coincide.

C. Convergence condition for time refraction: Time interface and smooth transition

Once we have obtained the full analytical expression of the modal amplitudes within the cavity, we can now proceed to compare the solutions when using a step function and a smooth transition of $\varepsilon(t)$ and $\mu(t)$. The aim here is to answer the question of what will happen to the phase of the FW wave, depending on the rise (fall) time of smoothly modulated $\varepsilon(t)$ and $\mu(t)$. As described in Eq. (9), the instantaneous frequency, $\Omega(t)$, for a smooth transition of $\varepsilon(t)$ and $\mu(t)$ becomes approximately the same as that of the step function for longer times after temporal modulation has been introduced. However, the question about the phase remains unanswered. Importantly, as shown below, the temporal evolution of the phase when considering smooth time variations and its dependence on the material parameters is not trivial and is yet to be explored (in comparison with spatially graded media [85,86]). This is the main focus of our work, as described above.

To calculate the conditions under which both solutions (smooth and step cases) may be the same, we take into account the following: first, it is considered that $t_0 > \Delta t/2$ to ensure the same value of $\varepsilon(t) = \varepsilon_1$ at the beginning of the observation time ($t = 0$) for both the smooth transition of $\varepsilon(t)$ and $\mu(t)$ and the time interface (see some illustrations related to the violation of these conditions in Appendix C where it is shown that the fields never converge, as such violation will produce a considerably different value of $\varepsilon(t)$ at $t = 0$ for the smooth transition compared to the value for the time interface). Next, it is also considered that the amplitude and phase matching of the oscillation produced by the smooth transition and the time interface is calculated outside the duration of the transition of $\varepsilon(t)$ and $\mu(t)$ (i.e., for times not involving the time interval from $t = t_0 - \Delta t/2$ to $t = t_0 + \Delta t/2$ or, in other words, where the permittivity of the medium is approximately either ε_1 or ε_2 but not in between). Importantly, this means that we need to ensure that $\varepsilon(t)$ for the smooth transition at times $t = t_0 - \Delta t/2$ and $t = t_0 + \Delta t/2$ is the same as the time interface (i.e., ε_1 or ε_2 , respectively, for the time interface at times $t < t_0$ and $t > t_0$, respectively). To enable this, let us evaluate the first time ($t = t_0 - \Delta t/2$). Using Eq. (1b) and considering the transition time as $\Delta t = \zeta/\gamma$, the permittivity becomes $\varepsilon(t)|_{t=t_0-\Delta t/2} = \frac{\varepsilon_2 + \varepsilon_1 e^{\gamma \Delta t/2}}{1 + e^{\gamma \Delta t/2}} = \frac{\varepsilon_2 + \varepsilon_1 e^{\zeta/2}}{1 + e^{\zeta/2}}$, meaning that $\varepsilon_2 e^{-\zeta/2} \ll \varepsilon_1$ to obtain $\varepsilon(t)|_{t=t_0-\Delta t/2} = \varepsilon_1$. This means that, with the mathematical definition of $\varepsilon(t)$ for the smooth transition from Eq. (1b), one can simply choose the value of ζ and then recalculate Δt to fulfil the condition for $\varepsilon(t)|_{t=t_0-\Delta t/2} = \varepsilon_1$. Based on this, here we use $\zeta = 20$ (so $\varepsilon_2 e^{-10} \ll \varepsilon_1$) for the initial studies. The same process can be applied for the second time, $t = t_0 + \Delta t/2$, where using Eq. (1b) we get $\varepsilon(t)|_{t=t_0+\Delta t/2} = \frac{\varepsilon_2 + \varepsilon_1 e^{-\zeta/2}}{1 + e^{-\zeta/2}}$, meaning that we should satisfy the condition $\varepsilon_1 e^{-\zeta/2} \ll \varepsilon_2$

for the smooth transition to have the same value as the time interface ($\varepsilon(t)|_{t=t_0+\Delta t/2} = \varepsilon_2$). These two conditions are not mutually exclusive. Moreover, note that as ζ is finite, the condition $\varepsilon_1 e^{-\zeta/2} \ll \varepsilon_2$ will then mean that $\varepsilon(t)|_{t=t_0+\Delta t/2} \approx \varepsilon_2$. In other words, the parameter ζ can be used to measure the accuracy of the final permittivity value (and hence, the overall solution of the modal amplitude) for the smooth transition compared to the time interface. A discussion about ζ is presented later in this manuscript to study its implications in terms of the accuracy of the results.

Now, with these considerations, let us analyze the expressions from Eqs. 9(a) and 9(b) to understand if the solutions for both smooth transition and time interface are similar for the time-refracted wave. Let us start with the expression for the amplitude, $A(t)$, from Eq. (9) and establish $A(t)^{\text{step}} = A(t)^{\text{smooth}}$ for longer times ($t > t_0$). The conditions that enable this equality to hold require $(\varepsilon_1/\varepsilon_2) \gg e^{-\gamma t_0}$ (see Appendix B). Similar to the analysis above for $\varepsilon(t)$ from Eqs. 1(a) and 1(b), the relationship $(\varepsilon_1/\varepsilon_2) \gg e^{-\gamma t_0}$ that satisfies $A(t)^{\text{step}} = A(t)^{\text{smooth}}$ will be true for times when $\varepsilon(t)$ for the smooth transition is the same as ε_2 for the step function. However, as discussed above, $\varepsilon(t)$ for the smooth transition at times $t > t_0 + \Delta t/2$ will be approximately equals to ε_2 . Now, regarding $\tau(t)^{\text{step}}$ and $\tau(t)^{\text{smooth}}$ from Eq. (9), they only differ by an additional term introduced by the smooth transition, i.e., $-\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} \left[\frac{1}{\gamma} \ln \left(\frac{\varepsilon_2 + \varepsilon_1 e^{-\gamma(t-t_0)}}{\varepsilon_2 e^{-\gamma t_0} + \varepsilon_1} \right) \right]$, which is time dependent, as expected. From Eq. (8a), the phase of the modal amplitude is $\omega'_1 \tau(t)^{\text{step}}$ and $\omega'_1 \tau(t)^{\text{smooth}}$, for each function of $\varepsilon(t)$ and $\mu(t)$. This means that there will be specific Δt values where the phase for the smooth transition will be approximately the same as that of the time interface; i.e., the modal amplitudes as a function of time will overlap. Considering the full expression for $\omega'_1 \tau(t)^{\text{smooth}}$, this occurs when $\frac{\omega'_1}{Z \varepsilon_2} \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} \left[\frac{1}{\gamma} \ln \left(\frac{\varepsilon_2 + \varepsilon_1 e^{-\gamma(t-t_0)}}{\varepsilon_2 e^{-\gamma t_0} + \varepsilon_1} \right) \right] = 2\pi j$, where $j = 1, 2, 3, \dots$. This yields the following phase condition for $t > t_0 + \Delta t/2$, which is simplified using the previous considerations $(\varepsilon_1/\varepsilon_2) \gg e^{-\gamma t_0}$ or $(\varepsilon_1/\varepsilon_2), (\varepsilon_2/\varepsilon_1) \gg e^{-\zeta/2}$, and $t_0 > \Delta t/2$, to mimic the time interface by the smooth transition:

$$\gamma = \frac{\omega'_1}{2\pi j} \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 \varepsilon_2 Z} \ln \left(\frac{\varepsilon_2}{\varepsilon_1} \right), j = 1, 2, 3, \dots \quad (10)$$

This means that if the duration of the smooth transition of $\varepsilon(t)$ and $\mu(t)$ satisfies Eq. (10), the temporal part of the electromagnetic field in the cavity produced by the smooth transition of $\varepsilon(t)$ and $\mu(t)$ will approximately match that produced by the time interface [step change of $\varepsilon(t)$ and $\mu(t)$].

III. RESULTS AND DISCUSSION

Here, theoretical calculations using smooth and step functions of $\varepsilon(t)$ and $\mu(t)$ [Eqs. (8) and (9)] are presented, showing both the amplitude and phase characteristics of the FW wave in the time domain (noting that, as mentioned above, we keep impedance unchanged through the process, so the FW wave is indeed the total field inside the cavity). These results are then compared with numerical simulations via COMSOL Multiphysics®, for completeness [87]. As shown in the previous section, Eqs. (8) and (9) fully describe the temporal behavior of the electromagnetic field within the cavity. Without loss of generality, we can consider a one-dimensional (1D) cavity with vertical parallel-plate metallic walls (PEC in our case), as shown in Fig. 1(b), by considering the corresponding 1D value of ω'_1 in Eqs. (8) and (9), i.e., $\omega'_1 = \pi \sqrt{\left(\frac{1}{l_x}\right)^2 / (\mu_0 \varepsilon_0)}$, where l_x is the distance between plates.

A. Changing $\varepsilon(t)$ and $\mu(t)$: From smaller to larger values

Let us first study the case when $\varepsilon(t)$ and $\mu(t)$ are changed from smaller to larger values. The time interface is introduced at t_0/T_1 . For the smooth transitions, we consider different rising times, $\Delta t/T_1$, with $T_1 = 2\pi Z\varepsilon_1/\omega'_1$ as the period of the field oscillation at times before $\varepsilon(t)$ and $\mu(t)$ are changed (with instantaneous frequency of electromagnetic field at these times defined as $\omega'_1/Z\varepsilon_1$). Specifically, for this study, we use $T_1 = 0.67$ ns (corresponding to a frequency of about 1.5 GHz) with $\Delta t = \zeta/\gamma = 20/\gamma$, $t_0 = 53$ ns, $\varepsilon_1 = 2$, and $\varepsilon_2 = 12$, as an example. Note that we make use of large changes of the material parameters to better appreciate the results and to provide a better understanding of the physics behind them; this is common in the field of time-varying media [48,56,63,88,89]. However, smaller values can also be used, and our theory will still be valid, as expected. As mentioned above, we also consider a time-dependent $\mu(t)$ with corresponding values of μ_1 and μ_2 chosen accordingly to keep a time-independent impedance of $Z = 0.85$. The numerical simulations were carried out using the time-domain solver from COMSOL Multiphysics® following the same configuration as in Ref. [18]; further details can be found in Appendix A.

Using this setup, the theoretical results [Eqs. (8) and (9)] of the mode amplitude, $e(t)^{\text{smooth}}$, as a function of transition time, $\Delta t/T_1$, and normalized time, t/T_1 , are shown in Fig. 2(a). To guide the eye, the vertical solid line represents the time when the time interface is applied ($t_0/T_1 = 79.1$), while the diagonal dashed lines represent the transition time of the smooth functions of $\varepsilon(t)$ and $\mu(t)$ ($\Delta t/T_1$). The duration of the smooth transition, $\Delta t/T_1$, is changed from 3 to 70 with a step of approximately 0.19 (theory) and 0.3 (simulations). Importantly, as we keep $\zeta = 20$,

the control parameter, γ , is first modified to define different slopes of the smooth transition with $\Delta t = \zeta/\gamma$. This means that Δt will provide information about the rising time of the smooth transition of $\varepsilon(t)$ and $\mu(t)$. This, however, will have a different meaning when γ is kept constant and ζ is changed, as it will be detailed later below.

From the theoretical results shown in Fig. 2(a), it can be observed how the frequency of the field inside the cavity is changed once the smooth transition is applied. Moreover, for times after the modulation has finished (results on the right-hand side of the right dashed diagonal line), the frequency seems to be similar for all values of $\Delta t/T_1$, as expected due to ε_2 (and correspondingly μ_2) being the same for all smooth transitions of $\varepsilon(t)$ and $\mu(t)$. To corroborate the validity of the theoretical calculations shown in Fig. 2(a) [calculated using Eqs. (8) and (9)], we provide the full-wave numerical simulations via COMSOL Multiphysics® in Fig. 2(c) where the results of the time-dependent electric field for the smooth transitions of $\varepsilon(t)$ and $\mu(t)$ [$E_y(t)^{\text{smooth}}$] are shown. Good qualitative agreement is observed between both theoretical and numerical results. For completeness, we extracted the values of $e(t)^{\text{smooth}}$ (theoretical) and $E_y(t)^{\text{smooth}}$ (numerical) for different values of $\Delta t/T_1$ from Figs. 2(a) and 2(c), respectively, and the results are shown as solid lines in Figs. 2(b) and 2(d), respectively, along with the values of $\varepsilon(t)$ and $\mu(t)$ for each case [see Fig. 2(b)]. The normalized amplitudes of $e(t)^{\text{smooth}}$ and $E_y(t)^{\text{smooth}}$ are in agreement. The small phase difference between the theoretical ($e(t)^{\text{smooth}}$) and numerical ($E_y(t)^{\text{smooth}}$) values comes from the fact that, in the latter case, the phase of the signal within the cavity is slightly different from that of the theoretical analysis when the smooth transition is introduced. Such a small difference, however, does not affect the analysis, as the aim here is to compare the results between the smooth and step functions of $\varepsilon(t)$ and $\mu(t)$, while the comparison between the theoretical and numerical results validates the calculations shown in the previous sections. Finally, and to have the full set of solutions, in Figs. 2(b) and 2(d) we provide the theoretical and numerical results for $e(t)^{\text{step}}$ and $E_y(t)^{\text{step}}$, respectively, as dotted lines, showing how the phase of the smooth transitions and time interface is different, while the amplitudes are similar. However, how can we verify that the results shown in Fig. 2 for a smooth transition of $\varepsilon(t)$ and $\mu(t)$ are similar to those produced by a time interface? As shown in Eq. (10), there should be values of rising time where $e(t)^{\text{step}} = e(t)^{\text{smooth}}$. To fully address this, the theoretical values of $e(t)^{\text{step}} - e(t)^{\text{smooth}}$ and numerical simulation results of $E_y(t)^{\text{step}} - E_y(t)^{\text{smooth}}$ as a function of $\Delta t/T_1$ and t/T_1 are shown in Figs. 3(c) and 3(d), respectively. Apart from the agreement between the results, it is interesting to note that there are values of $\Delta t/T_1$ where $e(t)^{\text{step}} - e(t)^{\text{smooth}} \approx 0$ and $E_y(t)^{\text{step}} - E_y(t)^{\text{smooth}} \approx 0$. This means that there are indeed values of

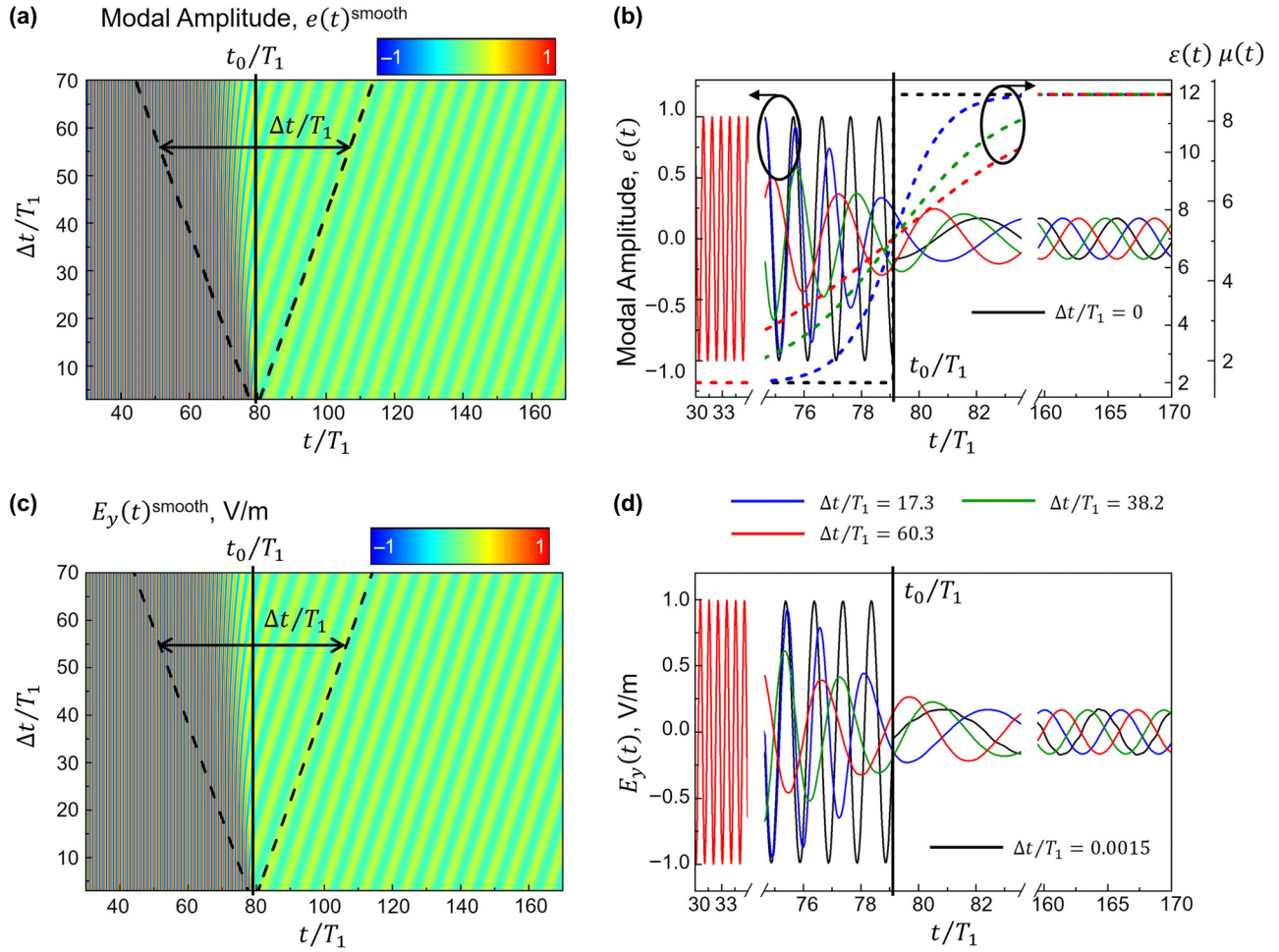


FIG. 2. Time-dependent fields inside the cavity using smooth transitions and time interfaces via $\varepsilon(t)$ and $\mu(t)$, from smaller to larger values. Theoretical results of the electric modal amplitude [calculated by Eq. (9)] (a) and numerical results of the electric field calculated using COMSOL Multiphysics® (c) as a function of rising time and normalized time using smooth transitions of permittivity from $\varepsilon_1 = 2$ to $\varepsilon_2 = 12$ when the impedance is time independent, $Z = 0.85$ (i.e., the permeability also changes with time). (b),(d) Theoretical [Eq. (9)] and numerical results (COMSOL Multiphysics®), respectively, extracted from (a),(c) using different values of rising time $\Delta t/T_1$ (solid lines) via a smooth transition of $\varepsilon(t)$ and $\mu(t)$ [dashed lines in panel (b)] along with the results of a single time interface (step function, black solid line). For the numerical simulations, the cavity has a length of $L_x = 117.57$ mm corresponding to a half wavelength inside the material filling the cavity at times before the time interface is applied. The vertical solid line in all panels corresponds to the value when the time interface is applied. The diagonal dashed lines from (a),(c) delimit the duration of the transitions of $\varepsilon(t)$ and $\mu(t)$ when implementing a smooth transition.

$\Delta t/T_1$ that may enable the full emulation of time refraction at a time interface using smooth time-dependent functions of $\varepsilon(t)$ and $\mu(t)$.

The implications of Eq. (10) are clearly seen in Figs. 3(c) and 3(d) where the values of $\Delta t/T_1$ (with $\Delta t = 20/\gamma$) corresponding to $j = 1, 2, 3, \dots$ are those where the modal amplitudes $(e(t)^{\text{step}} - e(t)^{\text{smooth}})$ and the electric field $(E_y(t)^{\text{step}} - E_y(t)^{\text{smooth}})$ are approximately zero. For completeness, the calculated values of $e(t)^{\text{step}} - e(t)^{\text{smooth}}$ for $j = 1, 2, 3$ as a function of t/T_1 are shown in Fig. 3(f) as solid lines. These results demonstrate that, after the smooth transition of $\varepsilon(t)$ and $\mu(t)$ is introduced, the modal amplitudes are approximately the same as that of the time

interface [step function of $\varepsilon(t)$ and $\mu(t)$]. Similarly, and to corroborate the theoretical calculations, we extracted $E_y(t)^{\text{step}} - E_y(t)^{\text{smooth}}$ from the numerical simulations as a function of time for the zeroes (or approximately zero values) shown in Fig. 3(d), and the results are shown in Fig. 3(g). Note that the numerical solutions give us approximately but not exactly zero values due to (i) the approximate value of $\Delta t/T_1$ for the smooth transitions, and (ii) the nonzero rising time for the time interface used in the simulations (see Appendix A for full details). However, it is clear how there is agreement with the theoretical calculations from Fig. 3(f) where the time-refracted signal produced by a time interface can be emulated using the

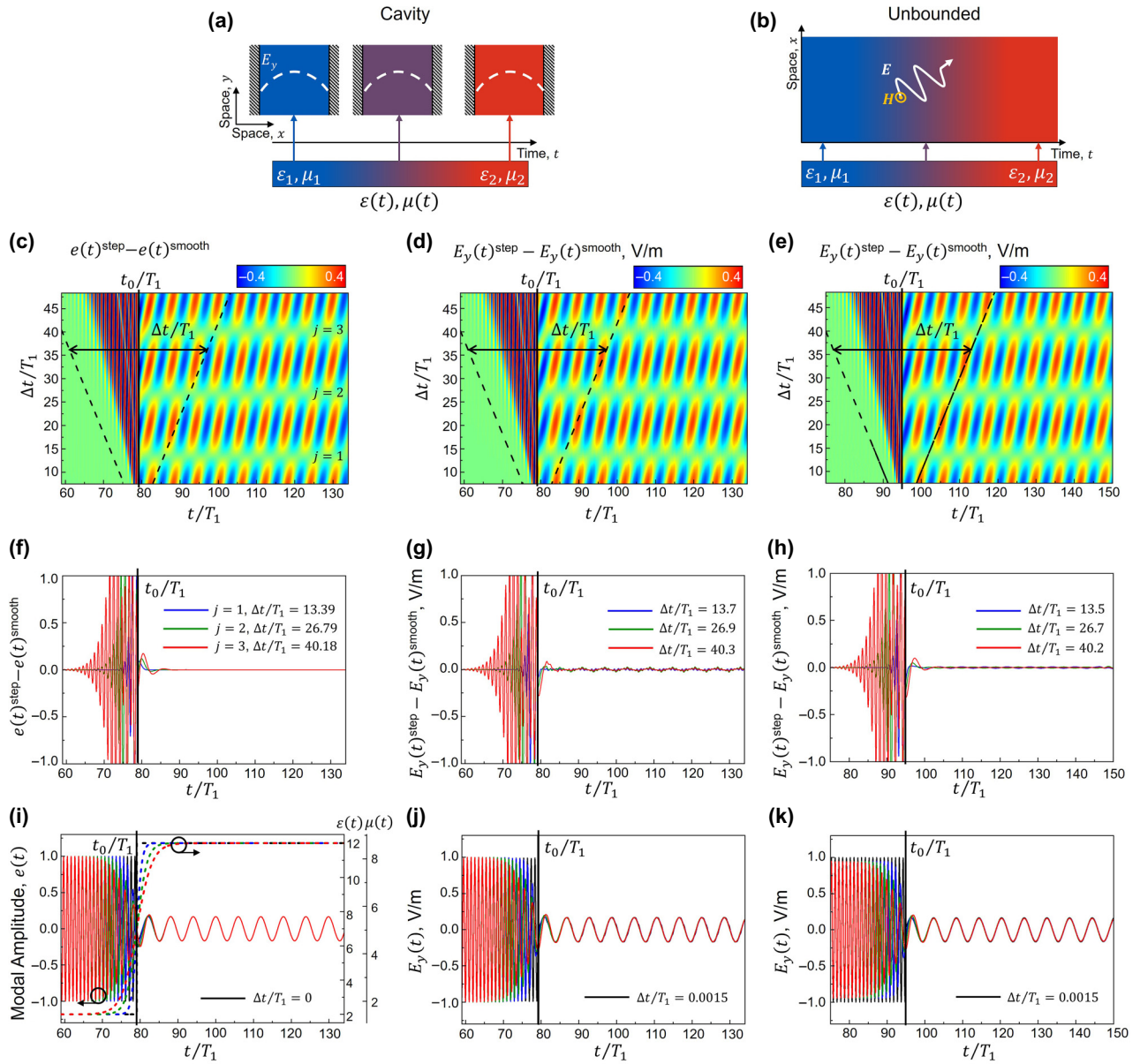
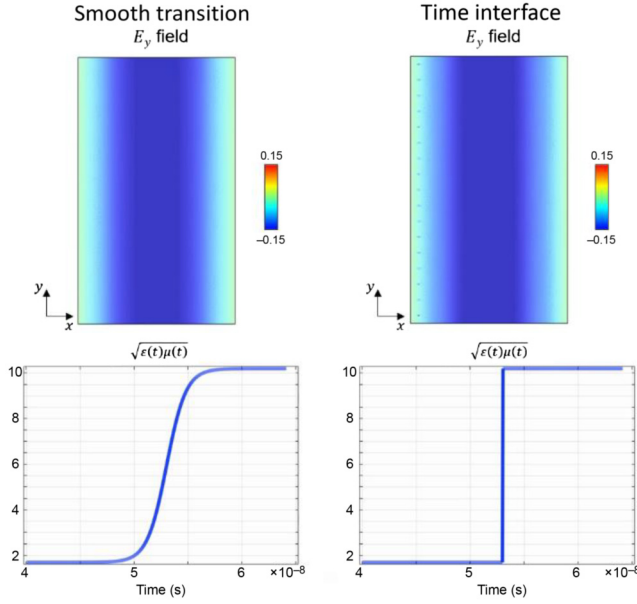


FIG. 3. Emulating time interfaces via smooth transitions, from smaller to larger values. Schematic representations of a cavity (a) and an unbounded medium (b) filled with a time-dependent medium. The results from the first (theoretical) and second (numerical) columns correspond to the scenario shown in panel (a). The results from the third column (numerical) correspond to the unbounded medium case from panel (b). Theoretical results [calculated by Eq. (9)] of the difference between the electric modal amplitudes produced by the step and smooth values of $\epsilon(t)$ and $\mu(t)$ (c) and numerical results (simulated using COMSOL Multiphysics®) of the difference in corresponding electric fields (d) as a function of rising time of the smooth transition and time for the scenario described in Fig. 2. (f),(g) Theoretical ($e^{\text{step}} - e^{\text{smooth}} \approx 0$) and numerical ($E_y^{\text{step}} - E_y^{\text{smooth}} \approx 0$) results, respectively, for the values of $\Delta t/T_1$ from (c),(d) where $e^{\text{step}} - e^{\text{smooth}} \approx 0$ and $E_y^{\text{step}} - E_y^{\text{smooth}} \approx 0$, respectively. (i),(j) Theoretical (electric modal amplitudes) and numerical (electric field) results, respectively (solid lines), produced by the time interface (step function, black solid lines in both panels), along with the results produced by the smooth transition of $\epsilon(t)$ and $\mu(t)$ (blue, red, and green solid lines). These results are calculated for the same values of $\Delta t/T_1$ as in panels (f),(g). Dashed lines in (i) represent the smooth transitions of $\epsilon(t)$ and $\mu(t)$. (e),(h),(k) Similar results to those in panels (d),(g),(j) but now considering an unbounded medium instead of a cavity. Solid vertical and dashed diagonal lines have the same meaning as those in Fig. 2.

smooth temporal transition. For completeness, the mode amplitudes and electric field (for both the step and smooth transitions) are shown in Figs. 3(i) and 3(j) along with the

corresponding functions of $\epsilon(t)$ and $\mu(t)$ [Fig. 3(i)], showing how the results (colored solid lines) overlap with those obtained with a time interface (black solid line). Also,



VIDEO 1. An example of a smooth transition (left) and a time interface (right). Both $\epsilon(t)$ and $\mu(t)$ change simultaneously. Top and bottom panels show E_y field distributions and time-dependent $\sqrt{\epsilon(t)\mu(t)}$, respectively.

an animation showing the numerical results for $E_y(t)^{\text{step}}$ and $E_y(t)^{\text{smooth}}$ for the case of $j = 2$ is presented in Video 1.

As mentioned before, the theoretical analysis was carried out using a closed cavity as we wanted to contain the electromagnetic signal within a closed space when introducing the time modulation of $\epsilon(t)$ and $\mu(t)$. However, it is expected that the theoretical and numerical results using a cavity [Figs. 3(c) and 3(d)] will also be valid for an unbounded medium. To confirm the validity of Eq. (10) obtained for the electromagnetic field in the cavity, we carried out numerical simulations using COMSOL Multiphysics® for the case when the time interface and the smooth transitions of $\epsilon(t)$ and $\mu(t)$ were implemented within the entire space where a wave propagated (unbounded medium). See Appendix A for the numerical setup. The numerical results for this scenario are shown in Figs. 3(e), 3(h), and 3(k). By comparing these results with those produced by the cavity, one can observe excellent agreement, showing how, indeed, the results are valid for an unbounded medium, as long as the electromagnetic wave remains inside the medium during the entire time when $\epsilon(t)$ and $\mu(t)$ is being changed, as expected.

B. Changing $\epsilon(t)$ and $\mu(t)$: From larger to smaller values

The results from Figs. 2 and 3 have shown the case when $\epsilon(t)$ and $\mu(t)$ are changed from smaller to larger values.

As a final study, we can also consider the scenario where $\epsilon(t)$ and, correspondingly, $\mu(t)$ are modified from larger to smaller values. Most of the parameters for this case are the same as those used in Figs. 2 and 3 but now with $\epsilon_1 = 12$ and $\epsilon_2 = 2$, with the same $Z = 0.85$. Moreover, the eigenfrequency of the cavity is initially $f_1 = 0.25$ GHz, considering the value of $\epsilon_1 = 12$ for times $t < t_0$ taking the time interface as a reference. This is different from the one used in Figs. 2 and 3 since we are keeping the same geometrical dimensions of the cavity so the eigenfrequency for times $t < t_0$ will change, as now $\epsilon_1 = 12$. Finally, $T_1 = 4.0$ ns and $t_0/T_1 = 13.25$. As before, we use $t_0 = 53$ ns as the time when the time interface is applied. With this configuration, the theoretical results for the modal amplitudes for the smooth transition of $\epsilon(t)$ and $\mu(t)$ as a function of $\Delta t/T_1$ and time t/T_1 are shown in Fig. 4(a) along with the values of $e(t)^{\text{step}} - e(t)^{\text{smooth}}$ in Fig. 4(c) (see Appendix D for the numerical simulations). As observed, there are also values of $\Delta t/T_1$ where the smooth transition of $\epsilon(t)$ and $\mu(t)$ emulates a time interface. To better observe this, the theoretical and numerical simulation results that satisfy Eq. (10) for $j = 1, 2, \dots, 5$ are shown in Figs. 4(b) and 4(d), respectively. These results demonstrate that, after the time interface is applied, the frequency of the signal inside the cavity is changed to a larger value [Figs. 4(a) and 4(b)], as expected, and $e(t)^{\text{step}} - e(t)^{\text{smooth}} \approx 0$ for $t > t_0$ [Figs. 4(c) and 4(d)]. As a final note, the results from Figs. 2–4 correspond to the case when specific values of ϵ_1 and ϵ_2 are used. As one would expect, the values of $\Delta t/T_1$ that fulfil Eq. (10) will depend on these parameters. Another example with different values of ϵ_1 and ϵ_2 is presented in Appendix E, for completeness.

C. Accuracy implications

As mentioned through this manuscript, the parameter ζ for the smooth transition of $\epsilon(t)$ and $\mu(t)$ in Eq. (1b) can be used to control how accurate a smooth transition mimics the response of a time interface. For completeness, in Fig. 5 we provide examples of how close the theoretical solution of the modal amplitude of the smooth transition is to that of the time interface (step function). For this, we first theoretically calculated $|e(t)^{\text{step}} - e(t)^{\text{smooth}}|$ at time $t = t_0 + \Delta t/2$. Here, differently from the results in Figs. 2–4, it is the parameter ζ that is changed (ranging from 1 to 80) while γ stays the same; i.e., the same smooth transition is used for all values of ζ . Due to this, Δt is recalculated for each value of ζ as $\Delta t = \zeta/\gamma$. The result of this is that now Δt does not provide information about the rising (falling) time of the smooth function of $\epsilon(t)$ and $\mu(t)$, but it gives information of how close $\epsilon(t)$ for the smooth transition is to ϵ_2 from the time interface case. In other words, when γ is constant, larger values of ζ will increase Δt and, hence, as the observation time increases ($t = t_0 + \Delta t/2$), $\epsilon(t)$ for the smooth transition will become a better approximation

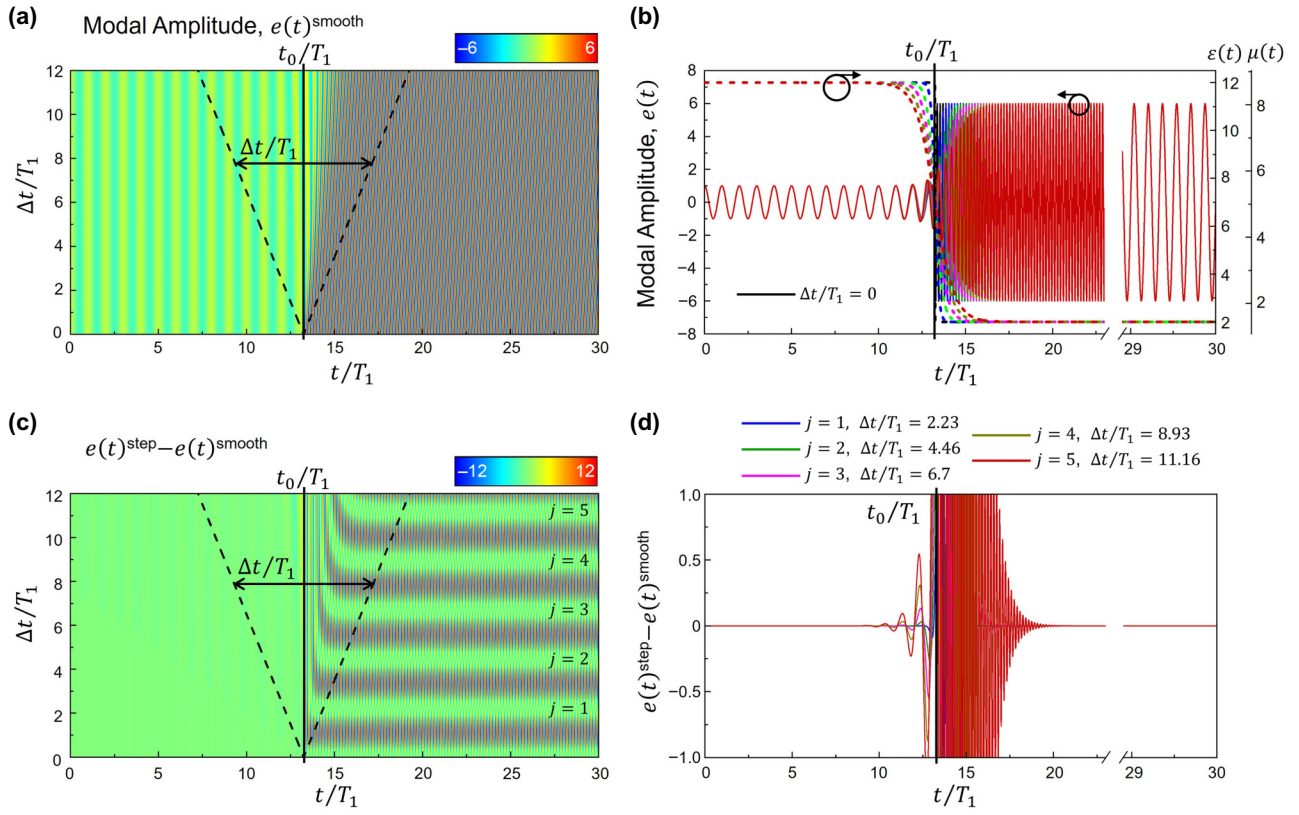


FIG. 4. Time-dependent fields inside the cavity using smooth transitions and time interfaces via $\varepsilon(t)$ and $\mu(t)$, from larger to smaller values. Theoretical results for the electric modal amplitude (a) as a function of falling time of the smooth transition and time using $\varepsilon_1 = 12$, $\varepsilon_2 = 2$, and time-independent $Z = 0.85$, as in Figs. 2 and 3. (b) Electric modal amplitudes related to the time interface (black solid line) and smooth transitions for several values of transition times that fulfil Eq. (10) (solid lines) with the related dependencies of $\varepsilon(t)$ and $\mu(t)$ (dashed lines). (c) Theoretical results for the difference between the electric modal amplitudes produced by step and smooth $\varepsilon(t)$ and $\mu(t)$. (d) Theoretical results for the values of $\Delta t/T_1$ from (c) where $e^{\text{step}} - e^{\text{smooth}} \approx 0$. The solid and vertical and dashed diagonal lines have the same meaning as those in Figs. 2 and 3.

of ε_2 at that time. With this configuration, the results from Figs. 5(a) and 5(b) correspond to the cases when $\varepsilon_1 = 2$, $\varepsilon_2 = 12$ and $\varepsilon_1 = 12$, $\varepsilon_2 = 2$, respectively, using the first three values of γ [Eq. (10)] that make both solutions (step and smooth cases) similar, i.e., $j = 1, 2, 3$. From Eq. (10) $\gamma = 2.24 \times 10^9 \text{ s}^{-1}$, $\gamma = 1.12 \times 10^9 \text{ s}^{-1}$ and $\gamma = 7.47 \times 10^8 \text{ s}^{-1}$ for $j = 1, 2, 3$, respectively. Meaning that each plot shown in Figs. 5(a) and 5(b) corresponds to different smooth transitions that fulfil Eq. (10), while γ is kept same for each case when ζ is modified. As observed in Figs. 5(a) and 5(b), the difference between the results of the modal amplitudes gets smaller as ζ increases. This is an expected result as $\zeta = \infty$ will make Eq. (1b) converge exactly into Eq. (1a) (as explained in the previous section). For the value of $\zeta = 20$ used in the results from Figs. 2–4, the difference between both modal amplitudes is on the order of approximately 1.8×10^{-5} and 3.1×10^{-4} for the cases shown in Figs. 5(a) and 5(b), respectively, at time $t = t_0 + \Delta t/2$. However, note that, while the difference between modal amplitudes reduces as ζ is increased, this difference starts to saturate for $\zeta > 60$ and $\zeta > 70$

from Figs. 5(a) and 5(b), which can be due to computational limitations on the decimal places when calculating the results. Nevertheless, one would expect the difference to asymptotically reach zero for $\zeta = \infty$.

As a final study, $t = t_0 + \Delta t/2$ is the time at which the results from Figs. 2–5(a) and 5(b) are obtained. This time enables us to compare the potential of smooth transitions to mimic a time interface. As one would expect, if this time were chosen such that $t = t_0 + \alpha$ with $\alpha > \Delta t/2$ and γ were fixed, the results of the modal amplitudes using both functions of $\varepsilon(t)$ would get much closer, as we are allowing more time for the smooth transition to converge into ε_2 . This can be observed in Figs. 5(c) and 5(d) where the influence of ζ on the accuracy of the emulation is presented, considering that the observation time is changed from $t = t_0 + \Delta t/2$ [as shown in Figs. 5(a) and 5(b)] to $t = t_0 + 2\Delta t$. The results from Figs. 5(c) and 5(d) correspond to the configurations of $\varepsilon(t)$ and $\mu(t)$ presented in Figs. 2–4 for $j = 1$. From Figs. 5(c) and 5(d), it can be seen how the modal amplitude for the smooth transitions of $\varepsilon(t)$ and $\mu(t)$ get much closer to the results

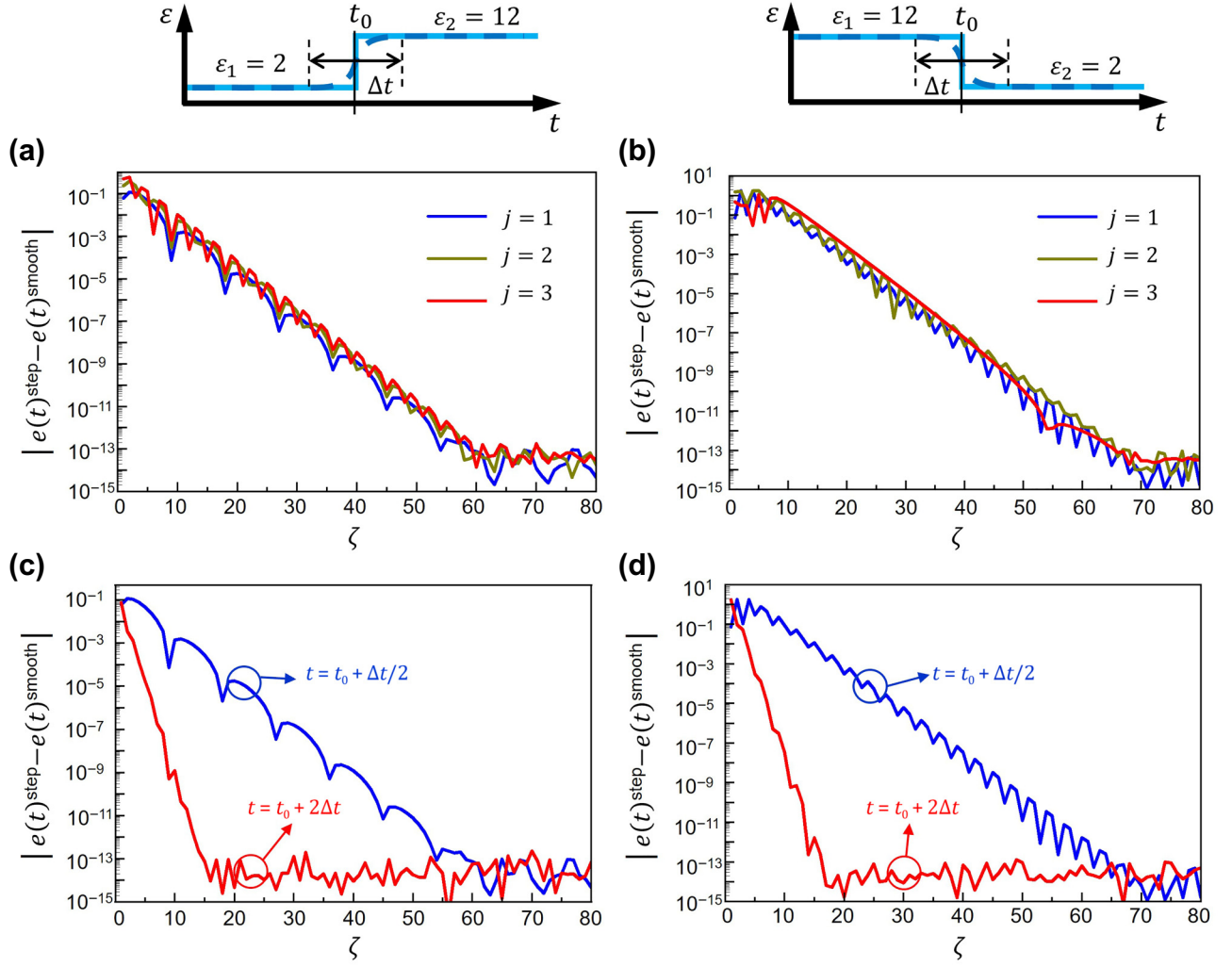


FIG. 5. Estimation of the accuracy of the emulation of time interfaces via smooth transitions. (a),(b) Theoretical results for the absolute value of the difference between the electric modal amplitudes produced by the step and smooth functions when $\varepsilon(t)$ and $\mu(t)$ are increased or reduced in time, respectively. These results are calculated at $t = t_0 + \Delta t/2$ for the condition $|e^{\text{step}} - e^{\text{smooth}}| \approx 0$ ($j = 1, 2, 3$). (c),(d) Values of $|e(t)^{\text{step}} - e(t)^{\text{smooth}}|$ produced by the step and smooth transitions for the same ε_1 and ε_2 and $j = 1$ as in panels (a),(b), respectively, as a function of the parameter ζ . These results are calculated using $t = t_0 + \Delta t/2$ (blue) and $t = t_0 + 2\Delta t$ (red).

of the time interface when $\alpha = 2\Delta t$ (i.e., $t = t_0 + 2\Delta t$). This means that the accuracy of the emulation can be further increased by changing α , as expected. These results demonstrate that, even when the smooth transition does not perfectly match the results of the time interface, it is possible to approximately emulate time refraction using smooth changes of $\varepsilon(t)$ and $\mu(t)$ by carefully choosing the rise (fall) time of the transition as well as the observation time. These results may open further experimental routes to exploit time interfaces being emulated by smooth modulations of $\varepsilon(t)$ and $\mu(t)$. As mentioned in Sec. I, experimental demonstrations of time interfaces are starting to be reported. While smooth temporal transitions of the material responses can also pose challenges, we hope that our study may be experimentally demonstrated by

leveraging all the research in this field. An example of this is, for instance, time-varying transmission lines using varactors [90,91], which can be designed to smoothly vary from one value to another. Research to develop platforms for time-varying media is a worldwide effort, and we hope our work will benefit from the efforts of the scientific community.

IV. CONCLUSIONS

We have discussed the potential of smooth changes of $\varepsilon(t)$ and $\mu(t)$ to mimic the response of the time-refracted signal produced by time interfaces. Full theoretical derivations have been carried out, demonstrating the conditions of the rise (fall) time of the smooth transition that can

enable this. Our findings may open further opportunities to explore the realm of space-time media. By enabling the emulation of time refraction via smooth modulations of $\varepsilon(t)$ and $\mu(t)$, the results presented here may serve as the basis to explore further experimental demonstrations by relaxing the need for ultrafast modulation of the properties of the material through which a signal travels.

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The authors declare no conflicts of interests.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: NUMERICAL SIMULATIONS

The numerical simulations were carried out using the time-domain solver of COMSOL Multiphysics® software with a similar setup as that in Ref. [18]. For the numerical calculations in Figs. 2–5 using the cavity, a total of 17 vertically polarized dipoles were used to excite the structure. They were placed $0.05\lambda_1$ away from the left PEC boundary of the cavity separated by a distance of $0.05\lambda_1$ along the y axis. Here, λ_1 is the wavelength of the signal inside the cavity when $\varepsilon(t) = \varepsilon_1$. To appreciate the influence of the time interface and the smooth transition of $\varepsilon(t)$ and $\mu(t)$, the dipoles are first switched OFF and then, after some time, $\varepsilon(t)$ and $\mu(t)$ of the medium within the cavity are changed simultaneously. For instance, for the results from Fig. 2, the dipoles are switched OFF at $t \approx 12T_1$ and the time interface is applied at $t \approx 12T_1$. A triangular mesh with the same maximum and minimum sizes as those reported in Ref. [18] was implemented. For the time interface, $\varepsilon(t)$ and $\mu(t)$ were implemented using analytical functions with a rise (fall) time of $1.5 \times 10^{-3}T_1$ with a smoothing of two continuous derivatives. The smooth transitions were implemented using the analytical expression described in Eq. (1b). A similar setup was implemented for the simulation results obtained using an unbounded medium, as shown in Figs. 3(b), 3(e), 3(h), and 3(k). For

this, a rectangular simulation space of $80\lambda_1 \times 3\lambda_1$ was implemented with top and bottom PECs and left and right scattering boundary conditions. Excitation was applied using the left scattering boundary condition. The source was switched OFF at $t \approx 60T_1$ and, after some time, the time interface was applied at $t \approx 95T_1$. The electric field shown in Figs. 3(e), 3(h), and 3(k) was recorded at a distance $x = 60\lambda_1$ from the source.

APPENDIX B: AMPLITUDE-MATCHING CONDITION

Consider the amplitude, $A(t)^{\text{smooth}}$, obtained for $t > t_0$ [Eq. (9b)] evaluated at longer times $t > t_0 + (\Delta t/2)$. Taking into account the relationship between the control parameter, γ , of the smooth transition and the estimation of the transition time, $\Delta t = \zeta/\gamma$, we get

$$\begin{aligned} A(t)^{\text{smooth}}|_{t=t_0+(\Delta t/2)} &= \frac{(\varepsilon_2 + \varepsilon_1 e^{\gamma t_0})(1 + e^{-\gamma \Delta t/2})}{(1 + e^{\gamma t_0})(\varepsilon_2 + \varepsilon_1 e^{-\gamma \Delta t/2})} \\ &= \frac{(\varepsilon_2 + \varepsilon_1 e^{\gamma t_0})(1 + e^{-\zeta/2})}{(1 + e^{\gamma t_0})(\varepsilon_2 + \varepsilon_1 e^{-\zeta/2})}. \quad (\text{B1}) \end{aligned}$$

At $t = t_0 + (\Delta t/2)$, Eq. (B1) becomes $A(t)^{\text{smooth}}|_{t=t_0+(\Delta t/2)} = \frac{(\varepsilon_2 + \varepsilon_1 e^{\gamma t_0})}{\varepsilon_2(1 + e^{\gamma t_0})}$. This means that the conditions that enable $A(t)^{\text{smooth}}|_{t=t_0+(\Delta t/2)} = A(t)^{\text{step}}|_{t>t_0} = \frac{\varepsilon_1}{\varepsilon_2}$ to hold require $e^{\gamma t_0} \gg 1$ and $\varepsilon_1 e^{\gamma t_0} \gg \varepsilon_2$ for the denominator and numerator, respectively, of $A(t)^{\text{smooth}}|_{t=t_0+(\Delta t/2)}$. The first condition can be reduced to $\gamma t_0 \gg 0$, which can be rewritten as $(\zeta/\Delta t)t_0 \gg 0$. The second condition corresponds to the relationship $(\varepsilon_1/\varepsilon_2) \gg e^{-\gamma t_0}$ shown in the main text.

APPENDIX C: EFFECT OF NOT FULFILLING THE CONDITION $t_0 > \Delta t/2$

Here, we present an example for when the condition $t_0 > \Delta t/2$ for the smooth transition is not fulfilled. For this, we consider the two examples from Figs. 2–4 in the main text. Time t_0 at which the time interface is applied is set as $t_0 = 10$ ns, or $t_0/T_1 = 15$ for the case when $\varepsilon(t)$ and $\mu(t)$ is increased [Figs. 6(a)–6(c)] and $t_0/T_1 = 2.5$ when $\varepsilon(t)$ and $\mu(t)$ is reduced [Figs. 6(d)–6(f)]. This value of t_0 is larger than $\Delta t/2$ for $j = 1, 2$ and smaller for $j = 3, 4, 5$. As observed from Fig. 6, modal amplitudes never converge as such a violation will produce a considerably different value of $\varepsilon(t)$ at $t = 0$ for the smooth transition compared to the value of the time interface.

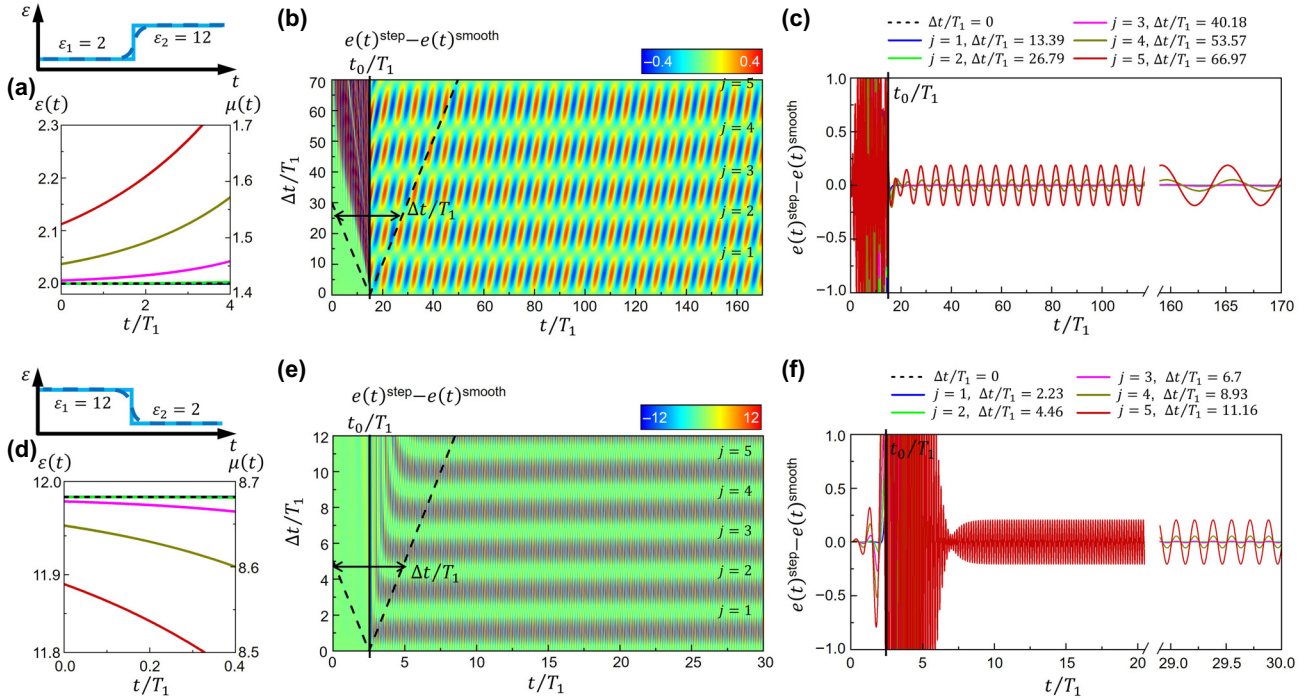


FIG. 6. (a),(d) Smooth transition of $\varepsilon(t)$ and $\mu(t)$ (solid lines) and time interface (dotted line) for rising and falling functions of $\varepsilon(t)$ and $\mu(t)$, respectively. (b),(e) Theoretical results for the difference between the electric modal amplitudes produced by the step and smooth values of $\varepsilon(t)$ and $\mu(t)$, for rising and falling functions, respectively. (c),(f) Theoretical results for the values of $\Delta t/T_1$ from (b),(e) where $e^{\text{step}} - e^{\text{smooth}}$ is at a minimum when $t > t_0$. Vertical solid lines in all panels correspond to the value when the time interface is applied. The diagonal dashed lines from (b),(e) delimit the duration of the transitions of $\varepsilon(t)$ and $\mu(t)$ when implementing a smooth transition.

APPENDIX D: NUMERICAL SIMULATION OF A SMOOTH TRANSITION: FROM LARGER TO SMALLER VALUES OF $\varepsilon(t)$ AND $\mu(t)$

The results from Fig. 7 correspond to the case discussed for Fig. 4(c) in the main text. As observed, there is good agreement between the results.

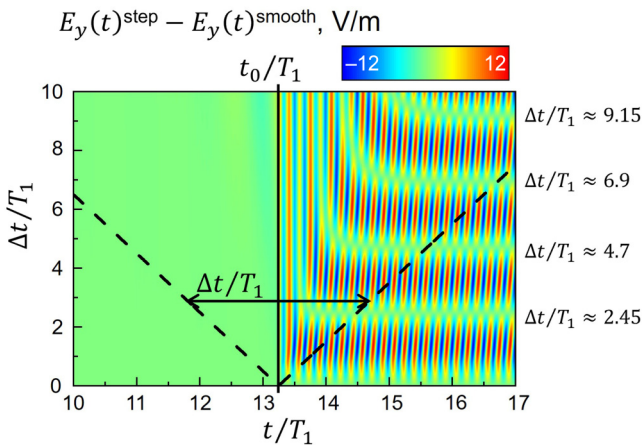


FIG. 7. Numerical simulation representing $E_y^{\text{step}} - E_y^{\text{smooth}}$ for the case discussed in Fig. 4(c).

APPENDIX E: CHANGING ε_1 AND ε_2

For completeness, in this section, we provide another example with different values of $\varepsilon(t)$ and $\mu(t)$. We consider a smooth transition and a time interface modified with $\varepsilon_1 = 2$ to $\varepsilon_2 = 6$ and from $\varepsilon_1 = 6$ to $\varepsilon_2 = 2$, with the same $Z = 0.85$, $\zeta = 20$, and time $t_0 = 53$ ns as the time when the time interface is applied.

With this configuration, the theoretical results for the modal amplitudes for the smooth transition of $\varepsilon(t)$ and $\mu(t)$, $e(t)^{\text{smooth}}$, as a function of $\Delta t/T_1$ and time t/T_1 are shown in Figs. 8(a), 8(b), 9(a), and 9(b) along with the values of $e(t)^{\text{step}} - e(t)^{\text{smooth}}$ in Figs. 8(c), 8(d), 9(c), and 9(d). For the case of rising $\varepsilon(t)$ and $\mu(t)$ (Fig. 8), T_1 and f_1 are same as those presented in Figs. 2 and 3 of the main text. For the case of falling $\varepsilon(t)$ and $\mu(t)$ (Fig. 9), $T_1 = 2.0$ ns, corresponding to a frequency of about 0.5 GHz and $t_0/T_1 = 26.5$. As expected, as we now have a smaller contrast ($|\varepsilon_1 - \varepsilon_2|$) between the two values of permittivity, there are only two values of $\Delta t/T_1$ that satisfy $e^{\text{step}} - e^{\text{smooth}} \approx 0$, considering the same range of Δt as in Figs. 2–4.

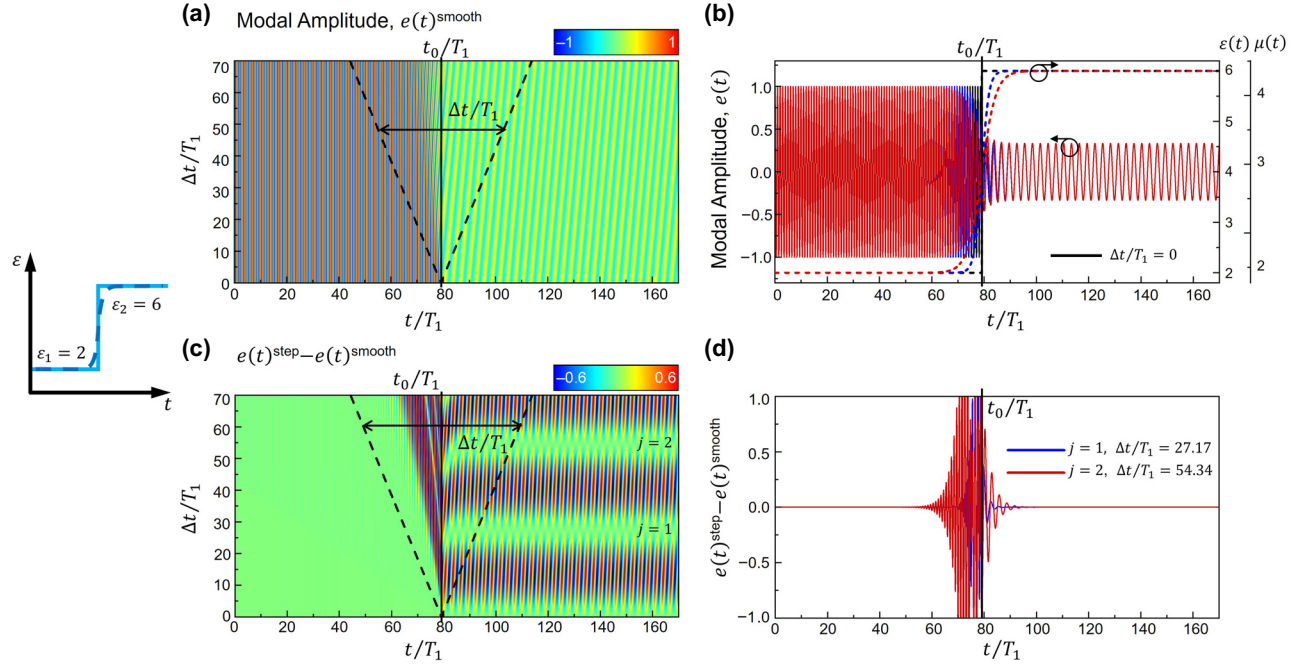


FIG. 8. Theoretical results for the electric modal amplitude (a) as a function of rising time of the smooth transition and time using $\varepsilon_1 = 2$, $\varepsilon_2 = 6$, and time-independent $Z = 0.85$. (b) Electric modal amplitudes related to the time interface (black solid line) and smooth transitions for several values of transition times that fulfil Eq. (10) (solid lines) with the related dependencies of the refractive index (dashed lines). (c) Difference between the electric modal amplitudes produced by step and smooth $\varepsilon(t)$ and $\mu(t)$. (d) Theoretical results for values of $\Delta t/T_1$ from (c) where $e^{\text{step}} - e^{\text{smooth}} \approx 0$.

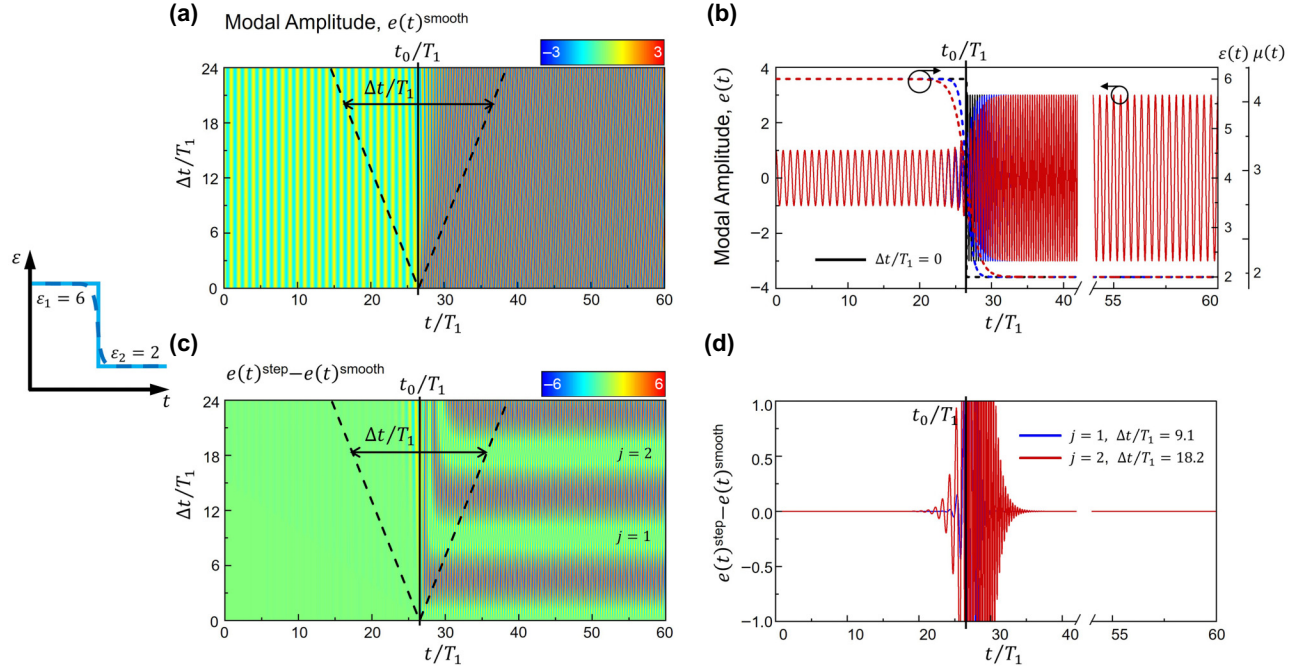


FIG. 9. Same caption as for Fig. 8 but now considering $\varepsilon_1 = 6$, $\varepsilon_2 = 2$.

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